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New approaches for periodic wave solutions of a non-Newtonian filtration equation with variable delay

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Abstract

A type of non-Newtonian filtration equations with variable delay is considered. Using a new approach which was established by Ge and Ren in (Nonlinear Anal. 58:477–488, 2004), we obtain the existence of periodic wave solutions for the non-Newtonian filtration equations. The methods of the present paper are markedly different from the existing ones.

MSC: 35C07

Keywords: Periodic wave solution; Continuation theorem; Existence

1 Introduction

This paper is devoted to studying the periodic wave solutions problem for a type of non-Newtonian filtration equation with variable delay as follows:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial y}{\partial x}\right|^{p-2}\frac{\partial y}{\partial x}\right) + f(y, y_{\tau(t)}) + g(t,x), \quad t \geq 0, x \in \mathbb{R}, \tag{1.1}$$

where $p > 1$, $f(y,z) = y^q(1-y)(z-a) + f(y)$, $q > 0$, $a \in (0,1)$ is a constant, $f(y) \in C(\mathbb{R},\mathbb{R})$ $y_{\tau(t)}(t,x) = y(t-\tau(t),x)$, $\tau(t) \in C(\mathbb{R},\mathbb{R})$.

Equation (1.1) is known as the evolutionary p-Laplacian equation. When $f(y) = 0$ and $g(t,x) = 0$ in Eq. (1.1), Eq. (1.1) is changed into

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial y}{\partial x}\right|^{p-2}\frac{\partial y}{\partial x}\right) + y^q(1-y)(z-a), \quad t \geq 0, x \in \mathbb{R}. \tag{1.2}$$

In 1967, Ladyzhenskaja [2] studied Eq. (1.2) for the description of incompressible fluids and solvability in the large boundary value of them, which is the first work for Eq. (1.2). After that, more related papers for non-Newtonian filtration equation and related nonlinear equation appeared, see e.g. [3–17].

In recent years, the solitary wave and periodic wave solutions for the non-Newtonian filtration equation have received great attention. Kong and Luo [18, 19] considered a non-Newtonian filtration equation with nonlinear sources and the variable (constant) delay and

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obtained the existence of solitary wave and periodic wave solutions for the above equation by using the extension of Mawhin's continuation theorem. Then, by the same way, Lian et al. [20] further studied a kind of singular non-Newtonian filtration equations.

As far as we know, the existence of periodic wave solutions for functional differential equations was obtained by the use of an extension of coincidence degree theory (see [21], Theorem 3.1). This paper aims to use a different method for obtaining the existence of periodic wave solutions for Eq. (1.1). In [1], Ge and Ren extended Mawhin's continuation theorem to a more general theorem for studying the p-Laplacian differential equations. Classic Mawhin's continuation theorem is used for a class of differential equations with weaker nonlinearity, see [22]. Hence, classic Mawhin's continuation theorem is no longer applicable for studying differential equations with stronger nonlinearity. In this paper, we use the theorem belonging to [1] to obtain the existence of periodic wave solutions for Eq. (1.1). To the best of our knowledge, there is no paper to use the theorem in [1] for studying the non-Newtonian filtration equations. The main purpose is to recommend a new method for the research of non-Newtonian filtration equations.

For Eq. (1.1), assume that there is a continuous function $h(t)$ such that $g(t,x) = h(x + ct)$, where $c \in \mathbb{R}$. Let $y(t,x) = u(s)$ with $s = x + ct$ be the solution of Eq. (1.1), then Eq. (1.1) is changed into the following equation:

$$cu'(s) = \left(\phi_p\left(u'(s)\right)\right)' + u^q(s)\left(1 - u(s)\right)\left[u\left(s - \tau(s)\right) - a\right] + f\left(u(s)\right) + h(s), \tag{1.3}$$

where $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $u \in \mathbb{R}$.

