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Existence of an approximate solution for a class of fractional multi-point boundary value problems with the derivative term

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Abstract

In this paper, we consider a class of fractional boundary value problems with the derivative term and nonlinear operator term. By establishing new mixed monotone fixed point theorems, we prove these problems to have a unique solution, and we construct the corresponding iterative sequences to approximate the unique solution.

MSC: 65R20; 34B15; 34B18

Keywords: Fractional equation; Mixed monotone operator; Partial order method; Derivative term

1 Introduction

In this paper, we consider the existence and uniqueness of nontrivial solutions of the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\beta}(D_{0^+}^{\alpha}u(t)) = f(t,u(t),D_{0^+}^{\nu}u(t)) + g(t,u(t),(\mathcal{K}u)(t)) - b, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0^+}^{\alpha}u(0) = (D_{0^+}^{\alpha}u)'(0) = \cdots = (D_{0^+}^{\alpha}u)^{(n-2)}(0) = 0, \\ D_{0^+}^{\gamma}u(1) = \sum_{i=1}^{m-2}\xi_i D_{0^+}^{\gamma}u(\eta_i), \qquad D_{0^+}^{\alpha}u(1) = \sum_{i=1}^{m-2}\zeta_i D_{0^+}^{\alpha}u(\eta_i), \end{cases} \tag{1.1}$$

where $b > 0$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$, $D_{0^+}^{\gamma}$, $D_{0^+}^{\nu}$ are the Riemann–Liouville fractional derivatives with $n-1 < \alpha, \beta \le n$, $n-2 < \gamma \le n-1$, $n \ge 2$ $(n \in \mathbb{N})$, $\alpha - \gamma - 1 > 0$, $0 < \nu \le \gamma$, $0 < \xi_i, \eta_i, \zeta_i < 1$, $i = 1,2,3,\ldots,m-2$, $m \ge 2$, $\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\gamma-1} < 1$, $\sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1} < 1$. $f,g : [0,1] \times (-\infty,+\infty) \times (-\infty,+\infty) \longrightarrow (-\infty,+\infty)$ are continuous.

In recent years, much attention has been paid to multi-point boundary value problems involving fractional order; see [1–29] and the references therein. We should mention related studies in [1–22], which motivated us to consider the problem (1.1). Lv [1] studied the existence of positive solutions of the following multi-point boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \qquad D_{0^+}^{\beta}u(1) = \sum_{i=1}^{m-2}\xi_i D_{0^+}^{\beta}u(\eta_i), \end{cases} \tag{1.2}$$

where $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 \leq \alpha - \beta - 1$, $0 < \xi_i, \eta_i < 1$ $(i = 1, 2, \dots, m-2)$, and $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} < 1$. $f : [0,1] \times [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous function. Lv [1] first obtained the Green's function of linear boundary value problem corresponding to the problem (1.2), which has been adopted in the proof of main theorems in [2–4]. Furthermore, Lv [2] studied the existence of solutions for nonlinear fractional m-point boundary value problems involving p-Laplacian operators by the fixed point index theorem. Li, Luo and Zhou [5] discussed Eq. (1.2) in the case of $m = 3$ by using some fixed point theorems.

Very recently, Wang, Zhang and Wang [6] considered the following nonlinear fractional boundary value problem:

$$\begin{cases} -D_{0^+}^{\alpha} u(t) = f(t, u(t), u(t)) + g(t, u(t)), & 0 < t < 1, n-1 < \alpha \leq n, \\ u^{(i)}(0) = 0, \quad i = 0, \dots, n-2, & D_{0^+}^{\nu} u(1) = b D_{0^+}^{\nu} u(\xi), \quad n-2 < \nu \leq n-1, \end{cases} \tag{1.3}$$

where $n - 1 < \alpha \leq n$ $(n > 2, n \in \mathbb{N})$, $n - 2 < \nu \leq n - 1$, $0 \leq b \leq 1$, $0 < \xi < 1$ satisfying $\alpha - \nu - 1 \geq 0$ and $0 \leq b\xi^{\alpha-\nu-1} < 1$. They established the existence and uniqueness of solutions of (1.3) by applying the properties of Green function and fixed point theorems for sum-type operator. On the other hand, they also gave the physical application of our system (1.1). The main feature of [6] is that the value of α is extended from $1 < \alpha \leq 2$ in (1.2) to $n - 1 < \alpha \leq n$. Liang and Zhang [7] considered the existence of solutions of the problem (1.3) when $n = 4$, $\nu = 2$ and $f(t, u(t), u(t)) = 0$. Moreover, Jleli and Samet [8] gave some sufficient conditions under which the problem (1.3) has a unique positive solution when $b = 0$.

We note that Wang [9] studied the existence and uniqueness of positive solutions for singular fractional differential equations as follows:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + p(t) f(t, u(t), D_{0^+}^{\beta} u(t)) + q(t) g(t, u(t), (Hu)(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & [D_{0^+}^{\gamma} u(t)]_{t=1} = k(u(1)), \end{cases} \tag{1.4}$$

where $n - 1 < \alpha \leq n$, $n > 3$, $1 \leq \beta \leq \gamma \leq n - 2$, $p, q \in C((0,1), [0,+\infty))$, $p(t)$ and $q(t)$ are allowed to be singular at $t = 0$ or $t = 1$. $f, g : (0,1) \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ are continuous, and $k : [0,1) \longrightarrow [0,+\infty)$ is also continuous. What attracts our attention is the nonlinear term contains not only the derivative term but also the operator term (Hu). Similarly, Zhang and Tian [10] also studied the problem (1.4) with derivative term, but the difference is that the function g does not include the operator term. In [11], Ji et al. also investigated positive solutions for the nonlinear fractional differential equation with a derivative term. Goodrich [12] first obtained the Green function of the problem (1.4) when $k(u(1)) = 0$. In [13–15], they considered the fractional differential equations with integer order derivative, and they did not consider the boundary condition $[D_{0^+}^{\gamma} u(t)]_{t=1}$. Zhang [16] considered the singular fractional differential equations with multiple derivative terms, and obtained the existence of positive solutions.

We should mention the work of Jong [3], which directly is related to our problem (1.1). Jong investigated the following nonlinear fractional m-point boundary value problem with

p-Laplacian operator:

$$\begin{cases} D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u))(t) = f(t,u(t)), \quad t \in (0,1), \\ u(0) = 0, \qquad D_{0^+}^{\alpha}u(0) = 0, \\ D_{0^+}^{\gamma}u(1) = \sum_{i=1}^{m-2}\xi_i D_{0^+}^{\gamma}u(\eta_i), \qquad \varphi_p(D_{0^+}^{\alpha}u)(1) = \sum_{i=1}^{m-2}\zeta_i\varphi_p(D_{0^+}^{\alpha}u)(\eta_i), \end{cases} \tag{1.5}$$

where $1 < \alpha, \beta \leq 2, 3 < \alpha + \beta \leq 4, 0 < \gamma \leq 1, \alpha - \gamma - 1 > 0, 0 < \xi_i, \eta_i, \zeta_i < 1, \sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\gamma-1} < 1$, $\sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1} < 1$, the p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. Jong obtained that the problem (1.5) has a unique solution which is given by $u(t) = \int_0^1 G(t,s) \times \varphi_p^{-1}(\int_0^1 H(s,\tau)f(\tau,u(\tau))\,d\tau)\,ds$. He first gave the Green function $H(s,\tau)$. The main tool of [3] is the Banach contraction mapping principle. Furthermore, he also showed the uniqueness of the problem (1.5) in [4] by the classic fixed point theorem of mixed monotone operators. Li and Qi [17] focused on p-Laplacian boundary value problems of higher order nonlinear differential equations. Tan and Li [18] used Kuratowski's noncompactness measure and Sadovskii's fixed point theorem to study the problem (1.5) when the boundary condition $\varphi_p(D_{0^+}^{\alpha}u)(1) = \sum_{i=1}^{m-2}\zeta_i\varphi_p(D_{0^+}^{\alpha}u)(\eta_i)$ is removed. Wang and Xiang [19] considered the problem (1.5) when all boundary conditions are replaced by $u(0) = 0$, $D_{0^+}^{\alpha}u(0) = 0$, $u(1) = au(\xi)$ and $D_{0^+}^{\nu}u(1) = bD_{0^+}^{\alpha}u(\eta)$. Wang, Xiang and Liu [20] investigated the problem (1.5) when the boundary conditions are replaced by $u(0) = 0$, $D_{0^+}^{\alpha}u(0) = 0$ and $u(1) = au(\xi)$.

We should point out that the main tools and methods adopted in [1–20] are cone mapping theory. Therefore, nonlinearities in the problems (1.2)–(1.5) are usually required to be non-negative. But more and more authors are beginning to remove this restriction imposed on nonlinear terms. Very recently, Sang and Ren [21] dealt with the following fractional boundary value problem:

$$\begin{cases} -D_{0^+}^{\alpha}u(t) = f(t,u(t),u(t)) + g(t,u(t),u(t)) - b, \quad 0 < t < 1, \\ u^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \qquad [D_{0^+}^{\beta}u(t)]_{t=1} = 0, \end{cases} \tag{1.6}$$

where $n - 1 < \alpha \leq n$, $1 \leq \beta \leq n - 2$, $n \geq 3$ ($n \in \mathbb{N}$), $b > 0$ is a constant, $f, g : [0,1] \times (-\infty, +\infty) \times (-\infty, +\infty) \longrightarrow (-\infty, +\infty)$ are two continuous functions. In fact, Zhai and Wang [22] have introduced φ–(h,e) operators, and established the existence and uniqueness of a nontrivial solution for a class of nonlinear fractional equations by using partial order method.

In this paper, the first goal is to obtain the fixed point of the solution of the operator equation $M(x,x) + N(x,x) + e = x$, where M and N are two mixed monotone operators. We will generalize the results of cone mapping to the non-cone case. Then we will provide some sufficient conditions under which the problem (1.1) has a unique solution and construct two iterative sequences of unique solution. Compared with [6, 9], we do not demand the assumption that nonlinearities are non-negative, and the more general boundary conditions are adopted.

Our paper is organized as follows. In Sect. 2, we will introduce some definitions and give some lemmas to prove the main conclusions. In Sect. 3, the existence of fixed point for the operator equation associated with the problem (1.1) is established. Then, based on our abstract results, the existence and uniqueness of the solution of the problem (1.1) are proved.

2 Preliminaries and related lemmas

In this section, we give some definitions and lemmas that are useful for the proof of our main results.

In this paper, $(E, \|\cdot\|)$ is a real Banach space. A partially ordered structure in E is induced by a cone $P \subset E$, i.e. $x \leq y$ if and only if $y - x \in P$. θ is the zero element. P is called normal if there exists $N > 0$ such that $\theta \leq x \leq y \Rightarrow \|x\| \leq N\|y\|$. Given $h > \theta$, we denote by P_h

$$P_h = \{x \in E \mid \text{there exist } \lambda > 0, \mu > 0 \text{ such that } \lambda h \leq x \leq \mu h\}.$$

Let $e \in P$ with $\theta \leq e \leq h$, denote

$$P_{h,e} = \{x \in E \mid x + e \in P_h\}.$$

Definition 2.1 ([30, 31]) If $A(x, y)$ is increasing in x, and decreasing in y, then $A : P_{h,e} \times P_{h,e} \to E$ is a mixed monotone operator. That is, for every $u_i, v_i \in P_{h,e}$ $(i = 1, 2)$ with $u_1 \geq v_1$, $u_2 \leq v_2$, we have $A(u_1, u_2) \geq A(v_1, v_2)$.

Definition 2.2 ([32, 33]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $h \in C[0, 1]$ is defined by

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t h(s)(t-s)^{n-\alpha-1}\, ds,$$

where $n = [\alpha] + 1$. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function h is given by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds.$$

Definition 2.3 ([32]) Let $\alpha > -1$, $\nu > 0$ and $t > 0$. Then

$$D_{0^+}^{\nu} t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\nu+1)} t^{\alpha-\nu}.$$

Lemma 2.1 ([34]) *Let $u \in C[0, 1] \cap L^1[0, 1]$, $\alpha > 0$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\alpha] + 1$.

