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Three new approaches for solving a class of strongly nonlinear two-point boundary value problems

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Abstract

Three new and applicable approaches based on guasi-linearization technique, wavelet-homotopy analysis method, spectral methods, and converting two-point boundary value problem to Fredholm–Urysohn integral equation are proposed for solving a special case of strongly nonlinear two-point boundary value problems, namely Troesch problem. A quasi-linearization technique is utilized to reduce the nonlinear boundary value problem to a sequence of linear equations in the first method. Second method is devoted to applying generalized Coiflet scaling functions based on the homotopy analysis method for approximating the numerical solution of Troesch equation. In the third method we use an interesting technique to convert the boundary value problem to Urysohn–Fredholm integral equation of the second kind; afterwards generalized Coiflet scaling functions and Simpson guadrature are employed for solving the obtained integral equation. Introduced methods are new and computationally attractive, and applications are demonstrated through illustrative examples. Comparing the results of the presented methods with the results of some other existing methods for solving this kind of equations implies the high accuracy and efficiency of the suggested schemes.

Keywords: Troesch problem; Quasi-linearization technique; Wavelet homotopy analysis method; Generalized Coiflet scaling functions; Galerkin method; Urysohn integral equation

1 Introduction

Troesch problem arises from a system of nonlinear ordinary differential equations which occur in the theory of gas porous electrodes [1] and investigation of the confinement of a plasma column by radiation pressure [2]. This problem is a special type of nonlinear two-point boundary value problems, defined as follows:

$$\nu''(x) - \lambda \sinh(\lambda \nu(x)) = 0, \quad 0 \le x \le 1, \tag{1}$$

subject to the following conditions:

$$\nu(0) = 0, \qquad \nu(1) = 1,$$
 (2)

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where the Troesch parameter λ is a positive constant. The existence of the solution of (1)–(2) was proved in [3] for $\lambda < 5$. The closed form solution to this problem in terms of the Jacobian elliptic function was given [4] as

$$\nu(x) = \frac{2}{\lambda} \sinh^{-1}\left(\frac{\nu'(0)}{2} sc \left(\lambda x \mid 1 - \frac{1}{4}\nu'(0)^2\right)\right),\,$$

where $\nu'(0) = 2\sqrt{1-\mu}$, and μ satisfies the transcendental equation

$$sc(\lambda \mid \mu) = \frac{\sinh(\frac{\lambda}{2})}{\sqrt{1-\mu}},$$

where the Jacobian elliptic function $sc(\lambda \mid \mu) = \tan \alpha$, and α and λ are related through the following integral:

$$\lambda = \int_0^\alpha \frac{1}{\cosh \theta - \mu} \, d\theta.$$

It is clear that v(x) has a singularity located at a pole of $sc(\lambda x | \mu)$ or approximately at [5, 6]

$$x_s = \frac{1}{\lambda} \ln \left(\frac{8}{\nu'(0)} \right),$$

which implies that, if $\nu'(0) > 8e^{-\lambda}$, then the singularity lies within the integration range. Because of difficulty in direct solving of the nonlinear boundary value problems, many researchers have paid considerable attention to numerical solving of these problems. For example, the numerical solution of one-dimensional Bratu's problem is solved by an operational matrix of the derivative of Chebyshev wavelets in [7]. Authors of [8] used a manipulation in the cubic splines to develop a collocation method and the generalized Newton method for solving the nonlinear Troesch problem. Sinc–Galerkin approach is applied to numerical solving of second-order nonlinear Dirichlet-type boundary value problems in [9]. In [10] and [11] some approaches based on B-splines are presented for solving the Troesch problem. The homotopy perturbation technique is presented in [4], and the reproducing kernel Hilbert space method is introduced in [12]. Sinc–Galerkin method based upon double exponential transformation is proposed to solve Troesch's problem in [13].

Wavelet theory is a relatively new and emerging area in mathematical research. In recent decades, wavelets have been extensively used in applied mathematics and a wide range of engineering disciplines, because they permit accurate representation of a variety of functions and operators. Wavelets were applied in numerically solving the integro-differential and differential equations [14, 15]. Also the hybrid methods, combination of some analytical and numerical methods, were successfully employed to achieve the solutions of initial and boundary value problems; for instance, see [16, 17, 26, 27]. In this paper we apply three different methods based on a quasi-linearization technique, the homotopy analysis method based Coiflet orthogonal scaling functions, and an efficient approach for transforming the aforementioned equation to an integral equation.

