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The applications of Sobolev inequalities in proving the existence of solution of the quasilinear parabolic equation

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Abstract

The aim of this paper is to show some applications of Sobolev inequalities in partial differential equations. With the aid of some well-known inequalities, we derive the existence of global solution for the quasilinear parabolic equations. When the blow-up occurs, we derive the lower bound of the blow-up solution.

MSC: 26D15; 26A51; 35K20; 35K55

Keywords: Sobolev inequalities; Partial differential equations; Global solution

1 Introduction

Sobolev inequalities, also called Sobolev imbedding theorems, belong to the issues of focus of current research, and play an important role in reality. Inequalities are often related to the dimension of space. We note two inequalities which are widely used in partial differential equations (see [6, 14, 15, 19]).

Lemma 1.1 *Assume that $\Omega \subset R^N$ is a bounded, sufficiently smooth, simply connected domain with boundary $\partial\Omega$ of bounded curvature and supposing $v \in C_0^1(\Omega)$. Then for $N > 2$*

$$\left(\int_\Omega |v|^\delta \, dx\right)^{\frac{1}{\delta}} \leq \Lambda\left(\int_\Omega |\nabla v|^2 \, dx\right)^{\frac{1}{2}},$$

where $\delta = \frac{2N}{N-2}$ and $\Lambda = [N(N-2)\pi]^{-\frac{1}{2}}[\frac{(N-1)!}{\Gamma(\frac{N}{2})}]^{\frac{1}{N}}$. This lemma has been proved in [18]. However, when $N = 2$, the Lemma 1.1 is no longer valid. Bandle [1] obtained a similar result which can be written as follows.

Lemma 1.2 *Let D be a plane domain with sufficiently smooth boundary ∂D, and let v be a sufficiently smooth function defined on the closure $\overline{D}$ of D. If $v = 0$ on ∂D, then*

$$\lambda_1 \int_D |v|^2 \, dx \leq \int_D |\nabla v|^2 \, dx,$$

where λ_1 *is the smallest positive eigenvalue of*

$$\Delta\omega + \lambda\omega = 0 \quad \text{in } D, \qquad \omega = 0 \quad \text{on } \partial D.$$

The aim of the present paper is to show how to use Lemma 1.1 and Lemma 1.2 to prove the nonexistence and global existence for the quasilinear parabolic system. Besides Lemma 1.1 and Lemma 1.2, we will also use the Young inequality, the arithmetic geometric mean inequality, and the Hölder inequality. These inequalities make it possible for us to obtain more optimal results than the literature. When one studied the global existence and blow-up of the solutions of parabolic equations, many papers always required the initial data to be sufficiently small (or sufficiently large) and/or compactly supported, and the dimension of space and the parameters of the equation satisfy certain restrictions (see e.g., [3, 8–10, 16, 20]). In this paper, it is only necessary to assume that the initial data belongs to $L^2(\Omega)$. When the dimension of space and the parameters of equation satisfy certain constraints, the global solution of quasilinear parabolic equation is proved. When blow-up occurs, we derive the lower bound of the blow-up time. Obviously, this approach fully shows that the differential inequality technique is very interesting. In next section, we introduce the quasilinear parabolic equation and give our main results.

2 The quasilinear parabolic equation

The global existence or nonexistence and the blow-up in finite time of solutions to semilinear or quasilinear parabolic equations and systems have received a lot of attention. Payne and Schaefer [15] considered the following problem of a semilinear heat equation:

$$u_t = \Delta u + f(u)$$

under homogeneous Dirichlet boundary conditions and appropriate constraints on the nonlinearity $f(u)$. By using a differential inequality technique, a lower bound on the blow-up time was determined if blow-up occurs.

Grillo et al. [9] considered the nonlinear evolution problem of the form

$$u_t = \Delta u^m + u^p,$$

in an N-dimensional complete, simply connected Riemannian manifold with nonpositive sectional curvatures (namely a Cartan–Hadamard manifold). Under some appropriate constraints on p, m and the initial data, they proved that the problem has a global in time solution or the solution of the problem blows up at a finite time.

