Ye *Boundary Value Problems* (2020) 2020:146
https://doi.org/10.1186/s13661-020-01444-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On locally superquadratic Hamiltonian systems with periodic potential

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Abstract

In this paper, we study the second-order Hamiltonian systems

$$\ddot{u} - L(t)u + \nabla W(t,u) = 0,$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, L and W depend periodically on t, 0 lies in a spectral gap of the operator $-d^2/dt^2 + L(t)$ and $W(t,x)$ is locally superquadratic. Replacing the common superquadratic condition that $\lim_{|x|\to\infty} \frac{W(t,x)}{|x|^2} = +\infty$ uniformly in $t \in \mathbb{R}$ by the local condition that $\lim_{|x|\to\infty} \frac{W(t,x)}{|x|^2} = +\infty$ a.e. $t \in J$ for some open interval $J \subset \mathbb{R}$, we prove the existence of one nontrivial homoclinic soluiton for the above problem.

MSC: 34C37; 37J45

Keywords: Homoclinic solutions; Hamiltonian systems; Strongly indefinite functional; Locally superquadratic; Linking theorem

1 Introduction and main results

Consider the second-order Hamiltonian systems

$$\ddot{u} - L(t)u + \nabla W(t,u) = 0, \tag{1.1}$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ satisfies the following basic conditions:

(W1) W is T-periodic in t and there exist constants $C_0 > 0$ and $p > 2$ such that

$$\bigl|\nabla W(t,x)\bigr| \le C_0\bigl(1 + |x|^{p-1}\bigr), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

(W2) $\nabla W(t,x) = o(|x|)$ as $x \to 0$ uniformly in t and $W(t,x) \ge 0$ for all (t,x).

Usually, a solution u of system (1.1) is said to be homoclinic to 0 if $u(t) \to 0$ as $|t| \to \infty$. Furthermore, if $u(t) \not\equiv 0$, then u is called a nontrivial homoclinic solution.

During the past two decades, there has been a remarkable amount of progress in the study of homoclinic motions of Hamiltonian systems, with many new ideas and methods being introduced; see, for instance, [2, 3, 5, 6, 8, 12–15, 20, 22–26] for results concerning the second-order systems, and [4, 7, 17–19] for the first-order systems. For

(1.1) with periodic potential, most results are obtained under the assumptions that $L(t)$ is positive definite for all $t \in \mathbb{R}$ and $W(t,x)$ is globally superquadratic in x; see, e.g., [2, 3, 5, 8, 10, 12, 14, 22, 24, 25]. A common feature of this work is that the following assumption, which is originally due to Ambrosetti and Rabinowitz [1], is imposed on the nonlinearity:

(AR) $\exists\, \mu > 2$ such that $0 < \mu W(t,x) \le (\nabla W(t,x), x)$ for all $(t,x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\})$.

It is well known that the crucial role of (AR) is to verify the mountain-pass geometry of the corresponding functional and also to ensure the boundedness of Palais–Smale (PS) sequences. It is also well known that many functions such as

$$W(t,x) = |x|^2 \ln\bigl(1 + |x|\bigr)$$

do not satisfy (AR). In recent years, many more natural conditions than (AR) have been proposed in the study of periodic or homoclinic solutions for superquadratic Hamiltonian systems. Ding and Lee [8] consider (1.1) with periodicity. Instead of (AR), they assume that

(S1) $W(t,x)/|x|^2 \to +\infty$ as $|x| \to \infty$ uniformly in t;

(S2) $\widetilde{W}(t,x) := \frac{1}{2}(\nabla W(t,x), x) - W(t,x) > 0$ if $x \neq 0$, and there exist $\varepsilon \in (0,1)$ and $r > 0$ such that $(\nabla W(t,x), x) \le c\widetilde{W}(t,x)|x|^{2-\varepsilon}$ for all $t \in \mathbb{R}$ and $|x| \ge r$;

and prove the existence of infinitely many geometrically distinct solutions for both asymptotically quadratic and superquadratic cases. See also [12, 22, 25] for the related results.

