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Local well-posedness of the inertial Qian–Sheng's Q-tensor dynamical model near uniaxial equilibrium

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Abstract

We consider the inertial Qian–Sheng's Q-tensor dynamical model for the nematic liquid crystal flow, which can be viewed as a system coupling the hyperbolic-type equations for the Q-tensor parameter with the incompressible Navier–Stokes equations for the fluid velocity. We prove the existence and uniqueness of local in time strong solutions to the system with the initial data near uniaxial equilibrium. The proof is mainly based on the classical Friedrich method to construct approximate solutions and the closed energy estimate.

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1 Introduction

Liquid crystals present a state of matter with properties between liquid and solid. The simplest form of liquid crystals is the nematic phase, which exhibits long-range orientational order but no positional order. Generally speaking, there are two primary continuum theories to describe nematic liquid crystal flow: the Ericksen–Lesile theory and the Landau–de Gennes theory. In the former one, the local alignment of molecules is described by a unit vector, which completely neglects molecular details. In contrast, the latter gives a more complex description of the local behavior of molecular alignments, such as line defects and biaxial configurations. This theory uses a symmetric and traceless tensor $Q(\mathbf{x})$ to characterize the alignment behavior of molecular orientations. Physically, $Q(\mathbf{x})$ can be defined as the second-order traceless moment of f:

$$Q(\mathbf{x}) = \int_{\mathbb{S}^2} \left(\mathbf{mm} - \frac{1}{3}\mathbf{I}\right) f(\mathbf{x}, \mathbf{m})\, \mathrm{d}\mathbf{m},$$

where $f(\mathbf{x}, \mathbf{m})$ is the density distribution function with the orientation parallel to $\mathbf{m}$ at material point $\mathbf{x}$. The tensor $Q(\mathbf{x})$ is said to be *isotropic* if all its eigenvalues are zero, *uniaxial* if it has only two different eigenvalues, and *biaxial* if its three eigenvalues are different from each other. When $Q(\mathbf{x})$ is uniaxial, it can be written as

$$Q(\mathbf{x}) = S\left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right), \quad \mathbf{n} \in \mathbb{S}^2,$$

where $S \in \mathbb{R}$ is the scalar order parameter. When $Q(\mathbf{x})$ is biaxial, it can be written as

$$Q(\mathbf{x}) = S\left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right) + R\left(\mathbf{n}'\mathbf{n}' - \frac{1}{3}\mathbf{I}\right), \quad \mathbf{n}, \mathbf{n}' \in \mathbb{S}^2, \mathbf{n} \cdot \mathbf{n}' = 0, S, R \in \mathbb{R}.$$

The Landau–de Gennes free energy functional is given as follows:

$$\begin{aligned}\mathcal{F}(Q, \nabla Q) &= \int_{\mathbb{R}^3} \left\{ -\frac{a}{2}\operatorname{Tr}(Q^2) - \frac{b}{3}\operatorname{Tr}(Q^3) + \frac{c}{4}\left(\operatorname{Tr}(Q^2)\right)^2 \right. \\ &\quad \left. + \frac{1}{2}\left(L_1|\nabla Q|^2 + L_2 Q_{ij,j}Q_{ik,k} + L_3 Q_{ij,k}Q_{ik,j}\right) \right\} \mathrm{d}\mathbf{x} \\ &\overset{\text{def}}{=} \int_{\mathbb{R}^3} \left(f_b(Q) + f_e(\nabla Q)\right) \mathrm{d}\mathbf{x}, \end{aligned} \tag{1.1}$$

where a, b, c are nonnegative coefficients depending on the material and temperature, and L_i $(i = 1, 2, 3)$ are material-dependent elastic coefficients. f_b is the bulk energy density describing the isotropic-nematic phase transition, while the elastic energy density f_e penalizes spatial non-homogeneities. For detailed introductions one is referred to [5, 13].

In the Landau–de Gennes framework, there exist two representative Q-tensor models, directly derived by a variational method, describing the hydrodynamics of nematic liquid crystals: the Beris–Edwards model [3] and the Qian–Sheng model [16]. The two models are, respectively, a system coupling the equation of Q-tensor order parameters with the time evolution equation of the fluid velocity. In this paper, we are concerned with the following Qian–Sheng model [16] with the inertial density:

$$J\ddot{Q} + \mu_1\dot{Q} = \mathbf{H} - \frac{\mu_2}{2}\mathbf{D} + \mu_1[\mathbf{\Omega}, Q], \tag{1.2}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot \left(\sigma + \sigma^d\right), \tag{1.3}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.4}$$

where J stands for the small *inertial coefficient*, and the inertial term $\ddot{Q} = (\partial_t + \mathbf{v} \cdot \nabla)\dot{Q}$ is the material derivative of $\dot{Q} = (\partial_t + \mathbf{v} \cdot \nabla)Q$. In addition, the viscous stress σ, the distortion stress σ^d and the molecular field $\mathbf{H}$ are, respectively, defined by

$$\begin{aligned}\sigma &= \beta_1 Q(Q : \mathbf{D}) + \beta_4\mathbf{D} + \beta_5\mathbf{D} \cdot Q + \beta_6 Q \cdot \mathbf{D} + \beta_7\left(\mathbf{D} \cdot Q^2 + Q^2 \cdot \mathbf{D}\right) \\ &\quad + \frac{\mu_2}{2}\left(\dot{Q} - [\mathbf{\Omega}, Q]\right) + \mu_1\left[Q, \left(\dot{Q} - [\mathbf{\Omega}, Q]\right)\right], \end{aligned} \tag{1.5}$$

$$\sigma^d_{ij} = -\frac{\partial \mathcal{F}}{\partial Q_{kl,j}}\partial_i Q_{kl}, \tag{1.6}$$

$$\mathbf{H}_{ij} = -\left(\frac{\delta \mathcal{F}(Q, \nabla Q)}{\delta Q}\right)_{ij} = -\frac{\partial \mathcal{F}}{\partial Q_{ij}} + \partial_k\left(\frac{\partial \mathcal{F}}{\partial Q_{ij,k}}\right) \overset{\text{def}}{=} -\mathcal{T}(Q) - \mathcal{L}(Q), \tag{1.7}$$

where the two operators $\mathcal{T}$ and $\mathcal{L}$ are, respectively, given by

$$\mathcal{T}(Q) = -aQ - bQ^2 + c|Q|^2 Q + \frac{1}{3}b|Q|^2\mathbf{I},$$
$$\big(\mathcal{L}(Q)\big)_{kl} = -\left(L_1 \Delta Q_{kl} + \frac{1}{2}(L_2 + L_3)\left(Q_{km,ml} + Q_{lm,mk} - \frac{2}{3}\delta_{kl}Q_{ij,ij}\right)\right).$$

The constants β_1, β_4, β_5, β_6, β_7, μ_1, and μ_2 in (1.5) are viscosity coefficients. The coefficients satisfy the following relation:

$$\beta_6 - \beta_5 = \mu_2. \tag{1.8}$$

It is worth emphasizing that, to be compared with the original Qian–Sheng model in [16], a new viscosity term $\beta_7(D_{ik}Q_{kl}Q_{lj} + Q_{ik}Q_{kl}D_{lj})$ in (1.5) is added to ensure that the energy of the system will always dissipate without assuming any relation between β_5 and β_6. The detailed discussion of the dissipative relation can be found in recent work [9].

For the Q-tensor dynamical model of liquid crystals, there has been published much analytical work. We only recall some relevant results here. Concerning the Beris–Edwards system, the well-posedness results on whole space and bounded domain can be found in [8, 14, 15] and [1, 2, 11], respectively. For the inertial Qian–Sheng model, De Anna and Zarnescu [4] investigated the local well-posdedness for bounded initial data and global well-posedness under the assumptions of the small initial data. For the non-viscous version of the Qian–Sheng model, Feireisl et al. [6] proved global existence of the dissipative solution which is inspired by that of the incompressible Euler equations. There is some interesting work, devoted to exploring the relation between different dynamical theories for liquid crystals. For example, by the Hilbert expansion method, Wang–Zhang–Zhang [19] rigorously justified that the strong solution to the non-inertial Beris–Edwards model converges to the solution to the Ericksen–Leslie model. In the same spirit, Li–Wang [9] extended this work, and rigorously proved the connection between the inertial Qian–Sheng model and the full inertial Ericksen–Leslie model. A unified formulation for liquid crystal modeling was put forward by Han et al. in [7] to establish relations between microscopic theories and macroscopic theories.

In [4], the well-posedness results rely on the assumption that the solution decays fast enough at infinity. However, during the physical modeling process, the liquid crystal system is not generally isotropic but certain nonzero uniaxial or biaxial equilibrium at infinity. Therefore, the main goal of this paper is to study the local well-posedness of the strong solution for the inertial Qian–Sheng system with the initial data near uniaxial equilibrium.

The rest of this paper is organized as follows. In Sect. 2, we state the notational conventions and some technical lemmas, and then present the main result of this paper. In Sect. 3, based on the classical Friedrich method and the closed energy estimate, we prove the local well-posedness of the inertial Qian–Sheng's Q-tensor dynamical model, when the solution to the system tends to the uniaxial equilibrium state at infinity.