Definition 1.1 Let $T > 0$ be a constant. Suppose that $u(s + T) = u(s)$ and $u(s)$ is a solution of Eq. (1.3) for $s \in \mathbb{R}$, then $u(s)$ is called a periodic wave solution of Eq. (1.3). Generally, the periodic solution of Eq. (1.3) is regarded as a periodic wave solution of Eq. (1.1).

The following sections are organized as follows: In Sect. 2, we give some useful lemmas and definitions. In Sect. 3, main results are obtained for the existence of periodic wave solutions to the non-Newtonian filtration equation (1.1). In Sect. 4, an example is given to show the feasibility of our results. Finally, some conclusions and discussions are given about this paper.

2 Preliminary

Definition 2.1 ([1]) Let $\mathcal{X}$ and $\mathcal{Z}$ be two Banach spaces with norms $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Z}}$, respectively. A continuous operator

$$\mathcal{M} : \mathcal{X} \cap \operatorname{dom} \mathcal{M} \to \mathcal{Z}$$

is called quasi-linear if

(i) $\operatorname{Im} \mathcal{M} := \mathcal{M}(\mathcal{X} \cap \operatorname{dom} \mathcal{M})$ is a closed subset of $\mathcal{Z}$;

(ii) $\operatorname{Ker} \mathcal{M} := \{x \in \mathcal{X} \cap \operatorname{dom} M : \mathcal{M}x = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,

where $\operatorname{dom} \mathcal{M}$ is the domain of $\mathcal{M}$.

Definition 2.2 ([1]) Let $\Omega \subset \mathcal{X}$ be an open and bounded set with the origin $\theta \in \Omega$. $N_\lambda : \bar{\Omega} \to \mathcal{Z}$, $\lambda \in [0,1]$ is said to be $\mathcal{M}$-compact in $\bar{\Omega}$ if there exist a subset $\mathcal{Z}_1$ of $\mathcal{Z}$ satisfying $\dim \mathcal{Z}_1 = \dim \operatorname{Ker} \mathcal{M}$ and an operator $R : \bar{\Omega} \times [0,1] \to \mathcal{X}_2$ being continuous and compact such that, for $\lambda \in [0,1]$,

(a) $(I-Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im}\mathcal{M} \subset (I-Q)\mathcal{Z}$,

(b) $QN_\lambda x = 0$, $\lambda \in (0,1) \Leftrightarrow QNx = 0$, $\forall x \in \Omega$,

(c) $R(\cdot,0) \equiv 0$ and $R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}$,

(d) $\mathcal{M}[Pu + R(\cdot,\lambda)] = (I-Q)N_\lambda$, $\lambda \in [0,1]$,

where $\mathcal{X}_2$ is a complement space of $\operatorname{Ker}\mathcal{M}$ in $\mathcal{X}$, i.e., $\mathcal{X} = \operatorname{Ker}\mathcal{M} \oplus \mathcal{X}_2$; P, Q are two projectors satisfying $\operatorname{Im}P = \operatorname{Ker}\mathcal{M}$, $\operatorname{Im}Q = \mathcal{Z}_1$, $N = N_1$, $\Sigma_\lambda = \{x \in \bar{\Omega} : \mathcal{M}x = N_\lambda x\}$.

Lemma 2.1 ([1]) *Let $\mathcal{X}$ and $\mathcal{Z}$ be two Banach spaces with norms $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Z}}$, respectively. Let $\Omega \subset X$ be an open and bounded nonempty set. Suppose that*

$$\mathcal{M} : \mathcal{X} \cap \operatorname{dom}\mathcal{M} \to \mathcal{Z}$$

is quasi-linear and $N_\lambda : \bar{\Omega} \to \mathcal{Z}$, $\lambda \in [0,1]$ is $\mathcal{M}$-compact in $\bar{\Omega}$. In addition, if the following conditions hold:

(A_1) $\mathcal{M}x \neq N_\lambda x$, $\forall (x,\lambda) \in \partial\Omega \times (0,1)$;

(A_2) $QNx \neq 0$, $\forall x \in \operatorname{Ker}\mathcal{M} \cap \partial\Omega$;

(A_3) $\deg\{JQN, \Omega \cap \operatorname{Ker}\mathcal{M}, 0\} \neq 0$, $J : \operatorname{Im}Q \to \operatorname{Ker}\mathcal{M}$ is a homeomorphism.