The outline of the paper is as follows: Sect. 2 is devoted to the brief definition of Coiflet scaling functions and function approximation by them. Suggested computational methods for solving Troesch equation are discussed in Sect. 3. Indeed, in this section three different



methods are introduced based on quasi-linearization, operational matrices of derivative and collocation and Galerkin methods and integral equation transform technique. Convergence analysis is discussed in Sect. 4. To show the accuracy and performance of the presented methods, some test problems are given in Sect. 5. The conclusion of the paper is described in Sect. 6.

2 Generalized Coiflet orthogonal wavelets

The basic concepts and preliminaries of the wavelet theory and multi-resolution analysis are given in literature [18, 19, 28–30]. Scaling function ϕ and wavelets of generalized Coiflet-type orthogonal wavelet possess the following properties [20, 21]:

(i)
$$\varphi(x) = \sum_{l \in \mathbb{Z}} p_k \varphi(2x - l),$$

(ii) $\psi(x) = \sum_{l \in \mathbb{Z}} (-1)^k \rho_{1-k} \psi(2x - l),$
(iii) $M_n = \int_{-\infty}^{\infty} x^n \varphi(x) \, dx = M_1^n, \quad 0 \le n < N,$
(iv) $\int_{-\infty}^{\infty} x^n \psi(x) \, dx = 0, \quad 0 \le n < N,$
(v) $\sum_{l \in \mathbb{Z}} \rho_k \varphi(2x - l) = 1,$
(i) (i) $\sum_{l \in \mathbb{Z}} \rho_k \varphi(2x - l) = 1,$

 p_k are the low-pass filter coefficients [22], N is the number of vanishing moment, and M_n is the nth-order moment of the scaling function. In this paper, we apply the generalized Coiflet type with N = 6 and $M_1 = 7$. The general difficulty for working with this family of wavelets is having no analytical expression for them. So the values of these functions are calculated [23]. Figure 1 shows the plot of Coiflet scaling function, and Table 1 represents some values of Coiflet scaling function in k = 1, ..., 16.

k	$\varphi(k)$
1	-3.66444437281758 × 10 ⁻⁸
2	1.53911669852067 × 10 ⁻⁵
3	-1.20347428401018 × 10 ⁻⁴
4	-6.01049457744450 × 10 ⁻³
5	3.83279869108597 × 10 ⁻²
6	-9.93321926649377 × 10-2
7	1.13897129589829 × 10 ⁰
8	-1.13651812655013 × 10 ⁻¹
9	5.48926083610606 × 10 ⁻²
10	-1.54279915381372 × 10 ⁻²
11	2.61977141546123 × 10 ⁻³
12	$-2.65205890511625 \times 10^{-4}$
13	-1.79596295480327 × 10 ⁻⁵
14	-1.01758785797000 × 10 ⁻⁶
15	4.86512575426836 × 10 ⁻⁹
16	$-1.48885946646455 \times 10^{-12}$

Table 1 Some values of generalized Coiflet scaling function for N = 6 and $M_1 = 7$

By the definition of Coiflet scaling function, we can construct a multi-resolution analysis for $L^2(R)$ with nested subspaces V_j as follows:

 $0 \subset \cdots \subset V_i \subset V_{i+1} \subset \cdots \subset L^2(R),$

where $V_j = \text{span}\{\varphi_{j,k} := \varphi(2^j x - k); j, k \in Z\}$. In this paper we introduce a classification for generalized Coiflet scaling functions on [0, 1], which have not been addressed previously. For this purpose we first define the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{11}{6} & -3 & \frac{3}{2} & -\frac{1}{3} \\ 2 & -5 & 4 & -1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$
(4)

and

$$\theta_{0,k} = \sum_{i=0}^{3} \frac{A_{i,k}}{i!} (2^{i}x)^{i}, \qquad \theta_{1,k} = \sum_{i=0}^{3} \frac{B_{i,k}}{i!} (2^{i}(x-1))^{i}.$$

Now we define left and right and interior scaling functions.