2 Preliminaries and the main result

2.1 Notations and convections

The Einstein summation convention is used in this paper. The configuration space of the Q-tensor is the set of symmetric, traceless 3×3-matrices, that is,

$$\mathbb{S}_0^3 \stackrel{\text{def}}{=} \{Q \in \mathbb{R}^{3\times 3} : Q_{ij} = Q_{ji}, Q_{ii} = 0\},$$

which is endowed with the inner product $Q_1 : Q_2 = Q_{1ij}Q_{2ij}$. The Frobenius norm on $\mathbb{S}_0^3$ is defined as $|Q| \stackrel{\text{def}}{=} \sqrt{\operatorname{Tr} Q^2} = \sqrt{Q_{ij}Q_{ij}}$. For two tensors $A, B \in \mathbb{S}_0^3$ we denote $(A \cdot B)_{ij} = A_{ik}B_{kj}$ and $A : B = A_{ij}B_{ij}$, and $[A, B] = A \cdot B - B \cdot A$. For any $Q_1, Q_2 \in L^2(\mathbb{R}^3)^{3\times 3}$, the corresponding inner product is defined as

$$\langle Q_1, Q_2\rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} Q_{1ij}(\mathbf{x}) : Q_{2ij}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

We denote by $\mathbf{n}_1 \otimes \mathbf{n}_2$ the tensor product of two vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and omit the symbol $\otimes$ for simplicity. We use $f_{,i}$ to denote $\partial_i f$ and $\mathbf{I}$ to denote the 3×3 identity tensor. In addition, the superscripted dot denotes the material derivative, i.e., $\dot{f} = (\partial_t + \mathbf{v} \cdot \nabla)f$, where the fluid velocity $\mathbf{v}$ can be understood from the context. We also define the commutator $[\![\nabla^s, f]\!]g = \nabla^s(fg) - f\nabla^s g$.

2.2 Useful lemmas

The following product estimates and commutator estimates are well-known, see [10, 17] for example, and they are frequently used in this paper.

Lemma 2.1 *Let $s \geq 0$. Then, for any multi-index α, β,*

$$\|\partial^\alpha f\partial^\beta g\|_{H^s} \leq C\big(\|f\|_{L^\infty}\|g\|_{H^{s+|\alpha|+|\beta|}} + \|g\|_{L^\infty}\|f\|_{H^{s+|\alpha|+|\beta|}}\big);$$
$$\|\partial^\alpha f\partial^\beta g\|_{H^s} \leq C\|f\|_{H^{s+|\alpha|+|\beta|}}\|g\|_{H^{s+|\alpha|+|\beta|}}, \quad \text{if } s + |\alpha| + |\beta| \geq 2.$$

In particular, we have

$$\|fg\|_{H^s} \leq C\big(\|f\|_{L^\infty}\|g\|_{H^s} + \|g\|_{L^\infty}\|f\|_{H^s}\big);$$
$$\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}, \quad \text{if } s \geq 2;$$
$$\|fg\|_{H^k} \leq C\min\big\{\|f\|_{H^k}\|g\|_{H^2}, \|f\|_{H^2}\|g\|_{H^k}\big\}, \quad \text{if } 0 \leq k \leq 2.$$

Lemma 2.2 *Let a be a multiple index. We have*

$$\big\|[\![\partial^a, g]\!]f\big\|_{L^2} \leq C\big(\|\nabla g\|_{L^\infty}\|f\|_{H^{|a|-1}} + \|\nabla g\|_{H^{|a|-1}}\|f\|_{L^\infty}\big).$$

In particular, if $|a| \geq 2$, we have

$$\big\|[\![\partial^a, g]\!]f\big\|_{L^2} \leq C\|g\|_{H^{|a|+1}}\|f\|_{H^{|a|-1}},$$
$$\big\|[\![\partial^{a+1}, g]\!]f\big\|_{L^2} \leq C\|g\|_{H^{|a|+1}}\|f\|_{H^{|a|-1}}.$$

The following energy dissipation relation can be found in [9].

Lemma 2.3 *Assume that* $\beta_1, \beta_4, \mu_1 > 0$, $\beta_7 \geq 0$, *and* $\beta_4 - \frac{\mu_2^2}{4\mu_1} > 0$. *Then, for any smooth solution* $(\mathbf{v}, Q)$ *of the inertial Qian–Sheng system* (1.2)–(1.4),

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^3} \frac{1}{2}\left(|\mathbf{v}|^2 + J|\dot{Q}|^2\right) \mathrm{d}\mathbf{x} + \mathcal{F}(Q, \nabla Q)\right) \\ &\quad = -\beta_1 \|Q : \mathbf{D}\|_{L^2}^2 - \left(\beta_4 - \frac{\mu_2^2}{4\mu_1}\right)\|\mathbf{D}\|_{L^2}^2 - (\beta_5 + \beta_6)\langle \mathbf{D} \cdot Q, \mathbf{D}\rangle \\ &\qquad - 2\beta_7 \|\mathbf{D} \cdot Q\|_{L^2}^2 - \mu_1 \left\|\dot{Q} - [\mathbf{\Omega}, Q] + \frac{\mu_2}{2\mu_1}\mathbf{D}\right\|_{L^2}^2. \end{aligned} \tag{2.1}$$

Moreover, if one of the following assumptions holds: (*i*) $\beta_5 + \beta_6 = 0$ *if* $\beta_7 = 0$, (*ii*) $(\beta_5 + \beta_6)^2 < 8\beta_7(\beta_4 - \frac{\mu_2^2}{4\mu_1})$ *if* $\beta_7 \neq 0$, *then the right hand side in* (2.1) *is non-positive.*

We give some results about critical points. A tensor Q_0 is called a critical point of $f_b(Q)$ if $\mathcal{T}(Q_0) := \frac{\partial f_b}{\partial Q}|_{Q=Q_0} = 0$. The following characterization of critical points can be obtained from [12, 19].

Lemma 2.4 $\mathcal{T}(Q) = 0$ *if and only if* $Q = S(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ *for some* $\mathbf{n} \in \mathbb{S}^2$, *where* $S = 0$ *or a solution of* $2cS^2 - bS + 3a = 0$, *that is,*

$$S_1 = \frac{b + \sqrt{b^2 + 24ac}}{4c} \quad \text{or} \quad S_2 = \frac{b - \sqrt{b^2 + 24ac}}{4c}.$$

Moreover, the critical point $Q_0 = S(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ *is stable if* $S = S_1$.

Given a critical point $Q_0 = S(\mathbf{nn} - \frac{1}{3}\mathbf{I})$, the linearized operator $\mathcal{H}_{Q_0}$ of $\mathcal{T}(Q)$ around Q_0 is given by

$$\mathcal{H}_{Q_0}(Q) = aQ - b(Q_0 \cdot Q + Q \cdot Q_0) + c|Q_0|^2 Q + 2(Q_0 : Q)\left(cQ_0 + \frac{b}{3}\mathbf{I}\right).$$

2.3 Main results

Throughout this paper, we assume that the viscosity coefficients satisfy $\beta_1, \beta_4, \mu_1 > 0$, $\beta_7 \geq 0$, and $\beta_4 - \frac{\mu_2^2}{4\mu_1} > 0$, and the elastic coefficients L_i $(i = 1, 2, 3)$ satisfy $L_1 > 0$, $L_1 + L_2 + L_3 > 0$, and the inertial coefficient J is positive, and $J \ll \mu_1$.

The main assertion of this paper is stated as follows.

Theorem 2.1 *Let* $s \geq 2$ *be an integer,* $\mathbf{n}^* \in \mathbb{S}^2$ *is a constant vector and* $Q^* = S(\mathbf{n}^*\mathbf{n}^* - \frac{1}{3}\mathbf{I})$. *If the initial data fulfills*

$$\mathbf{v}_I(\mathbf{x}) \in H^s(\mathbb{R}^3), \qquad Q_I(\mathbf{x}) - Q^* \in H^{s+1}(\mathbb{R}^3), \qquad \dot{Q}_I(\mathbf{x}) \in H^s(\mathbb{R}^3),$$

for all $\mathbf{x} \in \mathbb{R}^3$, *then there exist* $T > 0$ *and a unique solution* $(\mathbf{v}, Q)$ *of the inertial Qian–Sheng Q-tensor system* (1.2)–(1.4) *on* $[0, T]$, *such that* $\mathbf{v}(0, \mathbf{x}) = \mathbf{v}_I(\mathbf{x})$, $Q(0, \mathbf{x}) = Q_I(\mathbf{x})$, *and*

$$\mathbf{v} \in L^\infty([0, T]; H^s(\mathbb{R}^3)) \cap L^2([0, T]; H^{s+1}(\mathbb{R}^3)), \tag{2.2}$$

$$Q - Q^* \in L^\infty([0, T]; H^{s+1}(\mathbb{R}^3)), \qquad \dot{Q} \in L^\infty([0, T]; H^s(\mathbb{R}^3)). \tag{2.3}$$

3 Local well-posedness for the inertial Qian–Sheng model

This section is devoted to the proof of the local well-posedness result for the inertial Qian–Sheng model with the initial data near uniaxial equilibrium. The main framework of our proof follows the strategy in [18]. We divide the proof of Theorem 2.1 into four steps.

