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Infinite number of solutions for some elliptic eigenvalue problems of Kirchhoff-type with non-homogeneous material

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Abstract

In this paper, using variational method, we study the existence of an infinite number of solutions (some are positive, some are negative, and others are sign-changing) for a non-homogeneous elliptic Kirchhoff equation with a nonlinear reaction term.

MSC: 35J60; 35B32; 35J25; 35J62

Keywords: Kirchhoff equations; Sign-changing solution; Variational method

1 Introduction

In this paper, we consider the following nonlocal equation:

$$\begin{cases} -M(x, \|u\|^2)\triangle u = \lambda f(x,u), & x \in \Omega, \\ u|_{x\in\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary and

$$\begin{cases} f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}), \\ M(x,t) = a(x) + b(x)t, \quad \|u\| = \int_\Omega |\nabla u|^2\,dx, \end{cases}$$

with a, $b \in C^\gamma(\overline{\Omega})$, $\gamma \in (0,1)$, $a(x) \geq a_0 > 0$, $b(x) \geq 0$. Problem (1.1) is the steady-state problem associated with

$$\begin{cases} u_{tt} - M(x, \|u\|^2)\triangle u = f, & (x,t) \in (0,+\infty) \times \Omega, \\ u = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.2}$$

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which is an open problem proposed by Lions [25] as a generalization of

$$\begin{cases} u_{tt} - M(\|u\|^2)\triangle u = f, & (x,t) \in (0,+\infty) \times \Omega, \\ u = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $M(t) = a + bt$ with $a > 0$ and $b > 0$. In [15, 24], the authors noted that Problem (1.2) models small vertical vibrations of an elastic string with fixed ends when the density of the material is not constant. When $M(x,t)$ is independent of x, Problem (1.1) can be simplified to

$$\begin{cases} -M(\|u\|^2)\triangle u = \lambda f(x,u), & x \in \Omega, \\ u|_{x\in\partial\Omega} = 0. \end{cases} \tag{1.4}$$

The steady-state problem (1.4) associated with Problem (1.3) has received a lot of attention in the literature (usually using variational methods); see [2, 3, 10, 14, 16, 20, 22, 23, 29, 33–40] and the references therein.

There are many papers in the literature on sign-changing solutions for Dirichlet problems; see [4, 5, 7, 12, 19, 21] and their references. In [43], Zhang and Perera obtained sign-changing solutions for a class of Problem (1.4) using variational methods and invariant sets of descent flow; in [31] using minimax methods and invariant sets of descent flow, Mao and Zhang established the existence of sign-changing solutions; and in [35] combining the constraint variational method and the quantitative deformation lemma, Shuai proved that Problem (1.4) possesses one least energy sign-changing solution. Other results on the existence of sign-changing solutions for Kirchhoff equations can be found in [5, 9, 28, 30, 37] and their references.

Since $M(x,t)$ is dependent on x in Problem (1.1), the variational approach cannot be used to discuss it in a direct way, and fixed point theory and the Galerkin method were used to establish existence in [33] and [38]. In [15], Figueiredo et al. established the existence and uniqueness of a positive solution of Problem (1.1) via bifurcation theory, and in [17], Huy and Quan considered a generalization of Problem (1.1)

$$\begin{cases} -M(x,\|u\|^2)\triangle u = \lambda f(x,u,\nabla u) - g(x,u,\nabla u), & x \in \Omega, \\ u|_{x\in\partial\Omega} = 0, \end{cases}$$

and established existence results for both non-degenerate and degenerate cases of the function M using the fixed point index theory. We note, to the best of our knowledge, that there are no results in the literature on the existence of a sign-changing solution for Problem (1.1). In this paper (motivated by [21]) using the steepest descent method for gradient mappings of the isoperimetric variational problem (see [6]) and the method of invariant sets of descending flow in critical point theory (see [27]), we establish the existence of an infinite number of solutions (some are positive, some are negative, and others are sign-changing). Some ideas come from [18] and [42].

2 Main result

In this section, we suppose that f satisfies the following conditions:

(1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous;

(2) $f(x,t)t \geq 0$ and $f(x,t) \not\equiv 0$ in $\Omega \times (-\delta, 0) \cup \Omega \times (0, \delta)$;

(3) $|f(x,t)| \leq c_1|t|^p + c_2$, where $c_1, c_2 \in \mathbb{R}^+$, $1 \leq p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 \leq p < +\infty$ if $N = 1$ and $N = 2$.

Let $A := \mathbb{N}$ in our main result. The main theorem is as follows.

Theorem 2.1 *Suppose that f satisfies* (1), (2), *and* (3). *Then Problem* (1.1) *has an infinite number of positive solutions* $\{u_{1,\alpha}\}_{\alpha\in A}$, *an infinite number of negative solutions* $\{u_{2,\alpha}\}_{\alpha\in A}$, *and an infinite number of sign-changing solutions* $\{u_{3,\alpha}\}_{\alpha\in A}$.

First we establish the following lemma for Problem (1.1).