Then the abstract equation $\mathcal{M}x = Nx$ has at least one solution in $\operatorname{dom}\mathcal{M} \cap \bar{\Omega}$.

Lemma 2.2 ([23]) *If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Let $a > 0$ and $p > 1$ be two constants. Then, for $t \in \mathbb{R}$, the following inequality holds*:

$$\left|f(t)\right| \le (2a)^{-\frac{1}{p}} \left(\int_{t-a}^{t+a} \left|u(s)\right|^p ds \right)^{\frac{1}{p}} + a(2a)^{-\frac{1}{p}} \left(\int_{t-a}^{t+a} \left|u'(s)\right|^p ds \right)^{\frac{1}{p}}.$$

3 Main results

Denote

$$C_T = \left\{x | x \in C(\mathbb{R},\mathbb{R}), x(t+T) = x(t)\right\}, \qquad C_T^1 = \left\{x | x \in C^1(\mathbb{R},\mathbb{R}), x(t+T) = x(t)\right\}.$$

Let $\mathcal{X} = C_T^1$ with the norm $\|x\| = \max\{|x|_0, |x'|_0\}$. Let $\mathcal{Z} = C_T$ with the norm $|x|_0 = \max_{0 \le t \le T} |x(t)|$. Denote the operators $\mathcal{M}$, N_λ as follows:

$$\mathcal{M} : \operatorname{dom}\mathcal{M} \cap \mathcal{X} \to \mathcal{Z}, \qquad (\mathcal{M}u)(t) = \left(\phi_p\left(u'\right)\right)'(t), \quad t \in \mathbb{R}, \tag{3.1}$$

$N_\lambda : \mathcal{Z} \to \mathcal{Z}$,

$$\begin{aligned} (N_\lambda u)(s) = {} & c\lambda u'(s) - \lambda u^q(s)\left(1 - u(s)\right)\left[u\left(s - \tau(s)\right) - a\right] \\ & - \lambda f\left(u(s)\right) - \lambda h(s), \quad s \in \mathbb{R}, \lambda \in [0,1], \end{aligned} \tag{3.2}$$

where $\operatorname{dom}\mathcal{M} = \{u \in \mathcal{X} : \phi_p(u') \in C_T^1\}$. By (3.1) and (3.2), Eq. (1.1) is equivalent to the operator equation $Nx = \mathcal{M}x$, where $N_1 = N$. Then we have

$$\operatorname{Ker}\mathcal{M} = \left\{u \in \operatorname{dom}\mathcal{M} \cap X : u(t) = \tilde{a}, \tilde{a} \in \mathbb{R}, t \in \mathbb{R}\right\},$$

$$\operatorname{Im}\mathcal{M} = \left\{z \in \mathcal{Z} : \int_0^T z(s)\,ds = 0\right\}.$$

Obviously, $\operatorname{Ker}M \cong \mathbb{R}$, $\operatorname{Im}\mathcal{M}$ is a closed set in $\mathcal{Z}$, so we have the following lemma.

Lemma 3.1 *If the operator $\mathcal{M}$ is defined by (3.1), then $\mathcal{M}$ is a quasi-linear operator.*

Let

$$P:\mathcal{X}\to \operatorname{Ker}\mathcal{M},\qquad (Pu)(s)=u(0),\quad s\in\mathbb{R},$$
$$Q:\mathcal{Z}\to \mathcal{Z}/\operatorname{Im}\mathcal{M},\qquad (Qz)(t)=\frac{1}{T}\int_0^T z(s)\,ds,\quad t\in\mathbb{R}.$$

Now, we show that the operator N_λ is $\mathcal{M}$-compact.