Yang et al. [21] considered local quasilinear parabolic equation with a potential term

$$u_t = \Delta u^m - V(x)u^m + u^p.$$

By using the test function method and constructing a supersolution technique, they proved that every nontrivial solution blows up in finite time if $1 < p \leq p_c$ and there are both global and nonglobal solutions if $p > p_c$. For more results, see [5, 12, 17, 20].

In this paper, we consider a more interesting system of quasilinear parabolic equation

$$u_t = \Delta u^m - V(x)u + |x|^\alpha u^p \left(\int_\Omega \beta(x) u^q \, dx \right)^{\frac{r}{q}}, \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$u = 0, \quad \text{on } \partial\Omega \times (0, T), \tag{2}$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \tag{3}$$

where $\alpha, p, q, r > 0, m > 1$, $u_0 \in C_0(\Omega)$ is nonnegative, $V(x)$ is a positive function satisfying $V(x) \sim |x|^{-\sigma}, \sigma > 0$, $\beta(x)$ is positive. The model (1)–(3) describes the diffusion of concentration of some Newtonian fluids through a porous medium or the density of some biological species in many physical phenomena and biological species theories (see [2, 7]). Many methods (e.g., the Fourier coefficient method, the supersolution technique, the Green function method, the test function method, weighted energy arguments, the comparison method, and the concavity method) used to determine an upper bound for the blow-up time. The lower bound for the blow-up time is equally important and may be more difficult to obtain. In this paper, we first use the Sobolev inequalities to prove the existence of global solution. Our main results can be written as follows.

Theorem 2.1 *Letting $u(x,t)$ be a nonnegative solution of problem (1)–(3) in Ω, where Ω is a simply connected, bounded domain in R^N $(N \geq 2)$. Then, if $m + 3 > 4(p + r)$, then problem (1)–(3) has a solution that is global in time whether $N = 2$ or $N > 2$.*

Furthermore, if $m + 3 \leq 4(p + r)$, the solution of (1)–(3) maybe blows up in some finite time. In this case, it is necessary to derive the lower bound of blow-up time. Whether the blow-up occurs or not, such a lower bound is still meaningful. We can obtain the following result.

Theorem 2.2 *Letting $u(x,t)$ be a nonnegative solution of problem (1)–(3) in Ω, where Ω is a simply connected, bounded domain in R^N $(N \geq 2)$. If $u(x,t)$ blows up at some finite time t^*, then t^* can be bounded from below.*

More precisely,

if $2 < N < \frac{4(p+r)}{4(p+r)-(m+3)}$ and $2(p + r) < m + 3 < 4(p + r)$, then

$$t^* \geq C_4 \frac{2(m+3) - 4(p+r)}{4(p+r) - (m+3)} \big[\varphi(0)\big]^{\frac{(m+3)-4(p+r)}{2(m+3)-4(p+r)}},$$

where C_4 is a positive constant and $\varphi(0) = \int_\Omega u_0^2\,dx$.

If $N = 2$ and $2(p + r) < m + 1, m + 3 < 4(p + r)$, then

$$t^* \geq \frac{(m+1) - (p+r)}{p+r} C_6 \big[\varphi(0)\big]^{-\frac{p+r}{(m+1)-(p+r)}},$$

where C_6 is a positive constant.

3 The proof of Theorem 2.1

To prove Theorem 2.1, we establish an auxiliary function:

$$\varphi(t) = \int_\Omega u^2\,dx. \tag{4}$$

By using the divergence theorem and (1)–(3), we compute

$$\varphi'(t) = 2\int_\Omega u u_t\,dx = 2\int_\Omega u\left[\Delta u^m - V(x)u + |x|^\alpha u^p \left(\int_\Omega \beta(x) u^q\,dx\right)^{\frac{r}{q}}\right] dx$$

$$= -\frac{8m}{(m+1)^2}\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right|^2 dx - 2\int_\Omega V(x)u^2\,dx$$
$$+ 2\int_\Omega |x|^\alpha u^{p+1}\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}} dx. \quad (5)$$