Boundary scaling adaptation

· Left boundary Coiflet scaling functions

We define the boundary near functions at the left boundary for k = 0, ..., 3 by

$$\varphi_{j,k}(x) = \sum_{i=2-3N+M_1}^{-1} \theta_{0,k}\left(\frac{i}{2^j}\right) \varphi(2^i x - i + 7) + \varphi(2^j x - k + 7).$$
(5)

• Right boundary Coiflet scaling functions

At the right end of the interval, for $k = 2^{j} - 3, ..., 2^{j}$, we have

$$\varphi_{j,k}(x) = \sum_{i=2^{j}+1}^{2^{j}-1+M_{1}} \theta_{1,2^{j}-k}\left(\frac{i}{2^{j}}\right) \varphi\left(2^{j}x-i+7\right) + \varphi\left(2^{j}x-k+7\right).$$
(6)



Interior scalings

For interior Coiflet scaling functions, we have

$$\varphi_{j,k}(x) = \varphi(2^{j}x - k + 7), \quad k = 4, \dots 2^{j} - 4.$$
 (7)

Figure 2 shows the family of Coiflet scaling functions for j = 5.

2.1 Function approximation

A function $f(x) \in L^2[0, 1]$ can be approximated in the subspace V_i as [24]

$$f(x) \simeq P^{j} f(x) = \sum_{k=0}^{2^{j}} f_{k} \varphi_{j,k}(x) = \vec{F} \Phi_{j}(x),$$
(8)

where \vec{F} and Φ_i are row and column vectors defined by

$$\vec{F} = \left[f\left(\frac{k}{2^j}\right) \right]_{k=0}^{2^j}, \qquad \Phi_j = \left[\varphi_{j,k}(x) \right]_{k=0}^{2^j}.$$

It should be noted that by using the orthogonality property of Coiflet, the approximation coefficients could be obtained as follows:

$$f_k = \frac{1}{\xi_{j,k}} \langle f(x), \varphi_{j,k}(x) \rangle,$$

where $\xi_{j,k} = \langle \varphi_{j,k}(x), \varphi_{j,k}(x) \rangle$. For example, the function $\sin(x)$ is approximated by Coiflet in V_4 and V_5 , and the plots of absolute error of approximation are shown in Fig. 3.



Also the *n*-th derivative of function f(x) can be approximated by the *n*-th derivatives of Coiflet scaling functions as follows:

$$f^{(n)}(x) \simeq \frac{d^n [P^j f(x)]}{dx^n} = \sum_{k=0}^{2^j} f\left(\frac{k}{2^j}\right) \varphi_{j,k}^{(n)}(x).$$
(9)

Theorem 2.1 If $f(x) \in (L^2[0,1] \cap C^N[0,1])$ is approximated by generalized Coiflet scaling functions in subspace V_i , then

$$\left\| f(x) - P^{j} f(x) \right\|_{L^{2}} \le C 2^{-jN},\tag{10}$$

where *C* is a positive constant that depends only on the function f(x) and low-pass filter coefficients ρ_k .

Proof By using the Taylor expansion of *f* at *x*, we get

$$f(y) = \sum_{n=0}^{N-1} \frac{f^{(n)}(x)}{n!} (y-x)^n + \frac{f^{(N)}(\xi)}{N!} (y-x)^N.$$
(11)

Putting $y = \frac{k}{2^j}$ in equation (11) and substituting it in equation (8), we have

$$\begin{split} P_{j}f(x) &= \sum_{k=0}^{2^{j}} \left[\sum_{n=0}^{N-1} \frac{f^{(n)}(x)}{n!} \left(\frac{k}{2^{j}} - x\right)^{n} + \frac{f^{(N)}(\xi_{k})}{N!} \left(\frac{k}{2^{j}} - x\right)^{N} \right] \varphi(2^{j}x - k) \\ &= \sum_{k=0}^{2^{j}} \left[\sum_{n=1}^{N-1} \frac{f^{(n)}(x)}{n!} \left(\frac{k}{2^{j}} - x\right)^{n} + \frac{f^{(N)}(\xi_{k})}{N!} \left(\frac{k}{2^{j}} - x\right)^{N} + f(x) \right] \varphi(2^{j}x - k), \end{split}$$

we know that φ has *N* vanishing moments, that is,

$$\sum_{k=0}^{2^{j}} (k-2^{j}x)^{n} \varphi(2^{j}x-k) = 0, \quad n = 1, 2, \dots, N-1,$$

and $\sum_{k=0}^{2^{j}} \varphi(2^{j}x - k) = 1$, so we get

$$P_{j}f(x) = f(x) + \sum_{k=0}^{2^{j}} \frac{f^{(N)}(\xi_{k})}{2^{N_{j}}N!} (k - 2^{j}x)^{N} \varphi(2^{j}x - k).$$
(12)

Suppose Support f = [a, b], then $2^{j}b \le k \le 2^{j}a$ and define

$$\Lambda_k(x) = \frac{f^{(N)}(\xi_k)}{N!} (k - 2^j x)^N \varphi(2^j x - k).$$

We know Support $\varphi = [0, 3N - 1]$, thus Support $\Lambda_k = [k, k + 3N - 1]$, that is, Λ_k has compact support. So there exists a positive constant *C* such that

$$\left\|\sum_{k=0}^{2^{j}}\Lambda_{k}(x)\right\|_{L^{2}} < C,$$

therefore $||f(x) - P_i f(x)||_{L^2} < C2^{-Nn}$.