Lemma 3.2 *If $h,\tau\in C(\mathbb{R},\mathbb{R})$ with $h(s)=h(s+T)$ and $\tau(s)=\tau(s+T)$, then N_λ is $\mathcal{M}$-compact.*

Proof For convenience of proof, let

$$G(s,u):=cu'(s)-u^q(s)\bigl(1-u(s)\bigr)\bigl[u\bigl(s-\tau(s)\bigr)-a\bigr]-f\bigl(u(s)\bigr)-h(s). \tag{3.3}$$

Then (3.2) can be rewritten

$$(N_\lambda u)(s)=\lambda G(s,u).$$

Let $\mathcal{Z}_1=\operatorname{Im}Q$. For each bounded set $\overline{\Omega}\subset\mathcal{X}\neq\emptyset$, define the operator $R:\overline{\Omega}\times[0,1]\to\operatorname{Ker}P$ by

$$R(u,\lambda)(s)=\int_0^s\phi_{\tilde{p}}\left[a_u+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\right]dt,\quad s\in[0,T], \tag{3.4}$$

where $\tilde{p}>1$, $\frac{1}{\tilde{p}}+\frac{1}{p}=1$, G is defined by (3.3), and a_u is a constant which depends on u. We claim that a_u exists uniquely in (3.4). Using

$$\int_0^s\phi_{\tilde{p}}\left[a_u+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\right]dt\in C_T,$$

we can choose $a_u\in\mathbb{R}$ such that

$$\int_0^T\phi_{\tilde{p}}\left[a_u+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\right]dt=0. \tag{3.5}$$

Let

$$\mathcal{F}(\tilde{a})=\int_0^T\phi_{\tilde{p}}\left[\tilde{a}+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\right]dt,$$
$$\hat{A}=\sup_{\lambda\in[0,1],t\in[0,T]}\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr,$$
$$\check{A}=\inf_{\lambda\in[0,1],t\in[0,T]}\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr.$$

By (3.5), we have $\mathcal{F}(\hat{A})\geq 0$ and $\mathcal{F}(\check{A})\leq 0$. Thus, $\mathcal{F}(\tilde{a})$ is a monotone increasing function for $\tilde{a}$, and there exists uniquely $a_u\in[\hat{A},\check{A}]$ such that (3.5) holds. Hence, $R(u,\lambda)(s)$ is well defined.

Firstly, we show that $R(u,\lambda)$ is completely continuous on $\overline{\Omega}\times[0,1]$. From the assumptions of Lemma 3.2, $R(u,\lambda)$ is uniformly bounded on $\overline{\Omega}\times[0,1]$. Now we show that $R(u,\lambda)$ is equicontinuous on $\overline{\Omega}\times[0,1]$. Let each $s_1,s_2\in[0,T]$, $\varepsilon>0$ be sufficiently small. There is $\delta>0$, for $|s_1-s_2|<\delta$, we have

$$\begin{aligned}\bigl|R(u,\lambda)(s_1)-R(u,\lambda)(s_2)\bigr| &= \int_0^{s_1}\phi_{\tilde{p}}\biggl[a_u+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\biggr]\,dt\\ &\quad-\int_0^{s_2}\phi_{\tilde{p}}\biggl[a_u+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\biggr]\,dt\\ &=\int_{s_1}^{s_2}\phi_{\tilde{p}}\biggl[a_u+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\biggr]\,dt\\ &<\varepsilon.\end{aligned}$$

Hence, $R(u,\lambda)$ is equicontinuous on $\overline{\Omega}\times[0,1]$. Using the Arzelà–Ascoli theorem, $R(u,\lambda)$ is completely continuous on $\overline{\Omega}\times[0,1]$.

Next, we show that N_λ is $\mathcal{M}$-compact by four steps, i.e., four conditions of Definition 2.2 are all satisfied.

Step 1. In view of $Q^2=Q$, then $Q(I-Q)N_\lambda(\overline{\Omega})=0$ and $(I-Q)N_\lambda(\overline{\Omega})\subset\operatorname{Ker}Q=\operatorname{Im}\mathcal{M}$. Furthermore, $\forall z\in\operatorname{Im}\mathcal{M}$, then $Qz=0$ and $z=(I-Q)z$. Hence, condition (a) of Definition 2.2 holds.