Lemma 2.2 *Let $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\gamma-1} \neq 1$. If $y(t) \in C[0, 1]$, then the boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, n-1 < \alpha \leq n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0^+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0^+}^{\gamma} u(\eta_i), \quad n-2 < \gamma \leq n-1, \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s)\, ds,$$

where

$$G(t,s)=G_1(t,s)+G_2(t,s),$$

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}t^{\alpha-1}(1-s)^{\alpha-\gamma-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-\gamma-1}, & 0\le t\le s\le 1,\end{cases}$$

and

$$G_2(t,s)=\frac{1}{A\Gamma(\alpha)}\begin{cases}t^{\alpha-1}\sum_{0\le s\le\eta_i}\xi_i[\eta_i^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}-(\eta_i-s)^{\alpha-\gamma-1}], & 0\le t\le 1,\\ t^{\alpha-1}\sum_{\eta_i\le s\le 1}\xi_i\eta_i^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}, & 0\le t\le 1,\end{cases}$$

with

$$A=1-\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\gamma-1}.$$

Proof Using Lemma 2.1, we get

$$u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}=-I_{0^+}^{\alpha}y(t).$$

It follows from the condition $u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0$ that $c_n=c_{n-1}=\cdots=c_2=0$. Thus

$$u(t)=-I_{0^+}^{\alpha}y(t)-c_1t^{\alpha-1}.$$

The rest of our proof can be obtained from Lemma 2.1 in [1]. □

Lemma 2.3 *Let* $\sum_{i=1}^{m-2}\zeta_i\eta_i^{\beta-1}\neq 1$. *If* $f:[0,1]\times(-\infty,+\infty)\times(-\infty,+\infty)\to(-\infty,+\infty)$ *be a continuous function. Then the problem* (1.1) *has the following unique solution*:

$$u(t)=\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)\bigl(f\bigl(\tau,u(\tau),D_{0^+}^{\nu}u(\tau)\bigr)+g\bigl(\tau,u(\tau),(\mathcal{K}u)(\tau)\bigr)-b\bigr)\,d\tau\right)ds,$$

where

$$H(t,s)=H_1(t,s)+H_2(t,s),$$

in which

$$H_1(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}t^{\beta-1}(1-s)^{\beta-1}-(t-s)^{\beta-1}, & 0\le s\le t\le 1,\\ t^{\beta-1}(1-s)^{\beta-1}, & 0\le t\le s\le 1,\end{cases}$$

$$H_2(t,s)=\frac{1}{B\Gamma(\beta)}\begin{cases}t^{\beta-1}\sum_{0\le s\le\eta_i}\zeta_i[\eta_i^{\beta-1}(1-s)^{\beta-1}-(\eta_i-s)^{\beta-1}], & 0\le t\le 1,\\ t^{\beta-1}\sum_{\eta_i\le s\le 1}\zeta_i\eta_i^{\beta-1}(1-s)^{\beta-1}, & 0\le t\le 1,\end{cases}$$

where

$$B = 1 - \sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1}.$$

Proof Let $h \in C[0,1]$, consider the boundary value problem:

$$\begin{cases} D_{0^+}^{\beta} v(t) + h(t) = 0, & 0 < t < 1, n-1 < \beta \le n, \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, & v(1) = \sum_{i=1}^{m-2} \zeta_i v(\eta_i). \end{cases}$$

Similarly, using Lemma 2.1, we deduce

$$v(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_n t^{\beta-n} = -I_{0^+}^{\beta} h(t).$$

It follows from the condition $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0$ that $c_n = c_{n-1} = \cdots = c_2 = 0$. Thus

$$v(t) = -I_{0^+}^{\beta} h(t) - c_1 t^{\beta-1}.$$

The rest of our proof can be derived from Lemma 2.4 in [3]. □

Lemma 2.4 *Let*

$$C(s) = \frac{1}{A} \sum_{0 \le s \le \eta_i} \xi_i \bigl[\eta_i^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1} - (\eta_i - s)^{\alpha-\gamma-1}\bigr] + \sum_{s \ge \eta_i} \xi_i \eta_i^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1},$$

$$D = \frac{1}{A}\left(1 + \sum_{i=1}^{m-2} \xi_i \bigl(1 - \eta_i^{\alpha-\gamma-1}\bigr)\right).$$

Then the function $G(t,s)$ defined in Lemma 2.2 satisfies the following conditions:

$$C(s) t^{\alpha-1} \le \Gamma(\alpha) G(t,s) \le D t^{\alpha-1},$$

and

$$C(s) t^{\alpha-\nu-1} \le \Gamma(\alpha-\nu) D_{0^+}^{\nu} G(t,s) \le D t^{\alpha-\nu-1},$$

for every $t, s \in [0,1]$.

Proof Since $G_1(t,s) \ge 0$ for $t \in [0,1]$, $s \in [0,1]$, it follows that

$$G(t,s) \ge G_2(t,s) = \frac{C(s)}{\Gamma(\alpha)} t^{\alpha-1}.$$

At the same time, we have

$$\begin{aligned} G(t,s) &= G_1(t,s) + G_2(t,s) \\ &\leq \frac{1}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i t^{\alpha-1} \\ &= \frac{1}{A\Gamma(\alpha)} \left(1 + \sum_{i=1}^{m-2} \xi_i \bigl(1 - \eta_i^{\alpha-\gamma-1}\bigr)\right) t^{\alpha-1} \\ &= \frac{D}{\Gamma(\alpha)} t^{\alpha-1}. \end{aligned}$$

Consequently

$$C(s)t^{\alpha-1} \leq \Gamma(\alpha) G(t,s) \leq D t^{\alpha-1}.$$

Similarly

$$D_{0^+}^{\nu} G(t,s) \geq D_{0^+}^{\nu} G_2(t,s) = \frac{C(s)}{\Gamma(\alpha-\nu)} t^{\alpha-\nu-1}$$

and

$$\begin{aligned} D_{0^+}^{\nu} G(t,s) &= D_{0^+}^{\nu} G_1(t,s) + D_{0^+}^{\nu} G_2(t,s) \\ &\leq \frac{1}{\Gamma(\alpha-\nu)} t^{\alpha-\nu-1} + \frac{1}{A\Gamma(\alpha-\nu)} \sum_{i=1}^{m-2} \xi_i t^{\alpha-\nu-1} \\ &= \frac{1}{A\Gamma(\alpha-\nu)} \left(1 + \sum_{i=1}^{m-2} \xi_i \bigl(1 - \eta_i^{\alpha-\gamma-1}\bigr)\right) t^{\alpha-\nu-1} \\ &= \frac{D}{\Gamma(\alpha-\nu)} t^{\alpha-\nu-1}. \end{aligned}$$

Hence

$$C(s)t^{\alpha-\nu-1} \leq \Gamma(\alpha-\nu) D_{0^+}^{\nu} G(t,s) \leq D t^{\alpha-\nu-1}. \qquad \square$$

Lemma 2.5 ([4]) *Let*

$$E(s) = \frac{1}{B} \sum_{0 \leq s \leq \eta_i} \zeta_i \bigl[\eta_i^{\beta-1}(1-s)^{\beta-1} - (\eta_i - s)^{\beta-1}\bigr] + \sum_{s \geq \eta_i} \zeta_i \eta_i^{\beta-1} (1-s)^{\beta-1}$$

and

$$F = \frac{1}{B} \left(1 + \sum_{i=1}^{m-2} \zeta_i \bigl(1 - \eta_i^{\beta-1}\bigr)\right).$$

Then

$$E(s)t^{\beta-1} \leq \Gamma(\beta) H(t,s) \leq F t^{\beta-1},$$

for every $t, s \in [0,1]$.

Lemma 2.6 *Let P be a normal cone and $T : P_{h,e} \times P_{h,e} \longrightarrow E$ be a mixed monotone operator. Assume that the following conditions hold*:

(i) *for every $\lambda \in (0,1)$ and $u, v \in P_{h,e}$, there exists $\varphi(\lambda, u, v) > \lambda$ such that*

$$T\big(\lambda u + (\lambda - 1)e, \lambda^{-1} v + \big(\lambda^{-1} - 1\big)e\big) \geq \varphi(\lambda, u, v) T(u, v) + \big(\varphi(\lambda, u, v) - 1\big)e;$$

(ii) *for fixed $t \in (0,1)$ and $u \in P_{h,e}$, $\varphi(t, u, v)$ is decreasing in v, and for fixed $t \in (0,1)$ and $v \in P_{h,e}$, $\varphi(t, u, v)$ is increasing in u;*

(iii) *there exists $t_0 \in (0,1)$ such that*

$$\frac{t_0}{\varphi(t_0, h, h)} h + \left(\frac{t_0}{\varphi(t_0, h, h)} - 1\right) e \leq T(h, h) \leq \frac{1}{t_0} h + \left(\frac{1}{t_0} - 1\right) e.$$

Then:

(1) *T has a unique fixed point x^* in $P_{h,e}$;*

(2) *there exist initial values $u_0, v_0 \in P_{h,e}$, and $s \in (0,1)$ such that*

$$s v_0 \leq u_0 < v_0, \qquad u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0;$$

(3) *for any $x_0, y_0 \in P_{h,e}$, taking the iterative sequences as follows*:

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Proof By (i), we have

$$\begin{aligned}
&T\big(\lambda^{-1} u + \big(\lambda^{-1} - 1\big)e, \lambda v + (\lambda - 1)e\big) \\
&\quad \leq \big[\varphi\big(\lambda, \lambda^{-1} u + \big(\lambda^{-1} - 1\big)e, \lambda v + (\lambda - 1)e\big)\big]^{-1} T(u, v) \\
&\qquad + \big(\big[\varphi\big(\lambda, \lambda^{-1} u + \big(\lambda^{-1} - 1\big)e, \lambda v + (\lambda - 1)e\big)\big]^{-1} - 1\big)e,
\end{aligned} \tag{2.1}$$

for every $\lambda \in (0,1)$, $u, v \in P_{h,e}$. We can find a positive integer k with

$$\left(\frac{\varphi(t_0, h, h)}{t_0}\right)^k \geq \frac{1}{t_0}.$$

Let

$$\begin{aligned}
&u_n = T(u_{n-1}, v_{n-1}), \qquad v_n = T(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots. \\
&x_n = t_0^n h + \big(t_0^n - 1\big)e, \qquad y_n = t_0^{-n} h + \big(t_0^{-n} - 1\big)e, \quad n = 1, 2, \ldots.
\end{aligned}$$

Thus

$$x_n = t_0 x_{n-1} + (t_0 - 1)e, \qquad y_n = t_0^{-1} y_{n-1} + \big(t_0^{-1} - 1\big)e, \quad n = 1, 2, \ldots.$$