Step 1. Construction of approximate solutions. Based on the classical Friedrich method, we construct the approximate system of the inertial Qian–Sheng model (1.2)–(1.4). We define the mollification operator

$$\mathcal{J}_\varepsilon f(\xi) \overset{\text{def}}{=} \mathrm{F}^{-1}(\mathbf{1}_{|\xi|\le\frac{1}{\varepsilon}}\mathrm{F}f),$$

where F is the Fourier transform. Assume that $\mathbb{P}$ is the Leray projection operator from a vector field into the corresponding divergence-free field.

Then the approximate system associated with (1.2)–(1.4) is given by

$$\begin{cases} J\mathcal{J}_\varepsilon\ddot{Q}_\varepsilon + \mu_1\mathcal{J}_\varepsilon\dot{Q}_\varepsilon = -\mathcal{J}_\varepsilon(\mathcal{T}(\mathcal{J}_\varepsilon Q_\varepsilon) + \mathcal{L}(\mathcal{J}_\varepsilon Q_\varepsilon)) - \frac{\mu_2}{2}\mathcal{J}_\varepsilon\mathbf{D}_\varepsilon + \mu_1\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon\mathbf{\Omega}_\varepsilon, \mathcal{J}_\varepsilon Q_\varepsilon], \\ \mathcal{J}_\varepsilon\partial_t\mathbf{v}_\varepsilon + \mathcal{J}_\varepsilon\mathbb{P}(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon) \\ \quad = \nabla\cdot\mathcal{J}_\varepsilon\mathbb{P}(\beta_1\mathcal{J}_\varepsilon Q_\varepsilon(\mathcal{J}_\varepsilon Q_\varepsilon : \mathcal{J}_\varepsilon\mathbf{D}_\varepsilon) + \beta_4\mathcal{J}_\varepsilon\mathbf{D}_\varepsilon \\ \qquad + \beta_5\mathcal{J}_\varepsilon\mathbf{D}_\varepsilon\cdot\mathcal{J}_\varepsilon Q_\varepsilon + \beta_6\mathcal{J}_\varepsilon Q_\varepsilon\cdot\mathcal{J}_\varepsilon\mathbf{D}_\varepsilon + \beta_7(\mathcal{J}_\varepsilon\mathbf{D}_\varepsilon\cdot(\mathcal{J}_\varepsilon Q_\varepsilon)^2 + (\mathcal{J}_\varepsilon Q_\varepsilon)^2\cdot\mathcal{J}_\varepsilon\mathbf{D}_\varepsilon) \\ \qquad + \frac{\mu_2}{2}(\dot{Q}_\varepsilon - [\mathcal{J}_\varepsilon\mathbf{\Omega}_\varepsilon, \mathcal{J}_\varepsilon Q_\varepsilon]) + \mu_1[\mathcal{J}_\varepsilon Q_\varepsilon, (\mathcal{J}_\varepsilon\dot{Q}_\varepsilon - [\mathcal{J}_\varepsilon\mathbf{\Omega}_\varepsilon, \mathcal{J}_\varepsilon Q_\varepsilon])] \\ \qquad + \sigma^d(\mathcal{J}_\varepsilon Q_\varepsilon, \mathcal{J}_\varepsilon Q_\varepsilon)), \\ (\mathbf{v}_\varepsilon, Q_\varepsilon)|_{t=0} = (\mathcal{J}_\varepsilon\mathbf{v}_0, \mathcal{J}_\varepsilon Q_0), \end{cases}$$

where the material derivative $\dot{Q}_\varepsilon \overset{\text{def}}{=} \partial_t Q_\varepsilon + \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon\cdot\nabla\mathcal{J}_\varepsilon Q_\varepsilon)$, and $\mathcal{T}(\mathcal{J}_\varepsilon Q_\varepsilon)$ and $\mathcal{L}(\mathcal{J}_\varepsilon Q_\varepsilon)$ are, respectively, defined as

$$\begin{aligned} \mathcal{T}(\mathcal{J}_\varepsilon Q_\varepsilon) &= -a\mathcal{J}_\varepsilon Q_\varepsilon - b(\mathcal{J}_\varepsilon Q_\varepsilon)^2 + c|\mathcal{J}_\varepsilon Q_\varepsilon|^2\mathcal{J}_\varepsilon Q_\varepsilon + \frac{1}{3}b|\mathcal{J}_\varepsilon Q_\varepsilon|^2\mathbf{I}, \\ \mathcal{L}(\mathcal{J}_\varepsilon Q_\varepsilon)_{kl} &= -\bigg(L_1\Delta\mathcal{J}_\varepsilon(Q_\varepsilon)_{kl} + \frac{1}{2}(L_2+L_3)\bigg(\mathcal{J}_\varepsilon(Q_\varepsilon)_{km,ml} + \mathcal{J}_\varepsilon(Q_\varepsilon)_{lm,mk} \\ &\quad - \frac{2}{3}\delta_{kl}\mathcal{J}_\varepsilon(Q_\varepsilon)_{ij,ij}\bigg)\bigg). \end{aligned}$$

The above system can be regarded as an ODE system in $L^2(\mathbb{R}^3)$. Then, applying the Cauchy–Lipshitz theorem, there exist a strictly maximal time T_ε and a unique solution $(\mathbf{v}_\varepsilon, Q_\varepsilon)$, which is continuous in time with a value in $H^k(\mathbb{R}^3)$ for any $k\ge 0$. Since $\mathcal{J}_\varepsilon^2 = \mathcal{J}_\varepsilon$ and $\mathbb{P}$ is a self-adjoint operator in $L^2(\mathbb{R}^3)$, the pair $(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon, \mathcal{J}_\varepsilon Q_\varepsilon)$ is also a solution of the previous system. Therefore, the uniqueness of the solution leads to $(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon, \mathcal{J}_\varepsilon Q_\varepsilon) = (\mathbf{v}_\varepsilon, Q_\varepsilon)$, and thus $(\mathbf{v}_\varepsilon, Q_\varepsilon)$ satisfies the following system:

$$\begin{cases} J\ddot{Q}_\varepsilon + \mu_1\dot{Q}_\varepsilon = -\mathcal{J}_\varepsilon(\mathcal{T}(Q_\varepsilon) + \mathcal{L}(Q_\varepsilon)) - \frac{\mu_2}{2}\mathbf{D}_\varepsilon + \mu_1\mathcal{J}_\varepsilon[\mathbf{\Omega}_\varepsilon, Q_\varepsilon], \\ \partial_t\mathbf{v}_\varepsilon + \mathcal{J}_\varepsilon\mathbb{P}(\mathbf{v}_\varepsilon\cdot\nabla\mathbf{v}_\varepsilon) \\ \quad = \nabla\cdot\mathcal{J}_\varepsilon\mathbb{P}(\beta_1 Q_\varepsilon(Q_\varepsilon : \mathbf{D}_\varepsilon) + \beta_4\mathbf{D}_\varepsilon \\ \qquad + \beta_5\mathbf{D}_\varepsilon\cdot Q_\varepsilon + \beta_6 Q_\varepsilon\cdot\mathbf{D}_\varepsilon + \beta_7(\mathbf{D}_\varepsilon\cdot Q_\varepsilon^2 + Q_\varepsilon^2\cdot\mathbf{D}_\varepsilon) \\ \qquad + \frac{\mu_2}{2}(\dot{Q}_\varepsilon - [\mathbf{\Omega}_\varepsilon, Q_\varepsilon]) + \mu_1[Q_\varepsilon, (\dot{Q}_\varepsilon - [\mathbf{\Omega}_\varepsilon, Q_\varepsilon])] + \sigma^d(Q_\varepsilon, Q_\varepsilon)), \\ (\mathbf{v}_\varepsilon, Q_\varepsilon)|_{t=0} = (\mathcal{J}_\varepsilon\mathbf{v}_0, \mathcal{J}_\varepsilon Q_0). \end{cases} \tag{3.1}$$

Step 2. Uniform energy estimates. We define the energy functional $\mathcal{E}(t)$ by

$$\begin{aligned}\mathcal{E}(t) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^3}\bigg(|\mathbf{v}|^2 + \frac{J}{2}\big(\big|\dot{Q}+Q-Q^*\big|^2+|\dot{Q}|^2\big) + \frac{1}{2}(\mu_1-J)\big|Q-Q^*\big|^2 \\ &\quad + \mathcal{L}(Q):\big(Q-Q^*\big) + \frac{1}{2}\big|\nabla^s\mathbf{v}\big|^2 + \frac{J}{2}\big|\nabla^s\dot{Q}\big|^2 + \mathcal{L}\big(\nabla^s Q\big):\nabla^s Q\bigg)\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Recalling the fact that there exists a constant $L_0 = \min\{L_1, L_1 + L_2 + L_3\} > 0$ such that (see [19, Lemma 2.5])

$$\big\langle\mathcal{L}(Q), Q\big\rangle = \int_{\mathbb{R}^3} f_e(\nabla Q)\,\mathrm{d}\mathbf{x} \geq L_0\|\nabla Q\|_{L^2}^2.$$

By a Sobolev interpolation, we have

$$\mathcal{E}(t) \sim \big\|Q-Q^*\big\|_{L^2}^2 + \|\nabla Q\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2 + \|\dot{Q}\|_{H^s}^2.$$

