# Ground state solutions for Kirchhoff-type equations with general nonlinearity in low dimension 

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## Abstract

This paper is dedicated to studying the following Kirchhoff-type problem:

$$
\left\{\begin{array}{l}
-m\left(\|\nabla u\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}\right) \Delta u+V(x) u=f(u), \quad x \in \mathbb{R}^{N} ; \\
u \in H^{1}\left(\mathbb{R}^{N}\right),
\end{array}\right.
$$

where $N=1,2, m:[0, \infty) \rightarrow(0, \infty)$ is a continuous function, $V: \mathbb{R}^{N} \rightarrow \mathbb{R}$ is differentiable, and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. We obtain the existence of a ground state solution of Nehari-Pohozaev type and the least energy solution under some assumptions on $V$, $m$, and $f$. Especially, the existence of nonlocal term $m\left(\|\nabla u\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}\right)$ and the lack of Hardy's inequality and Sobolev's inequality in low dimension make the problem more complicated. To overcome the above-mentioned difficulties, some new energy inequalities and subtle analyses are introduced.

MSC: 35J20; 35J60
Keywords: Kirchhoff-type problem; Nehari-Pohozaev manifold; Least energy solution

## 1 Introduction

In this paper, we consider the following Kirchhoff-type equation:

$$
\left\{\begin{array}{l}
-m\left(\|\nabla u\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}\right) \Delta u+V(x) u=f(u), \quad x \in \mathbb{R}^{N}  \tag{1.1}\\
u \in H^{1}\left(\mathbb{R}^{N}\right)
\end{array}\right.
$$

where $N=1,2$, $m:[0, \infty) \rightarrow(0, \infty), V: \mathbb{R}^{N} \rightarrow \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Problem (1.1) has a profound physical meaning for it is related to the stationary analogue of the Kirchhoff equation, which arises in nonlinear vibrations, see Alves, Corréa, and Figueiredo [1] for more details. The following equation is a special case of (1.1) when $\mathbb{R}^{N}$ is replaced by a bounded domain $\Omega$ and $f(s)-V(x) s$ by $f(x, s)$ :

$$
\begin{equation*}
-m\left(\|\nabla u\|_{L^{2}(\Omega)}^{2}\right) \Delta u=f(x, u) \quad \text { in } \Omega . \tag{1.2}
\end{equation*}
$$

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Equation (1.2) appears when we search for a stationary solution to

$$
\begin{equation*}
u_{t t}-m\left(\int_{\Omega}|\nabla u|^{2} \mathrm{~d} x\right) \Delta u=f(x, t) \quad \text { in } \Omega \times(0, T) . \tag{1.3}
\end{equation*}
$$

Problem (1.3) was proposed by Kirchhoff [20] when $m(t)=a+b t$ and $N=1$. We refer the readers to $[2-4,9,10]$ for more mathematical and physical background on Kirchhoff-type problems.

Lions [25] proposed an abstract functional analysis framework to the Kirchhoff equation

$$
\begin{equation*}
u_{t t}-\left(a+b \int_{\Omega}|\nabla u|^{2} \mathrm{~d} x\right) \Delta u=f(x, u) \quad \text { in } \Omega \times(0, T) \tag{1.4}
\end{equation*}
$$

after which extensive attention to (1.4) was aroused. In recent years, the following Kirchhoff-type problem

$$
\left\{\begin{array}{l}
-\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} \mathrm{~d} x\right) \Delta u+V(x) u=f(x, u), \quad x \in \mathbb{R}^{N}  \tag{1.5}\\
u \in H^{1}\left(\mathbb{R}^{N}\right)
\end{array}\right.
$$

has been studied intensively by many researchers, where $V \in \mathcal{C}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ and $f \in \mathcal{C}\left(\mathbb{R}^{N} \times\right.$ $\mathbb{R}, \mathbb{R}), a, b>0$ are constants. By variational methods, a number of important results of the existence and multiplicity of solutions for problem (1.5) have been established with $f$ satisfying various conditions. In the meantime, $V$ is usually assumed as a constant, or periodic, or radial, or coercive, see for example [1, 5, 7, 8, 11, 14-16, 19, 21, 23, 24, 26-$28,30,31,34,36-38]$ and the references therein.

Recently, Ikoma [17] investigated the following Kirchhoff-type equations with power type nonlinearity:

$$
\left\{\begin{array}{l}
-m\left(\|\nabla u\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}\right) \Delta u+V(x) u=|u|^{q-1} u, \quad x \in \mathbb{R}^{N}  \tag{1.6}\\
u \in H^{1}\left(\mathbb{R}^{N}\right)
\end{array}\right.
$$

Here, $q \in(1, \infty)$ if $N=1,2$ and $q \in\left(1, \frac{N+2}{N-2}\right)$ if $N \geq 3$. Problem (1.6) in the general dimensions was considered, and the author proved that problem (1.6) has a ground state solution. Precisely, they assumed the following conditions on $m$ and $V$ when $N=1,2$ :
(M1) $m \in \mathcal{C}([0, \infty))$ and there exists $m_{0}>0$ such that $0<m_{0} \leq m(s)$ for all $s \in[0, \infty)$;
( $\tilde{M} 2)$ There exist $q_{0}>0$ and $\varepsilon_{0}>0$ such that $M(s)-\frac{1}{q_{0}+1} m(s) s \geq \varepsilon_{0} s$ in $(0, \infty)$;
( $\tilde{M} 3)$ The function $s^{-1} M(s)$ is nondecreasing in $(0, \infty)$;
(V1) $V \in \mathcal{C}^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ such that $\lim _{|x| \rightarrow \infty} V(x)=\sup _{x \in \mathbb{R}^{N}} V(x)=: V_{\infty}<+\infty$;
(V2) $0<\inf _{x \in \mathbb{R}^{N}} V(x)=: V_{0}$;
( $\tilde{V} 3)$ Let $q_{0}$ be the constant that appeared in ( $\left.\tilde{M} 2\right)$. When $1<q<2 q_{0}+1$, there exist $\alpha, \beta>0$ such that

$$
\begin{aligned}
& |x \cdot \nabla V(x)| \leq C(1+|x|)^{\alpha} \quad \text { for all } x \in \mathbb{R}^{N}, \\
& x \cdot \nabla V(x) \leq \beta V(x), \quad \beta \in\left(0, \frac{(q-1) 2-(N-2) q_{0}}{2 q_{0}+1-q}\right] \text { for all } x \in \mathbb{R}^{N} .
\end{aligned}
$$

Through refined topological analysis on energy functional of "limit problem", Ikoma compared the mountain pass level $c_{\nu, \lambda}$ with the one of the limit equation $c_{\infty, \lambda}$. Specially, in Ikoma's analysis, the situation of $N=1$ is different from the one when $N \geq 2$. To get a ground state solution of (1.6), the author used the standard arguments which combined the monotonicity trick to get a bounded $(P S)_{c_{v, \lambda}}$ sequence and the concentrationcompactness lemma to prove that the sequence has a convergent subsequence. It is worth noting that, in Ikoma's argument, he had to face the difficulties about obtaining the boundedness of the (PS) sequence and the strongly convergent subsequences when $1<q<2 q_{0}+1$.

Tang and Chen [32] dealt with the Schrödinger-Kirchhoff type equation

$$
\left\{\begin{array}{l}
-\left(a+b \int_{\mathbb{R}^{3}}|\nabla u|^{2} \mathrm{~d} x\right) \Delta u+V(x) u=f(u), \quad x \in \mathbb{R}^{3} ;  \tag{1.7}\\
u \in H^{1}\left(\mathbb{R}^{3}\right)
\end{array}\right.
$$

when the nonlinearity $f$ is more general. They proved that (1.7) possesses a ground state solution of Nehari-Pohozaev type and the least energy solution with the following assumptions about $f$, here $N=3$ :
(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exist constants $C_{0}>0$ and $p_{0} \in\left(2,2^{*}\right)$ such that

$$
|f(t)| \leq C_{0}\left(1+|t|^{p_{0}-1}\right), \quad \forall t \in \mathbb{R}
$$

(F2) $f(t)=o(t)$ as $t \rightarrow 0$;
(F3) $\lim _{|t| \rightarrow \infty} \frac{F(t)}{t^{2}}=+\infty$, where $F(t):=\int_{0}^{t} f(s) \mathrm{d} s$;
(F4) The function $\frac{f(t) t+N F(t)}{\mid t p^{p-N+1} t}$ is nondecreasing on $t \in(-\infty, 0) \cup(0, \infty)$, where $p>2$.
Precisely, they proved that there exists the least energy solution following the standard approach as Ikoma's result. The difference is that they developed a new trick and compared the level $c_{\lambda}$ with the energy $I_{\lambda}^{\infty}$ of the minimizer $u_{\lambda}^{\infty}$ more directly. Here, $c_{\lambda}$ is the limited energy of a bounded (PS) sequence $\left\{u_{n}(\lambda)\right\}$ for almost every $\lambda \in[1 / 2,1)$. By using their original highlight inequalities to obtain the comparison, they got a minimizer $u_{\lambda}^{\infty}$ on the Nehari-Pohozaev manifold which is also used in Li [22] and Ruiz [29] for $\lambda \in[1 / 2,1]$. Then, by using the global compactness lemma obtained in [22], they got a nontrivial critical point $u_{\lambda}$ which possesses energy $c_{\lambda}$. Their approach is applicable to the problems of other types, such as Schrödinger-Poisson problems [32], Choquard equations [33], and so on.