Denote $u_0 := x_k$, $v_0 := y_k$, then $u_0, v_0 \in P_{h,e}$,

$$u_0 \leq v_0, \qquad u_1 = T(u_0, v_0) \leq T(v_0, u_0) = v_1.$$

Since T is mixed monotone, we get

$$u_n \leq v_n, \quad n = 1, 2, \ldots.$$

By the conditions (ii) and (iii), combining with (2.1), we have

$$\begin{aligned}
u_1 &= T(u_0, v_0) \\
&= T\bigl(t_0^k h + \bigl(t_0^k - 1\bigr)e, t_0^{-k} h + \bigl(t_0^{-k} - 1\bigr)e\bigr) \\
&= T\bigl(t_0\bigl(t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e\bigr) + (t_0 - 1)e, t_0^{-1}\bigl(t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr) + \bigl(t_0^{-1} - 1\bigr)e\bigr) \\
&\geq \varphi\bigl(t_0, t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr) T\bigl(t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h \\
&\quad + \bigl(t_0^{-k+1} - 1\bigr)e\bigr) + \bigl(\varphi\bigl(t_0, t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr) - 1\bigr)e \\
&\geq \varphi\bigl(t_0, t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr)\bigl[\varphi\bigl(t_0, t_0^{k-2} h + \bigl(t_0^{k-2} - 1\bigr)e, \\
&\quad t_0^{-k+2} h + \bigl(t_0^{-k+2} - 1\bigr)e\bigr) T\bigl(t_0^{k-2} h + \bigl(t_0^{k-2} - 1\bigr)e, t_0^{-k+2} h + \bigl(t_0^{-k+2} - 1\bigr)e\bigr) \\
&\quad + \bigl(\varphi\bigl(t_0, t_0^{k-2} h + \bigl(t_0^{k-2} - 1\bigr)e, t_0^{-k+2} h + \bigl(t_0^{-k+2} - 1\bigr)e\bigr) - 1\bigr)e\bigr] \\
&\quad + \bigl(\varphi\bigl(t_0, t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr) - 1\bigr)e \\
&= \varphi\bigl(t_0, t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr)\varphi\bigl(t_0, t_0^{k-2} h + \bigl(t_0^{k-2} - 1\bigr)e, t_0^{-k+2} h \\
&\quad + \bigl(t_0^{-k+2} - 1\bigr)e\bigr) T\bigl(t_0^{k-2} h + \bigl(t_0^{k-2} - 1\bigr)e, t_0^{-k+2} h + \bigl(t_0^{-k+2} - 1\bigr)e\bigr) + \bigl[\varphi\bigl(t_0, t_0^{k-1} h \\
&\quad + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr)\varphi\bigl(t_0, t_0^{k-2} h + \bigl(t_0^{k-2} - 1\bigr)e, t_0^{-k+2} h \\
&\quad + \bigl(t_0^{-k+2} - 1\bigr)e\bigr) - 1\bigr]e \\
&\geq \cdots \geq \varphi\bigl(t_0, t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr)\varphi\bigl(t_0, t_0^{k-2} h + \bigl(t_0^{k-2} - 1\bigr)e, \\
&\quad t_0^{-k+2} h + \bigl(t_0^{-k+2} - 1\bigr)e\bigr) \ldots \varphi(t_0, h, h) T(h, h) + \bigl[\varphi\bigl(t_0, t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1} h \\
&\quad + \bigl(t_0^{-k+1} - 1\bigr)e\bigr)\varphi\bigl(t_0, t_0^{k-2} h + \bigl(t_0^{k-2} - 1\bigr)e, t_0^{-k+2} h + \bigl(t_0^{-k+2} - 1\bigr)e\bigr) \cdots \varphi(t_0, h, h) - 1\bigr]e \\
&\geq t_0^{k-1} \varphi(t_0, h, h) T(h, h) + \bigl(t_0^{k-1} \varphi(t_0, h, h) - 1\bigr)e \\
&\geq t_0^{k-1} \varphi(t_0, h, h)\left(\frac{t_0}{\varphi(t_0, h, h)} h + \left(\frac{t_0}{\varphi(t_0, h, h)} - 1\right)e\right) + \bigl(t_0^{k-1} \varphi(t_0, h, h) - 1\bigr)e \\
&= t_0^k h + \bigl(t_0^k - t_0^{k-1} \varphi(t_0, h, h)\bigr)e + \bigl(t_0^{k-1} \varphi(t_0, h, h) - 1\bigr)e \\
&= t_0^k h + \bigl(t_0^k - 1\bigr)e = x_k = u_0
\end{aligned}$$

and

$$\begin{aligned}
v_1 &= T(v_0, u_0) \\
&= T\bigl(t_0^{-k} h + \bigl(t_0^{-k} - 1\bigr)e, t_0^k h + \bigl(t_0^k - 1\bigr)e\bigr) \\
&\leq \varphi\bigl(t_0, t_0^{-k} h + \bigl(t_0^{-k} - 1\bigr)e, t_0^k h + \bigl(t_0^k - 1\bigr)e\bigr)^{-1} T\bigl(t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e, t_0^{k-1} h \\
&\quad + \bigl(t_0^{k-1} - 1\bigr)e\bigr) + \bigl(\varphi\bigl(t_0, t_0^{-k} h + \bigl(t_0^{-k} - 1\bigr)e, t_0^k h + \bigl(t_0^k - 1\bigr)e\bigr)^{-1} - 1\bigr)e \\
&\leq \varphi\bigl(t_0, t_0^{-k} h + \bigl(t_0^{-k} - 1\bigr)e, t_0^k h + \bigl(t_0^k - 1\bigr)e\bigr)^{-1} \varphi\bigl(t_0, t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e, t_0^{k-1} h \\
&\quad + \bigl(t_0^{k-1} - 1\bigr)e\bigr)^{-1} T\bigl(t_0^{-k+1} h + \bigl(t_0^{-k+1} - 1\bigr)e, t_0^{k-1} h + \bigl(t_0^{k-1} - 1\bigr)e\bigr) + \bigl[\varphi\bigl(t_0, t_0^{-k} h
\end{aligned}$$

$$
\begin{aligned}
&+ \left(t_0^{-k} - 1\right)e, t_0^k h + \left(t_0^k - 1\right)e\right)^{-1} \varphi\left(t_0, t_0^{-k+1} h + \left(t_0^{-k+1} - 1\right)e, t_0^{k-1} h \\
&+ \left(t_0^{k-1} - 1\right)e\right)^{-1} - 1\big]e \\
\le{}& \cdots \le \varphi\left(t_0, t_0^{-k} h + \left(t_0^{-k} - 1\right)e, t_0^k h + \left(t_0^k - 1\right)e\right)^{-1} \varphi\left(t_0, t_0^{-k+1} h + \left(t_0^{-k+1} - 1\right)e, \right. \\
&t_0^{k-1} h + \left(t_0^{k-1} - 1\right)e\right)^{-1} \cdots \varphi\left(t_0, t_0^{-1} h + \left(t_0^{-1} - 1\right)e, t_0 h + (t_0 - 1)e\right)^{-1} T(h,h) \\
&+ \big[\varphi\left(t_0, t_0^{-k} h + \left(t_0^{-k} - 1\right)e, t_0^k h + \left(t_0^k - 1\right)e\right)^{-1} \varphi\left(t_0, t_0^{-k+1} h + \left(t_0^{-k+1} - 1\right)e, t_0^{k-1} h \right. \\
&+ \left(t_0^{k-1} - 1\right)e\right)^{-1} \cdots \varphi\left(t_0, t_0^{-1} h + \left(t_0^{-1} - 1\right)e, t_0 h + (t_0 - 1)e\right)^{-1} - 1\big]e \\
\le{}& \varphi\left(t_0, t_0^{-1} h + \left(t_0^{-1} - 1\right)e, t_0 h + (t_0 - 1)e\right)^{-k} T(h,h) \\
&+ \big[\varphi\left(t_0, t_0^{-1} h + \left(t_0^{-1} - 1\right)e, t_0 h + (t_0 - 1)e\right)^{-k} - 1\big]e \\
\le{}& \varphi\left(t_0, t_0^{-1} h, t_0 h\right)^{-k} T(h,h) + \big[\varphi\left(t_0, t_0^{-1} h, t_0 h\right)^{-k} - 1\big]e \\
\le{}& \varphi(t_0, h, h)^{-k} T(h,h) + \big[\varphi(t_0, h, h)^{-k} - 1\big]e \\
\le{}& \varphi(t_0, h, h)^{-k} \big[t_0^{-1} h + \left(t_0^{-1} - 1\right)e\big] + \big[\varphi(t_0, h, h)^{-k} - 1\big]e \\
={}& \varphi(t_0, h, h)^{-k} t_0^{-1} h + \varphi(t_0, h, h)^{-k} t_0^{-1} e - e \\
\le{}& t_0^{-k} h + \left(t_0^{-k} - 1\right)e = y_k = v_0.
\end{aligned}
$$

Thus

$$u_0 \le T(u_0, v_0) \le T(v_0, u_0) \le v_0.$$

We deduce for all $n \in \mathbb{N}$ that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.2}$$

In addition

$$u_n \ge u_0 \ge s v_0 + (s-1)e \ge s v_n + (s-1)e, \quad n = 1, 2, \ldots.$$

Let

$$t_n = \sup\left\{t > 0 \mid u_n \ge t v_n + (t-1)e\right\}.$$

Thus we have $u_n \ge t_n v_n + (t_n - 1)e$, $n = 1, 2, \ldots$. Consequently $\{t_n\}$ is increasing with $\{t_n\} \subset (0, 1]$. Assume that $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. If not, $0 < t^* < 1$.

Next, we need to prove that $t^* = 1$. If $0 < t^* < 1$, we should discuss the following two cases.

Case 1: there is an integer N such that $t_N = t^*$. In this case, we have $t_n = t^*$ for all $n > N$. Then

$$
\begin{aligned}
u_{n+1} = T(u_n, v_n) &\ge T\left(t_n v_n + (t_n - 1)e, t_n^{-1} u_n + \left(t_n^{-1} - 1\right)e\right) \\
&= T\left(t^* v_n + \left(t^* - 1\right)e, \left(t^*\right)^{-1} u_n + \left(\left(t^*\right)^{-1} - 1\right)e\right) \\
&\ge \varphi\left(t^*, v_n, u_n\right) T(v_n, u_n) + \left(\varphi\left(t^*, v_n, u_n\right) - 1\right)e \\
&\ge \varphi\left(t^*, u_0, v_0\right) T(u_0, v_0) + \left(\varphi\left(t^*, u_0, v_0\right) - 1\right)e.
\end{aligned}
$$

We can get $t^* = t_{n+1} \geq \varphi(t^*) > t^*$ from the definition of t_{n+1}, which is a contradiction.

Case 2: for all n, $t_n < t^*$, we have

$$\begin{aligned}
u_{n+1} &= T(u_n, v_n) \geq T\bigl(t_n v_n + (t_n - 1)e, t_n^{-1} u_n + \bigl(t_n^{-1} - 1\bigr)e\bigr) \\
&= T\left(\frac{t_n}{t^*}\bigl(t^* v_n + \bigl(t^* - 1\bigr)e\bigr) + \left(\frac{t_n}{t^*} - 1\right)e, \left(\frac{t_n}{t^*}\right)^{-1}\left(\bigl(t^*\bigr)^{-1} u_n + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right)\right. \\
&\quad \left. + \left(\left(\frac{t_n}{t^*}\right)^{-1} - 1\right)e\right) \\
&\geq \varphi\left(\frac{t_n}{t^*}, t^* v_n + \bigl(t^* - 1\bigr)e, \bigl(t^*\bigr)^{-1} u_n + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right) T\left(t^* v_n + \bigl(t^* - 1\bigr)e, \bigl(t^*\bigr)^{-1} u_n\right. \\
&\quad \left. + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right) + \left(\varphi\left(\frac{t_n}{t^*}, t^* v_n + \bigl(t^* - 1\bigr)e, \bigl(t^*\bigr)^{-1} u_n + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right) - 1\right)e \\
&\geq \varphi\left(\frac{t_n}{t^*}, t^* v_n + \bigl(t^* - 1\bigr)e, \bigl(t^*\bigr)^{-1} u_n + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right)\bigl[\varphi\bigl(t^*, v_n, u_n\bigr) T(v_n, u_n) \\
&\quad + \bigl(\varphi\bigl(t^*, v_n, u_n\bigr) - 1\bigr)e\bigr] \\
&\quad + \left(\varphi\left(\frac{t_n}{t^*}, t^* v_n + \bigl(t^* - 1\bigr)e, \bigl(t^*\bigr)^{-1} u_n + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right) - 1\right)e \\
&\geq \varphi\left(\frac{t_n}{t^*}, t^* u_0 + \bigl(t^* - 1\bigr)e, \bigl(t^*\bigr)^{-1} v_0 + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right)\varphi\bigl(t^*, u_0, v_0\bigr) T(v_n, u_n) \\
&\quad + \left(\varphi\left(\frac{t_n}{t^*}, t^* u_0 + \bigl(t^* - 1\bigr)e, \bigl(t^*\bigr)^{-1} v_0 + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right)\varphi\bigl(t^*, u_0, v_0\bigr) - 1\right)e.
\end{aligned}$$

By the definition of t_{n+1}, we have

$$t_{n+1} \geq \varphi\left(\frac{t_n}{t^*}, t^* u_0 + \bigl(t^* - 1\bigr)e, \bigl(t^*\bigr)^{-1} v_0 + \left(\bigl(t^*\bigr)^{-1} - 1\right)e\right)\varphi\bigl(t^*, u_0, v_0\bigr) \geq \frac{t_n}{t^*}\varphi\bigl(t^*, u_0, v_0\bigr).$$

Let $n \to \infty$, we have $t^* \geq \varphi(t^*, u_0, v_0) > t^*$, which is a contradiction. Consequently $t^* = 1$. Since P is normal, we have

$$\|u_{n+p} - u_n\| \leq M(1 - t_n)\|v_0 + e\|, \qquad \|v_n - v_{n+p}\| \leq M(1 - t_n)\|v_0 + e\|,$$

where M is the normality constant. Let $n \to \infty$, we get

$$\|u_{n+p} - u_n\| \longrightarrow 0, \qquad \|v_n - v_{n+p}\| \longrightarrow 0.$$

