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On global dynamics of 2D convective Cahn–Hilliard equation

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Abstract

In this paper, we study the long time behavior of solution for the initial-boundary value problem of convective Cahn–Hilliard equation in a 2D case. We show that the equation has a global attractor in $H^4(\Omega)$ when the initial value belongs to $H^1(\Omega)$.

Keywords: Global attractor; Convective Cahn–Hilliard equation; Absorbing set

1 Introduction

The dynamic properties of diffusion equations ensure the stability of diffusion phenomena and provide the mathematical foundation for the study of diffusion dynamics. There are many studies on the existence of global attractors for diffusion equations. For the classical results, we refer the reader to [1–9].

The convective Cahn–Hilliard equation [10–16], which arises naturally as a continuous model for the formation of facets and corners in crystal growth, is a typical fourth order nonlinear parabolic equation. Let $\Omega = [0, L] \times [0, L]$, where $L > 0$, γ is a positive constant, $\vec{\beta}$ is a vector. We consider the convective Cahn–Hilliard equation in the 2D case:

$$u_t + \gamma \Delta^2 u = \Delta\varphi(u) - \vec{\beta} \cdot \nabla\psi(u), \quad x = (x_1, x_2) \in \mathbb{R}^2, t \geq 0. \tag{1}$$

Equation (1) is supplemented by the following boundary conditions:

$$u(x_1 + L, x_2, t) = u(x_1, x_2 + L, t) = u(x_1, x_2, t), \quad x \in \mathbb{R}^2, t \geq 0, \tag{2}$$

and the initial condition

$$u(x, 0) = u_0(x). \tag{3}$$

In this paper, we denote by $H = L^2(\Omega)$, $(\cdot, \cdot)$ the H-inner product and by $\|\cdot\|$ the corresponding H-norm, denote $A = -\Delta$, where Δ is the Laplace operator. Assume that the initial function has zero mean, i.e., $\int_\Omega u_0(x)\,dx = 0$, then it follows that $\int_\Omega u(x,t)\,dx = 0$ for $t > 0$. Here, as [3], we set

$$\dot{H}^k_{per} = \left\{ u \middle| u \in H^k_{per}(\Omega), \int_\Omega u(x,t)\,dx = 0, \right\}, \quad k = 1, 2, \ldots.$$

Using the same method as [13], we obtain the lemma on the existence of global weak solution to problem (1)–(3).

Lemma 1.1 *Suppose that $u_0 \in \dot{H}^1_{per}(\Omega)$ and the functions $\varphi(r) \in C^2(\mathbb{R})$, $\psi(r) \in C^1(\mathbb{R})$ satisfy*

$$\varphi'(r) > 0, \qquad \varphi^{(i)} \le c_0'|r|^{k-i} + c_1', \qquad \psi'(r) \le c_0 r\sqrt{\varphi'(r)} + c_1,$$

where $k \le 3$ is a positive constant and $i = 0, 1, 2$. Then there exists a unique solution u for problem (1)–(3) *such that*

$$u \in C\big(\mathbb{R}^+; \dot{H}^1_{per}(\Omega)\big) \cap L^2_{loc}\big(\mathbb{R}^+; \dot{H}^2_{per}(\Omega)\big).$$

By Lemma 1.1, we can define the operator semigroup $S(t)u_0 : \dot{H}^1_{per}(\Omega) \times \mathbb{R}^+ \to \dot{H}^1_{per}(\Omega)$, which is $(\dot{H}^1_{per}, \dot{H}^1_{per})$-continuous. In what follows, we always assume that $\{S(t)\}_{t\ge 0}$ is the semigroup generated by the weak solutions of problem (1). It is sufficient to see that the restriction of $\{S(t)\}$ on the affined space $\dot{H}^1_{per}(\Omega)$ is a well-defined semigroup.

Proposition 1.2 ([17–19]) *Suppose that $\mathcal{A}$ is an (H^1, H^1)-global attractor for $\{S(t)\}_{t\ge 0}$. Suppose further that $\{S(t)\}_{t\ge 0}$ has a bounded (H^1, H^4)-absorbing set and $\{S(t)\}_{t\ge 0}$ is (H^1, H^4)-asymptotically compact. Then $\mathcal{A}$ is also an (H^1, H^4)-global attractor.*

The main result of this paper will be stated in the following.

Theorem 1.3 *Suppose that $u_0 \in H^1_{per}(\Omega)$ and the functions $\varphi(r) \in C^3(\mathbb{R})$, $\psi(r) \in C^2(\mathbb{R})$ satisfy*

$$\varphi'(r) > 0, \qquad \varphi^{(i)} \le c_0'|r|^{k-i} + c_1', \qquad \psi'(r) \le c_0 r\sqrt{\varphi'(r)} + c_1, \tag{4}$$

where $k \le 3$ is a positive constant and $i = 0, 1, 2$. Then there exists an (H^1, H^4)-global attractor for the solution $u(x, t)$ of problem (1)–(3), *which is invariant and compact in $H^4(\Omega)$ and attracts every bounded subset of $H^1(\Omega)$ with respect to the norm topology of $H^4(\Omega)$.*

Remark 1.4 In the previous papers [18, 20, 21], my cooperators and I also studied the existence of global attractor for a 2D convective Cahn–Hilliard equation. There are two main differences between the previous results and Theorem 1.3. First, in [18, 20], we assumed that there exists double-well potential for the convective Cahn–Hilliard equation, which was replaced by the higher order polynomial in [21]. But, in this paper, this assumption is changed by (4), which seems more abroad than double-well potential and polynomial. Second, in [18], the existence of (H^2, H^2)-global attractor was obtained, and in [20, 21], the existence of (H^k, H^k)-global attractor was proved. In this paper, we only assume that the initial data belongs to $H^1(\Omega)$ and obtain the (H^1, H^4)-global attractor for the 2D convective Cahn–Hilliard equation.

The remaining parts are organized as follows. We begin by giving some uniform estimates of solutions for the 2D convective Cahn–Hilliard equation in Sect. 2. Then, in Sect. 3, we prove the main results on the existence of global attractor.

2 Uniform estimates of solutions

First of all, we establish the uniform estimates of solutions of problem (1) as $t \to \infty$. These estimates are necessary to prove the existence of global attractors.

Lemma 2.1 *Suppose that $u_0 \in L^2(\Omega)$ and the functions $\varphi(r) \in C^1(\mathbb{R})$, $\psi(r) \in C^1(\mathbb{R})$ satisfy*

$$\varphi'(r) > 0, \qquad \psi'(r) \le c_0 r\sqrt{\varphi'(r)} + c_1.$$

Then, for problem (1)–(3), we have

$$\|u(t)\| \le M_0, \quad \forall t \ge T_0,$$

and

$$\int_t^{t+1} \|Au(t)\|^2 \, d\tau \le M_0, \quad t \ge T_0.$$

Here, M_0 is a positive constant depending on γ and c_i $(i = 0, 1)$. T_0 depends on γ, c_i $(i = 0, 1)$ and R, where $\|u_0\|^2 \le R^2$.