Let $\widetilde{Q}_\varepsilon = Q_\varepsilon - Q^*$, then from the expression of $\mathcal{T}(Q)$ we have

$$\mathcal{T}\big(\widetilde{Q}_\varepsilon + Q^*\big) = \mathcal{T}\big(Q^*\big) + \mathcal{H}_{Q^*}(\widetilde{Q}_\varepsilon) + \mathcal{P}_3(\widetilde{Q}_\varepsilon), \tag{3.2}$$

where $\mathcal{H}_{Q^*}$ and $\mathcal{P}_3$ are, respectively, defined as

$$\begin{aligned}\mathcal{H}_{Q^*}(Q) &\stackrel{\text{def}}{=} -aQ - b\bigg(Q^*\cdot Q + Q\cdot Q^* - \frac{2}{3}(Q^*:Q)\mathbf{I}\bigg) + c\big(\big|Q^*\big|^2 Q + 2\big(Q^*:Q\big)Q^*\big), \\ \mathcal{P}_3(Q) &\stackrel{\text{def}}{=} -b\bigg(Q^2 + \frac{b}{3}|Q|^2\mathbf{I}\bigg) + c\big(|Q|^2 Q + |Q|^2 Q^* + 2\big(Q:Q^*\big)Q\big).\end{aligned}$$

Since for some constant vector $\mathbf{n}^* \in \mathbb{S}^2$, $Q^* = S(\mathbf{n}^*\mathbf{n}^* - \frac{1}{3}\mathbf{I})$ is a critical point of $\mathcal{T}(Q)$, from Lemma 2.4 we get $\mathcal{T}(Q^*) = 0$.

Multiplying the first equation in (3.1) by $Q_\varepsilon - Q^*$ and taking the L^2-inner product, we obtain

$$\begin{aligned}&\big\langle J\ddot{Q}_\varepsilon + \mu_1\dot{Q}_\varepsilon, Q_\varepsilon - Q^*\big\rangle + \big\langle\mathcal{L}(Q_\varepsilon), \mathcal{J}_\varepsilon\big(Q_\varepsilon - Q^*\big)\big\rangle \\ &\quad = \underbrace{\bigg\langle -\frac{\mu_2}{2}\mathbf{D}_\varepsilon + \mu_1[\boldsymbol{\Omega}_\varepsilon, Q_\varepsilon], \mathcal{J}_\varepsilon\big(Q_\varepsilon - Q^*\big)\bigg\rangle}_{I_1} \underbrace{-\big\langle\mathcal{T}(Q_\varepsilon), \mathcal{J}_\varepsilon\big(Q_\varepsilon - Q^*\big)\big\rangle}_{I_2}.\end{aligned} \tag{3.3}$$

Using the fact that $\langle[\boldsymbol{\Omega}, Q], Q\rangle = 0$, the estimate of I_1 can be calculated as

$$\begin{aligned}I_1 &= \bigg\langle -\frac{\mu_2}{2}\mathbf{D}_\varepsilon + \mu_1\big[\boldsymbol{\Omega}_\varepsilon, Q^*\big], \mathcal{J}_\varepsilon\big(Q_\varepsilon - Q^*\big)\bigg\rangle \\ &\leq C\|\nabla\mathbf{v}_\varepsilon\|_{L^2}\big\|Q_\varepsilon - Q^*\big\|_{L^2} \leq C_\delta\mathcal{E} + \delta\|\nabla\mathbf{v}_\varepsilon\|_{L^2}^2.\end{aligned}$$

The term I_2 can be handled as

$$\begin{aligned} I_2 &= -\langle \mathcal{T}(\widetilde{Q}_\varepsilon + Q^*), \mathcal{J}_\varepsilon \widetilde{Q}_\varepsilon \rangle = -\langle \mathcal{H}_{Q^*}(\widetilde{Q}_\varepsilon), \mathcal{J}_\varepsilon \widetilde{Q}_\varepsilon \rangle - \langle \mathcal{P}_3(\widetilde{Q}_\varepsilon), \mathcal{J}_\varepsilon \widetilde{Q}_\varepsilon \rangle \\ &\le -\langle \mathcal{H}_{Q^*}(\widetilde{Q}_\varepsilon), \mathcal{J}_\varepsilon \widetilde{Q}_\varepsilon \rangle + C\big(\|\nabla Q\|_{L^2} + \|\nabla Q\|_{L^2}^3\big)\|\widetilde{Q}_\varepsilon\|_{L^2} \\ &\le C\big(\mathcal{E} + \mathcal{E}^2\big). \end{aligned}$$

Noticing that, for $Q \in \mathbb{S}_0^3$ and a constant tensor Q^*, we have

$$\begin{aligned} \langle \ddot{Q}, Q - Q^* \rangle &= \int_{\mathbb{R}^3} (\partial_t + v_k \partial_k)\dot{Q}_{ij}\big(Q_{ij} - Q_{ij}^*\big)\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} \big(\partial_t\big(\dot{Q}_{ij}\big(Q_{ij} - Q_{ij}^*\big)\big) + v_k\partial_k\big(\dot{Q}_{ij}\big(Q_{ij} - Q_{ij}^*\big)\big) - \dot{Q}_{ij}\dot{Q}_{ij}\big)\,\mathrm{d}\mathbf{x} \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\big\|\dot{Q} + Q - Q^*\big\|_{L^2}^2 - \|\dot{Q}\|_{L^2}^2 - \big\|Q - Q^*\big\|_{L^2}^2\big) - \|\dot{Q}\|_{L^2}^2. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4) and the estimates of I_1 and I_2, we know that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(J\big\|\dot{Q}_\varepsilon + Q_\varepsilon - Q^*\big\|_{L^2}^2 - J\|\dot{Q}_\varepsilon\|_{L^2}^2 + (\mu_1 - J)\big\|Q_\varepsilon - Q^*\big\|_{L^2}^2\big) \\ &\quad + L_0\|\nabla Q_\varepsilon\|_{L^2}^2 + \langle \mathcal{H}_{Q^*}\big(Q_\varepsilon - Q^*\big), Q_\varepsilon - Q^* \rangle \\ &\quad \le C\big(\mathcal{E} + \mathcal{E}^2\big) + \delta\|\nabla \mathbf{v}_\varepsilon\|_{L^2}^2. \end{aligned} \tag{3.5}$$

The basic energy dissipation in Lemma 2.3 tells us that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_{\mathbb{R}^3} \frac{1}{2}\big(|\mathbf{v}_\varepsilon|^2 + J|\dot{Q}_\varepsilon|^2\big) + f_b\big(\widetilde{Q}_\varepsilon + Q^*\big) - f_b\big(Q^*\big) + f_e(\nabla Q_\varepsilon)\bigg)\,\mathrm{d}\mathbf{x} \\ &\quad = -\beta_1\|Q_\varepsilon : \mathbf{D}_\varepsilon\|_{L^2}^2 - \bigg(\beta_4 - \frac{\mu_2^2}{4\mu_1} - 3\delta\bigg)\|\mathbf{D}_\varepsilon\|_{L^2}^2 - (\beta_5 + \beta_6)\langle \mathbf{D}_\varepsilon \cdot Q_\varepsilon, \mathbf{D}_\varepsilon \rangle \\ &\qquad - 2\beta_7\|\mathbf{D}_\varepsilon \cdot Q_\varepsilon\|_{L^2}^2 - \mu_1\bigg\|\dot{Q}_\varepsilon - [\mathbf{\Omega}_\varepsilon, Q_\varepsilon] + \frac{\mu_2}{2\mu_1}\mathbf{D}_\varepsilon\bigg\|_{L^2}^2 - \delta\|\nabla \mathbf{v}_\varepsilon\|_{L^2}^2 \\ &\quad \le -\delta\|\nabla \mathbf{v}_\varepsilon\|_{L^2}^2. \end{aligned} \tag{3.6}$$

Thus, we multiply by 2 on (3.6) and then add it to (3.5), so that we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\|\mathbf{v}_\varepsilon\|_{L^2}^2 + \frac{J}{2}\big(\big\|\dot{Q}_\varepsilon + Q_\varepsilon - Q^*\big\|_{L^2}^2 + \|\dot{Q}_\varepsilon\|_{L^2}^2\big) \\ &\quad + \frac{1}{2}(\mu_1 - J)\big\|Q_\varepsilon - Q^*\big\|_{L^2}^2 + 2\mathcal{L}(Q) : \big(Q_\varepsilon - Q^*\big)\bigg) \\ &\quad \le -\delta\|\nabla \mathbf{v}_\varepsilon\|_{L^2}^2 + C\big(\mathcal{E} + \mathcal{E}^2\big). \end{aligned} \tag{3.7}$$