Therefore u_n and v_n are Cauchy sequences. Repeating the proof of Lemma 2.3 in Sang and Ren [21], we derive that our conclusions hold. □

Lemma 2.7 *Let P be a normal cone and $T : P_{h,e} \times P_{h,e} \longrightarrow E$ be a mixed monotone operator. Assume that the condition* (i) *in Lemma* 2.6 *is satisfied. In addition, $\varphi(t, u, v)$ is decreasing* *in u and increasing in v for every* $t \in (0,1)$. *Furthermore, there exists* $t_0 \in (0,1)$ *such that*

$$
\begin{aligned}
t_0 h + (t_0 - 1)e \leq T(h,h) & \\
& \leq \frac{1}{t_0}\varphi\bigl(t_0, t_0^{-1}h + \bigl(t_0^{-1} - 1\bigr)e, t_0 h + (t_0 - 1)e\bigr)h \\
& \quad + \biggl[\frac{1}{t_0}\varphi\bigl(t_0, t_0^{-1}h + \bigl(t_0^{-1} - 1\bigr)e, t_0 h + (t_0 - 1)e\bigr) - 1\biggr]e.
\end{aligned}
\tag{2.3}
$$

Then the conclusions (1), (2), (3) *in Lemma* 2.6 *hold.*

Proof As in the proof of Lemma 2.6, we only need to check that $u_1 = T(u_0, v_0) \geq u_0$ and $v_1 = T(v_0, u_0) \leq v_0$ hold. For every $t \in (0,1)$, since $\varphi(t,u,v)$ is decreasing in u and increasing in v, by (2.3), we have

$$
\begin{aligned}
u_1 &= T(u_0, v_0) \\
&= T\bigl(t_0^k h + \bigl(t_0^k - 1\bigr)e, t_0^{-k}h + \bigl(t_0^{-k} - 1\bigr)e\bigr) \\
&\geq \bigl[\varphi\bigl(t_0, t_0^{k-1}h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1}h + \bigl(t_0^{-k+1} - 1\bigr)e\bigr)\varphi\bigl(t_0, t_0^{k-2}h + \bigl(t_0^{k-2} - 1\bigr)e, t_0^{-k+2}h \\
&\quad + \bigl(t_0^{-k+2} - 1\bigr)e\bigr)\cdots\varphi(t_0, h, h)\bigr]T(h,h) + \bigl[\varphi\bigl(t_0, t_0^{k-1}h + \bigl(t_0^{k-1} - 1\bigr)e, t_0^{-k+1}h \\
&\quad + \bigl(t_0^{-k+1} - 1\bigr)e\bigr)\varphi\bigl(t_0, t_0^{k-2}h + \bigl(t_0^{k-2} - 1\bigr)e, t_0^{-k+2}h + \bigl(t_0^{-k+2} - 1\bigr)e\bigr)\cdots\varphi(t_0, h, h) - 1\bigr]e \\
&\geq \bigl[\varphi(t_0, h, h)\bigr]^k T(h,h) + \bigl(\bigl[\varphi(t_0, h, h)\bigr]^k - 1\bigr)e \\
&\geq \bigl[\varphi(t_0, h, h)\bigr]^k \bigl[t_0 h + (t_0 - 1)e\bigr] + \bigl(\bigl[\varphi(t_0, h, h)\bigr]^k - 1\bigr)e \\
&\geq t_0^k h + \bigl(t_0^k - 1\bigr)e = x_k = u_0
\end{aligned}
$$

and

$$
\begin{aligned}
v_1 &= T(v_0, u_0) \\
&= T\bigl(t_0^{-k}h + \bigl(t_0^{-k} - 1\bigr)e, t_0^k h + \bigl(t_0^k - 1\bigr)e\bigr) \\
&\leq \bigl[\varphi\bigl(t_0, t_0^{-k}h + \bigl(t_0^{-k} - 1\bigr)e, t_0^k h + \bigl(t_0^k - 1\bigr)e\bigr)^{-1}\varphi\bigl(t_0, t_0^{-k+1}h + \bigl(t_0^{-k+1} - 1\bigr)e, \\
&\quad t_0^{k-1}h + \bigl(t_0^{k-1} - 1\bigr)e\bigr)^{-1}\cdots\varphi\bigl(t_0, t_0^{-1}h + \bigl(t_0^{-1} - 1\bigr)e, t_0 h + (t_0 - 1)e\bigr)^{-1}\bigr]T(h,h) \\
&\quad + \bigl[\varphi\bigl(t_0, t_0^{-k}h + \bigl(t_0^{-k} - 1\bigr)e, t_0^k h + \bigl(t_0^k - 1\bigr)e\bigr)^{-1}\varphi\bigl(t_0, t_0^{-k+1}h + \bigl(t_0^{-k+1} - 1\bigr)e, \\
&\quad t_0^{k-1}h + \bigl(t_0^{k-1} - 1\bigr)e\bigr)^{-1}\cdots\varphi\bigl(t_0, t_0^{-1}h + \bigl(t_0^{-1} - 1\bigr)e, t_0 h + (t_0 - 1)e\bigr)^{-1} - 1\bigr]e \\
&\leq t_0^{-k+1}\varphi\bigl(t_0, t_0^{-1}h + \bigl(t_0^{-1} - 1\bigr)e, t_0 h + (t_0 - 1)e\bigr)^{-1}T(h,h) \\
&\quad + \bigl[t_0^{-k+1}\varphi\bigl(t_0, t_0^{-1}h + \bigl(t_0^{-1} - 1\bigr)e, t_0 h + (t_0 - 1)e\bigr)^{-1} - 1\bigr]e \\
&\leq t_0^{-k}h + \bigl(t_0^{-k} - 1\bigr)e = y_k = v_0.
\end{aligned}
$$

The rest of the proof is similar to that of Lemma 2.6, we omit it here. □

3 Main results

In this section, we will establish the existence and uniqueness of nontrivial solution for the problem (1.1). The main tools are fixed point theorems of an operator equation.

Theorem 3.1 *Let P be a normal cone in E, and let $M, N : P_{h,e} \times P_{h,e} \longrightarrow E$ be two mixed monotone operators, and the following conditions are satisfied*:

(L1) *for all $t \in (0,1)$ and $x, y \in P_{h,e}$, there exists $\psi(t,x,y) > t$ such that*

$$M\big(tx + (t-1)e, t^{-1}y + \big(t^{-1} - 1\big)e\big) \geq \psi(t,x,y)M(x,y) + \big(\psi(t,x,y) - 1\big)e;$$

(L2) *for fixed $t \in (0,1)$ and $x \in P_{h,e}$, $\psi(t,x,y)$ is decreasing in y, and for fixed $t \in (0,1)$ and $y \in P_{h,e}$, $\psi(t,x,y)$ is increasing in x*;

(L3) *for all $t \in (0,1)$ and $x, y \in P_{h,e}$*,

$$N\big(tx + (t-1)e, t^{-1}y + \big(t^{-1} - 1\big)e\big) \geq tN(x,y) + (t-1)e;$$

(L4) *there exists $t_0 \in (0,1)$ such that*

$$\frac{t_0}{\psi(t_0,h,h)}h + \left(\frac{t_0}{\psi(t_0,h,h)} - 1\right)e \leq M(h,h) \leq \frac{1}{t_0}h + \left(\frac{1}{t_0} - 1\right)e,$$
$$N(h,h) \in P_{h,e};$$

(L5) *for all $x, y \in P_{h,e}$, there exists a constant $\delta > 0$ such that*

$$M(x,y) \geq \delta N(x,y) + (\delta - 1)e.$$

Then the operator equation $M(x,x) + N(x,x) + e = x$ has a unique solution x^* in $P_{h,e}$, and for any initial values $x_0, y_0 \in P_{h,e}$, by setting the sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$x_n = M(x_{n-1}, y_{n-1}) + N(x_{n-1}, y_{n-1}) + e, \quad n = 1, 2, \ldots,$$
$$y_n = M(y_{n-1}, x_{n-1}) + N(y_{n-1}, x_{n-1}) + e, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ in E as $n \to \infty$.

Proof For every $x_i, y_i \in P_{h,e}$ $(i = 1,2)$ with $x_1 \geq x_2$, $y_1 \leq y_2$, the mixed monotone properties of $M(x,y)$ and $N(x,y)$ lead to

$$M(x_1,y_1) \geq M(x_2,y_2), \qquad N(x_1,y_1) \geq N(x_2,y_2).$$

Now we define the operator $T : P_{h,e} \times P_{h,e} \to E$ by

$$T(x,y) = M(x,y) + N(x,y) + e, \quad \text{for all } x, y \in P_{h,e}. \tag{3.1}$$

We have

$$T(x_1,y_1) = M(x_1,y_1) + N(x_1,y_1) + e \geq M(x_2,y_2) + N(x_2,y_2) + e = T(x_2,y_2).$$

Thus, T is a mixed monotone operator. Note that $N(h,h) \in P_{h,e}$, there exist $a_1, a_2 \in P_{h,e}$ such that

$$a_1 h + (a_1 - 1)e \leq N(h,h) \leq a_2 h + (a_2 - 1)e.$$

From (3.1), we have

$$T(h,h) = M(h,h) + N(h,h) + e.$$

By the condition (L4), we obtain

$$\begin{aligned} T(h,h) &\geq \frac{t_0}{\psi(t_0,h,h)}h + \left(\frac{t_0}{\psi(t_0,h,h)} - 1\right)e + a_1 h + (a_1 - 1)e \\ &= \left(\frac{t_0}{\psi(t_0,h,h)} + a_1\right)h + \left(\frac{t_0}{\psi(t_0,h,h)} + a_1 - 1\right)e \end{aligned}$$

and

$$\begin{aligned} T(h,h) &\leq \frac{1}{t_0}h + \left(\frac{1}{t_0} - 1\right)e + a_2 h + (a_2 - 1)e \\ &= \left(\frac{1}{t_0} + a_2\right)h + \left(\frac{1}{t_0} + a_2 - 1\right)e. \end{aligned}$$

Hence, the condition (iii) in Lemma 2.6 is proved. Next, by the condition (L4), we have

$$\begin{aligned} &M(x,y) + \delta M(x,y) \geq \delta N(x,y) + (\delta - 1)e + \delta M(x,y), \\ &M(x,y) \geq \frac{\delta}{1+\delta}T(x,y) - \frac{e}{1+\delta}. \end{aligned} \tag{3.2}$$

By conditions (L1), (L3), (3.1) and (3.2), for every $x, y \in P_{h,e}$, we obtain

$$\begin{aligned} &T\big(tx + (t-1)e, t^{-1}y + \big(t^{-1} - 1\big)e\big) - tT(x,y) \\ &\quad = M\big(tx + (t-1)e, t^{-1}y + \big(t^{-1} - 1\big)e\big) + N\big(tx + (t-1)e, t^{-1}y + \big(t^{-1} - 1\big)e\big) + e \\ &\qquad - t\big(M(x,y) + N(x,y) + e\big) \\ &\quad \geq \psi(t,x,y)M(x,y) + \big(\psi(t,x,y) - 1\big)e + tN(x,y) + (t-1)e + e - tM(x,y) \\ &\qquad - tN(x,y) - te \\ &\quad = \big(\psi(t,x,y) - t\big)M(x,y) + \big(\psi(t,x,y) - 1\big)e \\ &\quad \geq \big(\psi(t,x,y) - t\big)\left(\frac{\delta}{1+\delta}T(x,y) - \frac{e}{1+\delta}\right) + \big(\psi(t,x,y) - 1\big)e \\ &\quad = \frac{\delta(\psi(t,x,y) - t)}{1+\delta}T(x,y) + \left(\psi(t,x,y) - 1 - \frac{\psi(t,x,y) - t}{1+\delta}\right)e. \end{aligned}$$

Therefore

$$\begin{aligned} &T\big(tx + (t-1)e, t^{-1}y + \big(t^{-1} - 1\big)e\big) \\ &\quad \geq \left(\frac{\delta(\psi(t,x,y) - t)}{1+\delta} + t\right)T(x,y) + \left(\psi(t,x,y) - 1 - \frac{\psi(t,x,y) - t}{1+\delta}\right)e \\ &\quad = \frac{\delta\psi(t,x,y) + t}{1+\delta}T(x,y) + \left(\frac{\delta\psi(t,x,y) + t}{1+\delta} - 1\right)e. \end{aligned} \tag{3.3}$$

Let $\varphi(t,x,y) = \frac{\delta\psi(t,x,y)+t}{1+\delta}$. Then $\varphi(t,x,y) > t$, $t \in (0,1)$, together with (3.3), we obtain

$$T\bigl(tx + (t-1)e, t^{-1}y + \bigl(t^{-1}-1\bigr)e\bigr) \geq \varphi(t,x,y)T(x,y) + \bigl(\varphi(t,x,y)-1\bigr)e, \quad \forall x,y \in P_{h,e}.$$

Thus condition (i) in Lemma 2.6 is proved. By (L2), for every $x_1 \geq x_2$ and $y_1 \leq y_2$ with $x_i, y_i \in P_{h,e}$, $i = 1,2$, we have

$$\psi(t,x_1,y_1) \geq \psi(t,x_2,y_2).$$

Therefore

$$\varphi(t,x_1,y_1) = \frac{\delta\psi(t,x_1,y_1)+t}{1+\delta} \geq \frac{\delta\psi(t,x_2,y_2)+t}{1+\delta} = \varphi(t,x_2,y_2).$$

Thus we deduce condition (ii) in Lemma 2.6 to be met. According to Lemma 2.6, we get the conclusions of Theorem 3.1. □

In terms of Lemma 2.7, we can establish the following theorem, which is parallel with Theorem 3.1.