Theorem 2.2 For $f(x) \in (L^2[0,1] \cap C^N[0,1])$, the accuracy of derivative approximation *can be estimated by*

$$\left\| f^{(n)}(x) - \frac{d^n P^j f(x)}{dx^n} \right\|_{L^2} \le C 2^{-j(N-n)}$$

Proof It is sufficient to put $f^{(n)}(x) = g(x)$ and repeat the above proof.

3 Methods

In this section we introduce three computational methods for numerically solving a strongly nonlinear two-point boundary value problem, namely Troesch problem. In the first method, some quasi-linearization technique is applied to reduce nonlinear boundary value problem (1)-(2) to a sequence of linear equations. Generalized Coiflet scaling functions via HAM and Galerkin method are used to reduce the nonlinear equation to some algebraic system in the second approach. Utilizing an interesting technique, the third approach is divided to transform the given nonlinear boundary value problem to some Urysohn–Fredholm integral equation of the second kind, and the obtained integral equation is solved applying Coiflet scaling functions and the Simpson quadrature rule.

3.1 Approach I: quasi-linearization technique

Consider boundary value problem (1)–(2), in this method we use a quasi-linearization technique based on Taylor expansion for reducing the main nonlinear problem to a sequence of linear algebraic equations. For this purpose, we choose a reasonable initial approximation for the function u(x), call it u_0 , and we develop $\omega(x, v) = \sinh(\lambda v(x))$ by the Taylor expansion around v_0 , so we get

$$\omega(x,\nu_1) = \sum_{k=0}^{\infty} \left(\frac{\partial^k \omega(x,\nu)}{\partial \nu^k} \right)_{\nu_0} \frac{(\nu_1 - \nu_0)^k}{k!},$$

or in a general form we can rewrite the current equation for n = 0, 1, 2, ... (n = iteration index)

$$\omega(x,\nu_{n+1}) = \sum_{k=0}^{\infty} \left(\frac{\partial^k \omega(x,\nu)}{\partial \nu^k}\right)_{\nu_n} \frac{(\nu_{n+1} - \nu_n)^k}{k!}.$$
(13)

By truncating the infinite series in equation (13) for arbitrary M, we get

$$\omega(x, \nu_{n+1}) = \sum_{k=0}^{M} \left(\frac{\partial^{k} \omega(x, \nu)}{\partial \nu^{k}}\right)_{\nu_{n}} \frac{(\nu_{n+1} - \nu_{n})^{k}}{k!} + \frac{\partial^{M+1} \omega(x, \eta)}{\partial \nu^{M+1}} (\nu_{n+1} - \nu_{n})^{M+1},$$
(14)

where $\eta \in [\alpha, \beta]$, $\alpha = \min_{0 \le x \le 1} \{\nu_n(x), \nu_{n+1}(x)\}$, and $\beta = \max_{0 \le x \le 1} \{\nu_n(x), \nu_{n+1}(x)\}$. Substituting equation (14) in equation (1), we have

$$\nu_{n+1}'' + \Gamma_1(\lambda\nu_n)\nu_{n+1} + \Gamma_2(\lambda\nu_n)\nu_{n+1}^2 + \cdots = \Delta_\infty(\nu_n).$$

By truncating the obtained equation for arbitrary M, we get

$$\nu_{n+1}'' + \Gamma_1(\lambda \nu_n)\nu_{n+1} + \Gamma_2(\lambda \nu_n)\nu_{n+1}^2 + \dots + \Gamma_{M+1}(\nu_n)\nu_{n+1}^{M+1} = \Delta_M(\nu_n),$$
(15)

where

$$\begin{split} \Gamma_1(\lambda\nu_n) &= \sum_{j=1}^M \frac{(-1)^j \sinh^{(j)}(\lambda\nu_n)}{(j-1)!} \lambda^j \nu_n^{j-1} + \frac{(-1)^{M+1} \sinh^{(M+1)}(\lambda\xi_1)}{M!} \lambda^{M+1} \xi_1^M, \\ \Gamma_2(\lambda\nu_n) &= \sum_{j=2}^M \frac{(-1)^{j-1} \sinh^{(j-2)}(\lambda\nu_n)}{2 \times (j-2)!} \lambda^{j+2} \nu_n^{j-2} + \frac{(-1)^M \sinh^{(M-1)}(\lambda\xi_2)}{2 \times (M-1)!} \lambda^{M+3} \xi_2^{M-1}, \\ \vdots \end{split}$$