Let us point out that some arguments and tricks used in [32] fail to adapt directly to oneand two-dimensional cases for Hardy's and Sobolev's inequalities and do not work at this point. In this sense, it is more complicated than a three-dimensional case. To the best of our knowledge, there are few results concerning (1.1) in one- and two-dimensional case. Based on $[17,32]$, the main purpose of this paper is to extend and complement the corresponding existence results on (1.7) in a three-dimensional case and above to the lower dimensional situation. However, the general term $m(t)$ is more difficult to deal with than the special form $a+b t$, and the appearance of $m\left(\|\nabla u\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}\right)$ makes problem (1.1) more intricate. In addition, it is noticed that there is no need to distinguish $N=1$ and $N=2$ in our approach.
Now we state our hypotheses. Let $N=1,2$. For $V$, we suppose (V1), (V2) as above, and we use a mild hypothesis (V3) in place of ( $\tilde{\mathrm{V}} 3$ ). Furthermore, (V4) makes some subtle
inequalities hold, which helps us to prove the existence of ground states. For $f$, we suppose (F1)-(F4). For $m$, we assume (M1) above, (M2) instead of (M̃2), (M3), and (M4).
(V3) $(p-N) V(x)-\nabla V(x) \cdot x \geq 0$, where $p>2$;
(V4) The map $t \mapsto \frac{(N+2) V(t x)+\nabla V(t x) \cdot(t x)}{t^{p-N}}$ is nonincreasing on $(0,+\infty)$ for all $x \in \mathbb{R}^{N} \backslash\{0\}$, where $\cdot$ denotes the inner product in $\mathbb{R}^{N}, p>2$ holds here and after;
(M2) There exists $\varepsilon_{0}>0$ such that $M(s)-\frac{N}{2+N} m(s) s \geq \varepsilon_{0} s$ for $s \in(0, \infty)$, where $M(s):=\int_{0}^{s} m(t) \mathrm{d} t ;$
(M3) The map $t \mapsto \frac{m(t s) \cdot(t s)}{t^{2+p}}$ is nonincreasing in $(0, \infty)$ for all $s \in(0, \infty)$;
(M4) The function $m(s)$ is nondecreasing in $[0, \infty)$.
A simple example of $m$ satisfying all conditions (M1)-(M4) is the following:

$$
m(t)=a+b t^{\frac{1}{p}}
$$

where $a>0, b>0, p>2$. We can easily verify that $m(t)$ fits all the above hypotheses.
To state our results, we define the norm in $H^{1}\left(\mathbb{R}^{N}\right)$

$$
\begin{align*}
& \|u\|=\left(\int_{\mathbb{R}^{N}}\left(|\nabla u|^{2}+|u|^{2}\right) \mathrm{d} x\right)^{1 / 2},  \tag{1.8}\\
& \|u\|_{s}=\left(\int_{\mathbb{R}^{N}}|u|^{s} \mathrm{~d} x\right)^{1 / s}, \quad 1 \leq s<+\infty, \tag{1.9}
\end{align*}
$$

and the energy functional

$$
\begin{equation*}
E(u)=\frac{1}{2} M\left(\|\nabla u\|_{2}^{2}\right)+\frac{1}{2} \int_{\mathbb{R}^{N}} V(x) u^{2} \mathrm{~d} x-\int_{\mathbb{R}^{N}} F(u) \mathrm{d} x . \tag{1.10}
\end{equation*}
$$

Under assumptions (V1), (F1), and (F2), weak solutions to (1.1) correspond to critical points of $E$ and $E \in \mathcal{C}^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)$. For any $\varepsilon>0$, it follows from (F1) and (F2) that there exists $C_{\varepsilon}>0$ such that

$$
\begin{equation*}
|f(t)| \leq \varepsilon|t|+C_{\varepsilon}|t|^{p_{0}-1}, \quad \forall t \in \mathbb{R} \tag{1.11}
\end{equation*}
$$

Let us define the Pohozaev functional for (1.1) by

$$
\begin{align*}
\mathcal{P}(u):= & \frac{N-2}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}[N V(x)+(\nabla V(x), x)] u^{2} \mathrm{~d} x \\
& -N \int_{\mathbb{R}^{N}} F(u) \mathrm{d} x, \tag{1.12}
\end{align*}
$$

that is, $\mathcal{P}(u)=0$ if $u$ is a critical point of $E$, and define the following constraint set:

$$
\begin{equation*}
\mathcal{M}:=\left\{u \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}: J(u):=\left\langle E^{\prime}(u), u\right\rangle+\mathcal{P}(u)=0\right\}, \tag{1.13}
\end{equation*}
$$

clearly, $u \in \mathcal{M}$ if $u$ is a critical point of $E$.
Our main results are as follows.

Theorem 1.1 Suppose that $m, V$, and $f$ satisfy (M1)-(M3), (V1)-(V4), and (F1)-(F4). Then Problem (1.1) has a solution $\bar{u} \in H^{1}\left(\mathbb{R}^{N}\right)$ such that $E(\bar{u})=\inf _{\mathcal{M}} E>0$.

Theorem 1.2 Suppose that $m, V$, and $f$ satisfy (M1)-(M4), (V1)-(V3), and (F1)-(F4). Then Problem (1.1) has a least energy solution $\bar{u} \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$.

The paper is organized as follows. In Sect. 2, we give some preliminaries and the proof of Theorem 1.1. Section 3 is devoted to proving Theorem 1.2. In this paper, $C_{1}, C_{2}, \ldots$ denote positive constants possibly different in different places.

## 2 Preliminaries

To obtain the ground state solution of (1.1), we establish the key energy inequality related with $E$ and $J$. To this end, we first prove the following two inequalities.

Lemma 2.1 Suppose that (F1) and (F4) hold. Then

$$
\begin{equation*}
t^{N} F(t u)-F(u)+\frac{1-t^{2+p}}{2+p}[f(u) u+N F(u)] \geq 0, \quad \forall t \geq 0, u \in \mathbb{R} . \tag{2.1}
\end{equation*}
$$

Proof It is evident that (2.1) holds for $u=0$. For $u \neq 0$, let

$$
\begin{equation*}
g(t)=t^{N} F(t u)-F(u)+\frac{1-t^{2+p}}{2+p}[f(u) u+N F(u)], \quad t \geq 0 \tag{2.2}
\end{equation*}
$$

Then, from (F4), one has

$$
\begin{aligned}
g^{\prime}(t) & =N t^{N-1} F(t u)+t^{N} f(t u) u-t^{1+p}[N F(u)+f(u) u] \\
& =t^{1+p}|u|^{p}\left[\frac{f(t u) t^{N-1} u+N t^{N-2} F(t u)}{|t u|^{p}}-\frac{f(u) u+N F(u)}{|u|^{p}}\right] \\
& =t^{1+p}|u|^{2+p-N}\left[\frac{f(t u) t u+N F(t u)}{|t u|^{2+p-N}}-\frac{f(u) u+N F(u)}{|u|^{2+p-N}}\right] \\
& \begin{cases}\geq 0, & t \geq 1, \\
\leq 0, & 0<t<1 .\end{cases}
\end{aligned}
$$

It follows that $g(t) \geq g(1)=0$ for $t \geq 0$. This implies that (2.1) holds.

Lemma 2.2 Suppose that (V1) and (V4) hold. Then

$$
\begin{equation*}
V(x)-t^{N+2} V(t x)-\frac{1-t^{2+p}}{2+p}[(N+2) V(x)+\nabla V(x) \cdot x] \geq 0 . \tag{2.3}
\end{equation*}
$$

Proof Let

$$
h(t):=V(x)-t^{N+2} V(t x)-\frac{1-t^{2+p}}{2+p}[(N+2) V(x)+\nabla V(x) \cdot x] .
$$

By (V4), one has

$$
\begin{aligned}
h^{\prime}(t) & =-(N+2) t^{N+1} V(t x)-t^{N+1} \nabla V(t x) \cdot(t x)+t^{1+p}[(N+2) V(x)+\nabla V(x) \cdot x] \\
& =t^{1+p}\left[(N+2) V(x)+\nabla V(x) \cdot x-\frac{(N+2) V(t x)+\nabla V(t x) \cdot(t x)}{t^{p-N}}\right]
\end{aligned}
$$

$$
\begin{cases}\geq 0, & t \geq 1  \tag{2.4}\\ <0, & 0<t<1\end{cases}
$$

It follows that $h(t) \geq h(1)=0$ holds for $t \geq 0$.

Lemma 2.3 Suppose that (F1), (F2), (F4), (V1), (V4), and (M3) hold. Then, for all $u \in$ $H^{1}\left(\mathbb{R}^{N}\right)$ and $t>0$, the following key inequality holds:

$$
\begin{equation*}
E(u) \geq E\left(t u_{t}\right)+\frac{1-t^{2+p}}{2+p} J(u) \tag{2.5}
\end{equation*}
$$

where $u_{t}(x):=u\left(t^{-1} x\right)$ are fixed.