1. If $N > 2$, noting that $m+3 > 4(p+r)$, then, by using the Hölder inequality and Lemma 1.1, we are led to

$$\begin{aligned}
&2\int_\Omega |x|^\alpha u^{p+1}\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}} dx \\
&\quad = 2\left(\int_\Omega |x|^\alpha u^{p+1}\,dx\right)\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}} \\
&\quad \le 2\left(\int_\Omega (u^{\frac{m+1}{2}})^\delta\,dx\right)^{\delta_1}\left(\int_\Omega V(x)u^2\,dx\right)^{\delta_2+\frac{1}{2}} \\
&\qquad \times \left(\int_\Omega |x|^{\frac{\alpha}{\delta_3}}\beta^{\frac{r}{q\delta_3}}(x)V^{-\frac{\delta_2+\frac{1}{2}}{\delta_3}}(x)\,dx\right)^{\delta_3} \\
&\quad \le 2\Lambda^{\delta\delta_1}\left(\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right)\right|^2 dx)^{\frac{\delta\delta_1}{2}}\left(\int_\Omega V(x)u^2\,dx\right)^{\delta_2+\frac{1}{2}} \\
&\qquad \times \left(\int_\Omega |x|^{\frac{\alpha}{\delta_3}}\beta^{\frac{r}{q\delta_3}}(x)V^{-\frac{\delta_2+\frac{1}{2}}{\delta_3}}(x)\,dx\right)^{\delta_3},
\end{aligned} \quad (6)$$

where $\delta_1 = \frac{(p+r)(N-2)}{(m+3)N}$, $\delta_2 = \frac{p+r}{m+3}$ and $\delta_3 = \frac{4(p+r)-N[4(p+r)-(m+3)]}{2(m+3)N}$. So, we can get

$$\begin{aligned}
&2\int_\Omega |x|^\alpha u^{p+1}\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}} dx \\
&\quad \le 2C_1\Lambda^{\delta\delta_1}\left[\varepsilon_1^{-\frac{(p+r)}{m+3-(p+r)}}\left(\int_\Omega V(x)u^2\,dx\right)^{\frac{m+3+2(p+r)}{2(m+3)-2(p+r)}}\right]^{\frac{m+3-(p+r)}{(m+3)}} \\
&\qquad \times \left(\varepsilon_1\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right|^2 dx\right)^{\frac{p+r}{m+3}} \\
&\quad \le 2C_1\frac{m+3-(p+r)}{(m+3)}\Lambda^{\delta\delta_1}\varepsilon_1^{-\frac{(p+r)}{m+3-(p+r)}}\left(\int_\Omega V(x)u^2\,dx\right)^{\frac{m+3+2(p+r)}{2(m+3)-2(p+r)}} \\
&\qquad + 2C_1\frac{p+r}{m+3}\Lambda^{\delta\delta_1}\varepsilon_1\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right|^2 dx,
\end{aligned} \quad (7)$$

where $C_1 = (\int_\Omega |x|^{\frac{\alpha}{\delta_3}}\beta^{\frac{r}{q\delta_3}}(x)V^{-\frac{\delta_2+\frac{1}{2}}{\delta_3}}(x)\,dx)^{\delta_3}$ and ε_1 is a positive constant to be determined later. Inserting (7) into (5) and choosing $\varepsilon_1 = \frac{8m(m+3)}{2C_1(p+r)\Lambda^{\delta\delta_1}(m+1)^2}$, we obtain

$$\varphi'(t) \le C_2\left(\int_\Omega V(x)u^2\,dx\right)^{\frac{m+3+2(p+r)}{2(m+3)-2(p+r)}} - 2\int_\Omega V(x)u^2\,dx, \quad (8)$$

where $C_2 = 2C_1\frac{m+3-(p+r)}{m+3}\Lambda^{\delta\delta_1}\varepsilon_1^{-\frac{(p+r)}{m+3-(p+r)}}$. Since $m+3 > 4(p+r)$, we can deduce from (8)

$$\varphi'(t) \le \left(\int_\Omega V(x)u^2\,dx\right)^{\frac{m+3+2(p+r)}{2(m+3)-2(p+r)}}\left[C_2 - 2\left(\int_\Omega V(x)u^2\,dx\right)^{\frac{m+3-4(p+r)}{2(m+3)-2(p+r)}}\right]. \tag{9}$$

Inequality (9) shows that the solution of (1)–(3) cannot blow up in any finite time. Otherwise, if the solution of (1)–(3) becomes unbounded in a time $t^* < \infty$, there must be an interval $[t_0, t^*)$ in which $\varphi'(t) < 0$. So $\varphi(t^*) < \varphi(t_0)$. This is a contradiction.