If 0 lies in a spectral gap of the operator $-d^2/dt^2 + L(t)$, that is,

(L) $-\Lambda_1 := \sup[\sigma(A) \cap (-\infty, 0)] < 0 < \Lambda_2 := \inf[\sigma(A) \cap (0, +\infty)]$, where $A := -d^2/dt^2 + L(t)$ and σ denotes the spectrum,

then the negative space E^- of the quadratic form in the energy functional given by (2.5) is infinite dimensional. For this reason, we say that the problem is strongly indefinite. Up to now, few papers deal with this situation; see [2, 3, 10, 24]. Besides the conditions (L), (AR) and the restrictive assumption of $W(t,x) \ge c|x|^{\mu}$ and $|\nabla W(t,x)| \le c|x|^{\mu-1}$ for all (t,x), Arioli and Szulkin [2] prove the existence of one nontrivial homoclinic orbit by constructing subharmonics and passing to the limit. Recently, Chen [3] proves the existence of one nontrivial ground state homoclinic orbit under hypotheses (L), (W2), (S1) and

(S3) $\widetilde{W}(t,x) > 0$ if $x \in \mathbb{R}^N \setminus \{0\}$, and there exist $c, r > 0$ and $k < 1$ such that

$$\frac{|\nabla W(t,x)|^k}{|u|^k} \le c\widetilde{W}(t,x), \quad |x| \ge r.$$

Inspired by the works mentioned above and the recent paper [21], we are interested in the case where 0 lies in a gap of $\sigma(A)$ and $W(t,x)$ is locally superquadratic, i.e., it is allowed to be superquadratic at some $t \in \mathbb{R}$ and asymptotically quadratic at other $t \in \mathbb{R}$. The main ingredient is the observation that even in the strongly indefinite case, all Cerami sequences of the energy functional are bounded. Therefore, the existence of one homoclinic solution is proved by using the generalized linking theorem of Li and Szulkin (see [11]). Precisely, we further weaken (S1) to the following hypotheses:

(W3) There exists an open interval $J \subset \mathbb{R}$ such that $\lim_{|x|\to\infty} \frac{W(t,x)}{|x|^2} = +\infty$ a.e. $t \in J$.

(W4) $\widetilde{W}(t,x) \geq 0$ for all (t,x), and there exist $C_1 > 0$, $\delta \in (0, \Lambda_0)$ ($\Lambda_0 := \min\{\Lambda_1, \Lambda_2\}$), $\sigma \in (0,1)$ and $k \in (1, 2/(1-\sigma)]$ such that

$$\frac{|\nabla W(t,x)|}{|x|} \geq \Lambda_0 - \delta \quad \text{implies} \quad \left(\frac{|\nabla W(t,x)|}{|x|^\sigma}\right)^k \leq C_1 \widetilde{W}(t,x).$$

Our main result reads as follows.

Theorem 1.1 *Assume that* (L) *and* (W1)–(W4) *are satisfied. Then system* (1.1) *has at least one nontrivial homoclinic solution.*

Remark 1.1 (i) Comparing with the results of [2, 3, 10, 24], one advantage of Theorem 1.1 is that the globally superquadratic condition (cf. (AR) or (S1)) is replaced by the local one (W3). Thus our result applies to more general situations. Typical examples, which match our assumptions (W1)–(W4), but satisfying none of (AR), (S1), (S2) and (S3), are the following:

$$W(t,x) = \left(\sin\frac{2\pi t}{T} + \left|\sin\frac{2\pi t}{T}\right|\right)\bigl(2\bigl(|x|^2 - 1\bigr)\ln\bigl(1+|x|\bigr) + |x|\bigl(2-|x|\bigr)\bigr) \tag{1.2}$$

and

$$W(t,x) = |x|^{2+\alpha(t)}\left[1 - \frac{1}{\ln(e+|x|^2)}\right], \tag{1.3}$$

where $\alpha \in C(\mathbb{R},\mathbb{R})$, $F \subset \mathbb{R}$ is a closed set such that $\alpha(t) = 0$ for $t \in F$ and $\alpha(t) \in (0,2)$ for $t \in \mathbb{R}\backslash F$. One can easily check this fact for (1.2) by noting that

$$\nabla W(t,x) = \left(\sin\frac{2\pi t}{T} + \left|\sin\frac{2\pi t}{T}\right|\right)4x\ln\bigl(1+|x|\bigr),$$
$$\widetilde{W}(t,x) = \left(\sin\frac{2\pi t}{T} + \left|\sin\frac{2\pi t}{T}\right|\right)\bigl(|x|^2 + 2\ln\bigl(1+|x|\bigr) - 2|x|\bigr),$$

and for (1.3) by noting that

$$\nabla W(t,x) = \bigl(2+\alpha(t)\bigr)|x|^{\alpha(t)}x\left[1 - \frac{1}{\ln(e+|x|^2)}\right] + \frac{2|x|^{2+\alpha(t)}x}{(e+|x|^2)[\ln(e+|x|^2)]^2},$$
$$\widetilde{W}(t,x) = \frac{\alpha(t)}{2}|x|^{\alpha(t)+2}\left[1 - \frac{1}{\ln(e+|x|^2)}\right] + \frac{|x|^{4+\alpha(t)}}{(e+|x|^2)[\ln(e+|x|^2)]^2}.$$