Proof Note that

$$
\begin{equation*}
E\left(t u_{t}\right)=\frac{1}{2} M\left(t^{N}\|\nabla u\|_{2}^{2}\right)+\frac{t^{N+2}}{2} \int_{\mathbb{R}^{N}} V(t x) u^{2} \mathrm{~d} x-t^{N} \int_{\mathbb{R}^{N}} F(t u) \mathrm{d} x \tag{2.6}
\end{equation*}
$$

and

$$
\begin{align*}
J(u)= & \left\langle E^{\prime}(u), u\right\rangle+\mathcal{P}(u) \\
= & \frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}[(2+N) V(x)+\nabla V(x) \cdot x] u^{2} \mathrm{~d} x \\
& -\int_{\mathbb{R}^{N}}[f(u) u+N F(u)] \mathrm{d} x . \tag{2.7}
\end{align*}
$$

Thus, by (1.10), (2.1), (2.3), and (2.6), one has

$$
\begin{aligned}
E(u) & -E\left(t u_{t}\right) \\
= & \frac{1}{2} M\left(\|\nabla u\|_{2}^{2}\right)-\frac{1}{2} M\left(t^{N}\|\nabla u\|_{2}^{2}\right)+\frac{1}{2} \int_{\mathbb{R}^{N}}\left[V(x)-t^{N+2} V(t x)\right] u^{2} \mathrm{~d} x \\
& +\int_{\mathbb{R}^{N}}\left[t^{N} F(t u)-F(u)\right] \mathrm{d} x \\
= & \frac{1-t^{2+p}}{2+p} J(u)+\frac{1}{2} M\left(\|\nabla u\|_{2}^{2}\right)-\frac{1}{2} M\left(t^{N}\|\nabla u\|_{2}^{2}\right)-\frac{1-t^{2+p}}{2+p} \cdot \frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2} \\
& +\frac{1}{2} \int_{\mathbb{R}^{N}}\left\{V(x)-t^{N+2} V(t x)-\frac{1-t^{2+p}}{2+p}[(N+2) V(x)+\nabla V(x) \cdot x]\right\} u^{2} \mathrm{~d} x \\
& +\int_{\mathbb{R}^{N}}\left\{t^{N} F(t u)-F(u)+\frac{1-t^{2+p}}{2+p}[f(u) u+N F(u)]\right\} \mathrm{d} x \\
\geq & \frac{1-t^{2+p}}{2+p} J(u)+\frac{1}{2} M\left(\|\nabla u\|_{2}^{2}\right)-\frac{1}{2} M\left(t^{N}\|\nabla u\|_{2}^{2}\right)-\frac{1-t^{2+p}}{2+p} \cdot \frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2} .
\end{aligned}
$$

In fact, the following assertion holds:

$$
\begin{equation*}
L(t):=\frac{1}{2} M\left(\|\nabla u\|_{2}^{2}\right)-\frac{1}{2} M\left(t^{N}\|\nabla u\|_{2}^{2}\right)-\frac{1-t^{2+p}}{2+p} \cdot \frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2} \geq 0 \tag{2.8}
\end{equation*}
$$

From (M3), we have

$$
\begin{align*}
L^{\prime}(t) & =\frac{N}{2} t^{1+p} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}-\frac{N}{2} t^{N-1} m\left(t^{N}\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2} \\
& =\frac{N}{2} t^{1+p}\left[m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}-\frac{m\left(t^{N}\|\nabla u\|_{2}^{2}\right) t^{N}\|\nabla u\|_{2}^{2}}{t^{p+2}}\right] \\
& \begin{cases}\geq 0, \quad t \geq 1, \\
<0, & 0<t<1 .\end{cases} \tag{2.9}
\end{align*}
$$

Then $L(t) \geq L(1)=0$. That implies (2.5) holds.
From Lemma 2.3, we have the following corollary.
Corollary 2.4 Suppose that (F1), (F2), (F4), (V1), and (V4) hold. Then, for $u \in \mathcal{M}$,

$$
\begin{equation*}
E(u)=\max _{t>0} E\left(t u_{t}\right) . \tag{2.10}
\end{equation*}
$$

Lemma 2.5 Suppose that (F1)-(F3) and (M2) hold. Then, for any $u \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$, there exists unique $t_{u}>0$ such that $t_{u} u_{t_{u}} \in \mathcal{M}$.

Proof Let $u \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ be fixed. Clearly, for $E\left(t u_{t}\right)$ defined as (2.6) on $(0, \infty)$, we have

$$
\begin{align*}
\frac{\mathrm{d} E\left(t u_{t}\right)}{\mathrm{d} t}=0 \Leftrightarrow \quad & \frac{N}{2} m\left(t^{N}\|\nabla u\|_{2}^{2}\right) t^{N-1}\|\nabla u\|_{2}^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}\left[(N+2) t^{N+1} V(t x)\right. \\
& \left.+t^{N+1} \nabla V(t x) \cdot(t x)\right] u^{2} \mathrm{~d} x-t^{N-1} \int_{\mathbb{R}^{N}}[N F(t u)+f(t u) t u] \mathrm{d} x=0 \\
\Leftrightarrow \quad J\left(t u_{t}\right)=0 \Leftrightarrow & t u_{t} \in \mathcal{M} . \tag{2.11}
\end{align*}
$$

By (F1) and (F2), it is easy to verify that $E(0)=0$ when $t=0, E\left(t u_{t}\right)>0$ for $t>0$ small. By (M2), there exists $C_{1}>0$ such that $M(s) \leq M(1) s^{\frac{2+N}{N}}+C_{1} s$ for any $s \geq 0$, which yields

$$
\begin{align*}
E\left(t u_{t}\right) \leq & \frac{1}{2}\left\{M(1)\left(t^{N}\|\nabla u\|_{2}^{2}\right)^{\frac{2+N}{N}}+C_{1} t^{N}\|\nabla u\|_{2}^{2}\right\}+\frac{t^{N+2}}{2} \int_{\mathbb{R}^{N}} V(t x) u^{2} \mathrm{~d} x \\
& -t^{N} \int_{\mathbb{R}^{N}} F(t u) \mathrm{d} x, \quad \forall t u_{t} \in \mathcal{M}, \tag{2.12}
\end{align*}
$$

then $E\left(t u_{t}\right)<0$ for $t$ large follows from (F3). Therefore $\max _{t \in(0, \infty)} E\left(t u_{t}\right)$ is achieved at $t_{u}>0$ so that $\left.\frac{\mathrm{d} E\left(t u_{t}\right)}{\mathrm{d} t}\right|_{t=t_{u}}=0$ and $t_{u} u_{t_{u}} \in \mathcal{M}$.

Next we claim that $t_{u}$ is unique for any $u \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$. In fact, for any given $u \in$ $H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$, let $t_{1}, t_{2}>0$ such that $t_{1} u_{t_{1}}, t_{2} u_{t_{2}} \in \mathcal{M}$. Then $J\left(t_{1} u_{t_{1}}\right)=J\left(t_{2} u_{t_{2}}\right)=0$. Jointly with (2.5), we have

$$
\begin{equation*}
E\left(t_{1} u_{t_{1}}\right) \geq E\left(t_{2} u_{t_{2}}\right)+\frac{t_{1}^{2+p}-t_{2}^{2+p}}{(2+p) t_{1}^{2+p}} J\left(t_{1} u_{t_{1}}\right)=E\left(t_{2} u_{t_{2}}\right) . \tag{2.13}
\end{equation*}
$$

Also,

$$
\begin{equation*}
E\left(t_{2} u_{t_{2}}\right) \geq E\left(t_{1} u_{t_{1}}\right)-\frac{t_{2}^{2+p}-t_{1}^{2+p}}{(2+p) t_{2}^{2+p}} J\left(t_{2} u_{t_{2}}\right)=E\left(t_{1} u_{t_{1}}\right) . \tag{2.14}
\end{equation*}
$$

(2.13) and (2.14) imply $t_{1}=t_{2}$. Therefore, $t_{u}>0$ is unique for any $u \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$.

Lemma 2.6 Suppose that (F1)-(F4) hold. Then

$$
\inf _{u \in \mathcal{M}} E(u)=I=\inf _{u \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}} \max _{t>0} E\left(t u_{t}\right) .
$$

Proof Both Corollary 2.4 and Lemma 2.5 imply the above lemma.
Lemma 2.7 Suppose that (V1), (V4), and (M1) hold. Then there exists $\omega_{1}>0$ such that

$$
\begin{align*}
\omega_{1}\|u\|^{2} \leq & N m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2} \\
& +\int_{\mathbb{R}^{N}}[(N+2) V(x)+\nabla V(x) \cdot x] u^{2} \mathrm{~d} x, \quad \forall u \in H^{1}\left(\mathbb{R}^{N}\right) . \tag{2.15}
\end{align*}
$$

Proof From (2.3), one has

$$
\begin{align*}
& (N+2) V(x)+\nabla V(x) \cdot x \\
& \quad \geq(2+p) t^{N-p} V(t x)-(2+p) t^{-(2+p)} V(x), \quad \forall t>0, x \in \mathbb{R}^{N} . \tag{2.16}
\end{align*}
$$

Taking $t \rightarrow \infty$ in (2.16), we deduce that

$$
\begin{equation*}
(N+2) V(x)+\nabla V(x) \cdot x \geq 0, \quad \forall x \in \mathbb{R}^{N} . \tag{2.17}
\end{equation*}
$$

Arguing by contradiction, suppose that there exists a sequence $\left\{u_{n}\right\} \subset H^{1}\left(\mathbb{R}^{N}\right)$ such that $\left\|u_{n}\right\|=1$ and

$$
\begin{equation*}
N m\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)\left\|\nabla u_{n}\right\|_{2}^{2}+\int_{\mathbb{R}^{N}}[(N+2) V(x)+\nabla V(x) \cdot x] u_{n}^{2} \mathrm{~d} x=o(1) \tag{2.18}
\end{equation*}
$$