Lemma 2.1 *Problem* (1.1) *has a nontrivial solution if and only if there exists $r > 0$ such that the following problem*

$$\begin{cases} -\triangle u = \lambda \frac{1}{M(x,r^2)} f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2.1}$$

has a nontrivial solution u with $\|u\| = r$.

Proof Sufficiency. There exists $r > 0$ such that Problem (2.1) has a nontrivial solution u with $\|u\| = r$, and so u satisfies

$$\begin{cases} -\triangle u = \lambda \frac{1}{M(x,r^2)} f(x,u) = \lambda \frac{1}{M(x,\|u\|^2)} f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Clearly, u is a nontrivial solution of Problem (1.1).

Necessity. Problem (1.1) has a nontrivial solution u. Let $r = \|u\| > 0$. Then u satisfies

$$\begin{cases} -\triangle u = \lambda \frac{1}{M(x,\|u\|^2)} f(x,u) = \lambda \frac{1}{M(x,r^2)} f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial u, \end{cases}$$

that is, u is a nontrivial solution of Problem (2.1) with $\|u\| = r$.

The proof is completed. □

For given $r > 0$, set

$$S_r = \left\{ u \in H_0^1(\Omega) : \int_\Omega |\nabla u|^2 \, dx = r^2 \right\} \quad \text{and} \quad \overline{S}_r = S_r \cap C_0^1(\overline{\Omega}).$$

From Lemma 2.1, we only consider the existence of a nontrivial solution of Problem (2.1) in S_r.

Set

$$g(x,u) = \frac{1}{M(x,r)} f(x,u), \qquad G(x,u) = \int_0^u g(x,t) \, dt, \quad \forall u \in \mathbb{R},$$

$$\Psi(u) = -\int_\Omega G\bigl(x, u(x)\bigr) \, dx, \quad \forall u \in H_0^1(\Omega),$$

and

$$F = \Psi|_{S_r}, \quad \overline{F} = F|_{\overline{S}_r}.$$

Note that

$$F'(u) = \Psi'(u) - \frac{(\Psi'(u), u)}{\|u\|^2} u = -T(u)u - K\mathbb{G}(u),$$

where $(\cdot,\cdot)$ is the inner product in $H_0^1(\Omega)$ given by $(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, $K = (-\triangle)^{-1}$ with the Dirichlet boundary condition, $\mathbb{G}$ is the Nemitskii operator induced by g and

$$T(u) = \frac{(\Psi'(u), u)}{r^2}.$$

From condition (2), we have $(\Psi'(u), u) < 0$ for all $u \in S_r$ (see Lemma 1.0 in [19]), and we know that the solutions of Problem (2.1) correspond to the critical points of f.

In order to discuss Problem (2.1), for $r > 0$, we let (here $n \in \mathbb{N}$)

$$f_n(x,t) = \begin{cases} f(x,t), & \text{if } |t| \le n, \\ f(x,n) + t - n, & \text{if } t > n, \\ f(x,-n) + t + n, & \text{if } t < -n, \end{cases} \tag{2.2}$$

$$g_n(x,t) = \frac{1}{M(x,r^2)} f_n(x,t), \tag{2.3}$$

$$G_n(x,u) = \int_0^u g_n(x,t)\,dt, \quad \forall u \in \mathbb{R} \tag{2.4}$$

and consider

$$\begin{cases} -\triangle u = \lambda g_n(x,u), & x \in \Omega, \\ u|_{x\in\partial\Omega} = 0. \end{cases} \tag{2.5}$$

Let

$$\Psi_n(u) = -\int_\Omega G_n\bigl(x,u(x)\bigr)\,dx, \quad \forall u \in H_0^1(\Omega),$$

and

$$F_n = \Psi_n|_{S_r}, \quad \overline{F}_n = F_n|_{\overline{S}_r}. \tag{2.6}$$

We obtain that

$$F_n'(u) = \Psi_n'(u) - \frac{(\Psi_n'(u), u)}{\|u\|^2} u = -T_n(u)u - K\mathbb{G}_n(u), \tag{2.7}$$

where $\mathbb{G}_n$ is the corresponding Nemitskii operator to g_n and

$$T_n(u) = \frac{(\Psi_n'(u), u)}{r^2}.$$

From the definition of F_n in (2.6), we know that the solutions of Problem (2.5) correspond to the critical points of F_n.

From the definition of g_n and conditions (1), (2), and (3), it is easy to see that g_n also satisfies (1), (2), and (3) uniformly with respect to n and

(1)′ there exists $L_n > 0$ such that

$$\left|g_n(x,t_1) - g_n(x,t_2)\right| \le L_n|t_1 - t_2|, \quad \forall x \in \overline{\Omega}, t_1, t_2 \in \mathbb{R}. \tag{2.8}$$

We shall need the following results later.

Lemma 2.2 (see [1]) *Let Ω be a bounded, open subset of $\mathbb{R}^N$, and suppose that $\partial\Omega$ is C^1. Assume that $N < p \le +\infty$ and $u \in W^{k+1,p}(\Omega)$. Then there is $u^* \in C^{k,\gamma}(\overline{\Omega})$ with $u(x) = u^*(x)$ a.e. $x \in \Omega$ such that*

$$\left\|u^*\right\|_{C^{k,\gamma}} \le C\|u\|_{W^{k+1,p}};$$

here the constant C depends only on p, N, and Ω.