Proof Multiplying equation (1) by u and integrating the resulting relation over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \gamma\|\Delta u\|^2 + \int_\Omega \varphi'(u)|\nabla u|^2 \, dx = \beta \cdot \int_\Omega \psi'(u)u\nabla u \, dx. \tag{5}$$

Note that

$$\begin{aligned}\beta \cdot \int_\Omega \psi'(u)u\nabla u \, dx = \beta \cdot \int_\Omega \psi'(u)u\nabla u \, dx& \\ &\le c_2|\beta| \int_\Omega \left|u\nabla u\sqrt{\varphi'(u)}\right| dx + c_3|\beta| \int_\Omega |u| \, dx \\ &\le \frac{1}{2}\int_\Omega \varphi'(u)|\nabla u|^2 \, dx + \frac{c_2}{2}\|u\|^2 + \frac{c_3}{2}.\end{aligned}$$

Hence

$$\frac{d}{dt}\|u\|^2 + 2\gamma\|\Delta u\|^2 + \int_\Omega \varphi'(u)|\nabla u|^2 \, dx \le c_2\|u\|^2 + c_3. \tag{6}$$

Applying Poincaré's inequality, we arrive at

$$\|u\|^2 \le c'\|\nabla u\|^2.$$

Moreover,

$$c'\|\nabla u\|^2 = -c'\int_\Omega u\Delta u \, dx \le \frac{1}{2}\|u\|^2 + \frac{(c')^2}{2}\|\Delta u\|^2.$$

Therefore, the following inequality holds:

$$\|u\|^2 \le \left(c'\right)^2\|\Delta u\|^2.$$

Summing up, we get

$$\frac{d}{dt}\|u\|^2 + \left(\frac{2\gamma}{(c')^2} - c_4\right)\|u\|^2 \le c_5, \tag{7}$$

where γ satisfies $\frac{2\gamma}{(c')^2} - c_4 > 0$. Using Gronwall's inequality, we deduce that

$$\|u\|^2 \le e^{-(\frac{2\gamma}{(c')^2} - c_2)t}\|u_0\|^2 + \frac{c_3(c')^2}{2\gamma - c_2(c')^2} \le \frac{2c_3(c')^2}{2\gamma - c_2(c')^2} \tag{8}$$

for all $t \ge T^* = \frac{(c')^2}{2\gamma - c_2(c')^2} \ln \frac{[2\gamma - c_2(c')^2]R^2}{c_3(c')^2}$. Integrating (6) over $(t, t+1)$ with $t \ge T^*$ yields

$$\int_t^{t+1} \|\Delta u\|^2 \, d\tau \le c_4. \tag{9}$$

By using a mean value theorem for integrals, we obtain the existence of a time $t_0' \in (T^*, T^* + 1)$ such that

$$\left\|\Delta u(t_0')\right\|^2 \le c_5$$

holds uniformly, the proof is complete. □

Lemma 2.2 *Suppose that $u_0 \in H^1_{per}(\Omega)$ and the functions $\varphi(r) \in C^2(\mathbb{R})$, $\psi(r) \in C^1(\mathbb{R})$ satisfy*

$$\varphi'(r) > 0, \qquad \varphi^{(i)} \le c_0'|r|^{k-i} + c_1', \qquad \psi'(r) \le c_0 r\sqrt{\varphi'(r)} + c_1,$$

where $k \le 3$ is a positive constant and $i = 0, 1, 2$. Then, for problem (1)–(3), we have

$$\left\|\nabla u(t)\right\| \le M_1, \quad \forall t \ge T_1,$$

and

$$\int_t^{t+1} \left\|\nabla \Delta u(t)\right\|^2 d\tau \le M_1, \quad t \ge T_1.$$

Here, M_1 is a positive constant depending on γ and c_i, c_i' $(i = 0, 1)$. T_1 depends on γ, c_i, c_i' $(i = 0, 1)$ and R, where $\|u_0\|^2_{H^1_{per}} \le R^2$.

Proof Multiplying equation (1) by $-\Delta u$ and integrating the resulting relation over Ω yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \gamma\|\nabla\Delta u\|^2 &= -\int_\Omega \Delta\varphi(u)\Delta u\,dx - \beta\cdot\int_\Omega \nabla\psi(u)\Delta u\,dx \\
&= -\int_\Omega \varphi'(u)|\Delta u|^2\,dx - \int_\Omega \varphi''(u)|\nabla u|^2\Delta u\,dx \\
&\quad - \beta\cdot\int_\Omega \psi'(u)\nabla u\Delta u\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\gamma\|\nabla\Delta u\|^2+\int_\Omega\varphi'(u)|\Delta u|^2\,dx\\
&\quad=-\int_\Omega\varphi''(u)|\nabla u|^2\Delta u\,dx-\beta\cdot\int_\Omega\psi'(u)\nabla u\Delta u\,dx\\
&\quad\le c\int_\Omega|u\Delta u||\nabla u|^2\,dx+c|\beta|\int_\Omega\bigl|u^2\sqrt{\varphi'(u)}\nabla u\Delta u\bigr|\,dx+c\|\nabla u\|^2\\
&\quad\le\frac{c}{2}\int_\Omega|\nabla u|^4\,dx+\frac{c}{2}\int_\Omega|u\Delta u|^2\,dx+\int_\Omega\varphi'(u)|\Delta u|^2\,dx+\frac{c^2|\beta|^2}{4}\int_\Omega u^4|\nabla u|^2\,dx\\
&\qquad+\frac{c_6}{2}\|\nabla u\|^2.
\end{aligned}$$

By Nirenberg's inequality, we obtain

$$\begin{aligned}
&\|u\|_4\le c_1'\|\nabla\Delta u\|^{\frac{1}{6}}\|u\|^{\frac{5}{6}}+c_2'\|u\|,\qquad \|\nabla u\|_4\le c_1'\|\nabla\Delta u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}+c_2'\|u\|,\\
&\|u\|_8\le c_1'\|\nabla\Delta u\|^{\frac{1}{4}}\|u\|^{\frac{3}{4}}+c_2'\|u\|,\qquad \|\Delta u\|_4\le c_1'\|\nabla\Delta u\|^{\frac{5}{6}}\|u\|^{\frac{1}{6}}+c_2'\|u\|.
\end{aligned}$$

Thus, by Hölder's inequality and the above inequalities, we deduce that

$$\frac{c}{2}\int_\Omega|\nabla u|^4\,dx+\frac{c}{2}\int_\Omega|u\Delta u|^2\,dx+\frac{c^2|\beta|^2}{4}\int_\Omega u^4|\nabla u|^2\,dx\le\frac{\gamma}{2}\|\nabla\Delta u\|^2+\frac{c_7}{2}.$$