In addition, we point out that the function of (1.3) is asymptotically quadratic for $t \in F$ and superquadratic for $t \in \mathbb{R}\backslash F$.

(ii) Another advantage of this paper is that our argument is simpler. In [3], Chen discusses a family of perturbed functions

$$\varphi_\lambda(u) = \frac{1}{2}\|u^+\|^2 - \lambda\left(\frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} W(t,x)\,dt\right), \quad \lambda \in [1,2]$$

and apply a variant generalized weak linking theorem for strongly indefinite functionals developed by Schechter and Zou (see [16]). This approach is not very satisfactory, since working with a family of perturbed functionals makes things unnecessary complicated. In the present paper we will prove Theorem 1.1 by directly applying the usual variational method to the energy functional φ. The key point in our proof is that, although φ may has unbounded (PS) sequences, we can prove that all Cerami sequences of φ are bounded (see Lemma 2.4 below), and hence Theorem 1.1 follows directly from the generalized linking theorem (see [11]).

(iii) Homoclinics for locally superquadratic Hamiltonian systems has been studied in Wang [22] for periodic case. However, this paper *only* deals with the definite case, which is much simpler than the strongly indefinite case considered in the present paper.

Notation: "$\to$' and "$\rightharpoonup$", respectively, denote the strong convergence and the weak convergence. C and C_i $(i = 1, 2, \ldots)$ denote various positive constants which may vary from place to place.

2 Proof of Theorem 1.1

Let $A := -d^2/dt^2 + L(t)$. Then A is self-adjoint in $L^2(\mathbb{R}, \mathbb{R}^N)$ with domain $\mathcal{D}(A) = H^2(\mathbb{R}, \mathbb{R}^N)$. Let $\{\mathcal{E}(\lambda) : -\infty \leq \lambda \leq +\infty\}$, $|A|$ and $|A|^{1/2}$, respectively, be the spectral family, the absolute value of A and the square root of $|A|$. Take $U := \mathrm{id} - \mathcal{E}(0) - \mathcal{E}(0-)$. Then U commutes with A, $|A|$ and $|A|^{1/2}$, and $A = U|A|$ is the polar decomposition of A (see [9, Theorem IV 3.3]). Set

$$E = \mathcal{D}\bigl(|A|^{1/2}\bigr), \qquad E^- = \mathcal{E}(0-)E \quad \text{and} \quad E^+ = \bigl[\mathrm{id} - \mathcal{E}(0)\bigr]E.$$

For every $u \in E$, we see that

$$u^- := \mathcal{E}(0-)u \in E^-, \qquad u^+ := \bigl[\mathrm{id} - \mathcal{E}(0)\bigr]u \in E^+, \qquad u = u^- + u^+ \tag{2.1}$$

and

$$Au^- = -|A|u^-, \qquad Au^+ = |A|u^+. \tag{2.2}$$

Define on E the inner product and the norm

$$(u, v) = \bigl(|A|^{1/2}u, |A|^{1/2}v\bigr)_{L^2}, \qquad \|u\| = \bigl\||A|^{1/2}u\bigr\|_2, \tag{2.3}$$

where $(\cdot,\cdot)_{L^2}$ and $\|\cdot\|_s$ denote the inner product of $L^2(\mathbb{R}, \mathbb{R}^N)$ and the norm of $L^s(\mathbb{R}, \mathbb{R}^N)$ $(2 \leq s \leq +\infty)$, respectively. Then E is a Hilbert space. By (L), $E = H^1(\mathbb{R}, \mathbb{R}^N)$ with equivalent norms and E is continuously embedded in $L^s(\mathbb{R}, \mathbb{R}^N)$ for $2 \leq s \leq +\infty$. It is easy to check that E has the following decomposition $E = E^- \oplus E^+$ orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$. Furthermore, it follows from the definitions of Λ_1, Λ_2 and Λ_0 that

$$\Lambda_1 \bigl\|u^-\bigr\|_2^2 \leq \bigl\|u^-\bigr\|^2, \qquad \Lambda_2 \bigl\|u^+\bigr\|_2^2 \leq \bigl\|u^+\bigr\|^2, \qquad \Lambda_0 \|u\|_2^2 \leq \|u\|^2 \tag{2.4}$$

for all $u \in E$.