Lemma 2.3 (see [13]) *Let Ω be a bounded open subset of $\mathbb{R}^N$ with a C^1 boundary. Assume that $u \in W^{k,p}(\Omega)$.*

(1) *If*

$$k < \frac{n}{p},$$

then $u \in L^q(\Omega)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{N}.$$

Also

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{k,p}(\Omega)};$$

here the constant C depends only on k, p, N, and Ω.

(2) *If*

$$k > \frac{n}{p},$$

then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})$, where

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer,} \\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

Also

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})} \le C\|u\|_{W^{k,p}(\Omega)};$$

here the constant C depends only on k, p, N, and Ω.

Lemma 2.4 (see [11]) *Let p, $1 \le p \le p_0 = (N+2)/(N-2)$ (so that $2 \le p+1 \le 2^*$), and let $\beta = (2^*/N)(2^* - (p+1))$. Then, for each γ, $0 \le \gamma \le \beta$, there exists $c > 0$ such that*

$$\|u\|_{p+1}^{p+1} \le c\|\nabla u\|_2^{p+1-\gamma}\|u\|_2^{\gamma}$$

for all $u \in W_0^{1,2}(\Omega)$. (Here and henceforth $\|u\|_p$ denotes the norm of u in $L^p(\Omega)$.)

Lemma 2.5 (see [8]) *Let X be a Banach space and F be a closed subset in X. Assume that $V : X \to Y$ is locally continuous and*

$$\lim_{h\downarrow 0}\frac{d(u+hV(u),F)}{h} = 0 \tag{2.9}$$

for all $u \in \partial F$, where $d(\cdot,\cdot)$ is the distance on X. If $u_0 \in F$ and $\sigma(t)(0 \le t < \omega_+(u_0))$ is the solution of the initial value problem

$$\begin{cases} \frac{d\sigma}{dt} = V(\sigma), \\ \sigma(0,u_0) = u_0, \end{cases}$$

then $\sigma(t) \in F$ for all $t \in [0,\omega_+(u_0))$.

For each n, we consider

$$\begin{cases} \frac{d\sigma}{dt} = -F_n'(\sigma) = T_n(\sigma)\sigma + K\mathbb{G}_n(\sigma), \quad t \ge 0, \\ \sigma(0,u_0) = u_0 \end{cases} \tag{2.10}$$

in $H_0^1(\Omega)$ for $u_0 \in S_r$, where F_n' is defined in (2.7). Since (1)′, (2), and (3) hold, we have the following.

Lemma 2.6 (see [32]) *Let $c < b < 0$. For every $u \in F_n^{-1}([c,b])$, if $\sigma_n(t,u)$ is a solution of Problem (2.10) in $[0,+\infty)$ (see step 3 in Lemma 2.7), then either there is a unique $t(u) \in [0,+\infty)$ such that $F_n(\sigma_n(t(u),u)) = c$ or there is a critical point v of F_n in $F_n^{-1}([c,b])$ such that $\sigma_n(t,u) \to v$ as $t \to +\infty$.*

Lemma 2.7 *Under conditions (1), (2), and (3), Problem (2.10) has a unique solution $\sigma_n(t,u_0)$ on $[0,+\infty)$, which satisfies:*

(i) *$\sigma_n(t,u_0) \in S_r$ for all $u_0 \in S_r$; $\sigma_n(t,u_0) \in \overline{S_r}$ for all $u_0 \in \overline{S_r}$;*

(ii) *there exists $u_n \in S_r$ such that $\lim_{t\to+\infty}\sigma_n(t,u_0)) \overset{H_0^1}{=} u_n$ for $u_0 \in S_r$;*

(iii) *if $u_0 \in \overline{S}_r$, then $u_n \in \overline{S}_r$ and $\lim_{t\to+\infty}\sigma_n(t,u_0) \overset{C_0^1}{=} u_n$.*

Proof The proof is divided into six steps.

Step 1. We show that $F_n'(u) = -T_n(u)u - K\mathbb{G}_n(u)$ is globally Lipschitz continuous with respect to $H_0^1(\Omega)$, that is, there is $M > 0$ such that

$$\left\|F_n'(u_1) - F_n'(u_2)\right\|_{H_0^1} \le M\|u_1 - u_2\|_{H_0^1}, \quad \forall u_1, u_2 \in S_r.$$

Let $2^* = \frac{2N}{N-2}$. From (2.8), we have

$$\begin{aligned}\|\mathbb{G}_n(u_1) - \mathbb{G}_n(u_2)\|_{L^{2^*}} &= \left(\int_\Omega |g_n(x,u_1) - g_n(x,u_2)|^{2^*}\,dx\right)^{1/2^*} \\ &\le \left(\int_\Omega L_n^{2^*}|u_2(x) - u_1(x)|^{2^*}\,dx\right)^{1/2^*} \\ &= L_n\|u_1 - u_2\|_{L^{2^*}},\end{aligned}$$

i.e., $\mathbb{G}_n$ is globally Lipschitz in the L^{2^*} topology. Note that

$$H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega) \overset{\mathbb{G}}{\hookrightarrow} L^{2^*}(\Omega) \xrightarrow{K} H_0^1(\Omega),$$