Thus there exists $\bar{u} \in H^{1}\left(\mathbb{R}^{N}\right)$ such that $u_{n} \rightharpoonup \bar{u}$. Then $u_{n} \rightarrow \bar{u}$ in $L_{\mathrm{loc}}^{s}\left(\mathbb{R}^{N}\right)$ for $2 \leq s<\infty$ and $u_{n} \rightarrow \bar{u}$ a.e. in $\mathbb{R}^{N}$. By (2.17), (2.18), the weak semicontinuity of norm, and Fatou's lemma, we have

$$
\begin{align*}
0 & =\lim _{n \rightarrow \infty}\left\{N m\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)\left\|\nabla u_{n}\right\|_{2}^{2}+\int_{\mathbb{R}^{N}}[(N+2) V(x)+\nabla V(x) \cdot x] u_{n}^{2} \mathrm{~d} x\right\} \\
& \geq N m_{0}\|\nabla \bar{u}\|_{2}^{2}+\int_{\mathbb{R}^{N}}[(N+2) V(x)+\nabla V(x) \cdot x] \bar{u}^{2} \mathrm{~d} x, \tag{2.19}
\end{align*}
$$

which implies $\bar{u}=0$. Thus, from (V1) and (V4), one has

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left\{(N+2)\left[V(x)-V_{\infty}\right]+\nabla V(x) \cdot x\right\} u_{n}^{2} \mathrm{~d} x=o(1), \quad n \rightarrow \infty . \tag{2.20}
\end{equation*}
$$

Both (2.18) and (2.20) imply

$$
\begin{align*}
o(1) & =N m\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)\left\|\nabla u_{n}\right\|_{2}^{2}+\int_{\mathbb{R}^{N}}[(N+2) V(x)+\nabla V(x) \cdot x] u_{n}^{2} \mathrm{~d} x \\
& =N m\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)\left\|\nabla u_{n}\right\|_{2}^{2}+(N+2) V_{\infty}\left\|u_{n}\right\|_{2}^{2}+o(1) \\
& \geq \min \left\{N m_{0},(N+2) V_{\infty}\right\}\left\|u_{n}\right\|^{2}+o(1) \\
& =\min \left\{N m_{0},(N+2) V_{\infty}\right\}+o(1), \tag{2.21}
\end{align*}
$$

which is a contradiction, and it shows that there exists $\omega_{1}>0$ such that (2.15) holds.

Lemma 2.8 Suppose that (F1)-(F3), (M2), and (V2) hold. Then
(i). there exists $\rho_{0}>0$ such that $\|u\| \geq \rho_{0}, \forall u \in \mathcal{M}$;
(ii). $I=\inf _{u \in \mathcal{M}} E(u)>0$.

Proof (i). For all $u \in \mathcal{M}$, by (F1), (1.11), (2.7), (2.15), and the Sobolev embedding theorem, one has

$$
\begin{align*}
\frac{\omega_{1}}{2}\|u\|^{2} & \leq \frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}[(2+N) V(x)+\nabla V(x) \cdot x] u^{2} \mathrm{~d} x \\
& =\int_{\mathbb{R}^{N}}[f(u) u+N F(u)] \mathrm{d} x \\
& \leq \frac{\omega_{1}}{4}\|u\|^{2}+C_{2}\|u\|^{p_{0}} \tag{2.22}
\end{align*}
$$

This implies

$$
\begin{equation*}
\|u\| \geq \rho_{0}:=\left(\frac{\omega_{1}}{4 C_{2}}\right)^{1 /\left(p_{0}-2\right)}, \quad \forall u \in \mathcal{M} \tag{2.23}
\end{equation*}
$$

(ii). Together a direct calculation with (V2), it follows from (2.3) that

$$
\begin{align*}
(p & -N) V(x)-\nabla V(x) \cdot x \\
& \geq(2+p) t^{N+2} V(t x)-t^{2+p}[(N+2) V(x)+\nabla V(x) \cdot x] \\
& \geq(2+p) t^{N+2} V_{0}-t^{2+p}\left[(N+2)\|V\|_{\infty}+\|\nabla V(x) \cdot x\|_{\infty}\right] \tag{2.24}
\end{align*}
$$

for all $t \in \mathbb{R}, x \in \mathbb{R}^{N}$. There exists $t_{0}>0$ small enough such that

$$
\begin{equation*}
(2+p) t_{0}^{N+2} V_{0}-t_{0}^{2+p}\left[(N+2)\|V\|_{\infty}+\|\nabla V(x) \cdot x\|_{\infty}\right] \geq \frac{V_{0}}{4} \tag{2.25}
\end{equation*}
$$

then

$$
\begin{align*}
(p-N) V(x)-\nabla V(x) \cdot x & \geq(2+p) t_{0}^{N+2} V_{0}-t_{0}^{2+p}\left[(N+2)\|V\|_{\infty}+\|\nabla V(x) \cdot x\|_{\infty}\right] \\
& \geq \frac{V_{0}}{4}>0 \tag{2.26}
\end{align*}
$$

Let $t \rightarrow 0$ in (2.1), then

$$
\begin{equation*}
f(u) u-(2+p-N) F(u) \geq 0, \quad \forall u \in \mathbb{R} . \tag{2.27}
\end{equation*}
$$

From (M2), (2.26), and (2.27), for all $u \in \mathcal{M}$, one has

$$
\begin{aligned}
E(u)= & E(u)-\frac{1}{2+p} J(u) \\
= & \frac{1}{2} M\left(\|\nabla u\|_{2}^{2}\right)+\frac{1}{2(2+p)} \int_{\mathbb{R}^{N}}[(p-N) V(x)-\nabla V(x) \cdot x] u^{2} \mathrm{~d} x \\
& -\frac{1}{2+p} \cdot \frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{1}{2+p} \int_{\mathbb{R}^{N}}[f(u) u-(2+p-N) F(u)] \mathrm{d} x
\end{aligned}
$$

$$
\begin{align*}
& \geq \frac{1}{2} M\left(\|\nabla u\|_{2}^{2}\right)-\frac{N}{2(2+N)} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{V_{0}}{8(2+p)}\|u\|_{2}^{2} \\
& \geq \frac{\varepsilon_{0}}{2}\|\nabla u\|_{2}^{2}+\frac{V_{0}}{8(2+p)}\|u\|_{2}^{2} \\
& \geq \min \left\{\frac{\varepsilon_{0}}{2}, \frac{V_{0}}{8(2+p)}\right\}\|u\|^{2}:=C_{3}\|u\|^{2} \geq C_{3} \rho_{0}^{2}>0 . \tag{2.28}
\end{align*}
$$

This shows that $I=\inf _{u \in \mathcal{M}} E(u)>0$.

The following lemma has been proved in [12] and [32].
Lemma 2.9 Assume that (F1) and (F2) hold. If $u_{n} \rightharpoonup u$ in $H^{1}\left(\mathbb{R}^{N}\right)$, then along a subsequence of $\left\{u_{n}\right\}$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sup _{\varphi \in H^{1}\left(\mathbb{R}^{N}\right),\|\varphi\| \leq 1}\left|\int_{\mathbb{R}^{N}}\left[f\left(u_{n}\right)-f\left(u_{n}-u\right)-f(u)\right] \varphi \mathrm{d} x\right|=0 . \tag{2.29}
\end{equation*}
$$

Similar to [31, Lemma 2.10], by using Lemma 2.9, we have the following lemma.
Lemma 2.10 Assume that (F1)-(F4) hold. If $u_{n} \rightharpoonup u$ in $H^{1}\left(\mathbb{R}^{N}\right)$, then

$$
\begin{equation*}
E\left(u_{n}\right)=E(u)+E\left(u_{n}-u\right)+o(1) \tag{2.30}
\end{equation*}
$$

and

$$
\begin{equation*}
J\left(u_{n}\right)=J(u)+J\left(u_{n}-u\right)+o(1) . \tag{2.31}
\end{equation*}
$$

Similar to [31, Lemma 2.13], by using the key inequality (2.5), the deformation lemma, and the intermediate value theorem of continuous function, the following lemma is given. We omit the proof here.

Lemma 2.11 Suppose that (M1)-(M3), (V1)-(V4), and (F1)-(F4) hold. If $\bar{u} \in \mathcal{M}$ and $E(\bar{u})=I$, then $\bar{u}$ is a critical point of $E$.

To overcome the lack of the compactness of Sobolev embedding, we define its limit problem related to (1.1) by

$$
\left\{\begin{array}{l}
-m\left(\|\nabla u\|_{2}^{2}\right) \Delta u+V_{\infty} u=f(u), \quad x \in \mathbb{R}^{N}  \tag{2.32}\\
u \in H^{1}\left(\mathbb{R}^{N}\right)
\end{array}\right.
$$

Under assumptions (F1) and (F2), weak solutions to (2.32) correspond to critical points of the energy functional defined in $H^{1}\left(\mathbb{R}^{N}\right)$ by

$$
\begin{equation*}
E^{\infty}(u)=\frac{1}{2} M\left(\|\nabla u\|_{2}^{2}\right)+\frac{1}{2} \int_{\mathbb{R}^{N}} V_{\infty} u^{2} \mathrm{~d} x-\int_{\mathbb{R}^{N}} F(u) \mathrm{d} x . \tag{2.33}
\end{equation*}
$$

Similar to (1.13) and (2.7), we define the functional

$$
\begin{equation*}
J^{\infty}(u):=\frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{2+N}{2} \int_{\mathbb{R}^{N}} V_{\infty}|u|^{2}-\int_{\mathbb{R}^{N}}[f(u) u+N F(u)] \mathrm{d} x, \tag{2.34}
\end{equation*}
$$

the constraint set

$$
\begin{equation*}
\mathcal{M}^{\infty}:=\left\{u \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}: J^{\infty}(u)=0\right\} \tag{2.35}
\end{equation*}
$$

and the minimizer

$$
\begin{equation*}
I^{\infty}:=\inf _{u \in \mathcal{M}^{\infty}} E(u) \tag{2.36}
\end{equation*}
$$

Since $V(x) \equiv V_{\infty}$, which is well covered by (V1), (V2), and (V4), then all the above conclusions on $E$ are true for $E^{\infty}$. Through discussing the corresponding limit equation (2.32), we will get the proof of Theorem 1.1.