Step 2. $\forall u\in\Omega$, if $QN_\lambda(u)=\frac{\lambda}{T}\int_0^T G(s,u)\,ds=0$, $\lambda\in(0,1)$, then $\frac{1}{T}\int_0^T G(s,u)\,ds=0$, i.e., $QN(u)=0$. The inverse is true.

Step 3. If $\lambda=0$, then $a_u=0$. Thus, $R(u,0)=0$. Furthermore, $\forall u\in\Sigma_\lambda=\{u\in\bar{\Omega}:\mathcal{M}u=N_\lambda u\}$, we have

$$\bigl(\phi_p\bigl(u'\bigr)\bigr)'=\lambda G(s,u)\quad\text{and}\quad Q\bigl[G(s,u)\bigr]=0. \tag{3.6}$$

Choose $a_u=\phi_p(u'(0))$. It follows by (3.6) that

$$\begin{aligned}R(u,\lambda)(s)&=\int_0^s\phi_{\tilde{p}}\biggl[\phi_p\bigl(u'(0)\bigr)+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\biggr]\,dt\\ &=\int_0^s\phi_{\tilde{p}}\biggl[\phi_p\bigl(u'(0)\bigr)+\int_0^t\bigl(\phi_p\bigl(u'(r)\bigr)\bigr)'\,dr\biggr]\,dt\\ &=\int_0^s u'(s)\,ds\\ &=u(s)-u(0)\\ &=\bigl[(I-P)u\bigr](s).\end{aligned}$$

Step 4. $\forall u\in\overline{\Omega}$, $\lambda\in[0,1]$,

$$\begin{aligned}&\mathcal{M}\bigl[Pu+R(u,\lambda)\bigr](s)\\ &\quad=\biggl[\phi_p\biggl(\biggl[u(0)+\int_0^s\phi_{\tilde{p}}\biggl[a_u+\int_0^t\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\biggr]\,dt\biggr]'\biggr)\biggr]'\\ &\quad=\biggl[\phi_p(\phi_{\tilde{p}}\biggl(a_u+\int_0^s\lambda\bigl(G(r,u)-(QG)(r)\bigr)\,dr\biggr)\biggr]'\end{aligned}$$

$$= \left[a_u + \int_0^s \lambda\big(G(r,u) - (QG)(r)\big)\,dr \right]'$$
$$= \big[(I-Q)N_\lambda u\big](s).$$

The proof is completed. □

Remark 3.1 The proof of Lemma 3.2 is similar to the proof of [24]. For the convenience of readers, we give a detailed proof of Lemma 3.2.

In this section, we need the following assumptions:

(H_1) There exists a constant $b > 0$ such that

$$uf(u) \leq -b|u|^p, \quad \forall u \in \mathbb{R},$$

where $p > 1$ is defined by (1.1).

(H_2) There exist constants p and q satisfying the following inequality:

$$p > 2q + 4,$$

where $p > 1$ and $q > 0$ are defined by (1.1).

(H_3) If q is an even number, then there exists a constant $D > 1$ such that

$$f(u) < 0 \quad \text{for } u > D$$

and

$$f(u) < 0 \quad \text{for } u < -D.$$

If q is an odd number, then there exists a constant $D > 1$ such that

$$f(u) < 0 \quad \text{for } u > D$$

and

$$f(u) > 0 \quad \text{for } u < -D.$$

Theorem 3.1 *Suppose that* $\int_0^T h(s)\,ds = 0$, $\tau'(s) < 1$ *for* $s \in [0,T]$, *and assumptions* (H_1)–(H_3) *hold. Then Eq.* (1.1) *has at least one T-periodic wave solution.*

Proof We complete the proof by three steps.