Summing up, we obtain

$$\frac{d}{dt}\|\nabla u\|^2+\gamma\|\nabla\Delta u\|^2\le c_6\|\nabla u\|^2+c_7. \tag{10}$$

On the other hand,

$$\|\nabla u\|^2=-\int_\Omega u\Delta u\,dx\le\|u\|\|\Delta u\|\le\sqrt{\frac{2c_3(c')^2}{2\gamma-c_2(c')^2}}\|\Delta u\|$$

and

$$\|\Delta u\|^2=-\int_\Omega\nabla u\cdot\nabla\Delta u\,dx\le\|\nabla u\|\|\nabla\Delta u\|.$$

Adding the above two inequalities together gives

$$c_6\|\nabla u\|^2\le c\|\nabla\Delta u\|^{\frac{4}{3}}\le\frac{\gamma}{2}\|\nabla\Delta u\|^2+c_8. \tag{11}$$

It then follows from (10) and (11) that

$$\frac{d}{dt}\|\nabla u\|^2+\frac{\gamma}{2}\|\nabla u\|^2\le c_7+c_8.$$

Applying Gronwall's inequality yields

$$\|\nabla u\|^2\le e^{-\frac{\gamma}{2}t}\|\nabla u_0\|^2+\frac{2(c_7+c_8)}{\gamma}\le\frac{4(c_7+c_8)}{\gamma} \tag{12}$$

for all $t \geq T' = \max\{T^*, \frac{2}{\gamma}\ln\frac{\gamma R^2}{2(c_7+c_8)}\}$. Integrating (10) over $(t, t+1)$ with $t \geq T'$ gives

$$\int_t^{t+1} \|\nabla\Delta u\|^2 \, d\tau \leq c_9.$$

Using a mean value theorem for integrals, we obtain the existence of a time $t_0 \in (T', T'+1)$ such that

$$\|\nabla\Delta u(t_0)\|^2 \leq c_{10}$$

holds uniformly. Since we consider problem (1)–(3) in the 2D case, based on Sobolev's embedding theorem, we can get

$$\|u\|_p = \left(\int_\Omega u^p \, dx\right)^{\frac{1}{p}} \leq c_{11}, \quad 1 \leq p < \infty.$$

Set $T_1 = T'$, we complete the proof. □

Lemma 2.3 *Suppose that $u_0 \in H^1_{per}(\Omega)$ and the functions $\varphi(r) \in C^2(\mathbb{R})$, $\psi(r) \in C^1(\mathbb{R})$ satisfy*

$$\varphi'(r) > 0, \qquad \varphi^{(i)} \leq c_0'|r|^{k-i} + c_1', \qquad \psi'(r) \leq c_0 r\sqrt{\varphi'(r)} + c_1,$$

where $k \leq 3$ is a positive constant and $i = 0, 1, 2$. Then, for problem (1)–(3), we have

$$\|Au(t)\| \leq M_2, \quad \forall t \geq T_2,$$

and

$$\int_t^{t+1} \|u_t\|^2 \, d\tau \leq M_2, \quad t \geq T_2.$$

Here, M_2 is a positive constant depending on γ and c_i, c_i' $(i = 0, 1)$. T_2 depends on γ, c_i, c_i' $(i = 0, 1)$ and R, where $\|u_0\|^2_{H^1_{per}} \leq R^2$.

Proof Multiplying equation (1) by $\Delta^2 u$ and integrating the resulting relation over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \gamma\|\Delta^2 u\|^2 \\
&\quad = (\Delta\varphi(u), \Delta^2 u) + (\beta \cdot \nabla\psi(u), \Delta^2 u) \\
&\quad = (\varphi'(u)\Delta u + \varphi''(u)|\nabla u|^2, \Delta^2 u) + \beta \cdot (\psi(u)\nabla u, \Delta^2 u) \\
&\quad \leq \frac{\gamma}{2}\|\Delta^2 u\|^2 + \frac{2}{\gamma}\|\varphi'(u)\Delta u\|^2 + \frac{2}{\gamma}\|\varphi''(u)|\nabla u|^2\|^2 + \frac{|\beta|^2}{\gamma}\|\psi'(u)\nabla u\|^2.
\end{aligned}$$

Simple calculation shows that

$$\begin{aligned}
&\frac{d}{dt}\|\Delta u\|^2+\gamma\left\|\Delta^2 u\right\|^2\\
&\quad\le\frac{4}{\gamma}\int_\Omega\left|\varphi'(u)\Delta u\right|^2dx+\frac{4}{\gamma}\int_\Omega\left|\varphi''(u)|\nabla u|^2\right|^2dx+c\int_\Omega u^2\left|\varphi'(u)\right||\nabla u|^2dx+c\|\nabla u\|^2\\
&\quad\le c\left(\int_\Omega u^4|\Delta u|^2dx+\int_\Omega u^2|\nabla u|^4dx+\int_\Omega u^6|\nabla u|^2dx\right)\\
&\qquad+c\|\Delta u\|^2+c\|\nabla u\|_4^4+c\|\nabla u\|^2\\
&\quad\le c\left(\|u\|_8^4\|\Delta u\|_4^2+\|u\|_4^2\|\nabla u\|_8^4+\|u\|_{12}^6\|\nabla u|_4^2\right)+c\|\Delta u\|^2+c\|\nabla u\|_4^4+c\\
&\quad\le c\left(\|\Delta u\|_4^2+\|\nabla u\|_8^4++\|\nabla u\|_4^2+\|\nabla u\|_4^4\right)+c\|\Delta u\|^2+c.
\end{aligned}$$

By Sobolev's embedding theorem, we deduce that

$$\begin{aligned}
&\|\Delta u\|_4^2\le\left(c_1'\left\|\Delta^2u\right\|^{\frac{5}{8}}\|u\|^{\frac{3}{8}}+c_2'\|u\|\right)^2\le\frac{\varepsilon}{c}\left\|\Delta^2u\right\|^2+c_\varepsilon,\\
&\|\nabla u\|_4^2\le\left(c_1'\left\|\Delta^2u\right\|^{\frac{3}{8}}\|u\|^{\frac{5}{8}}+c_2'\|u\|\right)^2\le\frac{\varepsilon}{c}\left\|\Delta^2u\right\|^2+c_\varepsilon,\\
&\|\nabla u\|_4^4\le\left(c_1'\left\|\Delta^2u\right\|^{\frac{3}{8}}\|u\|^{\frac{5}{8}}+c_2'\|u\|\right)^4\le\frac{\varepsilon}{c}\left\|\Delta^2u\right\|^2+c_\varepsilon,
\end{aligned}$$

and

$$\|\nabla u\|_8^4\le\left(c_1'\left\|\Delta^2u\right\|^{\frac{7}{16}}\|u\|^{\frac{11}{16}}+c_2'\|u\|\right)^2\le\frac{\varepsilon}{c}\left\|\Delta^2u\right\|^2+c_\varepsilon.$$