2. If $N = 2$, noting that $m+3 > 4(p+r)$, we use the Hölder inequality and Lemma 1.2 to obtain

$$\begin{aligned}
&2\int_\Omega |x|^\alpha u^{p+1}\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}}dx\\
&\quad\le 2\left(\int_\Omega \left(u^{\frac{m+1}{2}}\right)^2 dx\right)^{\frac{p+r}{m+3}}\left(\int_\Omega V(x)u^2\,dx\right)^{\frac{p+r}{m+3}+\frac{1}{2}}\\
&\qquad\times\left(\int_\Omega |x|^{\frac{2(m+3)\alpha}{(m+3)-4(p+r)}}\beta^{\frac{2(m+3)r}{q[(m+3)-4(p+r)]}}(x)V^{-\frac{2(p+r)+(m+3)}{(m+3)-4(p+r)}}(x)\,dx\right)^{\frac{(m+3)-4(p+r)}{2(m+3)}}\\
&\quad\le 2\lambda_1^{\frac{p+r}{m+3}}\left[\varepsilon_2^{-\frac{p+r}{m+3-(p+r)}}\left(\int_\Omega V(x)u^2\,dx\right)^{\frac{(m+3)+2(p+r)}{2[(m+3)-(p+r)]}}\right]^{\frac{m+3-(p+r)}{m+3}}\\
&\qquad\times\left[\varepsilon_2\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right|^2 dx\right]^{\frac{p+r}{m+3}}\\
&\qquad\times\left(\int_\Omega |x|^{\frac{2(m+3)\alpha}{(m+3)-4(p+r)}}\beta^{\frac{2(m+3)r}{q[(m+3)-4(p+r)]}}(x)V^{-\frac{2(p+r)+(m+3)}{(m+3)-4(p+r)}}(x)\,dx\right)^{\frac{(m+3)-4(p+r)}{2(m+3)}}.
\end{aligned}$$

Similar to the computations in (6)–(9), we can prove the problem (1)–(3) has a solution that is global in time in this case. The proof of Theorem 2.1 is completed.

4 The proof for Theorem 2.2

First, we also use the function $\varphi(t)$ which we have defined in (4). However, if $N > 2$, we rewrite (6) as

$$\begin{aligned}
&2\int_\Omega |x|^\alpha u^{p+1}\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}}dx\\
&\quad\le 2\left(\int_\Omega u^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega \left(u^{\frac{m+1}{2}}\right)^\delta dx\right)^{\delta_1}\left(\int_\Omega V(x)u^2\,dx\right)^{\delta_2}\\
&\qquad\times\left(\int_\Omega |x|^{\frac{\alpha}{\delta_3}}\beta^{\frac{r}{q\delta_3}}(x)V^{-\frac{\delta_2}{\delta_3}}(x)\,dx\right)^{\delta_3}\\
&\quad\le 2\Lambda^{\delta\delta_1}\left(\int_\Omega u^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right|^2 dx\right)^{\frac{\delta\delta_1}{2}}\left(\int_\Omega V(x)u^2\,dx\right)^{\delta_2}\\
&\qquad\times\left(\int_\Omega |x|^{\frac{\alpha}{\delta_3}}\beta^{\frac{r}{q\delta_3}}(x)V^{-\frac{\delta_2}{\delta_3}}(x)\,dx\right)^{\delta_3}.
\end{aligned} \tag{10}$$

Since $2 < N < \frac{4(p+r)}{4(p+r)-(m+3)}$ and $2(p+r) < m+3 < 4(p+r)$, inequality (10) holds. Moreover, by the Young inequality we have