We shall apply the generalized linking theorem of Li and Szulkin to prove Theorem 1.1. First we introduce some notations. Let E be a real Hilbert space with $E = E^- \oplus E^+$ and

$E^- \perp E^+$. For $\varphi \in C^1(E, \mathbb{R}^N)$, φ is said to be weakly sequentially lower semi-continuous if $u_n \rightharpoonup u$ implies $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n)$, and φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty} \langle \varphi'(u_n), v\rangle = \langle \varphi'(u), v\rangle$ for any $v \in E$.

Theorem 2.1 (see [11, Theorem 2.1]) *Let $(E, \|\cdot\|)$ be a real Hilbert space with $E = E^- \oplus E^+$, and let $\varphi \in C^1(E, \mathbb{R})$ of the form*

$$\varphi(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\right) - \psi(u),$$

where $u = u^- + u^+ \in E^- \oplus E^+$. Suppose that

(i) *$\psi \in C^1(E, \mathbb{R})$ is bounded from below, weakly sequentially lower semi-continuous and ψ' is weakly sequentially continuous;*

(ii) *there exist $e \in E^+$ with $\|e\| = 1$ and $r > \rho > 0$ such that $\alpha := \inf \varphi(S_\rho^+) > \sup \varphi(\partial Q)$, where $S_\rho^+ = \{u \in E^+ : \|u\| = \rho\}$ and $Q = \{v + se : v \in E^-, s \ge 0, \|v + se\| \le r\}$.*

Then, for some $c > \alpha$, there is a sequence $(u_n) \subset E$ such that

$$\varphi(u_n) \to c, \qquad \left\|\varphi'(u_n)\right\|\left(1 + \|u_n\|\right) \to 0.$$

Such a sequence is called a Cerami sequence on level c, or a $(C)_c$ sequence.

Now we define the functional $\varphi : E \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}} \left[|\dot{u}|^2 + \left(L(t)u, u\right)\right] dt - \int_{\mathbb{R}} W(t, u)\, dt. \tag{2.5}$$

In view of (L) and (W1)–(W2), $\varphi \in C^1(E, \mathbb{R})$ and

$$\left\langle \varphi'(u), v\right\rangle = \int_{\mathbb{R}} \left[(\dot{u}, \dot{v}) + \left(L(t)u, v\right)\right] dt - \int_{\mathbb{R}} \left(\nabla W(t, u), v\right) dt, \quad u, v \in E.$$

Combining (2.1)–(2.3), we have

$$\varphi(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\right) - \int_{\mathbb{R}} W(t, u)\, dt \tag{2.6}$$

and

$$\left\langle \varphi'(u), v\right\rangle = \left(u^+, v\right) - \left(u^-, v\right) - \int_{\mathbb{R}} \left(\nabla W(t, u), v\right) dt, \quad \forall u, v \in E.$$

A standard argument shows that the critical points of φ are homoclinic solutions of (1.1) (see [5, 15]).

Let

$$\psi(u) = \int_{\mathbb{R}} W(t, u)\, dt.$$

Obviously, $\psi \ge 0$ and it follows from Fatou's lemma that ψ is weakly sequentially lower semi-continuous. By (W1) and (W2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|\nabla W(t, x)\right| \le \varepsilon|x| + C_\varepsilon|x|^{p-1}, \qquad \left|W(t, x)\right| \le \varepsilon|x|^2 + C_\varepsilon|x|^p \tag{2.7}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Since $u_n \rightharpoonup u$ yields $u_n \to u$ in $L^s_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ for all $s \in [1,\infty]$, it is easy to check that ψ' is weakly sequentially continuous. Thus (i) of Theorem 2.1 is satisfied.