$$\begin{split} \Gamma_{M+1}(\lambda\nu_n) &= \frac{(-1)\sinh^{(M)}(\lambda\nu_n)}{M!}\lambda^M,\\ \Delta_M(\lambda\nu_n) &= \sum_{j=0}^M \frac{(-\lambda\nu_n)^j\sinh^{(j)}(\lambda\nu_n)}{j!} + \frac{(-\lambda\xi)^{M+1}\sinh^{(M+1)}(\lambda\xi)}{(M+1)!}, \end{split}$$

where $\xi_1, \xi_2, ...$ and ξ are determined similarly to η . According to current equations, nonlinear boundary value problem (1) converts to equations (15), a sequence of semilinear differential equations. It is interesting to point out that for suitable choices of ν_0 equations (15) may be reduced to linear differential equations. The obtained system of semilinear differential equations with boundary conditions (2) can be solved directly or numerically.

3.2 Approach II: wavelet homotopy analysis method

In this approach we use HAM based on generalized Coiflet scaling functions for solving equation (1). In order to simplify the calculation, we put $u(x) = e^{\lambda v(x)}$, then strongly non-linear problem (1)–(2) can be written as

$$2(uu'' - u'^2) - \lambda^2(u^2 - u) = 0,$$
(16)

$$u(0) = 1, \qquad u(1) = e^{\lambda}.$$
 (17)

The basic idea of HAM is constructing a continuous variation between the known initial guess u_0 and the unknown solution u. Let $q \in [0, 1]$ denote the embedding parameter of homotopy, c_0 the convergence-control parameter, L the auxiliary linear and N the non-linear operators of HAM, respectively. Then a continuous variation (with respect to the

embedding parameter q from the initial guess $u_0(x)$ to the solution u(x)) can be built by means of the so-called zeroth-order deformation equation as

$$(1-q)L_0[\alpha(x;q) - V_0(x)] = c_0 q N[\alpha(x;q)],$$
(18)

with $\alpha(0;q) = 1, \alpha(1;q) = e^{\lambda}$, and

$$\alpha(x;0) = u_0(x), \qquad \alpha(x;1) = u(x).$$
 (19)

Now we write Maclaurin series of $\alpha(x; q)$ with respect to q, so

$$\alpha(x;q) = u_0(x) + \sum_{j=1}^{\infty} u_j(x)q^j,$$

where $u_j(x) = D_j[\alpha(x;q)] := \frac{1}{j!} \frac{\partial^j \alpha(x;q)}{\partial q^j}|_{q=0}$, and D_j is the *j*th homotopy derivative operator. By this assumption that the convergence control parameter c_0 and the auxiliary linear operator *L* are chosen that guarantee the convergence of series (19) at q = 1, we get

$$U(x) = u_0(x) + \sum_{j=1}^{\infty} u_j(x).$$
 (20)

If we truncate the infinite series (20) in suitable m, we obtain the mth-order approximation as

$$U_m(x) = u_0(x) + \sum_{j=1}^m u_j(x).$$
(21)

Taking the operator D_m on both sides of the zeroth-order deformation equation (18) and its boundary conditions, we have

$$L[U_m(x) - \chi_m U_{m-1}(x)] = c_0 R_m(x), \qquad (22)$$

and $U_m(0) = 0$, $U_m(1) = 0$, where

$$\begin{split} R_m(x) &= D_m\big(N\big[\alpha(x;q)\big]\big) = \sum_{j=0}^{m-1} \big(U_j''(x)U_{m-1-j}(x) - U_j'(x)U_{m-1-j}'(x)\big) \\ &- \frac{\lambda^2}{2}\sum_{j=0}^{m-1} \big(U_j(x)U_{m-1-j}(x) - U_j(x)\big), \end{split}$$

and $\chi_0 = \chi_1 = 0$ and $\chi_j = 1, j > 1$. Therefore nonlinear problem (1)–(2) reduces to a system of recursive equations. For solving this system, we utilize the Galerkin method with generalized Coiflet scaling functions as test and weight functions. For this purpose, first U_m and R_m are written in generalized Coiflet scaling functions terms in scale *j* as equation (8)

$$U_m(x) = \sum_{l=0}^{2^j} U_m\left(\frac{l}{2^j}\right)\varphi_{j,l}(x), \qquad R_m(x) = \sum_{l=0}^{2^j} R_m\left(\frac{l}{2^j}\right)\varphi_{j,l}(x).$$