K is a bounded linear operator, and so

$$\|K\mathbb{G}(u_1) - K\mathbb{G}(u_2)\| \le \overline{L}_n\|u_1 - u_2\|$$

for some positive constant $\overline{L}_n$, where $\|\cdot\|$ denotes the norm in $H_0^1(\Omega)$. Note

$$\begin{aligned}|T_n(u_1) - T_n(u_2)| &= \frac{1}{r^2}|(K\mathbb{G}_n(u_1), u_1) - (K\mathbb{G}_n)(u_2), u_2)| \\ &\le \frac{1}{r}\overline{L}_n\|u_1 - u_2\| + \frac{1}{r^2}\|K\mathbb{G}(u_2)\|\|u_1 - u_2\|,\end{aligned}$$

and

$$\|T_n(u_1)u_1 - T_n(u_2)u_2\| \le |T_n(u_1) - T_n(u_2)|\|u_1\| + |T_n(u_2)|\|u_1 - u_2\|.$$

Since $\|K\mathbb{G}_n(u)\|$ is bounded in S_r, so $T_n(u)$ is bounded also. Thus $F_n'(u)$ is globally Lipschitz continuous.

Step 2. We show that $F_n'(u) = -T_n(u)u - K\mathbb{G}_n(u)$ is globally Lipschitz continuous with respect to $C_0^1(\overline{\Omega})$, that is, there is $\overline{M} > 0$ such that

$$\|F_n'(u_1) - F_n'(u_2)\|_{C_0^1} \le \overline{M}\|u_1 - u_2\|_{C_0^1}, \quad \forall u_1, u_2 \in \overline{S_r}.$$

Let $l > N$. From (2.8), we have

$$\|G_n(u_1) - G_n(u_2)\|_{L^l} = \left(\int_\Omega |g_n(x,u_1) - g_n(x,u_2)|^l\,dx\right)^{1/l} \le L_n\|u_1 - u_2\|_{L^l},$$

i.e., $\mathbb{G}_n$ is globally Lipschitz in the $L^l(\Omega)$ topology. Note that

$$C_0^1(\overline{\Omega}) \hookrightarrow L^l(\Omega) \xrightarrow{\mathbb{G}} L^l(\Omega) \xrightarrow{K} W^{2,l}(\Omega) \cap W_0^{2,l}(\Omega) \hookrightarrow C_0^1(\overline{\Omega}),$$

K is a bounded linear operator, so there exists $\overline{L}_n' > 0$ such that

$$\|K\mathbb{G}_n(u_1) - K\mathbb{G}_n(u_2)\|_{C_0^1} \le \overline{L}_n'\|u_1 - u_2\|_{C_0^1}.$$

Note

$$
\begin{aligned}
&\left|T_n(u_1) - T_n(u_2)\right| \\
&\quad = \frac{1}{r^2}\left|\left(K\mathbb{G}_n(u_1), u_1\right) - (K\mathbb{G}_n)(u_2), u_2)\right| \\
&\quad = \frac{1}{r^2}\left|\left(K\mathbb{G}_n(u_1), u_1\right) - \left(K\mathbb{G}_n(u_2), u_1\right) + \left(K\mathbb{G}_n(u_2), u_1\right) - (K\mathbb{G}_n)(u_2), u_2)\right| \\
&\quad = \frac{1}{r^2}\left|\left(K\mathbb{G}_n(u_1) - K\mathbb{G}_n(u_2), u_1\right) + \left(K\mathbb{G}_n(u_2), u_1 - u_2\right)\right| \\
&\quad \leq \frac{1}{r}\overline{L}_n'\|u_1 - u_2\|_{C_0^1} + \frac{1}{r^2}\left\|K\mathbb{G}(u_2)\right\|_{C_0^1}\|u_1 - u_2\|_{C_0^1},
\end{aligned}
$$

and

$$
\left\|T_n(u_1)u_1 - T_n(u_2)u_2\right\|_{C_0^1} \leq \left|T_n(u_1) - T_n(u_2)\right|\|u_1\|_{C_0^1} + \left|T_n(u_2)\right|\|u_1 - u_2\|_{C_0^1}.
$$

Since $\|K\mathbb{G}_n(u)\|_{C_0^1}$ is bounded in $\overline{S}_r$, so $T_n(u)$ is bounded also. Thus $F_n'(u)$ is globally Lipschitz continuous.

Step 3. We show that Problem (2.10) has a unique solution $\sigma_n(t, u_0)$ with maximal interval $[0, +\infty)$ for $u_0 \in S_r$ and $\sigma_n(t, u_0) \in S_r$ for all $t \in [0, +\infty)$.