Lemma 2.12 Assume that (M1)-(M3) and (F1)-(F4) hold. Then $I^{\infty}:=\inf _{u \in \mathcal{M}} \infty E^{\infty}(u)$ is achieved.

Proof For any $u \in \mathcal{M}^{\infty}$, we have $E^{\infty}(u) \geq I^{\infty}$. Let $\left\{u_{n}\right\} \subset \mathcal{M}^{\infty}$ such that $E^{\infty}\left(u_{n}\right) \rightarrow I^{\infty}$ as $n \rightarrow \infty$. Since $J^{\infty}\left(u_{n}\right)=0$, then it follows from (2.27) that

$$
\begin{align*}
I^{\infty}+o(1)= & E^{\infty}\left(u_{n}\right)-\frac{1}{2+p} J^{\infty}\left(u_{n}\right) \\
= & \frac{1}{2} M\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)-\frac{N}{2(2+p)} m\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)\left\|\nabla u_{n}\right\|_{2}^{2} \\
& +\frac{p-N}{2(2+p)} V_{\infty}\left\|u_{n}\right\|_{2}^{2}+\frac{1}{2+p} \int_{\mathbb{R}^{N}}\left[f\left(u_{n}\right) u_{n}-(p+2-N) F\left(u_{n}\right)\right] \mathrm{d} x \\
\geq & \frac{1}{2} M\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)-\frac{N}{2(2+p)} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{p-N}{2(2+p)} V_{\infty}\left\|u_{n}\right\|_{2}^{2} \tag{2.37}
\end{align*}
$$

Similar to (2.28), this shows that $\left\{u_{n}\right\}$ is bounded in $H^{1}\left(\mathbb{R}^{N}\right)$.
With (2.8), the rest of the proof is similar to [31, Lemma 2.12], so we omit it.
The same as the case of dimension three in [31], we get the following relation between $I$ and $I^{\infty}$.

Lemma 2.13 Suppose that (M1)-(M3), (V1)-(V4), and (F1)-(F4) hold. Then $I<I^{\infty}$.
Lemma 2.14 Suppose that (M1)-(M3), (V1)-(V4), and (F1)-(F4) hold. Then I is achieved.
Proof Step 1. Choosing a minimizing sequence $\left\{u_{n}\right\} \subset \mathcal{M}$ of $I$ and showing that the sequence is bounded in $H^{1}\left(\mathbb{R}^{N}\right)$. This part of argument is the same as Lemma 2.12. So we omit it here.
Step 2. Showing that $\left\{u_{n}\right\}$ is convergent in $H^{1}\left(\mathbb{R}^{N}\right)$. By Lion's concentration compactness principle [35, Lemma 1.21], similar to [31, Lemma 3.2], one can easily obtain that result, so we also omit it here.

Proof of Theorem 1.1 Under Lemmas 2.6, 2.11, and 2.14, there exists $\bar{u} \in \mathcal{M}$ such that

$$
\begin{equation*}
E(\bar{u})=I=\inf _{u \in H^{1}\left(\mathbb{R}^{2}\right) \backslash\{0\}} \max _{t>0} E\left(t u_{t}\right), \quad E^{\prime}(\bar{u})=0 . \tag{2.38}
\end{equation*}
$$

This shows that $\bar{u}$ is a ground state solution of Nehari-Pohozaev type for (1.1).

## 3 The least energy solutions for (1.1)

In this section, we give the proof of Theorem 1.2.

Proposition 3.1 ([18]) Let $X$ be a Banach space and $\Lambda \subset \mathbb{R}^{+}$be an interval. We consider a family $\left\{\Phi_{\lambda}\right\}_{\lambda \in \Lambda}$ of $\mathcal{C}^{1}$-functionals on $X$ of the form

$$
\Phi_{\lambda}(u)=A(u)-\lambda B(u), \quad \forall \lambda \in \Lambda,
$$

where $B(u) \geq 0, \forall u \in X$, and such that either $A(u) \rightarrow+\infty$ or $B(u) \rightarrow+\infty$, as $\|u\| \rightarrow \infty$. We assume that there are two points $v_{1}, v_{2}$ in $X$ such that

$$
\begin{equation*}
c_{\lambda}:=\inf _{\gamma \in \Gamma} \max _{t \in[0,1]} \Phi_{\lambda}(\gamma(t))>\max \left\{\Phi_{\lambda}\left(v_{1}\right), \Phi_{\lambda}\left(v_{2}\right)\right\} \tag{3.1}
\end{equation*}
$$

where

$$
\Gamma=\left\{\gamma \in \mathcal{C}([0,1], X): \gamma(0)=v_{1}, \gamma(1)=v_{2}\right\} .
$$

Then, for almost every $\lambda \in \Lambda$, there is a bounded $(P S)_{c_{\lambda}}$ sequence for $\Phi_{\lambda}$, that is, there exists a sequence such that
(i). $\left\{u_{n}(\lambda)\right\}$ is bounded in $X$;
(ii). $\Phi_{\lambda}\left(u_{n}(\lambda)\right) \rightarrow c_{\lambda}$;
(iii). $\Phi_{\lambda}^{\prime}\left(u_{n}(\lambda)\right) \rightarrow 0$ in $X^{*}$, where $X^{*}$ is the dual of $X$.

To apply Proposition 3.1, we introduce two families of functionals defined by

$$
\begin{equation*}
E_{\lambda}(u)=\frac{1}{2} M\left(|\nabla u|^{2}\right)+\frac{1}{2} \int_{\mathbb{R}^{N}} V(x) u^{2} \mathrm{~d} x-\lambda \int_{\mathbb{R}^{N}} F(u) \mathrm{d} x \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{\lambda}^{\infty}(u)=\frac{1}{2} M\left(|\nabla u|^{2}\right)+\frac{1}{2} \int_{\mathbb{R}^{N}} V_{\infty} u^{2} \mathrm{~d} x-\lambda \int_{\mathbb{R}^{N}} F(u) \mathrm{d} x \tag{3.3}
\end{equation*}
$$

for $\lambda \in[1 / 2,1]$.

Lemma 3.2 ([13]) Suppose that (V1), (V2), (F1), and (F2) hold. Let u be a critical point of $E_{\lambda}$ in $H^{1}\left(\mathbb{R}^{N}\right)$, then we have the following Pohozaev-type identity:

$$
\begin{align*}
\mathcal{P}_{\lambda}(u):= & \frac{N-2}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}[N V(x)+(\nabla V(x), x)] u^{2} \mathrm{~d} x \\
& -N \lambda \int_{\mathbb{R}^{N}} F(u) \mathrm{d} x=0 . \tag{3.4}
\end{align*}
$$

We set $J_{\lambda}(u):=\left\langle E_{\lambda}^{\prime}(u), u\right\rangle+\mathcal{P}_{\lambda}(u)$, then

$$
\begin{align*}
J_{\lambda}(u)= & \frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}[(N+2) V(x)+(\nabla V(x), x)] u^{2} \mathrm{~d} x \\
& -\lambda \int_{\mathbb{R}^{N}}[f(u) u+N F(u)] \mathrm{d} x \tag{3.5}
\end{align*}
$$

for $\lambda \in[1 / 2,1]$. Correspondingly, we also let

$$
\begin{equation*}
J_{\lambda}^{\infty}(u)=\frac{N}{2} m\left(\|\nabla u\|_{2}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{N+2}{2} V_{\infty}\|u\|_{2}^{2}-\lambda \int_{\mathbb{R}^{N}}[f(u) u+N F(u)] \mathrm{d} x \tag{3.6}
\end{equation*}
$$

for $\lambda \in[1 / 2,1]$. Set

$$
\begin{equation*}
\mathcal{M}_{\lambda}^{\infty}:=\left\{u \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}: J_{\lambda}^{\infty}(u)=0\right\}, \quad I_{\lambda}^{\infty}:=\inf _{u \in \mathcal{M}_{\lambda}^{\infty}} E_{\lambda}^{\infty}(u) . \tag{3.7}
\end{equation*}
$$

By Lemma 2.3, we have the following lemma.