Theorem 3.1′ *Let P be a normal cone in E, and $M,N : P_{h,e} \times P_{h,e} \longrightarrow E$ be two mixed monotone operators. The assumptions* (L1), (L3) *and* (L5) *in Theorem* 3.1 *are satisfied. Furthermore, for fixed $t \in (0,1)$ and $x \in P_{h,e}$, $\psi(t,x,y)$ is increasing in y, and for fixed $t \in (0,1)$ and $y \in P_{h,e}$, $\psi(t,x,y)$ is decreasing in x. In addition, $N(h,h) \in P_{h,e}$, and there exists $t_0 \in (0,1)$ such that*

$$\begin{aligned} t_0h + (t_0-1)e &\leq M(h,h) \\ &\leq \frac{1}{t_0}\psi\bigl(t_0, t_0^{-1}h + \bigl(t_0^{-1}-1\bigr)e, t_0h + (t_0-1)e\bigr)h \\ &\quad + \biggl[\frac{1}{t_0}\psi\bigl(t_0, t_0^{-1}h + \bigl(t_0^{-1}-1\bigr)e, t_0h + (t_0-1)e\bigr) - 1\biggr]e. \end{aligned}$$

Then the conclusions of Theorem 3.1 *hold.*

Define $E = \{x \mid x \in C[0,1], D_{0^+}^{\nu}x \in C[0,1]\}$. Then E is a Banach space with an order relation $u \leq v$ if $u(t) \leq v(t)$, $D_{0^+}^{\nu}u(t) \leq D_{0^+}^{\nu}v(t)$. Let $P \subset E$ be defined by $P = \{x \in E \mid x(t) \geq 0, D_{0^+}^{\nu}x(t) \geq 0\}$ for all $t \in [0,1]$. It is clear that P is a normal cone. Let

$$e(t) = b\int_0^1 G(t,s)\biggl(\frac{s^{\beta-1}-s^{\beta}}{\Gamma(\beta+1)} + \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i^{\beta-1}-\eta_i^{\beta})s^{\beta-1}}{B\Gamma(\beta+1)}\biggr)\,ds, \quad t \in [0,1].$$

Theorem 3.2 *Assume that*

(H1) *$f,g : [0,1] \times [-e^*,+\infty) \times [-e^*,+\infty) \to (-\infty,+\infty)$ are continuous. For every $t \in [0,1]$, $g(t,0,\mathcal{K}(L)) \geq 0$ with $g(t,0,\mathcal{K}(L)) \not\equiv 0$ where $L \geq \frac{bDF}{\beta\Gamma(\alpha)\Gamma(\beta)}$ and $e^* = \max\{e(t) : t \in [0,1]\}$;*

(H2) *for fixed $t \in [0,1]$ and $y \in [-e^*,+\infty)$, $f(t,x,y)$, $g(t,x,y)$ are increasing in $x \in [-e^*,+\infty)$; for fixed $t \in [0,1]$ and $x \in [-e^*,+\infty)$, $f(t,x,y)$, $g(t,x,y)$ are decreasing in $y \in [-e^*,+\infty)$;*

(H3) *for all* $\lambda \in (0,1)$, *there exists* $\psi(\lambda, x, y) \in (\lambda, 1)$ *such that for all* $t \in [0,1]$,
  (a) $f(t, \lambda x + (\lambda - 1)\rho_1, \lambda^{-1} D_{0^+}^{\nu} y + (\lambda^{-1} - 1)\rho_2) \geq \psi(\lambda, x, y) f(t, x, D_{0^+}^{\nu} y)$,
  (b) $g(t, \lambda x + (\lambda - 1)\rho_3, \lambda^{-1} y + (\lambda^{-1} - 1)\rho_3) \geq \lambda g(t, x, y)$, *where* $x, y \in [-e^*, +\infty)$, $\rho_1, \rho_3 \in [0, e^*]$, *and* $\rho_2 \in [0, e_*]$ *with* $e_* = \max\{D_{0^+}^{\nu} e(t) : t \in [0,1]\}$,
  (c) *for fixed* $t \in [0,1]$ *and* $y \in P_{h,e}$, $\psi(\lambda, x, y)$ *are increasing in* $x \in P_{h,e}$ *and for fixed* $t \in [0,1]$ *and* $x \in P_{h,e}$, $\psi(\lambda, x, y)$ *are decreasing in* $y \in P_{h,e}$;

(H4) *for all* $t \in [0,1]$, $x, y \in [-e^*, +\infty)$, *there exists* $\delta > 0$ *such that*

$$f(t, x, y) \geq \delta g(t, x, 0);$$

(H5) $\mathcal{K} : C[0,1] \to C[0,1]$ *and satisfies*
  (a) $\mathcal{K} u \geq 0$ *for every* $u \in P_{h,e}$,
  (b) *for* $u, v \in P_{h,e}$, $u \leq v \Longrightarrow \mathcal{K} u \leq \mathcal{K} v$,
  (c) *for all* $\lambda \in (0,1)$ *and* $u \in P_{h,e}$,

$$\mathcal{K}\bigl(\lambda u + (\lambda - 1)\hat{e}\bigr) \geq \lambda \mathcal{K}(u) + (\lambda - 1)\hat{e}, \quad \hat{e} \in \bigl[0, e^*\bigr];$$

(H6) *there exists* $t_0 \in (0,1)$ *such that*

$$\begin{aligned} &\frac{t_0}{\psi(t_0, h, h)} h(t) + \frac{t_0}{\psi(t_0, h, h)} e(t) \\ &\quad \leq \int_0^1 G(t,s) \biggl(\int_0^1 H(s,\tau) f\bigl(\tau, h(\tau), D_{0^+}^{\nu} h(\tau)\bigr) \, d\tau \biggr) \, ds \\ &\quad \leq \frac{1}{t_0} h(t) + \frac{1}{t_0} e(t). \end{aligned}$$

Then the problem (1.1) *has a unique solution* u^* *in* $P_{h,e}$, *where* $h(t) = L t^{\alpha - 1}$, *for all* $t \in [0,1]$. *We can construct the following sequences*:

$$\begin{aligned} \omega_n(t) &= \int_0^1 G(t,s) \biggl(\int_0^1 H(s,\tau) f\bigl(\tau, \omega_{n-1}(\tau), D_{0^+}^{\nu} \sigma_{n-1}(\tau)\bigr) \, d\tau \biggr) \, ds \\ &\quad + \int_0^1 G(t,s) \biggl(\int_0^1 H(s,\tau) g\bigl(\tau, \omega_{n-1}(\tau), (\mathcal{K}\sigma_{n-1})(\tau)\bigr) \, d\tau \biggr) \, ds - e(t), \quad n = 1, 2, \ldots, \\ \sigma_n(t) &= \int_0^1 G(t,s) \biggl(\int_0^1 H(s,\tau) f\bigl(\tau, \sigma_{n-1}(\tau), D_{0^+}^{\nu} \omega_{n-1}(\tau)\bigr) \, d\tau \biggr) \, ds \\ &\quad + \int_0^1 G(t,s) \biggl(\int_0^1 H(s,\tau) g\bigl(\tau, \sigma_{n-1}, (\mathcal{K}\omega_{n-1})(\tau)\bigr) \, d\tau \biggr) \, ds - e(t), \quad n = 1, 2, \ldots, \end{aligned}$$

for every initial value $\omega_0, \sigma_0 \in P_{h,e}$, *we have* $\omega_n \to u^*$ *and* $\sigma_n \to u^*$ *as* $n \to \infty$.

Proof By [3], we have

$$\begin{aligned} \int_0^1 H(s,\tau) \, d\tau &= \int_0^1 H_1(s,\tau) \, d\tau + \int_0^1 H_2(s,\tau) \, d\tau \\ &= \frac{s^{\beta-1} - s^{\beta}}{\Gamma(\beta+1)} + \frac{\sum_{i=1}^{m-2} \zeta_i (\eta_i^{\beta-1} - \eta_i^{\beta}) s^{\beta-1}}{B\Gamma(\beta+1)}. \end{aligned}$$

Furthermore, it follows from Lemmas 2.4 and 2.5 that

$$\begin{aligned} 0 < e(t) &\leq b \int_0^1 \frac{Dt^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^1 \frac{Fs^{\beta-1}}{\Gamma(\beta)} \, d\tau \right) ds \\ &= \frac{bDF}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1} \int_0^1 s^{\beta-1} \, ds \\ &= \frac{bDF}{\beta\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1} \\ &\leq Lt^{\alpha-1} = h(t), \end{aligned}$$

where $L \geq \frac{bDF}{\beta\Gamma(\alpha)\Gamma(\beta)}$. Hence, $0 < e(t) \leq h(t)$. By Lemma 2.3, we find that the problem (1.1) has the following expression:

$$\begin{aligned} u(t) &= \int_0^1 G(t,s) \left(\int_0^1 H(s,\tau) \bigl(f\bigl(\tau, u(\tau), D_{0^+}^{\nu} u(\tau)\bigr) + g\bigl(\tau, u(\tau), (\mathcal{K}u)(\tau)\bigr) - b \bigr) \, d\tau \right) ds \\ &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), D_{0^+}^{\nu} u(\tau)\bigr) \, d\tau \, ds \\ &\quad + \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\bigl(\tau, u(\tau), (\mathcal{K}u)(\tau)\bigr) \, d\tau \, ds \\ &\quad - \int_0^1 G(t,s) b \int_0^1 H(s,\tau) \, d\tau \, ds \\ &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), D_{0^+}^{\nu} u(\tau)\bigr) \, d\tau \, ds - e(t) \\ &\quad + \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\bigl(\tau, u(\tau), (\mathcal{K}u)(\tau)\bigr) \, d\tau \, ds - e(t) + e(t). \end{aligned}$$

For every $t \in [0,1]$ and $u, v \in P_{h,e}$, we consider the following operators:

$$M(u,v)(t) = \int_0^1 G(t,s) \int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), D_{0^+}^{\nu} v(\tau)\bigr) \, d\tau \, ds - e(t) \tag{3.4}$$

and

$$N(u,v)(t) = \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\bigl(\tau, u(\tau), (\mathcal{K}v)(\tau)\bigr) \, d\tau \, ds - e(t). \tag{3.5}$$

Clearly, $u(t)$ is the solution of problem (1.1) is equivalent to u is the fixed point of $M(u,v)(t) + N(u,v)(t) + e$. By (3.4) and (3.5), we get

$$\begin{aligned} D_{0^+}^{\nu} M(u,v)(t) &= \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), D_{0^+}^{\nu} v(\tau)\bigr) \, d\tau \, ds - D_{0^+}^{\nu} e(t), \\ D_{0^+}^{\nu} N(u,v)(t) &= \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) g\bigl(\tau, u(\tau), (\mathcal{K}v)(\tau)\bigr) \, d\tau \, ds - D_{0^+}^{\nu} e(t). \end{aligned}$$