Moreover,

$$c\|\Delta u\|^2=-c\int_\Omega\nabla u\cdot\nabla\Delta u\,dx=c\int_\Omega u\Delta^2u\,dx\le\|u\|\left\|\Delta^2u\right\|\le\varepsilon\left\|\Delta^2u\right\|^2+c_\varepsilon.$$

Summing up and setting $\varepsilon=\frac{\gamma}{10}$ gives

$$\frac{d}{dt}\|\Delta u\|^2+\frac{\gamma}{2}\left\|\Delta^2u\right\|^2\le c_{12}.\tag{13}$$

By a Calderón–Zygmund type estimate, the following inequality holds:

$$\frac{d}{dt}\|\Delta u\|^2+\frac{\gamma c'}{2}\left(\|\Delta u\|^2+\|\nabla\Delta u\|^2\right)\le c_{12}.$$

Then, using Gronwall's inequality, we obtain

$$\|\Delta u\|^2\le e^{-\frac{\gamma c'}{2}(t-t_0')}\left\|\Delta u\left(t_0'\right)\right\|^2+\frac{2c_{12}}{\gamma c'}\le\frac{4c_{12}}{\gamma c'}\tag{14}$$

for all $t\ge T_0'=\max\{T_0,t_0'+\frac{2}{\gamma c'}\ln\frac{\gamma c'R^2}{2c_{12}}\}$. Setting $t\ge T_0'$, taking $s\in(t,t+1)$, integrating (14) over $(s,t+1)$, we derive that

$$\left\|\Delta u(t+1)\right\|^2\le c_{13}+\left\|\Delta u(s)\right\|^2.\tag{15}$$

Integrating (15) with respect to s in $(t, t+1)$, we can obtain

$$\left\|\Delta u(t+1)\right\|^2 \le c_{13} + \int_t^{t+1} \left\|\Delta u(s)\right\|^2 dx \le c_{14}, \quad \forall t \ge T_0'. \tag{16}$$

By (14), (12), (7), and Sobolev's embedding theorem, we conclude

$$\|u\|_\infty \le c_{15}, \qquad \|\nabla u\|_p \le c_{16}, \quad 1 \le p < \infty. \tag{17}$$

Multiplying equation (1) by u_t, integrating the resulting relation over Ω yields

$$\begin{aligned}
&\|u_t\|^2 + \frac{\gamma}{2}\frac{d}{dt}\|\Delta u\|^2 \\
&\quad = \int_\Omega \Delta\varphi(u) u_t\, dx + \beta \cdot \int_\Omega \nabla\psi(u) u_t\, dx \\
&\quad = \int_\Omega \varphi'(u)\Delta u u_t\, dx + \int_\Omega \varphi''(u)|\nabla u|^2 u_t\, dx + \beta \cdot \int_\Omega \psi'(u)\nabla u u_t\, dx \\
&\quad \le \left\|\varphi'(u)\right\|_\infty \|\Delta u\|\|u_t\| + \left\|\varphi''(u)\right\|_\infty \|\nabla u\|_4^2 \|u_t\| + |\beta| \left\|\psi'(u)\right\|_\infty \|\nabla u\|\|u_t\| \\
&\quad \le \frac{1}{2}\|u_t\|^2 + c\left(\left\|\varphi'(u)\right\|_\infty^2 \|\Delta u\|^2 + \left\|\varphi''(u)\right\|_\infty^2 \|\nabla u\|_4^4 + |\beta|^2 \left\|\psi'(u)\right\|_\infty^2 \|\nabla u\|^2\right) \\
&\quad \le \frac{1}{2}\|u_t\|^2 + \frac{c_{17}}{2},
\end{aligned}$$

that is,

$$\|u_t\|^2 + \frac{d}{dt}\|\Delta u\|^2 \le c_{17}. \tag{18}$$

Integrating (18) over $(t+1, t+2)$, using (14), we derive that

$$\int_{t+1}^{t+2} \|u_t\|^2\, dx \le c_{18}, \quad \forall t \ge T_0''.$$

Using a mean value theorem for integrals, we obtain the existence of a time $t_1 \in (T_0'' + 1, T_0'' + 2)$ such that the following estimate holds uniformly:

$$\left\|u_t(t_1)\right\|^2 \le c_{19}.$$

Then the proof is complete. □

Lemma 2.4 *Suppose that $u_0 \in H_{per}^1(\Omega)$ and the functions $\varphi(r) \in C^3(\mathbb{R})$, $\psi(r) \in C^2(\mathbb{R})$ satisfy*

$$\varphi'(r) > 0, \qquad \varphi^{(i)} \le c_0'|r|^{k-i} + c_1', \qquad \psi'(r) \le c_0 r\sqrt{\varphi'(r)} + c_1,$$

where $k \le 3$ is a positive constant and $i = 0, 1, 2$. Then, for problem (1)–(3), we have

$$\left\|\nabla\Delta u(t)\right\| \le M_3, \quad \forall t \ge T_3,$$

and

$$\int_t^{t+1} \left\|A^{\frac{1}{2}} u_t(t)\right\|^2 dt \le M_3, \quad \forall t \ge T_3.$$

Here, M_3 is a positive constant depending on γ, c_i, c_i' ($i = 0,1$). T_3 depends on γ, c_i, c_i' ($i = 0,1$) and R, where $\|u_0\|_{H^1}^2 \le R^2$.

Proof Multiplying (1) by $\Delta^3 u$ and integrating the resulting relation over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2 + \gamma\left\|\nabla\Delta^2 u\right\|^2 \\
&\quad = \int_\Omega \nabla\Delta\varphi(u)\nabla\Delta^2 u\,dx + \beta\cdot\int_\Omega \Delta\psi(u)\nabla\Delta^2 u\,dx \\
&\quad = \int_\Omega \varphi'(u)\nabla\Delta u\nabla\Delta^2 u\,dx + 3\int_\Omega \varphi''(u)\nabla u\Delta u\nabla\Delta^2 u\,dx \\
&\qquad + \int_\Omega \varphi'''(u)|\nabla u|^2\nabla u\nabla\Delta^2 u\,dx \\
&\qquad + \beta\cdot\int_\Omega \psi'(u)\Delta u\nabla\Delta^2 u\,dx + \beta\cdot\int_\Omega \psi''(u)|\nabla u|^2\nabla\Delta^2 u\,dx \\
&\quad \le \frac{\gamma}{2}\left\|\nabla\Delta^2 u\right\|^2 + c\Big(\left\|\varphi'(u)\right\|_\infty^2\|\nabla\Delta u\|^2 + 3\left\|\varphi''(u)\right\|_\infty^2\|\nabla u\|_4^2\|\Delta u\|_4^2 \\
&\qquad + \left\|\varphi''(u)\right\|_\infty^2\|\nabla u\|_6^6 + |\beta|^2\left\|\psi'(u)\right\|_\infty^2\|\Delta u\|^2 + |\beta|^2\left\|\psi''(u)\right\|_\infty^2\|\nabla u\|_4^4\Big).
\end{aligned} \tag{19}$$