We now turn to the estimates of the higher order derivative for $(Q_\varepsilon, \mathbf{v}_\varepsilon)$. On the one hand, we take ∇^s on the first equation of (3.1) and multiply it by $\nabla^s\dot{Q}_\varepsilon$, integrate over $\mathbb{R}^3$ and by parts, then we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{J}{2}\|\nabla^s\dot{Q}_\varepsilon\|_{L^2}^2+\mathcal{L}(\nabla^s Q_\varepsilon):\nabla^s Q_\varepsilon\right)+\mu_1\|\nabla^s\dot{Q}_\varepsilon\|_{L^2}^2\\
&=-J\langle\nabla^s(\mathbf{v}_\varepsilon\cdot\nabla\dot{Q}_\varepsilon),\nabla^s\dot{Q}_\varepsilon\rangle-\langle\nabla^s\mathcal{T}(Q_\varepsilon),\nabla^s\dot{Q}_\varepsilon\rangle-\langle\nabla^s\mathcal{L}(Q_\varepsilon),\nabla^s(\mathbf{v}_\varepsilon\cdot\nabla Q_\varepsilon)\rangle\\
&\quad-\frac{\mu_2}{2}\langle\nabla^s\mathbf{D}_\varepsilon,\nabla^s\dot{Q}_\varepsilon\rangle+\mu_1\langle\nabla^s[\mathbf{\Omega}_\varepsilon,Q_\varepsilon],\nabla^s\dot{Q}_\varepsilon\rangle\\
&\stackrel{\text{def}}{=}\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5.
\end{aligned}\tag{3.8}$$

Using Lemma 2.2 and $\nabla\cdot\mathbf{v}_\varepsilon=0$, we obtain

$$\begin{aligned}
\mathcal{I}_1&=-J\langle\nabla^s(\mathbf{v}_\varepsilon\cdot\nabla\dot{Q}_\varepsilon),\nabla^s\dot{Q}_\varepsilon\rangle+J\langle\mathbf{v}_\varepsilon\cdot\nabla\nabla^s\dot{Q}_\varepsilon,\nabla^s\dot{Q}_\varepsilon\rangle\\
&=-J\langle[\![\nabla^s,\mathbf{v}_\varepsilon]\!]\cdot\nabla\dot{Q}_\varepsilon,\nabla^s\dot{Q}_\varepsilon\rangle\\
&\le J\|[\![\nabla^s,\mathbf{v}_\varepsilon]\!]\cdot\nabla\dot{Q}_\varepsilon\|_{L^2}\|\nabla^s\dot{Q}_\varepsilon\|_{L^2}\\
&\le C\|\mathbf{v}_\varepsilon\|_{H^{s+1}}\|\dot{Q}_\varepsilon\|_{H^s}^2\le C_\delta\big(\mathcal{E}^{\frac{3}{2}}+\mathcal{E}^2\big)+\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2.
\end{aligned}$$

From $\mathcal{T}(Q^*)=0$ and Lemma 2.1, the term $\mathcal{I}_2$ can be derived,

$$\begin{aligned}
\mathcal{I}_2&=-\langle\nabla^s\big(\mathcal{T}(Q_\varepsilon)-\mathcal{T}(Q^*)\big),\nabla^s\dot{Q}_\varepsilon\rangle\\
&=-\langle\mathcal{H}_{Q^*}(\nabla^s\widetilde{Q}_\varepsilon),\nabla^s\dot{Q}_\varepsilon\rangle-\langle\nabla^s\mathcal{P}_3(\widetilde{Q}_\varepsilon),\nabla^s\dot{Q}_\varepsilon\rangle\\
&\le C\big(\|\nabla^s\widetilde{Q}_\varepsilon\|_{L^2}+\|\nabla^s\widetilde{Q}_\varepsilon\|_{L^2}^2+\|\nabla^s\widetilde{Q}_\varepsilon\|_{L^2}^3\big)\|\nabla^s\dot{Q}_\varepsilon\|_{L^2}\\
&\le C\big(\mathcal{E}+\mathcal{E}^{\frac{3}{2}}+\mathcal{E}^2\big).
\end{aligned}$$

We observe that, for any $Q\in\mathbb{S}_0^3$,

$$\begin{aligned}
&-\langle\mathcal{L}(Q),\mathbf{v}\cdot\nabla Q\rangle\\
&=\int_{\mathbb{R}^3}v_jQ_{kl,j}\left(L_1\Delta Q_{kl}+\frac{1}{2}(L_2+L_3)\left(Q_{km,ml}+Q_{lm,mk}-\frac{2}{3}\delta_{kl}Q_{ij,ij}\right)\right)\mathrm{d}\mathbf{x}\\
&=\int_{\mathbb{R}^3}\Big(-L_1v_jQ_{kl,mj}Q_{kl,m}-\frac{1}{2}(L_2+L_3)(v_jQ_{kl,lj}Q_{km,m}+v_jQ_{kl,kj}Q_{lm,m})\\
&\qquad-L_1v_{j,m}Q_{kl,j}Q_{kl,m}-\frac{1}{2}(L_2+L_3)(v_{j,l}Q_{kl,j}Q_{km,m}+v_{j,k}Q_{kl,j}Q_{lm,m})\Big)\mathrm{d}\mathbf{x}\\
&=\int_{\mathbb{R}^3}\left(-L_1v_{j,m}Q_{kl,j}Q_{kl,m}-\frac{1}{2}(L_2+L_3)(v_{j,l}Q_{kl,j}Q_{km,m}+v_{j,k}Q_{kl,j}Q_{lm,m})\right)\mathrm{d}\mathbf{x}\\
&\le C\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla Q\|_{L^2}^2.
\end{aligned}\tag{3.9}$$

By (3.9) and Lemma 2.1, the term $\mathcal{I}_3$ can be handled as follows:

$$\begin{aligned}
\mathcal{I}_3&=-\langle\nabla^s\mathcal{L}(Q_\varepsilon),\mathbf{v}_\varepsilon\cdot\nabla\nabla^sQ_\varepsilon\rangle-\langle\nabla^s\mathcal{L}(Q_\varepsilon),[\![\nabla^s,\mathbf{v}_\varepsilon]\!]\cdot\nabla Q_\varepsilon\rangle\\
&\le\mathcal{I}_3'+C_\delta\big(\mathcal{E}^2+\mathcal{E}^3+\mathcal{E}^4\big)+\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2.
\end{aligned}$$

The term $\mathcal{I}_5$ can be calculated as

$$\begin{aligned}
\mathcal{I}_5 &= \mu_1\langle\nabla^s[\boldsymbol{\Omega}_\varepsilon,\widetilde{Q}_\varepsilon],\nabla^s\dot{Q}_\varepsilon\rangle+\mu_1\langle[\nabla^s\boldsymbol{\Omega}_\varepsilon,Q^*],\nabla^s\dot{Q}_\varepsilon\rangle\\
&\le \mu_1\langle[\nabla^s\boldsymbol{\Omega}_\varepsilon,\widetilde{Q}_\varepsilon+Q^*],\nabla^s\dot{Q}_\varepsilon\rangle+C\|\widetilde{Q}_\varepsilon\|_{H^{s+1}}\|\mathbf{v}_\varepsilon\|_{H^s}\|\nabla^s\dot{Q}_\varepsilon\|_{L^2}\\
&\le \mu_1\langle[\nabla^s\boldsymbol{\Omega}_\varepsilon,Q_\varepsilon],\nabla^s\dot{Q}_\varepsilon\rangle+C\mathcal{E}^{\frac{3}{2}}.
\end{aligned}$$

On the other hand, we act the derivative operator ∇^s on the second equation of (3.1) and take L^2-inner product by multiplying $\nabla^s\mathbf{v}_\varepsilon$, then by integrating by parts we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^s\mathbf{v}_\varepsilon\|_{L^2}^2\\
&\quad=\langle\partial_t\nabla^s\mathbf{v}_\varepsilon,\nabla^s\mathbf{v}_\varepsilon\rangle\\
&\quad=-\langle\nabla^s(\mathbf{v}_\varepsilon\cdot\nabla\mathbf{v}_\varepsilon),\nabla^s\mathbf{v}_\varepsilon\rangle-\langle\nabla^s(\beta_1Q_\varepsilon(Q_\varepsilon:\mathbf{D}_\varepsilon)+\beta_4\mathbf{D}_\varepsilon\\
&\qquad+\beta_5\mathbf{D}_\varepsilon\cdot Q_\varepsilon+\beta_6Q_\varepsilon\cdot\mathbf{D}_\varepsilon+\beta_7(\mathbf{D}_\varepsilon\cdot Q_\varepsilon^2+Q_\varepsilon^2\cdot\mathbf{D}_\varepsilon)),\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\qquad-\frac{\mu_2}{2}\langle\nabla^s(\dot{Q}_\varepsilon-[\boldsymbol{\Omega}_\varepsilon,Q_\varepsilon]),\nabla^{s+1}\mathbf{v}_\varepsilon\rangle-\mu_1\langle\nabla^s[Q_\varepsilon,(\dot{Q}_\varepsilon-[\boldsymbol{\Omega}_\varepsilon,Q_\varepsilon])],\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\qquad-\langle\nabla^s\sigma^d(Q_\varepsilon,Q_\varepsilon),\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\quad\stackrel{\text{def}}{=}\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4+\mathcal{J}_5.
\end{aligned}\tag{3.10}$$

From Lemma 2.2, we can deduce that

$$\begin{aligned}
\mathcal{J}_1 &= \langle[\![\nabla^s,\mathbf{v}_\varepsilon]\!]\cdot\nabla\mathbf{v}_\varepsilon,\nabla^s\mathbf{v}_\varepsilon\rangle\le C\|\mathbf{v}_\varepsilon\|_{H^{s+1}}\|\mathbf{v}_\varepsilon\|_{H^s}^2\\
&\le C_\delta(\mathcal{E}^{\frac{3}{2}}+\mathcal{E}^2)+\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2.
\end{aligned}$$