Step 1. Let $\Omega_1 = \{u \in \operatorname{dom}\mathcal{M} : \mathcal{M}u = N_\lambda u, \lambda \in (0,1)\}$. We claim that Ω_1 is a bounded set. In fact, $\forall u \in \Omega_1$, then $\mathcal{M}u = N_\lambda u$, i.e.,

$$\big(\varphi_p(u')\big)'(s) = c\lambda u'(s) - \lambda u^q(s)\big(1-u(s)\big)\big[u\big(s-\tau(s)\big) - a\big] - \lambda f\big(u(s)\big) - \lambda h(s). \tag{3.7}$$

Multiply both sides of Eq. (3.7) by $u(s)$ and integrate over $[0,T]$, then

$$\begin{aligned} |u'|_p^p &= \lambda\int_0^T u^{q+1}(s)\big(1-u(s)\big)\big[u\big(s-\tau(s)\big)-a\big]\,ds \\ &\quad+\lambda\int_0^T f\big(u(s)\big)u(s)\,ds+\lambda\int_0^T h(s)u(s)\,ds. \end{aligned} \tag{3.8}$$

From (3.8), Hölder's inequality, assumptions (H_1) and (H_2), we have

$$\begin{aligned} &|u'|_p^p+\lambda b|u|_p^p \\ &\quad\le\lambda\int_0^T\big|u^{q+1}(s)u\big(s-\tau(s)\big)\big|\,ds+\lambda\int_0^T\big|u^{q+2}(s)u\big(s-\tau(s)\big)\big|\,ds \\ &\qquad+\lambda a\int_0^T\big|u^{q+2}(s)\big|\,ds+\lambda a\int_0^T\big|u^{q+1}(s)\big|\,ds+\lambda\int_0^T\big|h(s)u(s)\big|\,ds \\ &\quad\le\frac{\lambda}{2}\int_0^T|u|^{2(q+1)}(s)\,ds+\lambda\int_0^T\big|u\big(s-\tau(s)\big)\big|^2\,ds+\frac{\lambda}{2}\int_0^T|u|^{2(q+2)}(s)\,ds \\ &\qquad+\lambda aT^{\frac{p-q-2}{p}}|u|_p^{q+2}+\lambda aT^{\frac{p-q-1}{p}}|u|_p^{q+1}+\lambda|h|_{\tilde p}|u|_p \\ &\quad\le\frac{\lambda}{2}T^{\frac{p-2q-2}{p}}|u|_p^{2q+2}+\frac{\lambda}{2}T^{\frac{p-2q-4}{p}}|u|_p^{2q+4}+\lambda\Gamma T^{\frac{p-2}{p}}|u|_p^2 \\ &\qquad+\lambda aT^{\frac{p-q-2}{p}}|u|_p^{q+2}+\lambda aT^{\frac{p-q-1}{p}}|u|_p^{q+1}+\lambda|h|_{\tilde p}|u|_p, \end{aligned} \tag{3.9}$$

where $\Gamma=\max_{t\in[0,T]}\frac{1}{1-\tau'(\mu(t))}$, $\mu(t)$ is an inverse function of $t-\tau(t)$, $\frac{1}{\tilde p}+\frac{1}{p}=1$, $\tilde p>1$. By (3.9) we have

$$\begin{aligned} |u'|_p^p &\le\frac{1}{2}T^{\frac{p-2q-2}{p}}|u|_p^{2q+2}+\frac{1}{2}T^{\frac{p-2q-4}{p}}|u|_p^{2q+4}+\Gamma T^{\frac{p-2}{p}}|u|_p^2 \\ &\quad+aT^{\frac{p-q-2}{p}}|u|_p^{q+2}+aT^{\frac{p-q-1}{p}}|u|_p^{q+1}+|h|_{\tilde p}|u|_p \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |u|_p^p &\le\frac{1}{2b}T^{\frac{p-2q-2}{p}}|u|_p^{2q+2}+\frac{1}{2b}T^{\frac{p-2q-4}{p}}|u|_p^{2q+4}+\frac{1}{b}\Gamma T^{\frac{p-2}{p}}|u|_p^2 \\ &\quad+\frac{a}{b}T^{\frac{p-q-2}{p}}|u|_p^{q+2}+\frac{a}{b}T^{\frac{p-q-1}{p}}|u|_p^{q+1}+\frac{1}{b}|h|_{\tilde p}|u|_p. \end{aligned} \tag{3.11}$$