It follows form (17) that

$$\begin{aligned}
&\left\|\varphi'(u)\right\|_\infty^2\|\nabla\Delta u\|^2 \le \frac{c'}{2}\|\nabla\Delta u\|^2, \\
&3\left\|\varphi''(u)\right\|_\infty^2\|\nabla u\|_4^2\|\Delta u\|_4^2 \le \frac{c'}{2}\|\Delta u\|_4^2,
\end{aligned}$$

and

$$\left\|\varphi''(u)\right\|_\infty^2\|\nabla u\|_6^6 + |\beta|^2\left\|\psi'(u)\right\|_\infty^2\|\|\Delta u\|^2 + |\beta|^2\left\|\psi''(u)\right\|_\infty^2\|\nabla u\|_4^4 \le \frac{c_{19}}{2}.$$

Summing up, we find that

$$\frac{d}{dt}\|\nabla\Delta u\|^2 + \gamma\left\|\nabla\Delta^2 u\right\|^2 \le c'\left(\|\nabla\Delta u\|^2 + \|\Delta u\|_4^2 + c_{19}\right). \tag{20}$$

Using Nirenberg's inequality, we obtain

$$c'\|\Delta u\|_4^2 \le c'\left(c_1'\left\|\nabla\Delta^2 u\right\|^{\frac{1}{6}}\|\Delta u\|^{\frac{5}{6}} + c_2'\|\Delta u\|\right)^2 \le \frac{\gamma}{4}\left\|\nabla\Delta^2 u\right\|^2 + c_{20}.$$

On the other hand,

$$c'\|\nabla\Delta u\|^2 = c'\int_\Omega \nabla u\cdot\nabla\Delta^2 u\,dx \le c'\|\nabla u\|\left\|\nabla\Delta^2 u\right\| \le \frac{\gamma}{4}\left\|\nabla\Delta^2 u\right\|^2 + c_{21}.$$

Hence

$$\frac{d}{dt}\|\nabla\Delta u\|^2+\frac{\gamma}{2}\left\|\nabla\Delta^2 u\right\|^2\le c_{20}+c_{21}+c'c_{19}. \tag{21}$$

A simple calculation shows that

$$\frac{d}{dt}\|\nabla\Delta u\|^2+c_{22}\|\nabla\Delta u\|^2\le c_{23}. \tag{22}$$

By Gronwall's inequality, we immediately obtain

$$\left\|\nabla\Delta u(t)\right\|^2\le e^{-c_{22}(t-t_0)}\left\|\nabla u(t_0)\right\|^2+\frac{c_{23}}{c_{22}}\le\frac{2c_{23}}{c_{22}} \tag{23}$$

for all $t\ge T_1^*=\max\{T_1,t_0+\frac{1}{c_{22}}\ln\frac{c_{22}R^2}{2c_{23}}\}$. Combining (23), (14), (12), and (7) together gives

$$\|\nabla u\|_\infty\le c_{24},\qquad\|\Delta u\|_q\le c_{25},\quad 1\le q<\infty,\forall t\ge T_1^*. \tag{24}$$

Multiplying equation (1) by Au_t, integrating the resulting relation over Ω, we obtain

$$\begin{aligned}
&\|\nabla u_t\|^2+\frac{\gamma}{2}\frac{d}{dt}\|\nabla\Delta u\|^2\\
&\quad=\int_\Omega\nabla\Delta\varphi(u)\nabla u_t\,dx+\beta\cdot\int_\Omega\Delta\varphi(u)\nabla u_t\,dx\\
&\quad=\int_\Omega\left[\varphi'(u)\nabla\Delta u+3\varphi''(u)\nabla u\Delta u+\varphi'''(u)|\nabla u|^2\nabla u\right]\nabla u_t\,dx\\
&\qquad+\beta\cdot\int_\Omega\left[\psi'(u)\Delta u+\psi''(u)|\nabla u|^2\right]\nabla u_t\,dx\\
&\quad\le\left\|\varphi'(u)\right\|_\infty\|\nabla\Delta u\|\|\nabla u_t\|+3\left\|\varphi''(u)\right\|_\infty\|\nabla u\|_\infty\|\Delta u\|\|\nabla u_t\|\\
&\qquad+\left\|\varphi'''(u)\right\|_\infty\|\nabla u\|_\infty^2\|\nabla u\|\|\nabla u_t\|\\
&\qquad+|\beta|\left\|\psi'(u)\right\|_\infty\|\Delta u\|\|\nabla u_t\|+|\beta|\left\|\psi''(u)\right\|_\infty\|\nabla u\|_\infty\|\nabla u\|\|\nabla u_t\|\\
&\quad\le c\|\nabla u_t\|\le\frac{1}{2}\|\nabla u_t\|^2+\frac{c_{26}}{2}.
\end{aligned}$$

Summing up, using the result of (23) gives

$$\|\nabla u_t\|^2+\gamma\frac{d}{dt}\|\nabla\Delta u\|^2\le c_{26}. \tag{25}$$

Then

$$\gamma\frac{d}{dt}\|\nabla\Delta u\|^2\le c_{26}.$$

Setting $t\ge T_1^*$, taking $s\in(t,t+1)$, integrating the above inequality over $(s,t+1)$, we obtain

$$\left\|\nabla\Delta u(t+1)\right\|^2\le\frac{1}{\gamma}\left(c_{26}+\left\|\nabla\Delta u(s)\right\|^2\right).$$

Integrating the above inequality with respect to s in $(t,t+1)$, we have

$$\left\|\nabla\Delta u(t+1)\right\|^2 \le \frac{1}{\gamma}\left(c_{26}+\int_t^{t+1}\left\|\nabla\Delta u(s)\right\|^2 ds\right) \le c_{27}, \quad \forall t \ge T_1^*. \tag{26}$$

Integrating (25) over $(t+1,t+2)$, using (26) yields

$$\int_{t+1}^{t+2}\left\|A^{\frac{1}{2}}u_t\right\|^2 d\tau \le c_{28}, \quad \forall t \ge T_1^*.$$

Using a mean value theorem for integrals, we obtain the existence of a time $t_2 \in (T_1^*+1, T_1^*+2)$ such that the following estimate holds uniformly:

$$\left\|A^{\frac{1}{2}}u_t(t_2)\right\|^2 \le c_{29}.$$

Then we complete the proof. □

Lemma 2.5 *Suppose that $u_0 \in H^1_{per}(\Omega)$ and the functions $\varphi(r) \in C^3(\mathbb{R})$, $\psi(r) \in C^2(\mathbb{R})$ satisfy*

$$\varphi'(r) > 0, \qquad \varphi^{(i)} \le c_0'|r|^{k-i}+c_1', \qquad \psi'(r) \le c_0 r\sqrt{\varphi'(r)}+c_1,$$

where $k \le 3$ is a positive constant and $i = 0,1,2$. Then, for problem (1)–(3), we have

$$\|u_t\| \le M_4, \quad \forall t \ge T_4.$$

Here, M_4 is a positive constant depending on γ, c_i, c_i' $(i=0,1)$. T_4 depends on γ, c_i, c_i' $(i=0,1)$ and R, where $\|u_0\|^2_{H^1_{per}} \le R^2$.