The theory of Cauchy problems of ordinary differential equations together with step 1 implies that (2.10) has a unique solution $\sigma_n(t, u_0)$ with maximal interval $[0, \omega_+(u_0))$ for $u_0 \in S_r$. Note

$$
\begin{aligned}
&\sigma_n(t, u_0) = e^{-w(t)}\left\{u_0 + \int_0^t e^{w(s)}K\mathbb{G}\left(\sigma_n(s, u_0)\right)ds\right\} \\
&\quad \text{where } w(t) = -\int_0^t T_n\left(\sigma_n(s, u_0)\right)ds.
\end{aligned}
$$

Since $d\|\sigma_n(t, u_0)\|^2/dt \equiv 0$ for all $t \in [0, \omega_+(u_0))$, we have $\sigma_n(t, u_0) \in S_r$ for $t \in [0, \omega_+(u_0))$ if $u_0 \in S_r$.

Also, since $g_n(\sigma_n(t, u_0))$ is bounded in H_0^1 if $u_0 \in S_r$, then $\omega_+(u_0) = +\infty$ (see [32]).

Step 4. We show that Problem (2.10) has a unique solution $\sigma_n(t, u_0)$ with maximal interval $[0, +\infty)$ for $u_0 \in \overline{S}_r$ and $\sigma_n(t, u_0) \in \overline{S}_r$ for all $t \in [0, +\infty)$.

Since step 2 holds, the proof of step 4 is similar to that of step 3, so we omit it.

Step 5. For $u_0 \in S_r$, we show that there exists $u_n \in S_r$ such that

$$
\lim_{t\to+\infty} \sigma_n(t, u_0) = u_n \quad \text{in } H_0^1.
$$

First, since $F_n(u) < 0$ for $u \in S_r$, choose $b = F_n(u_0) < 0$. Since S_r is bounded and weakly convergent closed and F_n is weakly semi-continuous from below, we have $\inf_{u\in S_r} F_n(u) > -\infty$. Let $c < \inf_{u\in S_r} F_n(u)$. Then $u_0 \in F_n^{-1}([c, b])$. From Lemma 2.6, there exists $u_n \in S_r$ such that

$$
\lim_{t\to+\infty} \sigma_n(t, u_0) = u_n \quad \text{in } H_0^1.
$$

Step 6. For $u_0 \in \overline{S}_r$, there exists $u_n \in \overline{S}_r$ such that

$$\lim_{t\to+\infty} \sigma_n(t,u_0) = u_n \quad \text{in } C_0^1.$$

Using the proof of step 5, step 2 guarantees the conclusion is true. □

Let P be the positive cone in $C_0^1(\overline{\Omega})$ and $\mathring{P}$ be the interior set of P. The elements of $\mathring{P}$ are called positive and the elements of $-\mathring{P}$ are called negative.

Lemma 2.8 *Under condition* (1) *and* (2), *the flow in Lemma* 2.7 *has the following properties*:

$$\sigma_n(t,u_0) \in \pm\mathring{P} \quad \textit{for } u_0 \in \pm\mathring{P}\cap\overline{S}_r \textit{ and } t\in[0,+\infty).$$

Proof The proof follows the ideas in Lemma 1 and 6 in [26].

(1) We show that $K\mathbb{G}_n(u_0) \in \mathring{P}$ for $u_0 \in P - \{\theta\}$.

Let $v = K\mathbb{G}_n(u_0)$, and we have

$$-\triangle v = g_n(x,u_0) \geq\not\equiv 0, \quad \forall x\in\Omega, v|_{\partial\Omega} = 0.$$

The strong maximum principle implies that $v \in \mathring{P}$.

(2) We show that

$$K\mathbb{G}_n\big(\sigma_n(t,u_0)\big) \in \mathring{P} \quad \text{for } u_0 \in P\cap\overline{S}_r, \text{ and } t>0. \tag{2.11}$$

Now $\forall u \in P$, choose $\delta > 0$ small such that, for all $\delta > h > 0$, we have

$$u + h(\big(T_n(u)u + K\mathbb{G}_n(u)\big) = \big(1 + hT_n(u)\big)u + hK\mathbb{G}_n(u) \in P,$$

i.e., (2.9) is satisfied. Now Lemma 2.5 guarantees that the solution $\sigma_n(t,u_0)$ of the initial value problem (2.10) satisfies $\sigma_n(t,u_0) \in P$ for all $t \in [0,+\infty)$ (in fact $\sigma_n(t,u_0) \in P\cap\overline{S_r}$ since $u_0 \in P\cap\overline{S_r}$). Hence (as in (1)) (2.11) holds.