(1) Firstly, we show that $M,N : P_{h.e} \times P_{h.e} \to E$ are two mixed monotone operators. By (H1) and (H2), for every $u_i, v_i \in P_{h,e}$ $(i = 1,2)$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we have

$$\begin{aligned} M(u_1,v_1)(t) &= \int_0^1 G(t,s)\int_0^1 H(s,\tau)f\bigl(\tau,u_1(\tau),D_{0^+}^{\nu}v_1(\tau)\bigr)\,d\tau\,ds - e(t) \\ &\geq \int_0^1 G(t,s)\int_0^1 H(s,\tau)f\bigl(\tau,u_2(\tau),D_{0^+}^{\nu}v_2(\tau)\bigr)\,d\tau\,ds - e(t) = M(u_2,v_2)(t) \end{aligned}$$

and

$$\begin{aligned} D_{0^+}^{\nu}M(u_1,v_1)(t) &= \int_0^1 D_{0^+}^{\nu}G(t,s)\int_0^1 H(s,\tau)f\bigl(\tau,u_1(\tau),D_{0^+}^{\nu}v_1(\tau)\bigr)\,d\tau\,ds - D_{0^+}^{\nu}e(t) \\ &\geq \int_0^1 D_{0^+}^{\nu}G(t,s)\int_0^1 H(s,\tau)f\bigl(\tau,u_2(\tau),D_{0^+}^{\nu}v_2(\tau)\bigr)\,d\tau\,ds - D_{0^+}^{\nu}e(t) \\ &= D_{0^+}^{\nu}M(u_2,v_2)(t). \end{aligned}$$

Hence, M is a mixed monotone operator. Similarly, we deduce

$$\begin{aligned} N(u_1,v_1)(t) &= \int_0^1 G(t,s)\int_0^1 H(s,\tau)g\bigl(\tau,u_1(\tau),(\mathcal{K}v_1)(\tau)\bigr)\,d\tau\,ds - e(t) \\ &\geq \int_0^1 G(t,s)\int_0^1 H(s,\tau)g\bigl(\tau,u_2(\tau),(\mathcal{K}v_2)(\tau)\bigr)\,d\tau\,ds - e(t) = N(u_2,v_2)(t) \end{aligned}$$

and

$$\begin{aligned} D_{0^+}^{\nu}N(u_1,v_1)(t) &= \int_0^1 D_{0^+}^{\nu}G(t,s)\int_0^1 H(s,\tau)g\bigl(\tau,u_1(\tau),(\mathcal{K}v_1)(\tau)\bigr)\,d\tau\,ds - D_{0^+}^{\nu}e(t) \\ &\geq \int_0^1 D_{0^+}^{\nu}G(t,s)\int_0^1 H(s,\tau)g\bigl(\tau,u_2(\tau),(\mathcal{K}v_2)(\tau)\bigr)\,d\tau\,ds - D_{0^+}^{\nu}e(t) \\ &= D_{0^+}^{\nu}N(u_2,v_2)(t). \end{aligned}$$

Thus, N is also a mixed monotone operator.

(2) Next, by (H3), for every $t \in [0,1]$ and $\lambda \in (0,1)$, there exists $\psi(\lambda,u,v) \in (\lambda,1)$ such that, for every $u,v \in P_{h,e}$, we get

$$\begin{aligned} &M\bigl(\lambda u + (\lambda-1)e, \lambda^{-1}v + \bigl(\lambda^{-1}-1\bigr)e\bigr)(t) \\ &\quad= \int_0^1 G(t,s)\int_0^1 H(s,\tau)f\bigl(\tau,\lambda u + (\lambda-1)e, D_{0^+}^{\nu}\bigl(\lambda^{-1}v + \bigl(\lambda^{-1}-1\bigr)e\bigr)\bigr)\,d\tau\,ds - e(t) \\ &\quad= \int_0^1 G(t,s)\int_0^1 H(s,\tau)f\bigl(\tau,\lambda u + (\lambda-1)e, \lambda^{-1}D_{0^+}^{\nu}v + \bigl(\lambda^{-1}-1\bigr)D_{0^+}^{\nu}e\bigr)\,d\tau\,ds - e(t) \\ &\quad\geq \int_0^1 G(t,s)\int_0^1 H(s,\tau)\psi\bigl(\lambda,u(\tau),v(\tau)\bigr)f\bigl(\tau,u(\tau),D_{0^+}^{\nu}v(\tau)\bigr)\,d\tau\,ds - e(t) \\ &\qquad+ \psi(\lambda,u,v)e(t) - \psi(\lambda,u,v)e(t) \\ &\quad= \psi(\lambda,u,v)M(u,v)(t) + \bigl(\psi(\lambda,u,v)-1\bigr)e(t) \end{aligned}$$

and

$$\begin{aligned}
&D_{0^+}^{\nu} M\big(\lambda u + (\lambda - 1)e, \lambda^{-1}\nu + \big(\lambda^{-1} - 1\big)e\big)(t) \\
&\quad = \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) f\big(\tau, \lambda u + (\lambda - 1)e, D_{0^+}^{\nu}\big(\lambda^{-1}\nu + \big(\lambda^{-1} - 1\big)e\big)\big)\, d\tau\, ds \\
&\qquad - D_{0^+}^{\nu} e(t) \\
&\quad = \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) f\big(\tau, \lambda u + (\lambda - 1)e, \lambda^{-1} D_{0^+}^{\nu}\nu + \big(\lambda^{-1} - 1\big) D_{0^+}^{\nu} e\big)\, d\tau\, ds \\
&\qquad - D_{0^+}^{\nu} e(t) \\
&\quad \geq \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) \psi\big(\lambda, u(\tau), \nu(\tau)\big) f\big(\tau, u(\tau), D_{0^+}^{\nu}\nu(\tau)\big)\, d\tau\, ds - D_{0^+}^{\nu} e(t) \\
&\qquad + \psi(\lambda, u, \nu) D_{0^+}^{\nu} e(t) - \psi(\lambda, u, \nu) D_{0^+}^{\nu} e(t) \\
&\quad = \psi(\lambda, u, \nu) D_{0^+}^{\nu} M(u,\nu)(t) + \big(\psi(\lambda, u, \nu) - 1\big) D_{0^+}^{\nu} e(t).
\end{aligned}$$

Thus, $M(\lambda u + (\lambda - 1)e, \lambda^{-1}\nu + (\lambda^{-1} - 1)e) \geq \psi(\lambda, u, \nu) M(u,\nu) + (\psi(\lambda, u, \nu) - 1)e$.

In view of (H3)(b) and (H5), we derive

$$\begin{aligned}
&\mathcal{K}\big(\lambda^{-1}u + \big(\lambda^{-1} - 1\big)e\big) \leq \lambda^{-1}(\mathcal{K}u) + \big(\lambda^{-1} - 1\big)e, \\
&N\big(\lambda u + (\lambda - 1)e, \lambda^{-1}\nu + \big(\lambda^{-1} - 1\big)e\big)(t) \\
&\quad = \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\big(\tau, \lambda u + (\lambda - 1)e, \big(\mathcal{K}\big(\lambda^{-1}\nu + \big(\lambda^{-1} - 1\big)e\big)\big)\, d\tau\, ds - e(t) \\
&\quad \geq \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\big(\tau, \lambda u + (\lambda - 1)e, \lambda^{-1}(\mathcal{K}\nu) + \big(\lambda^{-1} - 1\big)e\big)\, d\tau\, ds - e(t) \\
&\quad \geq \int_0^1 G(t,s) \int_0^1 H(s,\tau) \lambda g\big(\tau, u(\tau), (\mathcal{K}\nu)(\tau)\big)\, d\tau\, ds - e(t) \\
&\quad = \lambda \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\big(\tau, u(\tau), (\mathcal{K}\nu)(\tau)\big)\, d\tau\, ds - e(t) + \lambda e(t) - \lambda e(t) \\
&\quad = \lambda N(u,\nu)(t) + (\lambda - 1)e(t),
\end{aligned}$$

and

$$\begin{aligned}
&D_{0^+}^{\nu} N\big(\lambda u + (\lambda - 1)e, \lambda^{-1}\nu + \big(\lambda^{-1} - 1\big)e\big)(t) \\
&\quad = \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) g\big(\tau, \lambda u + (\lambda - 1)e, \big(\mathcal{K}\big(\lambda^{-1}\nu + \big(\lambda^{-1} - 1\big)e\big)\big)\, d\tau\, ds - D_{0^+}^{\nu} e(t) \\
&\quad \geq \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) g\big(\tau, \lambda u + (\lambda - 1)e, \lambda^{-1}(\mathcal{K}\nu) + \big(\lambda^{-1} - 1\big)e\big)\, d\tau\, ds - D_{0^+}^{\nu} e(t) \\
&\quad \geq \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) \lambda g\big(\tau, u(\tau), (\mathcal{K}\nu)(\tau)\big)\, d\tau\, ds - D_{0^+}^{\nu} e(t) \\
&\quad = \lambda \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) g\big(\tau, u(\tau), (\mathcal{K}\nu)(\tau)\big)\, d\tau\, ds
\end{aligned}$$

$$
\begin{aligned}
&- D_{0^+}^{\nu} e(t) + \lambda D_{0^+}^{\nu} e(t) - \lambda D_{0^+}^{\nu} e(t) \\
&= \lambda D_{0^+}^{\nu} N(u,v)(t) + (\lambda - 1) D_{0^+}^{\nu} e(t).
\end{aligned}
$$

Thus, $N(\lambda u + (\lambda - 1)e, \lambda^{-1} v + (\lambda^{-1} - 1)e) \geq \lambda N(u,v) + (\lambda - 1)e$.

(3) In view of (H6), we have

$$
\begin{aligned}
M(h,h)(t) &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) f\bigl(\tau, h(\tau), D_{0^+}^{\nu} h(\tau)\bigr)\, d\tau\, ds - e(t) \\
&\leq \frac{1}{t_0} h(t) + \left(\frac{1}{t_0} - 1\right) e(t), \\
M(h,h)(t) &\geq \frac{t_0}{\psi(t_0,h,h)} h(t) + \left(\frac{t_0}{\psi(t_0,h,h)} - 1\right) e(t),
\end{aligned}
$$

and

$$
\begin{aligned}
D_{0^+}^{\nu} M(h,h)(t) &= \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau) f\bigl(\tau, h(\tau), D_{0^+}^{\nu} h(\tau)\bigr)\, d\tau\, ds - D_{0^+}^{\nu} e(t) \\
&\leq D_{0^+}^{\nu} \frac{1}{t_0} h(t) + \left(\frac{1}{t_0} - 1\right) D_{0^+}^{\nu} e(t), \\
D_{0^+}^{\nu} M(h,h)(t) &\geq D_{0^+}^{\nu} \frac{t_0}{\psi(t_0,h,h)} h(t) + \left(\frac{t_0}{\psi(t_0,h,h)} - 1\right) D_{0^+}^{\nu} e(t).
\end{aligned}
$$

Thus,

$$
\frac{t_0}{\psi(t_0,h,h)} h + \left(\frac{t_0}{\psi(t_0,h,h)} - 1\right) e \leq M(h,h) \leq \frac{\psi(t_0,h,h)}{t_0} h + \left(\frac{\psi(t_0,h,h)}{t_0} - 1\right) e.
$$