If $|u|_p>1$, by (3.11) and assumption (H_2), we have

$$\begin{aligned} |u|_p^{p-2q-4} &\le\frac{1}{2b}T^{\frac{p-2q-2}{p}}+\frac{1}{2b}T^{\frac{p-2q-4}{p}}+\frac{1}{b}\Gamma T^{\frac{p-2}{p}} \\ &\quad+\frac{a}{b}T^{\frac{p-q-2}{p}}+\frac{a}{b}T^{\frac{p-q-1}{p}}+\frac{1}{b}|h|_{\tilde p}, \end{aligned}$$

then there exists a constant $M_1>0$ such that

$$|u|_p\le M_1. \tag{3.12}$$

Thus,

$$|u|_p \le \max\{M_1, 1\} =: \widetilde{M}_1. \tag{3.13}$$

On the other hand, if $|u|_p > 1$, by (3.10) and (3.12) then there exists a constant $M_2 > 0$ such that

$$|u'|_p \le M_2.$$

Thus,

$$|u'|_p \le \max\{M_2, 1\} =: \widetilde{M}_2. \tag{3.14}$$

In Lemma 2.2, let $t = 0$, $a = T > 0$. If $f(t) \in C_T^1$, then

$$|f(t)| \le T^{-\frac{1}{p}} \left(\int_0^T |f(s)|^p \, ds \right)^{\frac{1}{p}} + T^{-\frac{1}{p}} \left(\int_0^T |f'(s)|^p \, ds \right)^{\frac{1}{p}}. \tag{3.15}$$

From (3.13)–(3.15), we have

$$|u(s)| \le T^{-\frac{1}{p}} \widetilde{M}_1 + T^{-\frac{1}{p}} \widetilde{M}_2 := \rho_1$$

and

$$|u|_0 = \max_{s\in[0,T]} |u(s)| \le \rho_1. \tag{3.16}$$

By (3.7) we have

$$|u'(s)|^{p-1} \le c|u'(s)| + \rho_1^q(1+\rho_1)[\rho_1 + a] + f_{\rho_1} + |h|_0, \quad s \in [0,T], \tag{3.17}$$

where $f_{\rho_1} = \max_{|u|\le\rho_1} |f(u)|$. If $|u'(s)| > 1$, by (3.17) we get

$$|u'(s)|^{p-2} \le c + \rho_1^q(1+\rho_1)[\rho_1 + a] + f_{\rho_1} + |h|_0, \quad s \in [0,T]. \tag{3.18}$$

In view of (3.18) and $p > 2$, there exists a positive constant L_1 such that

$$|u'(s)| \le \max\{1, L_1\} := \rho_2.$$

Thus,

$$|u'|_0 = \max_{s\in[0,T]} |u'(s)| \le \rho_2. \tag{3.19}$$

In view of (3.16) and (3.19), we have

$$\|u\| < \max\{\rho_1, \rho_2\} + 1 := \rho. \tag{3.20}$$

Step 2. Let $\Omega_2 = \{u \in \operatorname{Ker}\mathcal{M} : QNu = 0\}$. We claim that Ω_2 is a bounded set. In fact, $\forall u \in \Omega_2$, then $u = e_0$, $e_0 \in \mathbb{R}$. Then

$$\int_0^T \bigl[e_0^q(1-e_0)[e_0-a] + f(e_0)\bigr]\,ds = 0,$$

i.e.,

$$e_0^q(1-e_0)[e_0-a] + f(e_0) = 0. \tag{3.21}$$

If q is even and $|e_0| > D$, by assumption (H_3) then

$$e_0^q(1-e_0)[e_0-a] + f(e_0) < 0$$

which is a contradiction to (3.21). Thus, $|e_0| \leq D$. If q is odd, by assumption (H_3) we also have $|e_0| \leq D$. Hence, Ω_2 is a bounded set.