Substituting these approximations in equation (22), we get

$$\sum_{l=0}^{2^{j}} \left(U_{m} \left(\frac{l}{2^{j}} \right) - \chi_{m} U_{m-1} \left(\frac{l}{2^{j}} \right) \right) L(\varphi_{j,l}(x)) = c_{0} \sum_{l=0}^{2^{j}} R_{m} \left(\frac{l}{2^{j}} \right) \varphi_{j,l}(x).$$
(23)

Now we utilize the Galerkin method to solve the obtained system. By multiplying the both sides of equation (23) in $\varphi_{j,i}$, $i = 0, 1, ..., 2^{j}$, and integrating in the interval [0, 1], we get

$$\sum_{l=0}^{2^{j}} \left(U_{m} \left(\frac{l}{2^{j}} \right) - \chi_{m} U_{m-1} \left(\frac{l}{2^{j}} \right) \right) \Gamma_{i,l} = c_{0} \sum_{l=0}^{2^{j}} R_{m} \left(\frac{l}{2^{j}} \right) \Lambda_{i,l}, \tag{24}$$

where $\Gamma_{i,l} = \int_0^1 L(\varphi_{j,l}(x))\varphi_{j,i}(x) dx$, $\Lambda_{i,l} = \int_0^1 \varphi_{j,l}(x)\varphi_{j,i}(x) dx$. The matrix form of system (24) is as

$$\vec{\Gamma}(\vec{U}_m - \chi_m \vec{U}_{m-1}) = c_0 \vec{\Lambda} \vec{R}_m,\tag{25}$$

where

$$\vec{U}_k = \left[U_k\left(\frac{l}{2^j}\right)\right]_{l=0}^{2^j}, \qquad \vec{R}_k = \left[R_k\left(\frac{l}{2^j}\right)\right]_{l=0}^{2^j}, \qquad \vec{\Gamma} = \left[\Gamma_{i,l}\right]_{i,l=0}^{2^j}, \qquad \vec{\Lambda} = \left[\Lambda_{i,l}\right]_{i,l=0}^{2^j}.$$

From the boundary conditions it is clear that $U_k(0) = U_k(1) = 0$. It should be noticed that because of orthogonality and having compact support of generalized Coiflet scaling functions, the matrices $\vec{\Gamma}$ and $\vec{\Lambda}$ are so sparse. Figure 4 shows the gray scale plots of matrix $\vec{\Gamma}$ for j = 4, 5. By solving the system of equations (25) for m = 1, 2, 3, ... step by step and substituting obtained U_m in equation (21), we attain the approximate solution for u(x). Finally, in order to gain the solution of problem (1)–(2), we put $v(x) = \frac{1}{\lambda}Ln(u(x))$.

3.3 Approach III: converting to integral equation

In this method, using an interesting technique, nonlinear boundary value problem (1)-(2) is converted to a second kind nonlinear Fredholm integral equation of Urysohn type. For

this purpose, we integrate equation (1) two times, so we get

$$v(x) = xv'(0) + \int_0^x \int_0^x \sinh(\lambda v(t)) dt dt.$$

Reducing the double integral to a single integral, we have

$$\nu(x) = x\nu'(0) + \int_0^x (x-t)\sinh(\lambda\nu(t))\,dt.$$
 (26)

For evaluation $\nu'(0)$ in equation (26), we put x = 1 and get

$$\nu'(0) = 1 - \int_0^1 (1-t) \sinh(\lambda \nu(t)) dt.$$

By partitioning the integration interval to [0, x] and [x, 1] and substituting the result in equation (26), we have

$$\nu(x) = x + \int_0^1 \Delta(x, t, \nu(t)) dt,$$
(27)

where

$$\Delta(x, t, \nu(t)) = \sinh(\lambda\nu(t)) \begin{cases} t(x-1) & 0 \le t < x, \\ x(t-1) & x \le t < 1. \end{cases}$$