Next we show that $N(h,h) \in P_{h,e}$. It suffices to prove that $N(h,h) + e \in P_h$. From Lemma 2.4 and the condition (H2), we have

$$
\begin{aligned}
N(h,h)(t) + e(t) &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\bigl(\tau, h(\tau), (\mathcal{K}h)(\tau)\bigr)\, d\tau\, ds \\
&\leq \int_0^1 \frac{D t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 H(s,\tau) g\bigl(\tau, L\tau^{\alpha-1}, \mathcal{K}\bigl(L\tau^{\alpha-1}\bigr)\bigr)\, d\tau\, ds \\
&\leq \frac{D t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \int_0^1 H(s,\tau) g(\tau, L, 0)\, d\tau\, ds \\
&= \frac{D h(t)}{L\Gamma(\alpha)} \int_0^1 \int_0^1 H(s,\tau) g(\tau, L, 0)\, d\tau\, ds, \\
N(h,h)(t) + e(t) &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\bigl(\tau, h(\tau), (\mathcal{K}h)(\tau)\bigr)\, d\tau\, ds \\
&\geq \int_0^1 \frac{C(s)}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 H(s,\tau) g\bigl(\tau, L\tau^{\alpha-1}, \mathcal{K}\bigl(L\tau^{\alpha-1}\bigr)\bigr)\, d\tau\, ds \\
&\geq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 C(s) \int_0^1 H(s,\tau) g\bigl(\tau, 0, \mathcal{K}(L)\bigr)\, d\tau\, ds \\
&= \frac{h(t)}{L\Gamma(\alpha)} \int_0^1 C(s) \int_0^1 H(s,\tau) g\bigl(\tau, 0, \mathcal{K}(L)\bigr)\, d\tau\, ds,
\end{aligned}
$$

and

$$\begin{aligned}
D_{0^+}^{\nu}N(h,h)(t)+D_{0^+}^{\nu}e(t) &= \int_0^1 D_{0^+}^{\nu}G(t,s)\int_0^1 H(s,\tau)g\bigl(\tau,h(\tau),(\mathcal{K}h)(\tau)\bigr)\,d\tau\,ds \\
&\le \int_0^1 \frac{Dt^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\int_0^1 H(s,\tau)g\bigl(\tau,L\tau^{\alpha-1},\mathcal{K}\bigl(L\tau^{\alpha-1}\bigr)\bigr)\,d\tau\,ds \\
&\le \frac{Dt^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\int_0^1\int_0^1 H(s,\tau)g(\tau,L,0)\,d\tau\,ds \\
&= \frac{DD_{0^+}^{\nu}h(t)}{L\Gamma(\alpha)}\int_0^1\int_0^1 H(s,\tau)g(\tau,L,0)\,d\tau\,ds, \\
D_{0^+}^{\nu}N(h,h)(t)+D_{0^+}^{\nu}e(t) &= \int_0^1 D_{0^+}^{\nu}G(t,s)\int_0^1 H(s,\tau)g\bigl(\tau,h(\tau),(\mathcal{K}h)(\tau)\bigr)\,d\tau\,ds \\
&\ge \int_0^1 \frac{C(s)}{\Gamma(\alpha-\nu)}t^{\alpha-\nu-1}\int_0^1 H(s,\tau)g\bigl(\tau,L\tau^{\alpha-1},\mathcal{K}\bigl(L\tau^{\alpha-1}\bigr)\bigr)\,d\tau\,ds \\
&\ge \frac{t^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\int_0^1 C(s)\int_0^1 H(s,\tau)g\bigl(\tau,0,\mathcal{K}(L)\bigr)\,d\tau\,ds \\
&= \frac{D_{0^+}^{\nu}h(t)}{L\Gamma(\alpha)}\int_0^1 C(s)\int_0^1 H(s,\tau)g\bigl(\tau,0,\mathcal{K}(L)\bigr)\,d\tau\,ds.
\end{aligned}$$

Let

$$\begin{aligned}
l_1 &= \frac{D}{L\Gamma(\alpha)}\int_0^1\int_0^1 H(s,\tau)g(\tau,L,0)\,d\tau\,ds, \\
l_2 &= \frac{1}{L\Gamma(\alpha)}\int_0^1 C(s)\int_0^1 H(s,\tau)g\bigl(\tau,0,\mathcal{K}(L)\bigr)\,d\tau\,ds.
\end{aligned}$$

Then $l_2h \le N(h,h)+e \le l_1h$, thus $N(h,h)\in P_{h,e}$. Therefore, the condition (L4) of Theorem 3.1 is proved.

(4) For every $u,v\in P_{h,e}$ and $t\in[0,1]$, we derive that

$$\begin{aligned}
M(u,v)(t) &= \int_0^1 G(t,s)\int_0^1 H(s,\tau)f\bigl(\tau,u(\tau),D_{0^+}^{\nu}v(\tau)\bigr)\,d\tau\,ds - e(t) \\
&\ge \int_0^1 G(t,s)\int_0^1 H(s,\tau)\delta g\bigl(\tau,u(\tau),0\bigr)\,d\tau\,ds - e(t) \\
&\ge \int_0^1 G(t,s)\int_0^1 H(s,\tau)\delta g\bigl(\tau,u(\tau),(\mathcal{K}v)(\tau)\bigr)\,d\tau\,ds - e(t) \\
&= \delta\int_0^1 G(t,s)\int_0^1 H(s,\tau)g\bigl(\tau,u(\tau),(\mathcal{K}v)(\tau)\bigr)\,d\tau\,ds - e(t)+\delta e(t)-\delta e(t) \\
&= \delta N(u,v)(t)+(\delta-1)e(t)
\end{aligned}$$

and

$$\begin{aligned}
D_{0^+}^{\nu}M(u,v)(t) &= \int_0^1 D_{0^+}^{\nu}G(t,s)\int_0^1 H(s,\tau)f\bigl(\tau,u(\tau),D_{0^+}^{\nu}v(\tau)\bigr)\,d\tau\,ds - D_{0^+}^{\nu}e(t) \\
&\ge \int_0^1 D_{0^+}^{\nu}G(t,s)\int_0^1 H(s,\tau)\delta g\bigl(\tau,u(\tau),0\bigr)\,d\tau\,ds - D_{0^+}^{\nu}e(t)
\end{aligned}$$

$$
\begin{aligned}
&\ge \int_0^1 D_{0^+}^{\nu} G(t,s) \int_0^1 H(s,\tau)\delta g\big(\tau,u(\tau),(\mathcal{K}v)(\tau)\big)\,d\tau\,ds - D_{0^+}^{\nu} e(t)\\
&= \delta \int_0^1 G(t,s) \int_0^1 H(s,\tau) g\big(\tau,u(\tau),(\mathcal{K}v)(\tau)\big)\,d\tau\,ds - D_{0^+}^{\nu} e(t)\\
&+ \delta D_{0^+}^{\nu} e(t) - \delta D_{0^+}^{\nu} e(t)\\
&= \delta D_{0^+}^{\nu} N(u,v)(t) + (\delta-1) D_{0^+}^{\nu} e(t).
\end{aligned}
$$

Therefore, $M(u,v) \ge \delta N(u,v) + (\delta-1)e$. That is, the condition (L5) of Theorem 3.1 is satisfied. Consequently, all the conditions of Theorem 3.1 are satisfied, the conclusions of Theorem 3.2 hold. □

By the proof of Theorem 3.2, combining with Theorem 3.1′, we can obtain the following result.

Theorem 3.2′ *Assume that the conditions* (H1), (H2), (H3)(a)(b), (H4) *and* (H5) *in Theorem* 3.2 *are satisfied. Moreover, for fixed* $t \in [0,1]$ *and* $y \in P_{h,e}$, $\psi(\lambda,x,y)$ *are decreasing in* $x \in P_{h,e}$ *and for fixed* $t \in [0,1]$ *and* $x \in P_{h,e}$, $\psi(\lambda,x,y)$ *are increasing in* $y \in P_{h,e}$. *In addition, there exists* $t_0 \in (0,1)$ *such that*

$$
\begin{aligned}
&t_0 h(t) + (t_0-1)e(t)\\
&\quad\le \int_0^1 G(t,s)\left(\int_0^1 H(s,\tau) f\big(\tau,h(\tau),D_{0^+}^{\nu} h(\tau)\big)\,d\tau\right)ds - e(t)\\
&\quad\le \frac{1}{t_0}\psi\big(t_0, t_0^{-1}h + \big(t_0^{-1}-1\big)e, t_0 h + (t_0-1)e\big)h(t)\\
&\qquad + \left[\frac{1}{t_0}\psi\big(t_0, t_0^{-1}h + \big(t_0^{-1}-1\big)e, t_0 h + (t_0-1)e\big) - 1\right]e(t).
\end{aligned}
$$

Then the conclusions of Theorem 3.2 *hold.*

Lastly, let us give an example to illustrate our main results.

Example 3.1 Consider the following boundary value problem:

$$
\begin{cases}
D_{0^+}^{\frac{3}{2}}(D_{0^+}^{\frac{3}{2}} u)(t) = 2t^2 + 1 + (u(t) + \frac{15}{\Gamma^2(\frac{3}{2})} + 1)^{\frac{1}{3}} + (u(t) + \frac{15}{\Gamma^2(\frac{3}{2})} + 1)^{\frac{1}{2}}\\
\qquad\qquad + (D_{0^+}^{\frac{1}{8}} u(t) + \frac{15}{\Gamma^2(\frac{3}{2})} + 1)^{\frac{-1}{5}}\\
\qquad\qquad + (\int_0^t (u(s) + \frac{15}{\Gamma^2(\frac{3}{2})})\,ds + \frac{15}{\Gamma^2(\frac{3}{2})} + 1)^{-1} - 10,\\
u(0) = 0, \qquad D_{0^+}^{\frac{3}{2}} u(0) = 0,\\
D_{0^+}^{\frac{1}{4}} u(1) = \frac{1}{10} D_{0^+}^{\frac{1}{4}} u(\frac{1}{4}) + \frac{1}{10} D_{0^+}^{\frac{1}{4}} u(\frac{1}{2}) + \frac{1}{10} D_{0^+}^{\frac{1}{4}} u(\frac{3}{4}),\\
D_{0^+}^{\frac{3}{2}} u(1) = \frac{1}{10} D_{0^+}^{\frac{3}{2}} u(\frac{1}{4}) + \frac{1}{10} D_{0^+}^{\frac{3}{2}} u(\frac{1}{2}) + \frac{1}{10} D_{0^+}^{\frac{3}{2}} u(\frac{3}{4}).
\end{cases}
\tag{3.6}
$$

Then the problem (3.6) has a solution.

Proof The problem (1.1) becomes the problem (3.6) when we choose $n = 2$, $\alpha = \frac{3}{2}$, $\beta = \frac{3}{2}$, $\gamma = \frac{1}{4}$, $\nu = \frac{1}{8}$, $b = 10$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\eta_3 = \frac{3}{4}$, $\xi_1 = \xi_2 = \xi_3 = \frac{1}{10}$, and $\zeta_1 = \zeta_2 = \zeta_3 = \frac{1}{10}$. Then we have

$$
\begin{aligned}
A &= 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\gamma-1} \approx 0.7521 > 0, \\
B &= 1 - \sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1} \approx 0.7927 > 0, \\
F &= \frac{1}{B}\left(1 + \sum_{i=1}^{m-2} \zeta_i\bigl(1-\eta_i^{\beta-1}\bigr)\right) \approx 1.5571, \\
D &= \frac{1}{A}\left(1 + \sum_{i=1}^{m-2} \xi_i\bigl(1-\eta_i^{\alpha-\gamma-1}\bigr)\right) \approx 1.3988.
\end{aligned}
$$

A direct computation leads to

$$
\begin{aligned}
\int_0^1 H(s,\tau)\,d\tau &= \int_0^1 H_1(s,\tau)\,d\tau + \int_0^1 H_2(s,\tau)\,d\tau \\
&= \frac{s^{\frac{1}{2}} - s^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{\frac{3}{10}[\frac{1}{4}^{\frac{1}{2}} - \frac{1}{4}^{\frac{3}{2}} + \frac{1}{2}^{\frac{1}{2}} - \frac{1}{2}^{\frac{3}{2}} + \frac{3}{4}^{\frac{1}{2}} - \frac{3}{4}^{\frac{3}{2}}]s^{\frac{1}{2}}}{B\Gamma(\frac{5}{2})}, \\
e(t) &= 10\int_0^1 G(t,s)\int_0^1 H(s,\tau)\,d\tau\,ds \\
&= 10\int_0^1 G(t,s)\left[\frac{s^{\frac{1}{2}} - s^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{\frac{3}{10}[\frac{1}{4}^{\frac{1}{2}} - \frac{1}{4}^{\frac{3}{2}} + \frac{1}{2}^{\frac{1}{2}} - \frac{1}{2}^{\frac{3}{2}} + \frac{3}{4}^{\frac{1}{2}} - \frac{3}{4}^{\frac{3}{2}}]s^{\frac{1}{2}}}{B\Gamma(\frac{5}{2})}\right] ds \\
&\le 10\int_0^1 \frac{Dt^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\left[\frac{s^{\frac{1}{2}} - s^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{\frac{3}{10}[\frac{1}{4}^{\frac{1}{2}} - \frac{1}{4}^{\frac{3}{2}} + \frac{1}{2}^{\frac{1}{2}} - \frac{1}{2}^{\frac{3}{2}} + \frac{3}{4}^{\frac{1}{2}} - \frac{3}{4}^{\frac{3}{2}}]s^{\frac{1}{2}}}{B\Gamma(\frac{5}{2})}\right] ds \\
&\le \frac{15}{\Gamma^2(\frac{3}{2})} t^{\frac{1}{2}} = Lt^{\frac{1}{2}} = h(t),
\end{aligned}
$$

and

$$
e^* \le \frac{15}{\Gamma^2(\frac{3}{2})}, \qquad D_{0^+}^{\frac{1}{8}} e(t) \le \frac{15}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{8})}.
$$