Next we study the linking structure of φ. Without loss of generality, we may suppose that $J \subset \mathbb{R}$ is bounded. Choose $e \in C_0^\infty(J,\mathbb{R}^N)$ such that

$$\|e^+\|^2 - \|e^-\|^2 = \int_{\mathbb{R}} \big[|\dot{e}|^2 + \big(L(t)e,e\big)\big]\,dt = \int_J \big[|\dot{e}|^2 + \big(L(t)e,e\big)\big]\,dt \geq 1. \tag{2.8}$$

Lemma 2.1 *Let* (L) *and* (W1)–(W2) *be satisfied. Then there is* $\rho > 0$ *such that* $\alpha := \inf\varphi(S_\rho^+) > 0$, *where* $S_\rho^+ = \{u \in E^+ : \|u\| = \rho\}$.

Proof Since, by (2.7) and the Sobolev embedding inequality,

$$\psi(u) \leq \varepsilon\|u\|_2^2 + C_\varepsilon\|u\|_p^p = o\big(\|u\|^2\big) \quad \text{as } n \to \infty,$$

the conclusion follows from the form of φ (see (2.6)). □

Lemma 2.2 *Let* (L) *and* (W2)–(W3) *be satisfied. Then* $\sup\varphi(E^- \oplus \mathbb{R}^+e^+) < +\infty$ *and there exists* $R_e > 0$ *such that*

$$\varphi(u) \leq 0, \quad u \in E^- \oplus \mathbb{R}^+e^+, \|u\| \geq R_e.$$

Proof It is sufficient to show that $\varphi(u) \to -\infty$ as $\|u\| \to \infty$, $u \in E^- \oplus \mathbb{R}^+e^+$. Arguing indirectly, assume that, for some sequence $\{v_n + \theta_n e^+\} \subset E^- \oplus \mathbb{R}^+e^+$ with $\|v_n + \theta_n e^+\| \to \infty$ as $n \to \infty$, there is $C_2 > 0$ such that

$$\varphi\big(v_n + \theta_n e^+\big) \geq -C_2, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

Let $w_n = (v_n + \theta_n e^+)/\|v_n + \theta_n e^+\| = w_n^- + s_n e^+$. Then $\|w_n\| = 1$, and going if necessary to a subsequence, we may assume that

$$\begin{aligned} &s_n \to s_0, \qquad w_n^- \rightharpoonup w^- \quad \text{in } E, w_n^-(t) \to w^-(t) \text{ a.e. } t \in \mathbb{R}, \\ &w_n^- \to w^- \quad \text{in } L^2\big(J,\mathbb{R}^N\big) \quad \text{and} \quad \widehat{w_n^-} \rightharpoonup \widehat{w^-} \quad \text{in } L^2\big(\mathbb{R},\mathbb{R}^N\big). \end{aligned} \tag{2.10}$$

By (2.9), we have

$$-\frac{C_2}{\|v_n + \theta_n e^+\|^2} \leq \frac{\varphi(v_n + \theta_n e^+)}{\|v_n + \theta_n e^+\|^2} = \frac{s_n^2}{2}\big\|e^+\big\|^2 - \frac{1}{2}\big\|w_n^-\big\|^2 - \int_{\mathbb{R}} \frac{W(t, v_n + \theta_n e^+)}{\|v_n + \theta_n e^+\|^2}\,dt. \tag{2.11}$$

Since $W \geq 0$, it follows that

$$\frac{1}{2}\big\|w_n^-\big\|^2 \leq \frac{s_n^2}{2}\big\|e^+\big\|^2 + o(1),$$

and then

$$\frac{1}{2} = \frac{1}{2}\big(\big\|w_n^-\big\|^2 + s_n^2\big\|e^+\big\|^2\big) \leq s_n^2\big\|e^+\big\|^2 + o(1) = s_0^2\big\|e^+\big\|^2 + o(1),$$

which implies that $s_0 \neq 0$. We claim that

$$\left(w^- + s_0 e^+\right)|_J \neq 0. \tag{2.12}$$

Otherwise $(w^- + s_0 e^+)|_J = 0$. Hence, using the T-periodicity of $L(t)$, (2.10) and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \int_J \left(L(t)\left(w_n^- + s_n e^+\right), \left(w_n^- + s_n e^+\right)\right) dt &= \int_J \left(L(t)\left(w^- + s_0 e^+\right), \left(w^- + s_0 e^+\right)\right) dt + o(1) \\ &= o(1) \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \int_J \left|\widehat{w_n^-} + s_n \widehat{e^+}\right|^2 dt &= \int_J \left|\widehat{w_n^-} + s_0 \widehat{e^+}\right|^2 dt + o(1) \\ &= \int_J \left|\widehat{w_n^-} - \widehat{w^-}\right|^2 dt + o(1) \\ &= \int_J \left(\left|\widehat{w_n^-}\right|^2 - \left|\widehat{w^-}\right|^2\right) dt + o(1). \end{aligned} \tag{2.14}$$