Proof Setting $v = u_t$, differentiating (1) with respect to the time t, we deduce that

$$v_t + \gamma\Delta^2 v - \left[\Delta\varphi(u)\right]_t - \beta\cdot\left[\nabla\psi(u)\right]_t = 0. \tag{27}$$

Multiplying (27) by v, integrating the resulting relation over Ω yields

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \gamma\|\Delta v\|^2 - \int_\Omega\left[\Delta\varphi(u)\right]_t v\,dx - \int_\Omega \beta\cdot\left[\nabla\psi(u)\right]_t v\,dx = 0. \tag{28}$$

Using Sobolev's embedding theorem, we get

$$\begin{aligned}
&\int_\Omega\left[\Delta\varphi(u)\right]_t v\,dx + \beta\cdot\int_\Omega\left[\psi'(u)\nabla u\right]_t v\,dx \\
&\quad= \int_\Omega \varphi'(u)v\Delta v\,dx + \int_\Omega \varphi''(u)v^2\Delta u\,dx + \int_\Omega \varphi'''(u)|\nabla u|^2 v^2\,dx \\
&\qquad + 2\int_\Omega \varphi''(u)v\nabla u\nabla v\,dx + \beta\cdot\int_\Omega \psi'(u)v\nabla v\,dx + \beta\cdot\int_\Omega \psi''(u)v^2\nabla u\,dx \\
&\quad\le \left\|\varphi'(u)\right\|_\infty\|\Delta v\|\|v\| + \left\|\varphi''(u)\right\|_\infty\|\Delta u\|_6\|v\|_6^2 + \left\|\varphi'''(u)\right\|_\infty\|\nabla u\|_\infty^2\|v\|^2 \\
&\qquad + 2\left\|\varphi''(u)\nabla u\right\|_\infty\|v\|\|\nabla v\| + |\beta|\left\|\psi'(u)\right\|_\infty\|\nabla v\|\|v\| + |\beta|\left\|\psi''(u)\right\|_\infty\|\nabla u\|_\infty\|v\|^2
\end{aligned}$$

$$\begin{aligned}&\leq c\big(\|\Delta v\|\|v\|+\|\nabla v\|^2+\|\nabla v\|\|v\|\big)\\&\leq \frac{\gamma}{2}\|\Delta v\|^2+\frac{c_{30}}{2}\|v\|^2+\frac{c_{31}}{2}.\end{aligned}$$

Hence,

$$\frac{d}{dt}\|v\|^2+\gamma\|\Delta v\|^2\leq c_{30}\|v\|^2+c_{31}. \tag{29}$$

A simple calculation shows that

$$\|v\|^2\leq \frac{1}{c'}\|\Delta v\|^2.$$

It then follows from (29) and the above inequality that

$$\frac{d}{dt}\|v\|^2+\big(c'\gamma-c_{30}\big)\|v\|^2\leq c_{31},$$

where γ is sufficiently large, it satisfies $c'\gamma-c_{30}>0$. Using Gronwall's inequality, we derive that

$$\begin{aligned}\|v\|^2&\leq e^{-(c'\gamma-c_{30})(t-t_1)}\big\|v(t_1)\big\|^2+\frac{c_{31}}{c'\gamma-c_{30}}\\&\leq c_{19}e^{-(c'\gamma-c_{30})(t-t_1)}+\frac{c_{31}}{c'\gamma-c_{30}}\leq\frac{2c_{31}}{c'\gamma-c_{30}}\end{aligned} \tag{30}$$

for all $t\geq t_1+\frac{1}{c'\gamma-c_{30}}\ln\frac{c_{19}(c'\gamma-c_{30})}{c_{31}}$. Then the proof is complete. □

Lemma 2.6 *Suppose that* $u_0\in H^1_{per}(\Omega)$ *and the functions* $\varphi(r)\in C^3(\mathbb{R})$, $\psi(r)\in C^2(\mathbb{R})$ *satisfy*

$$\varphi'(r)>0,\qquad \varphi^{(i)}\leq c_0'|r|^{k-i}+c_1',\qquad \psi'(r)\leq c_0r\sqrt{\varphi'(r)}+c_1,$$

where $k\leq 3$ *is a positive constant and* $i=0,1,2$. *Then, for problem* (1)–(3), *we have*

$$\big\|A^{\frac{1}{2}}v_t(t)\big\|\leq M_5,\quad \forall t\geq T_5.$$

Here, M_5 *is a positive constant depending on* γ, c_i, c_i' $(i=0,1)$. T_5 *depends on* γ, c_i, c_i' $(i=0,1)$ *and* R, *where* $\|u_0\|^2_{H^1_{per}}\leq R^2$.

Proof Multiplying (27) by Av, integrating the resulting relation over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2+\gamma\|\nabla\Delta v\|^2=-\int_\Omega\big[\Delta\varphi(u)\big]_t\Delta v\,dx-\beta\cdot\int_\Omega\big[\nabla\psi(u)\big]_t\Delta v\,dx. \tag{31}$$

By Sobolev's embedding theorem, we get

$$
\begin{aligned}
&-\int_\Omega \big[\Delta\varphi(u)\big]_t \Delta v\,dx - \beta\cdot\int_\Omega \big[\nabla\psi(u)\big]_t \Delta v\,dx \\
&\quad = -\int_\Omega \varphi'(u)|\Delta v|^2\,dx - \int_\Omega \varphi''(u)v\Delta u\Delta v\,dx - \int_\Omega \varphi'''(u)v|\nabla u|^2\Delta v\,dx \\
&\qquad + 2\int_\Omega \varphi''(u)\nabla u\nabla v\Delta v\,dx + \beta\cdot\int_\Omega \psi'(u)\nabla v\Delta v\,dx + \beta\cdot\int_\Omega \psi''(u)v\nabla u\Delta v\,dx \\
&\quad \le \big\|\varphi'(u)\big\|_\infty\|\Delta v\|^2 + \big\|\varphi''(u)\big\|_\infty\|\Delta u\|\|\Delta v\|\|v\|_\infty + \big\|\varphi'''(u)\big\|_\infty\|\nabla u\|_\infty^2\|\Delta v\|\|v\| \\
&\qquad + 2\big\|\varphi''(u)\big\|_\infty\|\nabla u\|_\infty\|\nabla v\|\|\Delta v\| + |\beta|\big\|\psi'(u)\big\|_\infty\|\nabla v\|\|\Delta v\| \\
&\qquad + |\beta|\big\|\psi''(u)\big\|_\infty\|\nabla u\|_\infty\|\Delta v\|\|v\| \\
&\quad \le c\big(\|\Delta v\|^2 + \|\Delta v\|\|v\| + \|\nabla v\|\|\Delta v\|\big) \le \frac{\gamma}{2}\|\nabla\Delta v\|^2 + \frac{c_{32}}{2}\|\nabla v\|^2.
\end{aligned}
$$