Step 3. Let $\Omega = \{u \in \mathcal{X} : \|u\| < \rho\}$, where ρ is defined by (3.20), then $\Omega_1 \cup \Omega_2$ is a subset of Ω. Based on the above proof, $\forall (u,\lambda) \in \partial\Omega \times (0,1)$, $\mathcal{M}u \neq N_\lambda u$ holds. Due to the results of Step 2, condition (A_2) of Lemma 2.1 is also satisfied. We claim that condition (A_3) of Lemma 2.1 is satisfied. In fact, take the homotopy

$$H(u,\mu) = \mu u + (1-\mu)JQNu, \quad u \in \bar{\Omega} \cap \operatorname{Ker}\mathcal{M}, \mu \in [0,1],$$

where $J : \operatorname{Im} Q \to \operatorname{Ker}\mathcal{M}$ is a homeomorphism with $Je = e$, $e \in \mathbb{R}$. $\forall u \in \partial\Omega \cap \operatorname{Ker}\mathcal{M}$, then $u = e_1$, $|e_1| = \rho > D$, and

$$\begin{aligned} H(u,\mu) &= -e_1\mu + (1-\mu)\frac{1}{T}\int_0^T \bigl[-e_1^q(1-e_1)(e_1-a) - f(e_1)\bigr]\,ds \\ &= -e_1\mu - (1-\mu)\bigl[e_1^q(1-e_1)(e_1-a) + f(e_1)\bigr]. \end{aligned}$$

By using assumption (H_1), we have $H(u,\mu) \neq 0$. And then, by the degree theory,

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} &= \deg\bigl\{H(\cdot,0), \Omega \cap \operatorname{Ker}\mathcal{M}, 0\bigr\} \\ &= \deg\bigl\{H(\cdot,1), \Omega \cap \operatorname{Ker}\mathcal{M}, 0\bigr\} \\ &= \deg\{-I, \Omega \cap \operatorname{Ker}\mathcal{M}, 0\} \neq 0. \end{aligned}$$

Applying Lemma 2.1, we reach the conclusion. □

4 Example

As an application of Theorem 3.1, we consider the following equation:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial y}{\partial x}\right|^{p-2}\frac{\partial y}{\partial x}\right) + y^q(1-y)(y_{\tau(t)} - a) + g(t,x), \quad t \geq 0, x \in \mathbb{R}. \tag{4.1}$$

Let $g(t,x) = h(x+ct)$, $y(t,x) = u(x+ct) = u(s)$, then (4.1) is transformed into the following equation:

$$cu'(s) = \bigl(\phi_p\bigl(u'(s)\bigr)\bigr)' + u^q(s)\bigl(1-u(s)\bigr)\bigl[u\bigl(s-\tau(s)\bigr) - a\bigr] + f\bigl(u(s)\bigr) + h(s).$$

Let $p = 8$, $q = 1$, $f(u) = -u^9 e^{\sin u}$, $\tau(s) = \frac{1}{2}\cos s$, $h(s) = \sin s$. It is not difficult to verify that assumptions (H_1)–(H_3) hold. Therefore, Theorem 3.1 guarantees the existence of at least one periodic wave solution for Eq. (4.1).

5 Conclusion

In this article, we study non-Newtonian filtration equations with variable delay. By using a generalization of Mawhin's continuation theorem and some mathematic analysis methods, we obtain some existence results of periodic wave solutions for the non-Newtonian filtration equation with variable delay. The novelty of the present paper is that it is the first to discuss the existence of periodic wave solutions for the non-Newtonian filtration equations with time-varying delay. Our results improve and extend some corresponding results in the literature. However, many important questions about non-Newtonian filtration equations remain to be studied, such as exponential stability and asymptotic stability problems, non-Newtonian filtration equations with impulse effects and stochastic effects, etc. We hope to focus on the above issues in future studies.

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