Let

$$
\begin{aligned}
f(t,u,v) &= t^2 + 1 + \left(u(t) + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{\frac{1}{2}} + \left(v(t) + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{-\frac{1}{5}}, \\
g(t,u,v) &= t^2 + \left(u(t) + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{\frac{1}{3}} + \left(v(t) + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{-1}, \\
(\mathcal{K}u)(t) &= \int_0^t \left(u(s) + \frac{15}{\Gamma^2(\frac{3}{2})}\right) ds.
\end{aligned}
$$

For $\lambda \in (0,1)$ and $\hat{e} \in [0, \frac{15}{\Gamma^2(\frac{3}{2})}]$, we deduce

$$\begin{aligned}
\mathcal{K}\big(\lambda u + (\lambda-1)\hat{e}\big) &= \int_0^t \left(\lambda u + (\lambda-1)\hat{e} + \frac{15}{\Gamma^2(\frac{3}{2})}\right) ds \\
&= \lambda \int_0^t u\, ds + \int_0^t (\lambda-1)\hat{e}\, ds + \int_0^t \frac{15}{\Gamma^2(\frac{3}{2})}\, ds \\
&\geq \lambda(\mathcal{K}u)(t) + (\lambda-1)\hat{e},
\end{aligned}$$

and $(\mathcal{K}u)$ is increasing in u, thus (H5) is satisfied. It is easy to check that $f,g : [0,1] \times [-\frac{15}{\Gamma^2(\frac{3}{2})}, +\infty) \times [-\frac{15}{\Gamma^2(\frac{3}{2})}, +\infty) \to (-\infty,+\infty)$ are continuous, $f(t,u,v)$, $g(t,u,v)$ are both increasing in u and decreasing in v and $g(t,0,\mathcal{K}(L)) = t^2 + (\frac{15}{\Gamma^2(\frac{3}{2})}+1)^{\frac{1}{3}} + (\mathcal{K}(L) + \frac{15}{\Gamma^2(\frac{3}{2})}+1)^{-1} > 0$. Thus, (H1) and (H2) are satisfied.

For all $\lambda \in (0,1)$, $t \in [0,1]$, $u,v \in P_{h,e}$, $\rho_1, \rho_3 \in [0, \frac{15}{\Gamma^2(\frac{3}{2})}]$ and $\rho_2 \in [0, \frac{15}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{8})}]$, there exists $\psi(\lambda,u,v) = \lambda^{\frac{1}{2}}$ such that

$$\begin{aligned}
&f\big(t, \lambda u + (\lambda-1)\rho_1, \lambda^{-1} D_{0^+}^{\frac{1}{8}} v + \big(\lambda^{-1}-1\big)\rho_2\big) \\
&\quad = t^2 + 1 + \left(\lambda u + (\lambda-1)\rho_1 + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{\frac{1}{2}} \\
&\qquad + \left(\lambda^{-1} D_{0^+}^{\frac{1}{8}} v + \big(\lambda^{-1}-1\big)\rho_2 + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{-\frac{1}{5}} \\
&\quad = t^2 + 1 + \lambda^{\frac{1}{2}} \left(u + \big(1-\lambda^{-1}\big)\rho_1 + \lambda^{-1}\frac{15}{\Gamma^2(\frac{3}{2})} + \lambda^{-1}\right)^{\frac{1}{2}} \\
&\qquad + \lambda^{\frac{1}{5}} \left(D_{0^+}^{\frac{1}{8}} v + (1-\lambda)\rho_2 + \lambda\frac{15}{\Gamma^2(\frac{3}{2})} + \lambda\right)^{-\frac{1}{5}} \\
&\quad \geq t^2 + 1 + \lambda^{\frac{1}{2}} \left(u + \big(1-\lambda^{-1}\big)\frac{15}{\Gamma^2(\frac{3}{2})} + \lambda^{-1}\frac{15}{\Gamma^2(\frac{3}{2})} + \lambda^{-1}\right)^{\frac{1}{2}} \\
&\qquad + \lambda^{\frac{1}{5}} \left(D_{0^+}^{\frac{1}{8}} v + (1-\lambda)\frac{15}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{8})} + \lambda\frac{15}{\Gamma^2(\frac{3}{2})} + \lambda\right)^{-\frac{1}{5}} \\
&\quad \geq t^2 + 1 + \lambda^{\frac{1}{2}} \left(u + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{\frac{1}{2}} \\
&\qquad + \lambda^{\frac{1}{5}} \left(D_{0^+}^{\frac{1}{8}} v + (1-\lambda)\frac{15}{\Gamma^2(\frac{3}{2})} + \lambda\frac{15}{\Gamma^2(\frac{3}{2})} + \lambda\right)^{-\frac{1}{5}} \\
&\quad \geq t^2 + 1 + \lambda^{\frac{1}{2}} \left(u + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{\frac{1}{2}} + \lambda^{\frac{1}{5}} \left(D_{0^+}^{\frac{1}{8}} v + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{-\frac{1}{5}} \\
&\quad \geq \lambda^{\frac{1}{2}} \left[t^2 + 1 + \left(u + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{\frac{1}{2}} + \left(D_{0^+}^{\frac{1}{8}} v + \frac{15}{\Gamma^2(\frac{3}{2})} + 1\right)^{-\frac{1}{5}}\right] \\
&\quad = \psi(\lambda,u,v) f\big(t,u,D_{0^+}^{\frac{1}{8}} v\big).
\end{aligned}$$

Moreover, we deduce

$$\begin{aligned}
&g\bigl(t,\lambda u+(\lambda-1)\rho_3,\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)\rho_3\bigr)\\
&\quad=t^2+\left(\lambda u+(\lambda-1)\rho_3+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{3}}+\left(\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)\rho_3+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-1}\\
&\quad=t^2+\lambda^{\frac{1}{3}}\left(u+\bigl(1-\lambda^{-1}\bigr)\rho_3+\frac{15\lambda^{-1}}{\Gamma^2(\frac{3}{2})}+\lambda^{-1}\right)^{\frac{1}{3}}+\lambda\left(v+(1-\lambda)\rho_3+\frac{15\lambda}{\Gamma^2(\frac{3}{2})}+\lambda\right)^{-1}\\
&\quad\geq\lambda t^2+\lambda\left(u+\bigl(1-\lambda^{-1}\bigr)\frac{15}{\Gamma^2(\frac{3}{2})}+\frac{15\lambda^{-1}}{\Gamma^2(\frac{3}{2})}+\lambda^{-1}\right)^{\frac{1}{3}}\\
&\qquad+\lambda\left(v+(1-\lambda)\frac{15}{\Gamma^2(\frac{3}{2})}+\frac{15\lambda}{\Gamma^2(\frac{3}{2})}+\lambda\right)^{-1}\\
&\quad=\lambda t^2+\lambda\left(u+\frac{15}{\Gamma^2(\frac{3}{2})}+\lambda^{-1}\right)^{\frac{1}{3}}+\lambda\left(v+\frac{15}{\Gamma^2(\frac{3}{2})}+\lambda\right)^{-1}\\
&\quad\geq\lambda\left[t^2+\left(u+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{3}}+\left(v+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-1}\right]\\
&\quad=\lambda g(t,u,v).
\end{aligned}$$

Thus, (H3) is satisfied. Furthermore, for $u,v\in P_{h,e}$, we get

$$\begin{aligned}
f(t,u,v)&=t^2+1+\left(u(t)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{2}}+\left(v(t)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-\frac{1}{5}}\\
&\geq t^2+1+\left(u(t)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{3}}+\left(v(t)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-1}\\
&\geq t^2+\left(u(t)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{3}}+\left(\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-1}=g(t,u,0),
\end{aligned}$$

let $\delta=1$, we have $f(t,u,v)\geq\delta g(t,u,0)$. Thus (H4) is satisfied.

By Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
&\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f\bigl(\tau,h(\tau),D_{0^+}^{\nu}h(\tau)\bigr)\,d\tau\right)ds\\
&\quad\leq\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)\left(4+h(\tau)+\frac{15}{\Gamma^2(\frac{3}{2})}\right)d\tau\right)ds\\
&\quad=\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)h(\tau)\,d\tau\right)ds+\left[\frac{15}{\Gamma^2(\frac{3}{2})}+4\right]\frac{e(t)}{10}\\
&\quad\leq\int_0^1\frac{Dt^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\left(\int_0^1\frac{15Fs^{\frac{1}{2}}\tau^{\frac{1}{2}}}{\Gamma^3(\frac{3}{2})}\,d\tau\right)ds+\left[\frac{15}{\Gamma^2(\frac{3}{2})}+4\right]\frac{e(t)}{10}\\
&\quad=\frac{4DF}{9\Gamma^2(\frac{3}{2})}h(t)+\left[\frac{15}{\Gamma^2(\frac{3}{2})}+4\right]\frac{e(t)}{10}\\
&\quad=1.2325h(t)+2.3099e(t)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f\big(\tau,h(\tau),D_{0^+}^{\nu}h(\tau)\big)\,d\tau\right)ds \\
&\quad\geq \int_0^1 \frac{c(s)t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\left(\int_s^1 \frac{1}{\Gamma(\frac{3}{2})}s^{\frac{1}{2}}(1-\tau)^{\frac{1}{2}}\tau^{\frac{1}{4}}\,d\tau\right)ds+\frac{1}{10}e(t) \\
&\quad\geq \frac{4t^{\frac{1}{2}}}{5\Gamma^2(\frac{3}{2})}\int_0^1 c(s)s^{\frac{1}{2}}\big(1-s^{\frac{5}{4}}\big)\,ds+\frac{1}{10}e(t) \\
&\quad\geq \frac{4t^{\frac{1}{2}}}{5\Gamma^2(\frac{3}{2})}\int_{\frac{1}{2}}^1 \frac{1}{10}\left(\frac{1}{2}\right)^{\frac{1}{4}}(1-s)^{\frac{1}{4}}s^{\frac{1}{2}}\big(1-s^{\frac{5}{4}}\big)\,ds+\frac{1}{10}e(t) \\
&\quad\geq 0.0001h(t)+0.1e(t).
\end{aligned}$$

Choose $t_0 = 10^{-8}$, we deduce that the condition (H6) is satisfied. Therefore, all the assumptions of Theorem 3.2 are satisfied. We can construct the following iteration sequences:

$$\begin{aligned}
\omega_n(t) ={}& \int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)\left(\tau^2+\left(\omega_{n-1}(\tau)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{3}}\right.\right. \\
&+\left.\left.\left(\int_0^\tau\left(\sigma_{n-1}(x)+\frac{15}{\Gamma^2(\frac{3}{2})}\right)dx+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-1}\right)d\tau\right)ds \\
&+\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)\left(\tau^2+1+\left(\omega_{n-1}(\tau)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{2}}\right.\right. \\
&+\left.\left.\left(D_{0^+}^{\nu}\sigma_{n-1}(\tau)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-\frac{1}{5}}\right)d\tau\right)ds-e(t), \quad n=1,2,\ldots,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_n(t) ={}& \int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)\left(\tau^2+\left(\sigma_{n-1}(\tau)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{3}}\right.\right. \\
&+\left.\left.\left(\int_0^\tau\left(\omega_{n-1}(x)+\frac{15}{\Gamma^2(\frac{3}{2})}\right)dx+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-1}\right)d\tau\right)ds \\
&+\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)\left(\tau^2+1+\left(\sigma_{n-1}(\tau)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{\frac{1}{2}}\right.\right. \\
&+\left.\left.\left(\big(D_{0^+}^{\nu}\omega_{n-1}\big)(\tau)+\frac{15}{\Gamma^2(\frac{3}{2})}+1\right)^{-\frac{1}{5}}\right)d\tau\right)ds-e(t), \quad n=1,2,\ldots,
\end{aligned}$$

for any initial values $\omega_0,\sigma_0 \in P_{h,e}$,we have $\omega_n \to u^*$ and $\sigma_n \to u^*$ as $n \to \infty$. □

4 Conclusions

In this paper, we obtain two new mixed monotone fixed point theorems. By using our abstract results, we establish the existence and uniqueness theorems of the solution for a fractional m-point boundary value problem, which generalizes the well-known elastic beam equation. Furthermore, two iterative sequences to approximate the unique solution are also given.

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Availability of data and materials
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The authors declare that they have no competing interests.

Authors' contributions
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