Summing up gives

$$
\frac{d}{dt}\|\nabla v\|^2 + \gamma\|\nabla\Delta v\|^2 \le c_{32}\|\nabla v\|^2.
$$

Using Nirenberg's inequality, we obtain

$$
c_{32}\|\nabla v\|^2 \le c_{32}\big(c_1'\|\nabla\Delta v\|^{\frac{1}{3}}\|v\|^{\frac{2}{3}} + c_2'\|v\|\big)^2 \le \frac{\gamma}{2}\|\nabla\Delta v\|^2 + c_{33}.
$$

Adding the above two inequalities together gives

$$
\frac{d}{dt}\|\nabla v\|^2 + c_{32}\|\nabla v\|^2 \le 2c_{33}.
$$

By Gronwall's inequality, we can obtain

$$
\begin{aligned}
\|\nabla v\|^2 &\le e^{-c_{32}(t-t_2)}\big\|\nabla v(t_2)\big\|^2 + \frac{2c_{33}}{c_{32}} \\
&\le c_{29}e^{-c_{32}(t-t_2)} + \frac{2c_{33}}{c_{32}} \le \frac{4c_{33}}{c_{32}}
\end{aligned}
\tag{32}
$$

for all $t \ge t_2 + \frac{1}{c_{32}}\ln\frac{c_{29}c_{32}}{2c_{33}}$. Then the proof is complete. □

Lemma 2.7 *Suppose that $u_0 \in H^1_{per}(\Omega)$ and the functions $\varphi(r) \in C^3(\mathbb{R})$, $\psi(r) \in C^2(\mathbb{R})$ satisfy*

$$
\varphi'(r) > 0, \qquad \varphi^{(i)} \le c_0'|r|^{k-i} + c_1', \qquad \psi'(r) \le c_0 r\sqrt{\varphi'(r)} + c_1,
$$

where $k \le 3$ is a positive constant and $i = 0, 1, 2$. Then, for problem (1)–(3), we have

$$
\big\|A^2u(t)\big\| \le M_6, \quad \forall t \ge T_6.
$$

Here, M_6 is a positive constant depending on γ, c_i, c_i' $(i = 0, 1)$. T_6 depends on γ, c_i, c_i' $(i = 0, 1)$ and R, where $\|u_0\|^2_{H^1_{per}} \le R^2$.

Proof For equation (1), by Lemmas 2.1–2.6, we deduce that

$$\begin{aligned}\|\Delta^2 u\| &\le \frac{1}{\gamma}\big(\|u_t\| + \|\Delta\varphi(u)\| + |\beta|\|\nabla\psi(u)\|\big)\\ &\le c\big(\|u_t\| + \|\varphi'(u)\|_\infty\|\Delta u\| + \|\varphi''(u)\|_\infty\|\nabla u\|_\infty\|\nabla u\| + |\beta|\|\psi'(u)\|_\infty\|\nabla u\|\big)\\ &\le c_{34}, \quad \forall t \ge T.\end{aligned}$$

On the other hand, by Sobolev's embedding theorem, it yields that

$$\|\Delta u\|_\infty \le c_{35},$$

which completes the proof. □

3 Proof of Theorem 1.3

Suppose that M_1 and M_6 are the constants in Lemma 2.2 and Lemma 2.7, respectively. Denote

$$B_1 = \{u \in \dot{H}^1_{per} : \|A^{\frac{1}{2}}u\| \le M_1\}, \tag{33}$$

$$B_2 = \{u \in \dot{H}^4_{per} : \|A^2 u\| \le M_6\}. \tag{34}$$

Using Lemmas 2.2 and 2.7, we easily obtain that B_1 is a bounded $(\dot{H}^1_{per}, \dot{H}^1_{per})$-absorbing set for $\{S(t)\}_{t\ge0}$ and B_2 is a bounded $(\dot{H}^1_{per}, \dot{H}^4_{per})$-absorbing set for $\{S(t)\}_{t\ge0}$. Note that the embedding $\dot{H}^4_{per} \hookrightarrow \dot{H}^1_{per}$ is compacted. Applying Lemma 2.3, we obtain $\{S(t)\}_{t\ge0}$ is $(\dot{H}^1_{per}, \dot{H}^1_{per})$-asymptotically compact. Hence, $\{S(t)\}_{t\ge0}$ has an $(\dot{H}^1_{per}, \dot{H}^1_{per})$-global attractor $\mathcal{A}$. In the following, we show that $\mathcal{A}$ is actually an $(\dot{H}^1_{per}, \dot{H}^4_{per})$-global attractor for $\{S(t)\}_{t\ge0}$.

Lemma 3.1 *Suppose that $u_0 \in H^1_{per}(\Omega)$ and the functions $\varphi(r) \in C^3(\mathbb{R})$, $\psi(r) \in C^2(\mathbb{R})$ satisfy*

$$\varphi'(r) > 0, \qquad \varphi^{(i)} \le c_0'|r|^{k-i} + c_1', \qquad \psi'(r) \le c_0 r\sqrt{\varphi'(r)} + c_1,$$

where $k \le 3$ is a positive constant and $i = 0, 1, 2$. Then, for the solution $u(x,t)$ of problem (1)–(3), the dynamical system $\{S(t)\}_{t\ge0}$ is $(\dot{H}^1_{per}, \dot{H}^4_{per})$-asymptotically compact.