Similar to (2.13), we also obtain

$$\int_J \left(L(t)\left(w_n^- - s_n e^-\right), \left(w_n^- - s_n e^-\right)\right) dt = \int_J \left(L(t)\left(w^- - s_0 e^-\right), \left(w^- - s_0 e^-\right)\right) dt + o(1). \tag{2.15}$$

Now, combining (2.13)–(2.15), (2.11), (2.8) and using the fact $e|_{\mathbb{R}\setminus J} \equiv 0$, we deduce that

$$\begin{aligned} 0 &\leq 2 \int_{\mathbb{R}} \frac{W(t, v_n + \theta_n e^+)}{\|v_n + \theta_n e^+\|^2} dt \\ &= s_n^2 \left\|e^+\right\|^2 - \left\|w_n^-\right\|^2 + \frac{2C_2}{\|v_n + \theta_n e^+\|^2} \\ &= \int_{\mathbb{R}} \left[\left|\widehat{w_n^-} + s_n \widehat{e^+}\right|^2 + \left(L(t)\left(w_n^- + s_n e^+\right), \left(w_n^- + s_n e^+\right)\right)\right] dt + o(1) \\ &= \int_J \left[\left|\widehat{w_n^-} + s_n \widehat{e^+}\right|^2 + \left(L(t)\left(w_n^- + s_n e^+\right), \left(w_n^- + s_n e^+\right)\right)\right] dt \\ &\quad + \int_{\mathbb{R}\setminus J} \left[\left|\widehat{w_n^-} + s_n \widehat{e^+}\right|^2 + \left(L(t)\left(w_n^- + s_n e^+\right), \left(w_n^- + s_n e^+\right)\right)\right] dt + o(1) \\ &= \int_J \left(\left|\widehat{w_n^-}\right|^2 - \left|\widehat{w^-}\right|^2\right) dt + o(1) \\ &\quad + \int_{\mathbb{R}} \left[\left|\widehat{w_n^-} - s_n \widehat{e^-}\right|^2 + \left(L(t)\left(w_n^- - s_n e^-\right), \left(w_n^- - s_n e^-\right)\right)\right] dt \\ &\quad - \int_J \left[\left|\widehat{w_n^-} - s_n \widehat{e^-}\right|^2 + \left(L(t)\left(w_n^- - s_n e^-\right), \left(w_n^- - s_n e^-\right)\right)\right] dt \\ &= \int_J \left(\left|\widehat{w_n^-}\right|^2 - \left|\widehat{w^-}\right|^2\right) dt - \left\|w_n^- - s_n e^-\right\|^2 \\ &\quad - \int_J \left|\widehat{w_n^-} - s_n \widehat{e^-}\right|^2 dt - \int_J \left(L(t)\left(w^- - s_0 e^-\right), \left(w^- - s_0 e^-\right)\right) dt + o(1) \end{aligned}$$

$$
\begin{aligned}
&= -\left\|w_n^- - s_n e^-\right\|^2 - \int_J \left[\left|\hat{\dot{w^-}} - s_0 \hat{\dot{e^-}}\right|^2 + \left(L(t)\left(w^- - s_0 e^-\right), \left(w^- - s_0 e^-\right)\right)\right] dt + o(1) \\
&= -\left\|w_n^- - s_n e^-\right\|^2 - s_0^2 \int_J \left[|\dot{e}|^2 + \left(L(t)e, e\right)\right] dt + o(1) \\
&\leq -s_0^2 + o(1),
\end{aligned}
$$

which is a contradiction. Consequently, it follows from (2.11), (2.12), (W2)–(W3) and Fatou's lemma that