$$\begin{aligned}
&2\int_\Omega |x|^\alpha u^{p+1}\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}}dx \\
&\quad\leq 2C_3\Lambda^{\delta\delta_1}\left[\varepsilon_3^{-\frac{2(p+r)}{m+3-2(p+r)}}\left(\int_\Omega u^2\,dx\right)^{\frac{m+3}{2(m+3)-4(p+r)}}\right]^{\frac{m+3-2(p+r)}{m+3}} \\
&\qquad\times\left[\varepsilon_3\left(\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega V(x)u^2\,dx\right)^{\frac{1}{2}}\right]^{\frac{2(p+r)}{m+3}} \\
&\quad\leq 2C_3\Lambda^{\delta\delta_1}\frac{m+3-2(p+r)}{m+3}\varepsilon_3^{-\frac{2(p+r)}{m+3-2(p+r)}}\left(\int_\Omega u^2\,dx\right)^{\frac{m+3}{2(m+3)-4(p+r)}} \\
&\qquad+C_3\Lambda^{\delta\delta_1}\frac{2(p+r)}{m+3}\varepsilon_3\left[\int_\Omega\left|\nabla u^{\frac{m+1}{2}}\right|^2dx+\int_\Omega V(x)u^2\,dx\right],
\end{aligned}\tag{11}$$

where $C_3 = (\int_\Omega |x|^{\frac{\alpha}{\delta_3}}\beta^{\frac{r}{q\delta_3}}(x)V^{-\frac{\delta_2}{\delta_3}}(x)\,dx)^{\delta_3}$ and ε_3 is a positive constant to be determined later. Now inserting (11) into (5), we have

$$\begin{aligned}
\varphi'(t) &\leq -\left[\frac{8m}{(m+1)^2}-C_3\Lambda^{\delta\delta_1}\frac{2(p+r)}{m+3}\varepsilon_3\right]\int_\Omega\left|\nabla u^{\frac{m+1}{2}}\right|^2dx \\
&\quad-\left[2-C_3\Lambda^{\delta\delta_1}\frac{2(p+r)}{m+3}\varepsilon_3\right]\int_\Omega V(x)u^2\,dx \\
&\quad+2C_3\Lambda^{\delta\delta_1}\frac{m+3-2(p+r)}{m+3}\varepsilon_3^{-\frac{2(p+r)}{m+3-2(p+r)}}\left(\int_\Omega u^2\,dx\right)^{\frac{m+3}{2(m+3)-4(p+r)}}.
\end{aligned}\tag{12}$$

After choosing ε_3 small enough such that

$$\frac{8m}{(m+1)^2}-C_3\Lambda^{\delta\delta_1}\frac{2(p+r)}{m+3}\varepsilon_3>0,\qquad 2-C_3\Lambda^{\delta\delta_1}\frac{2(p+r)}{m+3}\varepsilon_3>0,$$

we have from (12)

$$\varphi'(t)\leq\frac{1}{C_4}\left[\varphi(t)\right]^{\frac{m+3}{2(m+3)-4(p+r)}},\tag{13}$$

where

$$C_4=\left[2C_3\Lambda^{\delta\delta_1}\frac{m+3-2(p+r)}{p+r}\varepsilon_3^{-\frac{2(p+r)}{m+3-2(p+r)}}\right]^{-1}.$$

If the solution of problem (1)–(3) blows up at some finite time t^*, we may derive from (13)

$$t^*\geq C_4\frac{2(m+3)-4(p+r)}{4(p+r)-(m+3)}\left[\varphi(0)\right]^{\frac{(m+3)-4(p+r)}{2(m+3)-4(p+r)}}.\tag{14}$$

2. If $N=2$, we use the Hölder inequality and Lemma 1.2 to obtain

$$2\int_\Omega |x|^\alpha u^{p+1}\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}}dx$$

$$
\begin{aligned}
&\leq 2\left(\int_\Omega u^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega \left(u^{\frac{m+1}{2}}\right)^2 dx\right)^{\frac{p+r}{m+1}} \\
&\quad\times\left(\int_\Omega |x|^{\frac{2(m+1)\alpha}{(m+1)-2(p+r)}}\beta^{\frac{2(m+1)r}{q[(m+1)-2(p+r)]}}(x)\,dx\right)^{\frac{(m+1)-2(p+r)}{2(m+1)}} \\
&\leq 2\lambda_1^{\frac{p+r}{m+1}}\left[\varepsilon_4^{-\frac{p+r}{m+1-(p+r)}}\left(\int_\Omega u^2\,dx\right)^{\frac{(m+1)}{(m+1)-(p+r)}}\right]^{\frac{m+1-(p+r)}{m+1}} \\
&\quad\times\left[\varepsilon_4\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right|^2 dx\right]^{\frac{p+r}{m+1}} \\
&\quad\times\left(\int_\Omega |x|^{\frac{2(m+1)\alpha}{(m+1)-2(p+r)}}\beta^{\frac{2(m+1)r}{q[(m+1)-2(p+r)]}}(x)\,dx\right)^{\frac{(m+1)-2(p+r)}{2(m+1)}},
\end{aligned}
$$