Lemma 3.3 Suppose that (M1)-(M3), (V1), (F1), (F2), and (F4) hold. Then

$$
\begin{equation*}
E_{\lambda}^{\infty}(u) \geq E_{\lambda}^{\infty}\left(t u_{t}\right)+\frac{1-t^{2+p}}{2+p} J_{\lambda}^{\infty}(u), \quad \forall u \in H^{1}\left(\mathbb{R}^{N}\right), t>0, \lambda \geq 0 \tag{3.8}
\end{equation*}
$$

In view of Theorem 1.1, $E_{1}^{\infty}$ has a minimizer $u_{1}^{\infty}$ on $\mathcal{M}_{1}^{\infty}$, i.e.,

$$
\begin{equation*}
u_{1}^{\infty} \in \mathcal{M}_{1}^{\infty}, \quad\left(E_{1}^{\infty}\right)^{\prime}\left(u_{1}^{\infty}\right)=0, \quad \text { and } \quad I_{1}^{\infty}=E_{1}^{\infty}\left(u_{1}^{\infty}\right) \tag{3.9}
\end{equation*}
$$

Lemma 3.4 Suppose that (M1)-(M3), (V1), (V2), and (F1)-(F3) hold. Then
(i). there exists $T>0$ independent of $\lambda$ such that $E_{\lambda}\left(T\left(u_{1}^{\infty}\right)_{T}\right)<0$ for all $\lambda \in[1 / 2,1]$;
(ii). there exists $\kappa_{0}>0$ independent of $\lambda$ such that, for all $\lambda \in[1 / 2,1]$,

$$
\begin{equation*}
c_{\lambda}:=\inf _{\gamma \in \Gamma} \max _{t \in[0,1]} E_{\lambda}(\gamma(t)) \geq \kappa_{0}>\max \left\{E_{\lambda}(0), E_{\lambda}\left(T\left(u_{1}^{\infty}\right)_{T}\right)\right\}, \tag{3.10}
\end{equation*}
$$

where

$$
\Gamma=\left\{\gamma \in \mathcal{C}\left([0,1], H^{1}\left(\mathbb{R}^{N}\right)\right): \gamma(0)=0, \gamma(1)=T\left(u_{1}^{\infty}\right)_{T}\right\} ;
$$

(iii). $c_{\lambda}$ and $I_{\lambda}^{\infty}$ are nonincreasing on $\lambda \in[1 / 2,1]$.

The proof of Lemma 3.4 is standard, the reader can refer to [6, Lemma 4.4].
We use the ingenious assumptions on $V$ borrowed from [31], that is, for $V \in \mathcal{C}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ and $V(x) \leq V_{\infty}$ but $V(x) \not \equiv V_{\infty}$, there exist $\bar{x} \in \mathbb{R}^{N}$ and $\bar{r}>0$ such that

$$
\begin{equation*}
V_{\infty}>V(x) \quad \text { and } \quad\left|\left(u_{1}^{\infty}\right)(x)\right|>0, \quad \text { a.e. }|x-\bar{x}| \leq \bar{r} . \tag{3.11}
\end{equation*}
$$

Lemma 3.5 Suppose that (M1)-(M3), (V1), (V2), and (F1)-(F4) hold. Then there exists $\bar{\lambda} \in[1 / 2,1)$ such that $c_{\lambda}<I_{\lambda}^{\infty}$ for $\lambda \in[\bar{\lambda}, 1]$.

Proof It is easy to see that $E_{\lambda}\left(t\left(u_{1}^{\infty}\right)_{t}\right)$ is continuous on $t \in(0, \infty)$. Hence, for any $\lambda \in$ $[1 / 2,1)$, we can choose $t_{\lambda} \in(0, T)$ such that $E_{\lambda}\left(t_{\lambda}\left(u_{1}^{\infty}\right)_{t_{\lambda}}\right)=\max _{t \in[0, T]} E_{\lambda}\left(t\left(u_{1}^{\infty}\right)_{t}\right)$. Define

$$
\gamma_{0}(t)= \begin{cases}t T\left(u_{1}^{\infty}\right)_{t T}, & \text { for } t>0 \\ 0, & \text { for } t=0\end{cases}
$$

Then $\gamma_{0} \in \Gamma$ defined by Lemma 3.4(ii), i.e., $\gamma_{0}(0)=0, \gamma_{0}(1)=T\left(u_{1}^{\infty}\right)_{T}$. Moreover,

$$
\begin{equation*}
E_{\lambda}\left(t_{\lambda}\left(u_{1}^{\infty}\right)_{t_{\lambda}}\right)=\max _{t \in[0,1]} E_{\lambda}\left(\gamma_{0}(t)\right) \geq c_{\lambda} \tag{3.12}
\end{equation*}
$$

It follows from (2.27) that the function $F(t) /|t|^{2+p-N}$ is nondecreasing on $t \in(-\infty, 0) \cup$ $(0,+\infty)$. Since $t_{\lambda} \in(0, T)$, then we have

$$
\frac{F\left(t_{\lambda} u_{1}^{\infty}\right)}{t_{\lambda}^{\frac{2+p-N}{2}}} \leq \frac{F\left(T u_{1}^{\infty}\right)}{T^{\frac{2+p-N}{2}}}
$$

Let

$$
\begin{equation*}
\zeta_{0}:=\min \{3 \bar{r} / 8(1+|\bar{x}|), 1 / 4\} . \tag{3.13}
\end{equation*}
$$

Then it follows from (3.11) and (3.13) that

$$
\begin{equation*}
|x-\bar{x}| \leq \frac{\bar{r}}{2} \quad \text { and } \quad \tau \in\left[1-\zeta_{0}, 1+\zeta_{0}\right] \quad \Rightarrow \quad|\tau x-\bar{x}| \leq \bar{r} \tag{3.14}
\end{equation*}
$$

Let

$$
\begin{align*}
\bar{\lambda}:= & \max \left\{\frac{1}{2}, 1-\frac{\left(1-\zeta_{0}\right)^{N+2} \min _{\tau \in\left[1-\zeta_{0}, 1+\zeta_{0}\right]} \int_{\mathbb{R}^{N}}\left[V_{\infty}-V(\tau x)\right]\left|u_{1}^{\infty}\right|^{2} \mathrm{~d} x}{2 T^{2} \int_{\mathbb{R}^{N}} F\left(T u_{1}^{\infty}\right) \mathrm{d} x},\right.  \tag{3.15}\\
& \left.1-\frac{L(T)}{T^{2} \int_{\mathbb{R}^{N}} F\left(T u_{1}^{\infty}\right) \mathrm{d} x}\right\},
\end{align*}
$$

where $L(t)$ is defined in (2.8). Then it follows from (3.11) and (3.14) that $1 / 2 \leq \bar{\lambda}<1$. We have two cases to distinguish:

Case (i). $t_{\lambda} \in\left[1-\zeta_{0}, 1+\zeta_{0}\right]$. From (3.2), (3.3), (3.8)-(3.12), (3.14), (3.15), and Lemma 3.4(iii), we have

$$
\begin{aligned}
I_{\lambda}^{\infty} \geq & I_{1}^{\infty}=E_{1}^{\infty}\left(u_{1}^{\infty}\right) \geq E_{1}^{\infty}\left(t_{\lambda}\left(u_{1}^{\infty}\right)_{t_{\lambda}}\right) \\
= & E_{\lambda}\left(t_{\lambda}\left(u_{1}^{\infty}\right)_{t_{\lambda}}\right)-(1-\lambda) t_{\lambda}^{2} \int_{\mathbb{R}^{N}} F\left(t_{\lambda} u_{1}^{\infty}\right) \mathrm{d} x+\frac{t_{\lambda}^{N+2}}{2} \int_{\mathbb{R}^{N}}\left[V_{\infty}-V\left(t_{\lambda} x\right)\right]\left|u_{1}^{\infty}\right|^{2} \mathrm{~d} x \\
> & c_{\lambda}-(1-\lambda) T^{2} \int_{\mathbb{R}^{N}} F\left(T u_{1}^{\infty}\right) \mathrm{d} x \\
& +\frac{\left(1-\zeta_{0}\right)^{N+2}}{2} \min _{\tau \in\left[1-\zeta_{0}, 1+\zeta_{0}\right]} \int_{\mathbb{R}^{2}}\left[V_{\infty}-V(\tau x)\right]\left|u_{1}^{\infty}\right|^{2} \mathrm{~d} x \\
\geq & c_{\lambda}, \quad \forall \lambda \in[\bar{\lambda}, 1] .
\end{aligned}
$$

Case (ii). $t_{\lambda} \in\left(0,1-\zeta_{0}\right) \cup\left(1+\zeta_{0}, T\right]$. From (2.5), (3.2), (3.3), (3.8), (3.11), (3.12), (3.15), Assertion 1, and Lemma 3.4(iii),

$$
\begin{aligned}
I_{\lambda}^{\infty} & \geq I_{1}^{\infty}=E_{1}^{\infty}\left(u_{1}^{\infty}\right) \\
& \geq E_{1}^{\infty}\left(t_{\lambda}\left(u_{1}^{\infty}\right)_{t_{\lambda}}\right)+\frac{1}{2} M\left(\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right)-\frac{1}{2} M\left(t_{\lambda}^{N}\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1-t_{\lambda}^{2+p}}{2+p} \cdot \frac{N}{2} m\left(\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right)\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2} \\
\geq & E_{\lambda}\left(t_{\lambda}\left(u_{1}^{\infty}\right)_{t_{\lambda}}\right)-(1-\lambda) t_{\lambda}^{2} \int_{\mathbb{R}^{N}} F\left(t_{\lambda} u_{1}^{\infty}\right) \mathrm{d} x+\frac{t_{\lambda}^{N+2}}{2} \int_{\mathbb{R}^{N}}\left[V_{\infty}-V\left(t_{\lambda} x\right)\right]\left|u_{1}^{\infty}\right|^{2} \mathrm{~d} x \\
& +\frac{1}{2} M\left(\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right)-\frac{1}{2} M\left(t_{\lambda}^{N}\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right)-\frac{1-t_{\lambda}^{2+p}}{2+p} \cdot \frac{N}{2} m\left(\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right)\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2} \\
> & c_{\lambda}-(1-\lambda) T^{2} \int_{\mathbb{R}^{N}} F\left(T u_{1}^{\infty}\right) \mathrm{d} x+\frac{1}{2} M\left(\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right)-\frac{1}{2} M\left(T^{N}\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right) \\
& -\frac{1-T^{2+p}}{2+p} \cdot \frac{N}{2} m\left(\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2}\right)\left\|\nabla u_{1}^{\infty}\right\|_{2}^{2} \\
\geq & c_{\lambda}, \quad \forall \lambda \in(\bar{\lambda}, 1] .
\end{aligned}
$$

Combining both the above cases, we have $c_{\lambda}<I_{\lambda}^{\infty}$ for $\lambda \in(\bar{\lambda}, 1]$.