The term $\mathcal{J}_2$ can be derived from Lemma 2.2,

$$\begin{aligned}
\mathcal{J}_2 &= -\langle\beta_1Q_\varepsilon(Q_\varepsilon:\nabla^s\mathbf{D}_\varepsilon)+\beta_4\nabla^s\mathbf{D}_\varepsilon+\beta_5\nabla^s\mathbf{D}_\varepsilon\cdot Q_\varepsilon+\beta_6Q_\varepsilon\cdot\nabla^s\mathbf{D}_\varepsilon\\
&\quad+\beta_7(\nabla^s\mathbf{D}_\varepsilon\cdot Q_\varepsilon^2+Q_\varepsilon^2\cdot\nabla^s\mathbf{D}_\varepsilon),\nabla^s\mathbf{D}_\varepsilon\rangle-\beta_1\langle[\![\nabla^s,\widetilde{Q}_\varepsilon\widetilde{Q}_\varepsilon:]\!]\mathbf{D}_\varepsilon,\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\quad-\beta_1\langle[\![\nabla^s,Q^*\widetilde{Q}_\varepsilon:]\!]\mathbf{D}_\varepsilon,\nabla^{s+1}\mathbf{v}_\varepsilon\rangle-\beta_1\langle[\![\nabla^s,\widetilde{Q}_\varepsilon Q^*:]\!]\mathbf{D}_\varepsilon,\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\quad-\beta_5\langle[\![\nabla^s,\widetilde{Q}_\varepsilon]\!]\cdot\mathbf{D}_\varepsilon,\nabla^{s+1}\mathbf{v}_\varepsilon\rangle-\beta_6\langle\mathbf{D}_\varepsilon\cdot[\![\nabla^s,\widetilde{Q}_\varepsilon]\!],\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\quad-\beta_7\langle[\![\nabla^s,\widetilde{Q}_\varepsilon^2]\!]\cdot\mathbf{D}_\varepsilon+\mathbf{D}_\varepsilon\cdot[\![\nabla^s,\widetilde{Q}_\varepsilon^2]\!],\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\quad-\beta_7\langle[\![\nabla^s,\widetilde{Q}_\varepsilon\cdot Q^*]\!]\cdot\mathbf{D}_\varepsilon+\mathbf{D}_\varepsilon\cdot[\![\nabla^s,\widetilde{Q}_\varepsilon\cdot Q^*]\!],\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\quad-\beta_7\langle[\![\nabla^s,Q^*\cdot\widetilde{Q}_\varepsilon]\!]\cdot\mathbf{D}_\varepsilon+\mathbf{D}_\varepsilon\cdot[\![\nabla^s,Q^*\cdot\widetilde{Q}_\varepsilon]\!],\nabla^{s+1}\mathbf{v}_\varepsilon\rangle\\
&\le-\beta_1\|Q_\varepsilon:\nabla^s\mathbf{D}_\varepsilon\|_{L^2}^2-\beta_4\|\nabla^s\mathbf{D}_\varepsilon\|_{L^2}^2-(\beta_5+\beta_6)\langle\nabla^s\mathbf{D}_\varepsilon\cdot Q_\varepsilon,\nabla^s\mathbf{D}_\varepsilon\rangle\\
&\quad-2\beta_7\|\nabla^s\mathbf{D}_\varepsilon\cdot Q_\varepsilon\|_{L^2}^2-\frac{\mu_2}{2}\langle[\nabla^s\boldsymbol{\Omega}_\varepsilon,Q_\varepsilon],\nabla^s\mathbf{D}_\varepsilon\rangle\\
&\quad+C_\delta\|\mathbf{v}_\varepsilon\|_{H^s}^2(\|\widetilde{Q}_\varepsilon\|_{H^{s+1}}^2+\|\widetilde{Q}_\varepsilon\|_{H^{s+1}}^4)+\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^2}^2.
\end{aligned}$$

For $\mathcal{J}_3$, we get

$$\begin{aligned}\mathcal{J}_3 &= -\frac{\mu_2}{2}\langle\nabla^s\dot{Q}_\varepsilon - \nabla^s[\boldsymbol{\Omega}_\varepsilon, \widetilde{Q}_\varepsilon], \nabla^{s+1}\mathbf{v}_\varepsilon\rangle - \frac{\mu_2}{2}\langle\nabla^s\dot{Q}_\varepsilon - [\nabla^s\boldsymbol{\Omega}_\varepsilon, Q^*], \nabla^{s+1}\mathbf{v}_\varepsilon\rangle \\ &\leq -\frac{\mu_2}{2}\langle\nabla^s\dot{Q}_\varepsilon - [\nabla^s\boldsymbol{\Omega}_\varepsilon, Q_\varepsilon], \nabla^s\mathbf{D}_\varepsilon\rangle + C_\delta\|\mathbf{v}_\varepsilon\|_{H^s}^2\|\widetilde{Q}_\varepsilon\|_{H^{s+1}}^2 + \delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2.\end{aligned}$$

In the same way, the term $\mathcal{J}_4$ can be estimated,

$$\begin{aligned}\mathcal{J}_4 &= -\mu_1\langle\nabla^s[\widetilde{Q}_\varepsilon, (\dot{Q}_\varepsilon - [\boldsymbol{\Omega}_\varepsilon, \widetilde{Q}_\varepsilon])], \nabla^{s+1}\mathbf{v}_\varepsilon\rangle - \mu_1\langle\nabla^s[\widetilde{Q}_\varepsilon, (\dot{Q}_\varepsilon - [\boldsymbol{\Omega}_\varepsilon, Q^*])], \nabla^{s+1}\mathbf{v}_\varepsilon\rangle \\ &\quad - \mu_1\langle[Q^*, (\nabla^s\dot{Q}_\varepsilon - \nabla^s[\boldsymbol{\Omega}_\varepsilon, \widetilde{Q}_\varepsilon])], \nabla^{s+1}\mathbf{v}_\varepsilon\rangle - \mu_1\|[Q^*, \nabla^s\boldsymbol{\Omega}_\varepsilon]\|_{L^2}^2 \\ &\leq -\mu_1\langle[\widetilde{Q}_\varepsilon, (\nabla^s\dot{Q}_\varepsilon - [\nabla^s\boldsymbol{\Omega}_\varepsilon, \widetilde{Q}_\varepsilon])], \nabla^{s+1}\mathbf{v}_\varepsilon\rangle - \mu_1\|[Q^*, \nabla^s\boldsymbol{\Omega}_\varepsilon]\|_{L^2}^2 \\ &\quad + C_\delta(\|\mathbf{v}_\varepsilon\|_{H^s}^2 + \|\dot{Q}_\varepsilon\|_{H^s}^2 + \|\mathbf{v}_\varepsilon\|_{H^s}^2\|\widetilde{Q}_\varepsilon\|_{H^{s+1}}^2)\|\widetilde{Q}_\varepsilon\|_{H^{s+1}}^2 + \delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2 \\ &\leq -\mu_1\langle[Q_\varepsilon, \nabla^s\boldsymbol{\Omega}_\varepsilon], \nabla^s\dot{Q}_\varepsilon - [\nabla^s\boldsymbol{\Omega}_\varepsilon, Q_\varepsilon]\rangle + C_\delta(\mathcal{E}^2 + \mathcal{E}^3) + \delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2.\end{aligned}$$