Proof For (1), we have

$$\gamma A^2 u = -u_t + \Delta\varphi(u) + \beta\cdot\nabla\psi(u). \tag{35}$$

Assume that $\{u_{0,n}\}_{n=1}^\infty$ is bounded in $\dot{H}^1_{per}(\Omega)$ and $t_n \to \infty$. In the following we prove that $\{S(t_n)u_{0,n}\}_{n=1}^\infty$ has a convergent subsequence in $\dot{H}^4_{per}(\Omega)$. Denote

$$u_n(t) = S(t)u_{0,n} \quad \text{and} \quad v_n(t_n) = \frac{du_n}{dt}\bigg|_{t=t_n}.$$

Note that $\{u_{0,n}\}_{n=1}^\infty$ is bounded in $\dot{H}^1_{per}$. Then there exists $R > 0$ such that

$$\|u_{0,n} + A^{\frac{1}{2}}u_{0,n}\| \le R, \quad \forall n = 1, 2, \ldots.$$

By Lemmas 2.6 and 2.7, there exists $T > 0$ such that

$$\|v_n\|_{D(A^{\frac{1}{2}})} \le M_5, \qquad \|u_n\|_{D(A^2)} \le M_6, \quad \forall t \ge T, n = 1, 2, \ldots. \tag{36}$$

Since $t_n \to \infty$, there exists $N > 0$ such that $t_n \ge T$ for all $n \ge N$. Therefore, by (36), we get

$$\big\|v_n(t_n)\big\|_{D(A^{\frac{1}{2}})} \le M_5, \qquad \big\|u_n(t_n)\big\|_{D(A^2)} \le M_6, \quad \forall n \ge N. \tag{37}$$

Note that the embedding $D(A^{\frac{1}{2}}) \hookrightarrow H$ and $D(A^2) \hookrightarrow D(A)$ are compacted. Hence, by (36), there exist $v \in D(A^{\frac{1}{2}})$, $\Delta u \in D(A)$, $\nabla u \in \dot{H}^3_{per}$, and $u \in \dot{H}^4_{per}$ such that, up to a subsequence,

$$\begin{cases} v_n(t_n) \to v \quad \text{strongly in } H, \\ \Delta u_n(t_n) \to \Delta u \quad \text{strongly in } D(A^{\frac{1}{2}}), \\ \nabla u_n(t_n) \to \nabla u \quad \text{strongly in } D(A), \\ u_n(t_n) \to u \quad \text{strongly in } \dot{H}^3_{per}. \end{cases} \tag{38}$$

By (37) and Sobolev's embedding theorem, we obtain

$$\big\|u_n(t_n)\big\|_{W^{2,\infty}} \le C, \quad \forall n \ge N.$$

It then follows from (36) and (38) that

$$\big\|u_n(t_n) - u\big\| \to 0, \qquad \big\|v_n(t_n) - v\big\|^2 \to 0, \qquad \big\|\Delta u_n(t_n) - \Delta u\big\|^2 \to 0,$$

and

$$\begin{aligned}
&\big\|\Delta\varphi\big(u_n(t_n)\big) - \Delta\varphi(u)\big\| \\
&\quad = \big\|\varphi'\big(u_n(t_n)\big)\Delta u_n(t_n) - \varphi'(u)\Delta u + \varphi''\big(u_n(t_n)\big)\big|\nabla u_n(t_n)\big|^2 - \varphi''(u)|\nabla u|^2\big\| \\
&\quad \le c\big(\big\|\varphi'\big(u_n(t_n)\big)\big[\Delta u_n(t_n) - \Delta u\big]\big\| + \big\|\Delta u\big[\varphi'\big(u_n(t_n)\big)\Delta u_n(t_n) - \varphi'(u)\big]\big\| \\
&\qquad + \big\|\varphi''\big(u_n(t_n)\big)\big[\big|\nabla u_n(t_n)\big|^2 - |\nabla u|^2\big]\big\| + \big\||\nabla u|^2\big[\varphi''\big(u_n(t_n)\big) - \varphi''(u)\big]\big\|\big) \\
&\quad \le c\big\|\varphi'\big(u_n(t_n)\big)\big\|_\infty\big\|\Delta u_n(t_n) - \Delta u\big\| + c\|\Delta u\|_\infty\big\|\varphi'\big(u_n(t_n)\big)\Delta u_n(t_n) - \varphi'(u)\big\| \\
&\qquad + c\big\|\varphi''\big(u_n(t_n)\big)\big\|_\infty\big\|\nabla u_n(t_n) + \nabla u\big\|_\infty\big\|\nabla u_n(t_n) - \nabla u\big\| \\
&\qquad + c\|\nabla u\|^2_\infty\big\|\varphi''\big(u_n(t_n)\big) - \varphi''(u)\big\| \\
&\quad \le c\big\|\varphi'\big(u_n(t_n)\big)\big\|_\infty\big\|\Delta u_n(t_n) - \Delta u\big\| \\
&\qquad + c\|\Delta u\|_\infty\big\|\varphi''\big(\theta_1 u_n(t_n) + (1-\theta_1)u\big)\big\|_\infty\big\|u_n(t_n) - u\big\| \\
&\qquad + c\big\|\varphi''\big(u_n(t_n)\big)\big\|_\infty\big\|\nabla u_n(t_n) + \nabla u\big\|_\infty\big\|\nabla u_n(t_n) - \nabla u\big\| \\
&\qquad + c\|\nabla u\|^2_\infty\big\|\varphi'''\big(\theta_2 u_n(t_n) + (1-\theta_2)u\big)\big\|_\infty\big\|u_n(t_n)u\big\| \\
&\quad \le c\big(\big\|\Delta u_n(t_n) - \Delta u\big\| + \big\|\nabla u_n(t_n) - \nabla u\big\| + \big\|u_n(t_n) - u\big\|\big) \\
&\quad \to 0,
\end{aligned} \tag{39}$$

where $\theta_1, \theta_2 \in (0,1)$. Using the same method as above, we also have

$$\left\| \nabla\psi\big(u_n(t_n)\big) - \nabla\psi(u) \right\| \to 0.$$

Therefore

$$\gamma A^2 u_n(t_n) \to -u_t + \Delta\varphi(u) + \beta\cdot\nabla\psi(u), \quad \text{strongly in } H,$$

that is, $\{u_n(t_n)\}_{n=1}^{\infty}$ converges to $A^{-2}(-v + \Delta\varphi(u) + \beta\cdot\nabla\psi(u))$ in $\dot{H}^4_{per}(\Omega)$. Then we complete the proof. □

Now we give the proof of the main result.

Proof of Theorem 1.3 Note that $\{S(t)\}_{t\geq 0}$ has an $(\dot{H}^1_{per}, \dot{H}^1_{per})$-global attractor $\mathcal{A}$. By Lemma 2.7, B_2 is a bounded $(\dot{H}^1_{per}, \dot{H}^4_{per})$-absorbing set for $\{S(t)\}_{t\geq 0}$. On the other hand, by Lemma 3.1, we can obtain $\{S(t)\}_{t\geq 0}$ is $(\dot{H}^1_{per}, \dot{H}^4_{per})$-asymptotically compact. Then, by Proposition 1.2, $\mathcal{A}$ is actually an $(\dot{H}^1_{per}, \dot{H}^4_{per})$-global attractor for $\{S(t)\}_{t\geq 0}$. The proof of Theorem 1.3 is complete. □

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Authors' contributions
The main idea of this paper was proposed by XZ. XZ prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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