Equation (27) is a nonlinear second-order Fredholm integral equation of the Urysohn type. For solving this equation, we apply generalized Coiflet scaling functions. Using equation (8), we approximate the unknown function as $v(x) = \vec{v} \Phi_j(x)$ and substitute it in equation (27). Now we utilize the collocation method with mesh points $\tau_i = \frac{i}{2^j}$, $i = 0, 1, ..., 2^j$, so we have

$$\vec{\nu}\Phi_j(\tau_i) = \tau_i + \int_0^1 \Delta(\tau_i, t, \vec{\nu}\Phi_j(t)) dt.$$
(28)

The integral term in equation (28) can be found by applying some quadrature rule. We use the Simpson integration rule for evaluating the mentioned term. For this purpose, we consider the following partition for integration interval [0, 1]:

$$P=\{0=\gamma_0,\gamma_1,\ldots,\gamma_{2N}=1\},\qquad \gamma_k=\frac{k}{2N},$$

so we can write

$$\begin{split} &\int_0^1 \Delta\big(\tau_i, t, \vec{\nu} \Phi_j(t)\big) \, dt \\ &\simeq \frac{1}{6N} \Big[\Delta\big(\tau_i, \gamma_0, \vec{\nu} \Phi_j(\gamma_0)\big) + \Delta\big(\tau, \gamma_{2N}, \vec{\nu} \Phi_j(\gamma_{2N})\big) \Big] \\ &\quad + \frac{2}{3N} \sum_{k=1}^N \Delta\big(\tau, \gamma_{2k-1}, \vec{\nu} \Phi_j(\gamma_{2k-1})\big) + \frac{1}{3N} \sum_{k=1}^N \Delta\big(\tau, \gamma_{2k}, \vec{\nu} \Phi_j(\gamma_{2k})\big). \end{split}$$

Therefore nonlinear boundary value problem (1)-(2) converts to some linear or nonlinear algebraic system. In this paper the obtained system is solved using *Wolframe Mathematica Programming 10.*

4 Convergence analysis

In this section, by using Theorem 2.1, we present a theorem for error bound of approximate solution obtained by approach III.

Theorem 4.1 Suppose $v \in (L^2[0,1] \cap C^N[0,1])$. If v and v_* are the exact and approximated solutions of equation (27) obtained by approach III, respectively, then

$$\|v(x) - v_*(x)\|_{L^2} \le \alpha 2^{-jN},\tag{29}$$

where α is constant and depends on the function v, and j is the resolution scale of generalized Coiflet scaling functions.

Proof Substituting v_* in equation (27) and subtracting two relations, we get

$$\|v(x) - v_*(x)\|_{L^2} = \|(\Delta(x, t, v(t)) - \Delta(x, t, P^j v(t)))\|_{L^2}.$$

By considering the definition of Δ , we prove the theorem for two following cases.

• Suppose $t \in [0, x]$, then

$$\|v(x) - v_*(x)\|_{L^2} \le (x-1)^2 \int_0^x \|t^2(\sinh(\lambda v(t)) - \sinh(\lambda P^j v(t)(t)))\|_{L^2} dt.$$

By the mean value theorem we can write

$$\left\|\sinh\left(\lambda\nu(t)\right)-\sinh\left(\lambda P^{j}\nu(t)\right)\right\|_{L^{2}}\leq A\left\|\nu(t)-P^{j}\nu(t)(t)\right\|_{L^{2}},$$

where $A = \sup\{|\lambda \nu'(t) \cosh(\lambda \nu'(t))|; 0 \le t \le 1\}$, so we have

$$\|v(x) - v_*(x)\|_{L^2} \le A(x-1)^2 \int_0^x \|t^2\|_{L^2} \|v(t) - P^i v(t)\|_{L^2} dt.$$

Considering Theorem 2.1, we can write

$$\|v(x) - v_*(x)\|_{L^2} \le AC2^{-jN}(x-1)^2 \frac{x^3}{3}$$

It is clear that

$$\|\nu(x) - \nu_*(x)\|_{L^2} \le \frac{AC}{3} 2^{-jN}.$$
 (30)

• Suppose $t \in [x, 1]$, then

$$\|v(x) - v_*(x)\|_{L^2} \le (x-1)^2 \int_x^1 \|t^2 (\sinh(\lambda v(t)) - \sinh(\lambda P^j v(t)(t)))\|_{L^2} dt.$$

Similar to the proof of the pervious case, we can write

$$\|v(x) - v_*(x)\|_{L^2} \le A(x-1)^2 \int_x^1 \|t^2\|_{L^2} \|v(t) - P^j v(t)\|_{L^2} dt.$$

Again considering Theorem 2.1, we can write

$$\|v(x) - v_*(x)\|_{L^2} \le AC2^{-jN}(x-1)^2 \frac{(1-x^3)}{3}.$$

It is clear that

$$\|v(x) - v_*(x)\|_{L^2} \le \frac{AC}{3} 2^{-jN}.$$
 (31)

Therefore, by putting $\alpha = \frac{AC}{3}$, the proof is complete.