where we have used the condition $2(p+r) < m+1, m+3 < 4(p+r)$. By the Young inequality, we have

$$
\begin{aligned}
&2\int_\Omega |x|^\alpha u^{p+1}\left(\int_\Omega \beta(x)u^q\,dx\right)^{\frac{r}{q}} dx \\
&\quad\leq 2\lambda_1^{\frac{p+r}{m+1}}C_5\frac{m+1-(p+r)}{m+1}\varepsilon_4^{-\frac{p+r}{m+1-(p+r)}}\left(\int_\Omega u^2\,dx\right)^{\frac{m+1}{(m+1)-(p+r)}} \\
&\qquad+2\lambda_1^{\frac{p+r}{m+1}}C_5\frac{(p+r)}{m+1}\varepsilon_4\int_\Omega \left|\nabla u^{\frac{m+1}{2}}\right|^2 dx,
\end{aligned}
\tag{15}
$$

where $C_5 = (\int_\Omega |x|^{\frac{2(m+1)\alpha}{(m+1)-2(p+r)}}\beta^{\frac{2(m+1)r}{q[(m+1)-2(p+r)]}}(x)\,dx)^{\frac{(m+1)-2(p+r)}{2(m+1)}}$. Inserting (15) into (5) and choosing ε_4 small enough such that

$$
\frac{8m}{(m+1)^2} - 2\lambda_1^{\frac{p+r}{m+1}}C_5\frac{(p+r)}{m+1}\varepsilon_4 = 0,
$$

we have

$$
\varphi'(t) \leq \frac{1}{C_6}\left[\varphi(t)\right]^{\frac{m+1}{(m+1)-(p+r)}},
\tag{16}
$$

where $C_6 = [2\lambda_1^{\frac{p+r}{m+1}}C_5\frac{m+1-(p+r)}{m+1}\varepsilon_4^{-\frac{p+r}{m+1-(p+r)}}]^{-1}$. Integrating (16) from 0 to t^*, we obtain

$$
t^* \geq \frac{(m+1)-(p+r)}{p+r}C_6\left[\varphi(0)\right]^{-\frac{p+r}{(m+1)-(p+r)}}.
\tag{17}
$$

The proof of Theorem 2.2 is completed.

5 Conclusion

From the proofs of Theorem 2.1 and Theorem 2.2, we can see that Lemmas 1.1 and 1.2 play a key role. In general, Sobolev inequalities are not only related to the dimension of space, but also to the boundary conditions. For example, if the solution of (1) does not vanish on the boundary of Ω, Lemmas 1.1 and 1.2 do not hold. However, in this case, Brezis (see [4]) has obtained for $N > 2$

$$\int_\Omega v^{\frac{2N}{N-2}}\,dx \le C\left[\int_\Omega v^2\,dx + \int_\Omega |\nabla v|^2\,dx\right]^{\frac{N}{N-2}},$$

where C is a positive constant which depends on Ω and N. For $N = 2$, Li [11] has proved

$$\left(\int_\Omega v^4\,dx\right)^{\frac{1}{2}} \le C\left[\int_\Omega v^2\,dx + \delta\int_\Omega |\nabla v|^2\,dx\right], \tag{18}$$

where δ is a positive arbitrary constant and Ω is a rectangular area. Combining with the methods in Appendix B of [13], it is possible to get a result similar to (18), when Ω is a bounded star-shaped domain in R^2. Predictably, such an inequality will also be widely used. For example, when Neumann or nonlinear conditions are prescribed on the boundary rather than Dirichlet conditions (2), the problem (1) becomes more complicated and interesting. We will study this problem in a future paper.

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Authors' contributions
YL proposed the main idea of this paper and wrote the whole paper. LG prepared the manuscript initially. PZ performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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