$$
\begin{aligned}
0 &\leq \limsup_{n\to\infty} \left[\frac{s_n^2}{2}\left\|e^+\right\|^2 - \frac{1}{2}\left\|w_n^-\right\|^2 - \int_{\mathbb{R}} \frac{W(t, v_n + \theta_n e^+)}{\|v_n + \theta_n e^+\|^2}\, dt \right] \\
&\leq \frac{s_0^2}{2}\left\|e^+\right\|^2 - \liminf_{n\to\infty} \int_{\mathbb{R}} \frac{W(t, v_n + \theta_n e^+)}{|v_n + \theta_n e^+|^2} \left|w_n^- + s_n e^+\right|^2 dt \\
&\leq \frac{s_0^2}{2}\left\|e^+\right\|^2 - \int_J \liminf_{n\to\infty} \frac{W(t, v_n + \theta_n e^+)}{|v_n + \theta_n e^+|^2} \left|w^- + s_0 e^+\right|^2 dt \\
&= -\infty,
\end{aligned}
$$

a contradiction. □

Corollary 2.1 *Let* (L) *and* (W2)–(W3) *be satisfied and* $\rho > 0$ *be given by Lemma* 2.1. *Then there exists* $r > \rho$ *such that* $\sup \varphi(\partial Q) < 0$, *where* $Q = \{v + se^+ : v \in E^-, s \geq 0, \|v + se^+\| \leq r\}$.

Combining Lemma 2.1, Corollary 2.1 and Theorem 2.1, we have the following.

Lemma 2.3 *Assume that* (L) *and* (W1)–(W3) *are satisfied. Then there exist a constant* $c > 0$ *and a sequence* $(u_n) \subset E$ *such that*

$$
\varphi(u_n) \to c, \qquad \left\|\varphi'(u_n)\right\|\left(1 + \|u_n\|\right) \to 0 \quad \text{as } n \to \infty. \tag{2.16}
$$

Lemma 2.4 *Assume that* (L) *and* (W1)–(W4) *are satisfied. Then the sequence* (u_n) *obtained in Lemma* 2.3 *is bounded.*

Proof Arguing by contradiction, suppose that $\|u_n\| \to \infty$ as $n \to \infty$ and set $w_n = u_n/\|u_n\|$. Then $\|w_n\| = 1$. By (2.16), we have

$$
\begin{aligned}
o(1) &= \left\langle \varphi'(u_n), u_n^+ \right\rangle = \left\|u_n^+\right\|^2 - \int_{\mathbb{R}} \left(\nabla W(t, u_n), u_n^+\right) dt, \\
o(1) &= \left\langle \varphi'(u_n), u_n^- \right\rangle = -\left\|u_n^-\right\|^2 - \int_{\mathbb{R}} \left(\nabla W(t, u_n), u_n^-\right) dt.
\end{aligned}
$$

Since $\|w_n^+\|^2 + \|w_n^-\|^2 = \|w_n\|^2 = 1$, one has

$$
1 + o(1) = \int_{\mathbb{R}} \frac{(\nabla W(t, u_n), u_n^+ - u_n^-)}{\|u_n\|^2}\, dt. \tag{2.17}
$$

Letting $\Omega_n = \{t \in \mathbb{R} : |\nabla W(t,u_n)| \leq (\Lambda_0 - \delta)|u_n|\}$, we obtain using the relation $(w^+, w^-) = 0$, (2.4) and the Hölder inequality

$$\begin{aligned}\int_{\Omega_n} \frac{|\nabla W(t,u_n)||u_n^+ - u_n^-|}{\|u_n\|^2}\,dt &\leq \int_{\Omega_n} \frac{|\nabla W(t,u_n)|}{|u_n|}\big|w_n^+ - w_n^-\big||w_n|\,dt \\ &\leq (\Lambda_0 - \delta)\|w_n\|_2^2 \\ &\leq (\Lambda_0 - \delta)\frac{1}{\Lambda_0}\|w_n\|^2 \\ &\leq 1 - \frac{\delta}{\Lambda_0}. \end{aligned} \tag{2.18}$$

Moreover, it follows from (2.16) that

$$\int_{\mathbb{R}} \widetilde{W}(t,u_n)\,dt = \varphi(u_n) - \frac{1}{2}\big\langle \varphi'(u_n), u_n\big\rangle \leq C, \quad \forall n \in \mathbb{N}.$$