(3) We show that

$$\sigma_n(t,u_0) \in \mathring{P} \quad \text{for } u_0 \in \mathring{P}\cap\overline{S}_r \text{ and } t\in[0,+\infty).$$

Let $w(t) = -\int_0^t T_n(\sigma_n(s,u_0))\,ds$. We have $w'(t) > 0$, $w(t) > 0$, and $w(t)$ is strictly increasing. Let $w^{-1}(t)$ be the inverse function of $w(t)$. It follows from (2.11), for $u_0 \in P\cap\overline{S}_r$, that

$$\big(1/w'(t)\big)K\mathbb{G}_n\big(\sigma_n(t,u_0)\big) \in \mathring{P}. \tag{2.12}$$

Let $A(t) = (1/w'(t))K\mathbb{G}_n(\sigma(t,u_0))$ and $E_t = \{A(s) : 0 \leq s \leq t\}$. Note that E_t is a compact set in $C_0^1(\overline{\Omega})$ and (2.12) implies that $E_t \subseteq \mathring{P}$ and hence $\overline{co}E_t \subseteq \mathring{P}$, where $\overline{co}E_t$ is the closed convex set hull of E_t in $C_0^1(\overline{\Omega})$. Note

$$\begin{aligned}\frac{1}{e^{w(t)}-1}\int_0^t e^{w(s)}K\mathbb{G}_n\big(\sigma_n(s,u_0)\big)\,ds &= \frac{1}{e^{w(t)}-1}\int_1^{e^{w(t)}} \frac{K\mathbb{G}_n(\sigma_n(w^{-1}(\ln(s)),u_0))}{w'(w^{-1}(\ln(s)))}\,ds \\ &= \lim_{m\to+\infty}\frac{1}{m}\sum_{i=1}^m A\left(w^{-1}\left(\ln\left(1+\frac{i}{m}\big(e^{w(t)}-1\big)\right)\right)\right).\end{aligned}$$

Therefore

$$\frac{1}{e^{w(t)}-1}\int_0^t e^{w(t)}K\mathbb{G}\big(\sigma_n(s,u_0)\big)\,ds \in \overline{co}F_t \in \mathring{P},$$

and this together with

$$\sigma_n(t,u_0) = e^{-w(t)}\left\{u_0 + \int_0^t e^{w(s)}K\mathbb{G}\big(\sigma_n(s,u_0)\big)\,ds\right\}, \quad t \in [0,+\infty)$$

and

$$e^{-w(t)} = \big(1-e^{-w(t)}\big)\frac{1}{e^{w(t)}-1}$$

yields

$$\sigma_n(t,u_0) \in \mathring{P} \quad \text{for } u_0 \in \mathring{P} \cap \overline{S}_r \text{ and } t \in [0,+\infty).$$

For the case $u_0 \in (-\mathring{P})$, the proof is similar, so we omit it.

The proof is completed. □

Lemma 2.9 *Under conditions* (1), (2), *and* (3), *Problem* (2.5) *has at least one positive solution $u_{1,n} \in \overline{S}_r \cap P$, one negative solution $u_{2,n} \in \overline{S} \cap (-P)$, and one sign-changing solution $u_{3,n} \in \overline{S}_r \cap (C_0^1 - (-P \cup P))$.*

Proof Let e_1 be an eigenfunction corresponding to the first eigenvalue of the Dirichlet eigenvalue problem: $-\Delta u = \lambda u$ in Ω, $u|_{\partial\Omega} = 0$, e_2 be an eigenfunction corresponding to the second one with $\|e_1\| = \|e_2\| = r$. Let $\Lambda = \operatorname{span}\{e_1,e_2\} \cap S_r$. Note that $\Psi_n(u) < 0$ for each $n > 0$ if $u \not\equiv 0$ and

$$\Lambda = \{\cos\theta e_1 + \sin\theta e_2 : 0 \le \theta \le 2\pi\}$$

is a compact set in S_r. Then there exists $\alpha_n > 0$ such that

$$\max\big\{\Psi_n(u) : u \in \Lambda\big\} = \max\big\{\Psi(u) : u \in \Lambda\big\} < -\alpha_n. \tag{2.13}$$

Set

$$\Lambda^{\pm} = \big\{u \in \Lambda : \sigma_n(t,u) \in \pm\mathring{P} \text{ for some } t > 0\big\}.$$

(1) We show that $\Lambda^{\pm} \neq \emptyset$.

Since $e_1 \in \overline{S}_r \cap \mathring{P}$, $-e_1 \in \overline{S}_r \cap (-\mathring{P})$, Lemma 2.8 guarantees that $\sigma_n(t,e_1) \in \mathring{P}$ and $\sigma_n(t,-e_1) \in (-\mathring{P})$ for $t \in [0,+\infty)$. Therefore, $\Lambda^{\pm} \neq \emptyset$.

(2) We show that Problem (2.5) has at least one positive solution $u_{1,n}$ and one negative solution $u_{2,n}$.

Consider $\sigma_n(t,e_1)$, $t \in [0,+\infty)$. Lemma 2.7 guarantees that there exists $u_{1,n} \in \overline{S}_r \cap P$ such that

$$\lim_{t\to+\infty} \sigma_n(t,e_1) \overset{C_0^1}{=} u_{1,n},$$

and $u_{1,n}$ is a critical point of F_n in $\overline{S}_r \cap P$. Then $u_{1,n}$ is a solution of Problem (2.5) and $u_{1,n} \in \overline{S}_r$. By using the strong maximum principle, we have $u_{1,n} \in \mathring{P}$.

For $\sigma_n(t,-e_1)$, $t \in [0,+\infty)$, a similar argument to that of $\sigma_n(t,e_1)$ shows that there exists $u_{2,n} \in \overline{S}_r \cap (-\mathring{P})$ such that $u_{2,n}$ is a solution of Problem (2.5).