Lemma 3.6 Suppose that (V1), (V2), and (F1)-(F3) hold. Let $\left\{u_{n}\right\}$ be a bounded (PS) $)_{c_{\lambda}}$ sequence for $E_{\lambda}$ with $\lambda \in[1 / 2,1]$. Then there exist a subsequence of $\left\{u_{n}\right\}$, still denoted by $\left\{u_{n}\right\}$, an integer $l \in \mathbb{N}$, and $u_{0} \in H^{1}\left(\mathbb{R}^{N}\right)$ such that
(i) $A_{\lambda}^{2}:=\lim _{n \rightarrow \infty}\left\|\nabla u_{n}\right\|_{2}^{2}, u_{n} \rightharpoonup u_{\lambda}$ in $H^{1}\left(\mathbb{R}^{N}\right)$ and $\mathcal{E}_{\lambda}^{\prime}\left(u_{\lambda}\right)=0$;
(ii) there exist $w^{1}, \ldots, w^{l} \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ such that $\left(\mathcal{E}_{\lambda}^{\infty}\right)^{\prime}\left(w^{k}\right)=0$ for $1 \leq k \leq l$;
(iii)

$$
\begin{aligned}
& c+\frac{1}{4} m\left(A_{\lambda}^{2}\right) A_{\lambda}^{2}=\mathcal{E}_{\lambda}\left(u_{\lambda}\right)+\sum_{k=1}^{l} \mathcal{E}_{\lambda}^{\infty}\left(w^{k}\right) ; \\
& A_{\lambda}^{2}=\left\|\nabla u_{\lambda}\right\|_{2}^{2}+\sum_{k=1}^{l}\left\|\nabla w^{k}\right\|_{2}^{2}
\end{aligned}
$$

where

$$
\begin{equation*}
\mathcal{E}_{\lambda}(u)=\frac{1}{2} m\left(A_{\lambda}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}} V(x) u^{2} \mathrm{~d} x-\lambda \int_{\mathbb{R}^{N}} F(u) \mathrm{d} x \tag{3.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{E}_{\lambda}^{\infty}(u)=\frac{1}{2} m\left(A_{\lambda}^{2}\right)\|\nabla u\|_{2}^{2}+\frac{V_{\infty}}{2} \int_{\mathbb{R}^{N}} u^{2} \mathrm{~d} x-\lambda \int_{\mathbb{R}^{N}} F(u) \mathrm{d} x . \tag{3.17}
\end{equation*}
$$

We agree that in the case $l=0$ the above holds without $w^{k}$.

Analogous to the proof of Lemma 2.3 in [22], we can prove Lemma 3.6. We omit it here.

Lemma 3.7 Suppose that (V1), (V2), (V3), (M4), and (F1)-(F3) hold. Then, for almost every $\lambda \in[\bar{\lambda}, 1]$, there exists $u_{\lambda} \in H^{1}\left(\mathbb{R}^{2}\right) \backslash\{0\}$ such that

$$
\begin{equation*}
E_{\lambda}^{\prime}\left(u_{\lambda}\right)=0, \quad E_{\lambda}\left(u_{\lambda}\right)=c_{\lambda} . \tag{3.18}
\end{equation*}
$$

Proof Under (F1)-(F3), Lemma 3.4 implies that $E_{\lambda}(u)$ satisfies the assumptions of Proposition 3.1 with $X=H^{1}\left(\mathbb{R}^{N}\right)$ and $\Phi_{\lambda}=E_{\lambda}$. So, for almost every $\lambda \in[1 / 2,1]$, there exists a
bounded sequence $\left\{u_{n}(\lambda)\right\} \subset H^{1}\left(\mathbb{R}^{N}\right)$ (for simplicity, we denote $\left\{u_{n}\right\}$ instead of $\left\{u_{n}(\lambda)\right\}$ ) such that

$$
\begin{equation*}
E_{\lambda}\left(u_{n}\right) \rightarrow c_{\lambda}>0, \quad\left\|E_{\lambda}^{\prime}\left(u_{n}\right)\right\| \rightarrow 0 \tag{3.19}
\end{equation*}
$$

By Lemma 3.6, there exist a subsequence of $\left\{u_{n}\right\}$, still denoted by $\left\{u_{n}\right\}$, and $u_{\lambda} \in H^{1}\left(\mathbb{R}^{N}\right)$ such that $A_{\lambda}^{2}:=\lim _{n \rightarrow \infty}\left\|\nabla u_{n}\right\|_{2}^{2}$ exists, $u_{n} \rightharpoonup u_{\lambda}$ in $H^{1}\left(\mathbb{R}^{N}\right)$ and $\left(\mathcal{E}_{\lambda}\right)^{\prime}\left(u_{\lambda}\right)=0$.

If (ii) occurs, i.e., there exist $l \in \mathbb{N}$ and $w^{1}, \ldots, w^{l} \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ such that $\left(\mathcal{E}_{\lambda}^{\infty}\right)^{\prime}\left(w^{k}\right)=0$ for $1 \leq k \leq l$,

$$
\begin{equation*}
c_{\lambda}+\frac{1}{4} m\left(A_{\lambda}^{2}\right)\left\|\nabla u_{\lambda}\right\|_{2}^{2}=\mathcal{E}_{\lambda}\left(u_{\lambda}\right)+\sum_{k=1}^{l} \mathcal{E}_{\lambda}^{\infty}\left(w^{k}\right) \tag{3.20}
\end{equation*}
$$

and

$$
\begin{equation*}
A_{\lambda}^{2}=\left\|\nabla u_{\lambda}\right\|_{2}^{2}+\sum_{k=1}^{l}\left\|\nabla w^{k}\right\|_{2}^{2} \tag{3.21}
\end{equation*}
$$

Since $\left(\mathcal{E}_{\lambda}\right)^{\prime}\left(u_{\lambda}\right)=0$, then we have the Pohozaev identity of the functional $\mathcal{E}_{\lambda}$

$$
\begin{align*}
\tilde{\mathcal{P}}_{\lambda}\left(u_{\lambda}\right) & :=\frac{N-2}{2} m\left(A_{\lambda}^{2}\right)\left\|\nabla u_{\lambda}\right\|_{2}^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}[N V(x)+\nabla V(x) \cdot x] u_{\lambda}^{2} \mathrm{~d} x-N \lambda \int_{\mathbb{R}^{2}} F\left(u_{\lambda}\right) \mathrm{d} x \\
& =0 \tag{3.22}
\end{align*}
$$

It follows from (2.27), (3.16), (3.22), and (V3) that

$$
\begin{align*}
\mathcal{E}_{\lambda}\left(u_{\lambda}\right)= & \mathcal{E}_{\lambda}\left(u_{\lambda}\right)-\frac{1}{2+p}\left[\left\langle\mathcal{E}_{\lambda}^{\prime}\left(u_{\lambda}\right), u_{\lambda}\right\rangle+\tilde{\mathcal{P}}_{\lambda}\left(u_{\lambda}\right)\right] \\
= & \frac{2+p-N}{2(2+p)} m\left(A_{\lambda}^{2}\right)\left\|\nabla u_{\lambda}\right\|_{2}^{2}+\frac{1}{2(2+p)} \int_{\mathbb{R}^{N}}[(p-N) V(x)-\nabla V(x) \cdot x] u_{\lambda}^{2} \mathrm{~d} x \\
& +\frac{\lambda}{2+p} \int_{\mathbb{R}^{N}}\left[f\left(u_{\lambda}\right) u_{\lambda}-(2+p-N) F\left(u_{\lambda}\right)\right] \mathrm{d} x \\
\geq & \frac{1}{4} m\left(A_{\lambda}^{2}\right)\left\|\nabla u_{\lambda}\right\|_{2}^{2} . \tag{3.23}
\end{align*}
$$

Since $\left(\mathcal{E}_{\lambda}^{\infty}\right)^{\prime}\left(w^{k}\right)=0$, then we have the Pohozaev identity of the functional $\mathcal{E}_{\lambda}^{\infty}$

$$
\begin{equation*}
\tilde{\mathcal{P}}_{\lambda}^{\infty}\left(w^{k}\right):=\frac{N-2}{2} m\left(A_{\lambda}^{2}\right)\left\|\nabla w^{k}\right\|_{2}^{2}+\frac{N}{2} V_{\infty} \int_{\mathbb{R}^{N}}\left(w^{k}\right)^{2} \mathrm{~d} x-N \lambda \int_{\mathbb{R}^{N}} F\left(w^{k}\right) \mathrm{d} x=0 \tag{3.24}
\end{equation*}
$$