Combining this with (W4) and Hölder's inequality, we deduce that

$$\begin{aligned}&\left|\int_{\mathbb{R}\setminus\Omega_n} \frac{(\nabla W(t,u_n), u_n^+ - u_n^-)}{\|u_n\|^2}\,dt\right| \\ &\quad\leq \frac{1}{\|u_n\|^{1-\sigma}}\left[\int_{\mathbb{R}\setminus\Omega_n}\left(\frac{|\nabla W(t,u_n)|}{|u_n|^\sigma}\right)^k dt\right]^{\frac{1}{k}} \big\|w_n^+ - w_n^-\big\|_{\frac{k(1+\sigma)}{k-1}} \|w_n\|^\sigma_{\frac{k(1+\sigma)}{k-1}} \\ &\quad\leq \frac{C}{\|u_n\|^{1-\sigma}}\left(\int_{\mathbb{R}\setminus\Omega_n}\widetilde{W}(t,u_n)\,dt\right)^{\frac{1}{k}} = o(1).\end{aligned} \tag{2.19}$$

Hence, by (2.17), (2.18) and (2.19),

$$1 + o(1) = \int_{\mathbb{R}} \frac{(\nabla W(t,u_n), u_n^+ - u_n^-)}{\|u_n\|^2}\,dt \leq 1 - \frac{\delta}{\Lambda_0} + o(1),$$

a contradiction. □

Proof of Theorem 1.1 According to Lemmas 2.3 and 2.4, there is a bounded $(C)_c$ sequence (u_n) with $c > 0$. Since (u_n) is bounded, there exists $M > 0$ such that

$$\sqrt{\Lambda_0}\|u_n\|_2 \leq \|u_n\| \leq M, \quad \forall n \in \mathbb{N}. \tag{2.20}$$

By (2.7), for $\varepsilon = \frac{c\Lambda_0}{2M^2}$, there is $C_3 > 0$ such that

$$\big|\widetilde{W}(t,x)\big| \leq \frac{c\Lambda_0}{2M^2}|x|^2 + C_3|x|^p, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{2.21}$$

If (u_n) is vanishing, that is, for each $r > 0$, $\lim_{n\to\infty}\sup_{a\in\mathbb{R}}\int_{a-r}^{a+r}|u_n|^2\,dt = 0$, then, by [25, Lemma 2.3], $u_n \to 0$ in $L^s(\mathbb{R},\mathbb{R}^N)$ for $2 \leq s \leq +\infty$. Hence, using (2.20) and (2.21), we deduce that

$$c + o(1) = \varphi(u_n) - \frac{1}{2}\big\langle \varphi'(u_n), u_n\big\rangle$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \widetilde{W}(t,u_n)\,dt \\
&\leq \frac{c\Lambda_0}{2M^2}\int_{\mathbb{R}} |u_n|^2\,dt + C_3\int_{\mathbb{R}} |u_n|^p\,dt \\
&\leq \frac{c}{2} + o(1),
\end{aligned}
$$

a contradiction.

Hence (u_n) is nonvanishing, i.e., there are $r, \sigma > 0$ and $(a_n) \subset \mathbb{Z}$ such that

$$\liminf_{n\to\infty} \int_{a_n-r}^{a_n+r} |u_n|^2\,dt \geq \frac{\sigma}{2}.$$

Setting $\tilde{u}_n(t) = u_n(t + a_n T)$, we have

$$\liminf_{n\to\infty} \int_{-r}^{r} |\tilde{u}_n|^2\,dt \geq \frac{\sigma}{2}. \tag{2.22}$$

Noticing L and W are T-periodic in t, we get $\|\tilde{u}_n\| = \|u_n\|$, $\varphi(\tilde{u}_n) \to c$ and $\|\varphi'(\tilde{u}_n)\|(1 + \|\tilde{u}_n\|) \to 0$ as $n \to \infty$. Passing to a subsequence, we assume that $\tilde{u}_n \rightharpoonup \tilde{u}$, $\varphi'(\tilde{u}) = 0$ and $\tilde{u} \neq 0$ by (2.22). This completes the proof. □

Acknowledgements
The author would like to thank the handling editors and the anonymous reviewers.

Funding
This work is supported by the National Natural Science Foundation of China (No. 11971393) and the Science and Technology Research Program of Chongqing Municipal Education Commission (No. KJQN201900501).

Availability of data and materials
Not applicable.

Competing interests
The author declares that there are no competing interests.

Authors' contributions
The author conceived of the study, drafted the manuscript, and approved the final manuscript.

Publisher's Note

Received: 10 January 2019 Accepted: 23 August 2020 Published online: 01 September 2020

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