(3) We show that Problem (2.5) has at least one sign-changing solution $u_{3,n} \in \overline{S}_r \cap (C_0^1 - (P \cup (-P)))$.

From the proof of step 2, $e_1 \in \Lambda^+$, $-e_1 \in \Lambda^-$. Note that both Λ^+ and λ^- are open sets of Λ since $\sigma_n(t,u)$ depends continuously on u (see [32]). From Lemma 2.8, we have $\Lambda^+ \cap \Lambda^- = \emptyset$, and the connectedness of Λ implies that there is $u_0 \in \Lambda - (\Lambda^+ \cup \Lambda^-)$. By Lemma 2.7, $\sigma_n(t,u_0) \to u_{3,n}$, a critical point of F_n, in $H_0^1(\Omega)$ as $t \to +\infty$. Then $u_{3,n}$ is a solution of Problem (2.5) and $u_{3,n} \in \overline{S}_r$. From (iii) of Lemma 2.7, we have that $\lim_{t\to+\infty} \sigma_n(t,u_0) = u_{3,n}$ in the $C_0^1(\overline{\Omega})$-topology. Therefore $u_{3,n} \notin \mathring{P} \cap (-\mathring{P})$ since $\sigma_n(t,u_0) \notin \mathring{P} \cap (-\mathring{P})$. Then $u_{3,n} \notin P \cap (-P)$ by using the strong maximum principle. Hence $u_{3,n}$ changes its sign in Ω. Also $\Psi_n(u_{3,n}) < \Psi_n(\sigma_n(t,u_0)) < -\alpha_n$, $\forall t \in [0,+\infty)$ since σ_n is a negative descent flow. □

Proof (Theorem 2.1) We only prove the existence of sign-changing solutions of Problem (1.1) since the proofs of the existence of positive solutions and negative solutions are similar, so we omit them.

From Lemma 2.9, for given $r > 0$, Problem (2.5) has at least one sign-changing solution $u_{3,n}$ with $u_{3,n} \in \overline{S}_r \cap (C_0^1 - (P \cup (-P)))$, where $\lambda_{3,n}^{-1} = \int_\Omega g_n(x,u_{3,n})u_n\,dx/r^2 > 0$ and g_n is defined in (2.3) for each $n \in \mathbb{N}$.

(1) We first prove that $\{\lambda_{3,n}\}$ is bounded.

Since $\{u_{3,n}\}$ is bounded in the $H_0^1(\Omega)$ topology, we may assume that it converges weakly to u^* in $H_0^1(\Omega)$. Then $u_{3,n} \to u^*$ in $L^{p+1}(\Omega)$ since $1 \le p < \frac{N+2}{N-2}$. There exists a number $c > 0$ such that, for all $t \in \mathbb{R}$ and for all $n = 1,2,\dots$,

$$\bigl|g_n(x,t)\bigr| \le c\bigl(1 + |t|^p\bigr) \quad \text{and} \quad \bigl|G_n(x,t)\bigr| \le c\bigl(1 + |t|^{p+1}\bigr), \tag{2.14}$$

where G_n is defined in (2.4).

From Lemma A.1 in [41], there exists a subsequence of $\{u_{3,n}\}$, denoted also by $\{u_{3,n}\}$, and there exists $h \in L^{p+1}(\Omega)$ such that $u_n \to u^*$ a.e. in Ω, $|u^*(x)| \le h(x)$, $|u_{3,n}(x)| \le h(x)$ a.e. in Ω. From the Lebesgue dominated convergence theorem, we have

$$\int_\Omega G_n(x,u_{3,n})\,dx \to \int_\Omega G\bigl(x,u^*\bigr)\,dx \quad \text{as } n \to +\infty. \tag{2.15}$$

Let $n > k_0 = \max\{\|u\|_{C(\overline{\Omega})} : u \in \Lambda\}$. By the definitions of Ψ_n and Ψ, if $n > k_0$, we have $\Psi_n(u) = \Psi(u)$ for $u \in \Lambda$.

From (2.2), (2.13), there exists a positive constant $\alpha > 0$ such that

$$-\Psi_n(u) = \int_\Omega G_n(x,u)\,dx = -\Psi(u) = \int_\Omega G(x,u)\,dx > \alpha > 0 \quad \text{for } u \in \Lambda, n > k_0.$$

Since $u_{3,n}$ is obtained along a descending flow, it follows that

$$\int_\Omega G_n(x,u_{3,n})\,dx \ge \int_\Omega G_n\bigl(x,\sigma(0,u_{0,n})\bigr)\,dx = \int_\Omega G_n(x,u_{0,n})\,dx > \alpha > 0, \quad \forall n > k_0,$$

for some $u_{0,n} \in \Lambda$, where $\sigma(t, u_{0,n})$ is a solution of Problem (2.5). From (2.15), we have $\int_\Omega G(x, u^*)\,dx > 0$. Hence $u^* \not\equiv 0$. Similar to (2.15), we have

$$\int_\Omega g_n(x, u_{3,n})u_{3,n}\,dx \to \int_\Omega g(x, u^*)u^*\,dx \stackrel{\triangle}{=} 2\beta > 0.$$

Thus $0 < \lambda_{3,n} = \frac{r^2}{\int_\Omega g_n(x,u_{3,n})u_n\,dx/r^2} < r^2/\beta$ for n large enough.