Thus, from (3.6), (3.17), (3.21), and (3.24), we have

$$
\begin{align*}
0 & =\left\langle\left(\mathcal{E}_{\lambda}^{\infty}\right)^{\prime}\left(w^{k}\right), w^{k}\right\rangle+\tilde{\mathcal{P}}_{\lambda}^{\infty}\left(w^{k}\right) \\
& =\frac{N}{2} m\left(A_{\lambda}^{2}\right)\left\|\nabla w^{k}\right\|_{2}^{2}+\frac{N+2}{2} V_{\infty}\left\|w^{k}\right\|_{2}^{2}-\lambda \int_{\mathbb{R}^{N}}\left[f\left(w^{k}\right) w^{k}+N F\left(w^{k}\right)\right] \mathrm{d} x \\
& \geq J_{\lambda}^{\infty}\left(w^{k}\right) \tag{3.25}
\end{align*}
$$

Since $w^{k} \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$, in view of Lemma 2.5, there exists $t_{k}>0$ such that $t_{k}\left(w^{k}\right)_{t_{k}} \in \mathcal{M}_{\lambda}^{\infty}$. From (3.3), (3.6), (3.8), (3.17), and (3.21), one has

$$
\begin{align*}
\mathcal{E}_{\lambda}^{\infty}\left(w^{k}\right)= & \mathcal{E}_{\lambda}^{\infty}\left(w^{k}\right)-\frac{1}{2+p}\left[\left\langle\left(\mathcal{E}_{\lambda}^{\infty}\right)^{\prime}\left(w^{k}\right), w^{k}\right\rangle+\tilde{\mathcal{P}}_{\lambda}^{\infty}\left(w^{k}\right)\right] \\
= & \frac{2+p-N}{2(2+p)} m\left(A_{\lambda}^{2}\right)\left\|\nabla w^{k}\right\|_{2}^{2}+\frac{N}{2(2+p)} m\left(\left\|\nabla w^{k}\right\|_{2}^{2}\right)\left\|\nabla w^{k}\right\|_{2}^{2}-\frac{1}{2} M\left(\left\|\nabla w^{k}\right\|_{2}^{2}\right) \\
& +E_{\lambda}^{\infty}\left(w^{k}\right)-\frac{1}{2+p} J_{\lambda}^{\infty}\left(w^{k}\right) \\
\geq & \frac{2+p-N}{2(2+p)} m\left(A_{\lambda}^{2}\right)\left\|\nabla w^{k}\right\|_{2}^{2}+\frac{N}{2(2+p)} m\left(\left\|\nabla w^{k}\right\|_{2}^{2}\right)\left\|\nabla w^{k}\right\|_{2}^{2}-\frac{1}{2} M\left(\left\|\nabla w^{k}\right\|_{2}^{2}\right) \\
& +E_{\lambda}^{\infty}\left(t_{k}\left(w^{k}\right)_{t_{k}}\right)-\frac{t_{k}^{2+p}}{2+p} J_{\lambda}^{\infty}\left(w^{k}\right) . \tag{3.26}
\end{align*}
$$

Let

$$
\begin{equation*}
H(t)=\frac{2+p-N}{2(2+p)} m\left(A_{\lambda}^{2}\right) t+\frac{N}{2(2+p)} m(t) t-\frac{1}{2} M(t) \tag{3.27}
\end{equation*}
$$

then by (M4) one has

$$
\begin{align*}
& H^{\prime}(t)=\frac{2+p-N}{2(2+p)} m\left(A_{\lambda}^{2}\right)+\frac{N}{2(2+p)} m^{\prime}(t) t-\frac{2+p-N}{2(2+p)} m(t) \geq 0,  \tag{3.28}\\
& H(0)=-\frac{1}{2} M(0)=0 \tag{3.29}
\end{align*}
$$

Then, from (3.21), (3.25), and (3.26), one has

$$
\begin{equation*}
\mathcal{E}_{\lambda}^{\infty}\left(w^{k}\right) \geq I_{\lambda}^{\infty} \tag{3.30}
\end{equation*}
$$

It follows from (3.20), (3.21), (3.23), and (3.26) that

$$
\begin{aligned}
c_{\lambda}+\frac{1}{4} m\left(A_{\lambda}^{2}\right)\left\|\nabla u_{\lambda}\right\|_{2}^{2} & =\mathcal{E}_{\lambda}\left(u_{\lambda}\right)+\sum_{k=1}^{l} \mathcal{E}_{\lambda}^{\infty}\left(w^{k}\right) \\
& \geq l I_{\lambda}^{\infty}+\frac{1}{4} m\left(A_{\lambda}^{2}\right)\left\|\nabla u_{\lambda}\right\|_{2}^{2} \\
& \geq I_{\lambda}^{\infty}+\frac{1}{4} m\left(A_{\lambda}^{2}\right)\left\|\nabla u_{\lambda}\right\|_{2}^{2}, \quad \forall \lambda \in[\bar{\lambda}, 1]
\end{aligned}
$$

which together with Lemma 3.5 implies that $l=0$ and $\mathcal{E}_{\lambda}\left(u_{\lambda}\right)=c_{\lambda}+\frac{1}{4} m\left(A_{\lambda}^{2}\right)\left\|\nabla u_{\lambda}\right\|_{2}^{2}$. Thus, it follows from (3.21) that $A_{\lambda}=\left\|u_{\lambda}\right\|_{2}, u_{n} \rightarrow u_{\lambda}$ in $H^{1}\left(\mathbb{R}^{N}\right)$ and $E_{\lambda}\left(u_{\lambda}\right)=c_{\lambda}$.

Proof of Theorem 1.2 In view of Lemma 3.7, there exist two sequences of $\left\{\lambda_{n}\right\} \subset[\bar{\lambda}, 1]$ and $\left\{u_{\lambda_{n}}\right\} \subset H^{1}\left(\mathbb{R}^{N}\right)$, denoted by $\left\{u_{n}\right\}$, such that

$$
\begin{equation*}
\lambda_{n} \rightarrow 1, \quad E_{\lambda_{n}}^{\prime}\left(u_{n}\right)=0, \quad E_{\lambda_{n}}\left(u_{n}\right)=c_{\lambda_{n}} . \tag{3.31}
\end{equation*}
$$

From Lemma 3.4(iii), (2.27), (3.2), (3.5), and (3.31), one has

$$
\begin{align*}
c_{1 / 2} \geq & c_{\lambda_{n}}=E_{\lambda_{n}}\left(u_{n}\right)-\frac{1}{2+p} J_{\lambda_{n}}\left(u_{n}\right) \\
= & \frac{1}{2} M\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)+\frac{1}{2(2+p)} \int_{\mathbb{R}^{N}}[(p-N) V(x)-\nabla V(x) \cdot x] u_{n}^{2} \mathrm{~d} x \\
& -\frac{N}{2(2+p)} m\left(\left\|\nabla u_{n}\right\|_{2}^{2}\right)\left\|\nabla u_{n}\right\|_{2}^{2}+\frac{\lambda_{n}}{2+p} \int_{\mathbb{R}^{N}}\left[f\left(u_{n}\right) u_{n}-(2+p-N) F\left(u_{n}\right)\right] \mathrm{d} x \\
\geq & \frac{\varepsilon_{0}}{2}\left\|\nabla u_{n}\right\|_{2}^{2}+\frac{V_{0}}{8(2+p)}\left\|u_{n}\right\|_{2}^{2} \\
\geq & C_{4}\left\|u_{n}\right\|^{2} . \tag{3.32}
\end{align*}
$$

This shows that $\left\{u_{n}\right\}$ is bounded in $H^{1}\left(\mathbb{R}^{N}\right)$. Since $c_{\lambda_{n}} \rightarrow c_{1}$, then similar to the proof of Lemma 3.7, there exists $\tilde{u} \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ such that

$$
\begin{equation*}
E^{\prime}(\tilde{u})=0, \quad 0<E(\tilde{u})=c_{1} . \tag{3.33}
\end{equation*}
$$

Let

$$
\Sigma:=\left\{u \in H^{1}\left(\mathbb{R}^{2}\right) \backslash\{0\}: E^{\prime}(u)=0\right\}, \quad \hat{I}=\inf _{u \in \Sigma} E(u)
$$

it follows from (3.33) that $\Sigma \neq \emptyset$ and $\hat{I} \leq c_{1}$. For any $u \in \Sigma$, Lemma 3.2 yields $\mathcal{P}_{\lambda}(u)=$ $\mathcal{P}_{1}(u)=0$. Therefore, it follows from (3.23) that $E(u)=E_{1}(u)>0$, thus $\hat{I} \geq 0$. Set $\left\{u_{n}\right\} \subset \Sigma$ such that

$$
\begin{equation*}
E^{\prime}\left(u_{n}\right)=0, \quad E\left(u_{n}\right) \rightarrow \hat{I} . \tag{3.34}
\end{equation*}
$$

By Lemma 3.5, we have $\hat{I} \leq c_{1}<I_{1}^{\infty}$. Through a similar argument as in the proof of Lemma 3.7, we can certify that there exists $\bar{u} \in H^{s}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ such that

$$
\begin{equation*}
E^{\prime}(\bar{u})=0, \quad E(\bar{u})=\hat{I} \tag{3.35}
\end{equation*}
$$

This shows that $\bar{u}$ is a nontrivial least energy solution of (1.1).

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## Authors' contributions

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