Therefore, from (3.8) and (3.10), noting $\mathcal{I}_3' + \mathcal{J}_5 = 0$ and gathering the previous estimates yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla^s\mathbf{v}_\varepsilon\|_{L^2}^2 + \frac{J}{2}\|\nabla^s\dot{Q}_\varepsilon\|_{L^2}^2 + \mathcal{L}(\nabla^s Q_\varepsilon) : \nabla^s Q_\varepsilon\right) \\ &\quad\leq -\mu_1\langle\nabla^s\dot{Q}_\varepsilon - [\nabla^s\boldsymbol{\Omega}_\varepsilon, Q_\varepsilon], \nabla^s\dot{Q}_\varepsilon\rangle - \frac{\mu_2}{2}\langle\nabla^s\mathbf{D}_\varepsilon, \nabla^s\dot{Q}_\varepsilon\rangle - \beta_1\|Q_\varepsilon : \nabla^s\mathbf{D}_\varepsilon\|_{L^2}^2 \\ &\qquad - \beta_4\|\nabla^s\mathbf{D}_\varepsilon\|_{L^2}^2 - (\beta_5 + \beta_6)\langle\nabla^s\mathbf{D}_\varepsilon\cdot Q_\varepsilon, \nabla^s\mathbf{D}_\varepsilon\rangle - 2\beta_7\|\nabla^s\mathbf{D}_\varepsilon\cdot Q_\varepsilon\|_{L^2}^2 \\ &\qquad - \frac{\mu_2}{2}\langle[\nabla^s\boldsymbol{\Omega}_\varepsilon, Q_\varepsilon], \nabla^s\mathbf{D}_\varepsilon\rangle - \frac{\mu_2}{2}\langle\nabla^s\dot{Q}_\varepsilon - [\nabla^s\boldsymbol{\Omega}_\varepsilon, Q_\varepsilon], \nabla^s\mathbf{D}_\varepsilon\rangle \\ &\qquad - \mu_1\langle[Q_\varepsilon, \nabla^s\boldsymbol{\Omega}_\varepsilon], \nabla^s\dot{Q}_\varepsilon - [\nabla^s\boldsymbol{\Omega}_\varepsilon, Q_\varepsilon]\rangle + C(\mathcal{E}^{\frac{3}{2}} + \mathcal{E}^2 + \mathcal{E}^3) + 6\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2 \\ &\quad= -\beta_1\|Q_\varepsilon : \nabla^s\mathbf{D}_\varepsilon\|_{L^2}^2 - \left(\beta_4 - \frac{\mu_2^2}{4\mu_1} - 7\delta\right)\|\nabla^s\mathbf{D}_\varepsilon\|_{L^2}^2 - (\beta_5 + \beta_6)\langle\nabla^s\mathbf{D}_\varepsilon\cdot Q_\varepsilon, \nabla^s\mathbf{D}_\varepsilon\rangle \\ &\qquad - 2\beta_7\|\nabla^s\mathbf{D}_\varepsilon\cdot Q_\varepsilon\|_{L^2}^2 - \mu_1\left\|\nabla^s\dot{Q}_\varepsilon - [\nabla^s\boldsymbol{\Omega}_\varepsilon, Q_\varepsilon] + \frac{\mu_2}{2\mu_1}\nabla^s\mathbf{D}_\varepsilon\right\|_{L^2}^2 \\ &\qquad - \delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2 + C(\mathcal{E} + \mathcal{E}^{\frac{3}{2}} + \mathcal{E}^2 + \mathcal{E}^3 + \mathcal{E}^4) \\ &\quad\leq -\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2 + C(\mathcal{E} + \mathcal{E}^{\frac{3}{2}} + \mathcal{E}^2 + \mathcal{E}^3 + \mathcal{E}^4).\end{aligned} \tag{3.11}$$

Then, combining (3.7) and (3.11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2 \leq F(\mathcal{E}(t)), \tag{3.12}$$

where F is an increasing function with $F(0) = 0$, and is given by

$$F(\mathcal{E}(t)) = C(\mathcal{E}(t) + \mathcal{E}^{\frac{3}{2}}(t) + \mathcal{E}^2(t) + \mathcal{E}^3(t) + \mathcal{E}^4(t)).$$

Step 3. Existence of the solution. For $s \geq 2$, by virtue of (3.12), there exists $T > 0$ depending only on $\mathcal{E}(0)$ such that, for any $t \in [0, \min(T, T_\varepsilon)]$,

$$\mathcal{E}(t) + \delta\|\nabla\mathbf{v}_\varepsilon\|_{H^s}^2 \leq 2\mathcal{E}(0),$$

where $\mathcal{E}(0)$ depends only on the initial data $(\mathbf{v}_I, Q_I)$. By a continuous argument we deduce that $T_\varepsilon \geq T$. Therefore, we get a uniform estimate for the approximate solution on $[0, T]$. Furthermore, the existence of the solution can be obtained by the standard compactness argument.

Step 4. Uniqueness of the solution. Assume that $(\mathbf{v}_1, Q_1)$ and $(\mathbf{v}_2, Q_2)$ are two strong solutions with the same initial data. We denote

$$\delta_Q = Q_1 - Q_2, \qquad \delta_{\dot{Q}} = \dot{Q}_1 - \dot{Q}_2, \qquad \delta_{\mathbf{v}} = \mathbf{v}_1 - \mathbf{v}_2,$$
$$\delta_{\mathbf{D}} = \mathbf{D}_1 - \mathbf{D}_2, \qquad \delta_{\boldsymbol{\Omega}} = \boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_2,$$

where

$$\delta_{\dot{Q}} = \partial_t \delta_Q + \mathbf{v}_1 \cdot \nabla \delta_Q + \delta_{\mathbf{v}} \cdot \nabla Q_2.$$

Taking the difference between the equations of the two solutions, we observe that $(\delta_Q, \delta_{\mathbf{v}})$ satisfies the following system:

$$J(\partial_t \delta_{\dot{Q}} + \mathbf{v}_1 \cdot \nabla \delta_{\dot{Q}}) = -\mu_1\big(\delta_{\dot{Q}} - [\delta_{\boldsymbol{\Omega}}, Q_2]\big) - \mathcal{L}(\delta_Q) - \frac{\mu_2}{2}\delta_{\mathbf{D}} - J\delta_{\mathbf{v}} \cdot \nabla \dot{Q}_2 + \delta_{\mathbf{F}_1}, \tag{3.13}$$

$$\begin{aligned}\partial_t \delta_{\mathbf{v}} + \mathbf{v}_1 \cdot \nabla \delta_{\mathbf{v}} = -\nabla p + \nabla \cdot \bigg(&\beta_1 Q_2(Q_2 : \delta_{\mathbf{D}}) + \beta_4 \delta_{\mathbf{D}} + \beta_5 \delta_{\mathbf{D}} \cdot Q_2 \\ &+ \beta_6 Q_2 \cdot \delta_{\mathbf{D}} + \beta_7\big(\delta_{\mathbf{D}} \cdot Q_2^2 + Q_2^2 \cdot \delta_{\mathbf{D}}\big) + \frac{\mu_2}{2}\big(\delta_{\dot{Q}} - [\delta_{\boldsymbol{\Omega}}, Q_2]\big) \\ &+ \mu_1\big[Q_2, \delta_{\dot{Q}} - [\delta_{\boldsymbol{\Omega}}, Q_2]\big]\bigg) + \nabla \cdot \delta_{\mathbf{F}_2},\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\delta_{\mathbf{F}_1} &= \mu_1[\boldsymbol{\Omega}_1, \delta_Q] + a\delta_Q + b\bigg(Q_1 \cdot \delta_Q + \delta_Q \cdot Q_2 - \frac{1}{3}(Q_1 : \delta_Q + \delta_Q : Q_2)\mathbf{I}\bigg) \\ &\quad - c\big(|Q_1|^2 \delta_Q + (Q_1 : \delta_Q + \delta_Q : Q_2)Q_2\big), \\ \delta_{\mathbf{F}_2} &= \beta_1\big(\delta_Q(Q_1 : \mathbf{D}_1) + Q_2(\delta_Q : \mathbf{D}_1)\big) + \beta_5 \mathbf{D}_1 \cdot \delta_Q + \beta_6 \mathbf{D}_1 \cdot \delta_Q + \frac{\mu_2}{2}[\boldsymbol{\Omega}_1, \delta_Q] \\ &\quad + \mu_1\big[\delta_Q, \dot{Q}_1 - [\boldsymbol{\Omega}_1, Q_1]\big] - \mu_1\big[Q_2, [\boldsymbol{\Omega}_1, \delta_Q]\big] + \sigma^d(\delta_Q, Q_i) - \delta_{\mathbf{v}} \otimes \mathbf{v}_2.\end{aligned}$$

We denote $\widetilde{Q}_i = Q_i - Q^*$, then a direct calculation leads to the following estimates:

$$\begin{aligned}\|\delta_{\mathbf{F}_1}\|_{L^2} &\leq C\big(1 + \|\nabla \mathbf{v}_1\|_{L^3} + \big\|(\widetilde{Q}_1, \widetilde{Q}_2)\big\|_{L^\infty} + \big\|(\widetilde{Q}_1, \widetilde{Q}_2)\big\|_{L^\infty}^2\big)\big(\|\delta_Q\|_{H^1} + \|\delta_{\mathbf{v}}\|_{L^2}\big), \\ \|\delta_{\mathbf{F}_2}\|_{L^2} &\leq C\big(\|\mathbf{v}_2\|_{L^\infty} + \|\nabla \mathbf{v}_1\|_{L^3} + \big\|(\widetilde{Q}_1, \widetilde{Q}_2)\big\|_{L^\infty}\|\nabla \mathbf{v}_1\|_{L^3} + \|\dot{Q}_1\|_{H^1} \\ &\quad + \big\|(\nabla Q_1, \nabla Q_2)\big\|_{L^\infty}\big)\big(\|\delta_Q\|_{H^1} + \|\delta_{\mathbf{v}}\|_{L^2}\big).\end{aligned}$$