(2) We prove that $\{u_{3,n}\}$ is bounded in the C_0^1-topology.

Choose a sequence of numbers $\{q_i\}$ satisfying

$$q_1 = \frac{2N}{N-2} < q_2 < \cdots < q_{m-1} < q_m = 2Np$$

and

$$\frac{1}{q_{i+1}} \geq \frac{p}{q_i} - \frac{2}{N}, \quad i = 1, 2, \ldots, m-1.$$

Let $p_i = q_i/p$ $(i = 1, 2, \ldots, m)$. From Lemma 2.4, we have

$$H_0^1 \hookrightarrow L^{q_1}(\Omega). \tag{2.16}$$

From (2.14), one has (note $p_1\,p = q_1$)

$$\left\|\lambda_{3,n}\mathbb{G}_n(u_{3,n})\right\|_{L^{p_1}} \leq \frac{1}{\beta}\left(\int_\Omega \left|g_n(x, u_{3,n})\right|^{p_1}\,dx\right)^{\frac{1}{p_1}} \leq \frac{2c}{\beta}\left(|\Omega|^{\frac{1}{p_1}} + \|u_{3,n}\|_{L^{q_1}}^p\right) \tag{2.17}$$

for large n. Since K is a bounded linear operator, one has together with Lemma 2.3

$$L^{p_1}(\Omega) \xrightarrow{K} W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega) \hookrightarrow L^{q_2}(\Omega). \tag{2.18}$$

Combining (2.16), (2.17), and (2.18), we have

$$\{u_{3,n}\}_{n\geq k} \subseteq L^{q_2}(\Omega) \text{ is bounded}, \quad \text{for large } k. \tag{2.19}$$

Repeating the progress of (2.17), (2.18), and (2.19) for $i = 2, 3, \ldots, m$, we have

$$\{u_{3,n}\}_{n\geq k} \subseteq L^{q_m}(\Omega) \text{ is bounded}, \quad \text{for large } k.$$

We have (note $p_m\,p = q_m$)

$$\left\|\lambda_{3,n}\mathbb{G}_n(u_{3,n})\right\|_{L^{p_m}} \leq \frac{1}{\beta}\left(\int_\Omega \left|g_n(x, u_{3,n})\right|^{p_m}\,dx\right)^{\frac{1}{p_m}} \leq \frac{2c}{\beta}\left(|\Omega|^{\frac{1}{p_m}} + \|u_{3,n}\|_{L^{q_m}}^p\right)$$

for large n, which together with boundedness of the linear operator K guarantees that

$$\{u_{3,n}\}_{n\geq k} \subseteq W^{2,p_m} \cap W_0^{1,p_m} \text{ is bounded}, \quad \text{for large } k.$$

Now Lemma 2.2 implies (note $p_m = 2N > N$) that

$$\{u_{3,n}\}_{n\geq k} \subseteq W^{2,p_m} \cap W_0^{1,p_m} \hookrightarrow C_0^1(\overline{\Omega}) \text{ is bounded,} \quad \text{for large } k. \tag{2.20}$$

(3) We consider sign-changing solutions of Problem (1.1) in S_r.

From (2.20), set $L > 0$ such that

$$\|u_{3,n}\|_{C_0^1(\overline{\Omega})} \leq L, \quad n \in \{1, 2, \dots\}. \tag{2.21}$$

Choose $n_0 > L$. From the definitions of f_{n_0} and g_{n_0} in (2.2) and (2.3), we have together with (2.21) that

$$f_{n_0}\big(x, u_{3,n_0}(x)\big) = f\big(x, u_{3,n_0}(x)\big), \quad x \in \overline{\Omega}$$

and

$$g_{n_0}\big(x, u_{3,n_0}(x)\big) = \frac{1}{M(x, r^2)} f\big(x, u_{3,n_0}(x)\big), \quad x \in \overline{\Omega},$$

which implies that u_{3,n_0} is a sign-changing solution of Problem (2.1) with

$$\lambda = r^2 \Big/ \int_\Omega g\big(x, u_{3,n_0}(x)\big) u_{3,n_0}(x)\, dx.$$

For $r > 0$ given above, write $u_{3,r}(x) = u_{3,n_0}(x)$ for $x \in \overline{\Omega}$. Lemma 2.1 guarantees that $u_{3,r}$ is a sign-changing solution of Problem (1.1) with

$$\lambda = r^2 \Big/ \int_\Omega g\big(x, u_{3,r}(x)\big) u_{3,r}(x)\, dx.$$

Similarly, we obtain a set $\{u_{1,r}\}_{r\in A}$ of positive solutions of Problem (1.1) and a set $\{u_{2,r}\}_{r\in A}$ of negative solutions of Problem (1.1).

The proof is completed. □

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