For the system (3.13)–(3.14), making an L^2-energy estimate for $(\delta_{\dot{Q}}, \delta_{\mathbf{v}})$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\delta_{\mathbf{v}}\|_{L^2}^2 + J\|\delta_{\dot{Q}}\|_{L^2}^2 + \langle \mathcal{L}(\delta_Q), \delta_Q\rangle\big) \\ &\quad = -\mu_1\langle \delta_{\dot{Q}} - [\delta_{\boldsymbol{\Omega}}, Q_2], \delta_{\dot{Q}}\rangle - \langle \mathcal{L}(\delta_Q), \mathbf{v}_1 \cdot \nabla \delta_Q + \delta_{\mathbf{v}} \cdot \nabla Q_2\rangle\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mu_2}{2}\langle\delta_{\mathbf{D}},\delta_{\dot Q}\rangle - J\langle\delta_{\mathbf{v}}\cdot\nabla\dot Q_2,\delta_{\dot Q}\rangle + \langle\delta_{\mathbf{F}_1},\delta_{\dot Q}\rangle - \beta_1\|Q_2:\delta_{\mathbf{D}}\|_{L^2}^2 \\
&- \beta_4\|\delta_{\mathbf{D}}\|_{L^2}^2 - \langle\beta_5\delta_{\mathbf{D}}\cdot Q_2 + \beta_6 Q_2\cdot\delta_{\mathbf{D}},\nabla\delta_{\mathbf{v}}\rangle - 2\beta_7\big\|\delta_{\mathbf{D}}\cdot Q_2^2\big\|_{L^2}^2 \\
&- \frac{\mu_2}{2}\big\langle\delta_{\dot Q} - [\delta_{\mathbf{\Omega}},Q_2],\nabla\delta_{\mathbf{v}}\big\rangle - \mu_1\big\langle\big[Q_2,\delta_{\dot Q} - [\delta_{\mathbf{\Omega}},Q_2]\big],\nabla\delta_{\mathbf{v}}\big\rangle - \langle\delta_{\mathbf{F}_2},\nabla\delta_{\mathbf{v}}\rangle \\
={}& -\beta_1\|Q_2:\delta_{\mathbf{D}}\|_{L^2}^2 - \left(\beta_4 - \frac{\mu_2^2}{4\mu_1}\right)\|\delta_{\mathbf{D}}\|_{L^2}^2 - (\beta_5+\beta_6)\langle\delta_{\mathbf{D}}\cdot Q_2,\delta_{\mathbf{D}}\rangle \\
&- 2\beta_7\|\delta_{\mathbf{D}}\cdot Q_2\|_{L^2}^2 - \mu_1\left\|\delta_{\dot Q} - [\delta_{\mathbf{\Omega}},Q_2] + \frac{\mu_2}{2\mu_1}\delta_{\mathbf{D}}\right\|_{L^2}^2 + \langle\delta_{\mathbf{F}_1},\delta_{\dot Q}\rangle \\
&- \big\langle\mathcal{L}(\delta_Q),\mathbf{v}_1\cdot\nabla\delta_Q + \delta_{\mathbf{v}}\cdot\nabla Q_2\big\rangle - J\langle\delta_{\mathbf{v}}\cdot\nabla\dot Q_2,\delta_{\dot Q}\rangle - \langle\delta_{\mathbf{F}_2},\nabla\delta_{\mathbf{v}}\rangle. \qquad (3.15)
\end{aligned}
$$

From (3.9) we get

$$
\begin{aligned}
&-\big\langle\mathcal{L}(\delta_Q),\mathbf{v}_1\cdot\nabla\delta_Q + \delta_{\mathbf{v}}\cdot\nabla Q_2\big\rangle \\
&\quad\le C\|\nabla\mathbf{v}_1\|_{L^\infty}\|\nabla\delta_Q\|_{L^2}^2 + C_\delta\|\nabla Q_2\|_{L^\infty}^2\|\nabla\delta_Q\|_{L^2}^2 + \delta\|\nabla\delta_{\mathbf{v}}\|_{L^2}^2.
\end{aligned}
$$

Using the Sobolev embeddings $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$, we find

$$
\begin{aligned}
-J\langle\delta_{\mathbf{v}}\cdot\nabla\dot Q_2,\delta_{\dot Q}\rangle &\le C\|\delta_{\mathbf{v}}\|_{L^3}\|\nabla\dot Q_2\|_{L^6}\|\delta_{\dot Q}\|_{L^2} \le C\|\delta_{\mathbf{v}}\|_{H^1}\|\nabla\dot Q_2\|_{H^1}\|\delta_{\dot Q}\|_{L^2} \\
&\le C\big(1+\|\dot Q_2\|_{H^2}\big)\big(\|\delta_{\mathbf{v}}\|_{L^2}^2 + \|\delta_{\dot Q}\|_{L^2}^2\big) + \delta\|\nabla\delta_{\mathbf{v}}\|_{L^2}^2.
\end{aligned}
$$

Consequently, from (3.15) and the above estimates and using the dissipation relation, for $i = 1,2$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\delta_{\mathbf{v}}\|_{L^2}^2 + J\|\delta_{\dot Q}\|_{L^2}^2 + \big\langle\mathcal{L}(\delta_Q),\delta_Q\big\rangle\big) + \delta\|\nabla\delta_{\mathbf{v}}\|_{L^2}^2 \\
&\quad\le C(\mathbf{v}_i,\widetilde{Q}_i,\delta)\big(\|\delta_{\mathbf{v}}\|_{L^2}^2 + \|\delta_Q\|_{H^1}^2 + \|\delta_{\dot Q}\|_{L^2}^2\big). \qquad (3.16)
\end{aligned}
$$

In addition, multiplying Eq. (3.13) by δ_Q and taking the L^2-inner product, using integration by parts, then we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(J\langle\delta_{\dot Q},\delta_Q\rangle + \frac{\mu_1}{2}\|\delta_Q\|_{L^2}^2\right) + \big\langle\mathcal{L}(\delta_Q),\delta_Q\big\rangle \\
&\quad= J\|\delta_{\dot Q}\|_{L^2}^2 - J\langle\delta_{\dot Q},\delta_{\mathbf{v}}\cdot\nabla Q_2\rangle - \mu_1\big\langle\delta_{\mathbf{v}}\cdot\nabla Q_2 - [\delta_{\mathbf{\Omega}},Q_2],\delta_Q\big\rangle \\
&\qquad - J\langle\delta_{\mathbf{v}}\cdot\nabla\dot Q_2,\delta_Q\rangle - \frac{\mu_2}{2}\langle\delta_{\mathbf{D}},\delta_Q\rangle + \langle\delta_{\mathbf{F}_1},\delta_Q\rangle \\
&\quad\le C_\delta\big(1+\|\nabla Q_2\|_{L^\infty}^2 + \|\widetilde{Q}_2\|_{L^\infty} + \|\dot Q_2\|_{H^2}^2\big)\big(\|\delta_{\mathbf{v}}\|_{L^2}^2 + \|\delta_{\dot Q}\|_{L^2}^2 + \|\delta_Q\|_{H^1}^2\big) \\
&\qquad + \frac{\delta}{2}\|\nabla\delta_{\mathbf{v}}\|_{L^2}^2 + \|\delta_{\mathbf{F}_1}\|_{L^2}\|\delta_Q\|_{L^2} \\
&\quad\le C(\delta,\widetilde{Q}_i,\mathbf{v}_i)\big(\|\delta_{\mathbf{v}}\|_{L^2}^2 + \|\delta_{\dot Q}\|_{L^2}^2 + \|\delta_Q\|_{H^1}^2\big) + \frac{\delta}{2}\|\nabla\delta_{\mathbf{v}}\|_{L^2}^2. \qquad (3.17)
\end{aligned}
$$

Hence, combining (3.16) and (3.17) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\delta_{\mathbf{v}}\|_{L^2}^2 + J\|\delta_{\dot{Q}} + \delta_Q\|_{L^2}^2 + (\mu_1 - J)\|\delta_Q\|_{L^2}^2 + \langle\mathcal{L}(\delta_Q), \delta_Q\rangle\right) \\
&\quad + \langle\mathcal{L}(\delta_Q), \delta_Q\rangle + \frac{\delta}{2}\|\nabla\delta_{\mathbf{v}}\|_{L^2}^2 \\
&\quad \le C(\delta, \widetilde{Q}_i, \mathbf{v}_i)\left(\|\delta_{\mathbf{v}}\|_{L^2}^2 + \|\delta_{\dot{Q}}\|_{L^2}^2 + \|\delta_Q\|_{H^1}^2\right).
\end{aligned}$$

Since $J \ll \mu_1$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\delta_{\mathbf{v}}\|_{L^2}^2 + \|\delta_{\dot{Q}}\|_{L^2}^2 + \frac{\mu_1}{2}\|\delta_Q\|_{L^2}^2 + \langle\mathcal{L}(\delta_Q), \delta_Q\rangle\right) \\
&\quad \le C(\delta, \widetilde{Q}_i, \mathbf{v}_i)\left(\|\delta_{\mathbf{v}}\|_{L^2}^2 + \|\delta_{\dot{Q}}\|_{L^2}^2 + \|\delta_Q\|_{H^1}^2\right),
\end{aligned}$$

thus, the Gronwall inequality implies that $\delta_{\mathbf{v}}(t) = 0$ and $\delta_Q(t) = 0$ on $[0, T]$.

Combining the above four steps, we complete the proof of Theorem 2.1.

4 Conclusions

In this paper, we are mainly concerned with the inertial Qian–Sheng Q-tensor model describing the nematic liquid crystal flow. The inertial term J is responsible for the hyperbolic feature of the equation describing molecular orientation. Under the assumption of the initial data near uniaxial equilibrium, we investigate the existence and uniqueness of local in time strong solutions to the system. However, the global in time existence around the uniaxial equilibrium is rather difficult because the energy of the system is not strong enough to estimate the L^2-norm of the difference $Q - Q^*$. This will be left for our future work.

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Authors' contributions
XW, SL and TW participated in the theoretical research and drafted the manuscript. All authors read and approved the final manuscript.

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