5 Test problem

In this section, to show the accuracy and efficiency of the described methods, we obtain the numerical solution of the Troesch problem for different values of λ , initial solution ν_0 , iteration index *m*, and Coiflet scale *j* and compare the results with the result of some other methods for solving this class of equations. All of the results are gained using the same desktop, Asus DESKTOP-M0F5LBS, Intel(R) Core(TM) i7-6700HQ CPU@ 2.60 GHz, 16GB memory.

5.1 Case 1. Approach I for $v_0 = 0$

Numerical solutions of equation (1) for different values of λ , initial solution $\nu_0 = 0$, and iteration index *n* are evaluated by approach I and are shown in Figs. 5–10. Results for $\lambda \leq 1$ are so close together that it is impossible to detect them by plot. Thus in Fig. 5 we show the error plot of numerical solutions for some arbitrary $\lambda \leq 1$. Suppose that $\nu_{\lambda_i}(x)$ is a relevant solution of equation (1) for the Troesch parameter λ_i , then

 $\varepsilon_{\lambda_i,\lambda_i}(x) = \nu_{\lambda_i}(x) - \nu_{\lambda_i}(x).$

Figure 6 shows the numerical solutions of equation (1) obtained by approach I for the Troesch parameter $1 \le \lambda \le 5$. For $5 \le \lambda \le 10$, numerical results are so close together in the interval [0, 0.5] that their diagnosis in the plot is very hard. So in order to have geometric understanding of relevant numerical solutions for the Troesch parameter $5 \le \lambda \le 10$, we show the approximated solution in two intervals [0, 0.5] and [0.5, 1] by Figs. 7 and 8. Also, for $10 \le \lambda \le 50$, numerical results are so close together that variation of solutions is visible only in the interval [0.85, 1] in Fig. 9. Therefore, for these values of the Troesch parameter, we show the error plot: $\varepsilon_{10,20}(x)$, $\varepsilon_{20,30}(x)$, $\varepsilon_{30,40}(x)$, and $\varepsilon_{40,50}(x)$ in Fig. 10.

5.2 Case 2. Approach I for $v_0 = 1$

Numerical solutions of equation (1) for different values of λ , initial solution $\nu_0 = 1$, and iteration index *n* is evaluated by approach I, and results are given in Figs. 11–13. As the pervious case results for $\lambda \leq 1$ are so close together that it is impossible detect them by plot, thus in Fig. 11 we show the error plot of $\varepsilon_{0.1,0.25}(x)$, $\varepsilon_{0.25,0.5}(x)$, $\varepsilon_{0.5,0.75}(x)$ and $\varepsilon_{0.75,1}(x)$





for iteration index n = 1. In Fig. 12 the numerical solutions of equation (1) obtained by approach I are shown for the Troesch parameter $\lambda = 1, 2, 3, 4, 5$. Figure 13 is useful for geometric understanding of the numerical solutions for $\lambda = 6, ..., 20$. As we can see in this plot for $5 \le \lambda$, almost approximated solutions have a linear form and are close together. Also it is considerable that the solutions function v(x) is increasing with respect to the Troesch parameter λ .

5.3 Numerical results of approach II

The performance of this approach is analyzed for the auxiliary linear operator $L(\alpha(x;q)) = \frac{\partial^2 \alpha(x;q)}{\partial^2 x}$, the resolution level j = 3, 4, 5, the iteration index m = 4, 5, and the convergencecontrol parameter $c_0 = -0.5$. The attained results of approach II for $\lambda = 0.5$ are given in Table 2 and are compared with the solutions of decomposition method (DM) of [4] and the spline collocation method (SCM) of [8]. Table 3 contains the numerical solutions for





scales m = 4, 5 and the Troesch parameter $\lambda = 1$ and also the results of variational iteration method (VIM) of [4] and the coupled Laplace transform and modified decomposition method (LT-MDM) of [11]. In Table 4 numerical solutions of problem (1)–(2) for $\lambda = 5$ are given in some arbitrary points and compared with the results of Fortran code (FD) [25] and B-spline collocation method (B-SCM) [25]. Also the effect of resolution scale *j* and iteration index *m* for $\lambda = 1$ are presented in Tables 5–6 by means averaged square error E(m, j), defined as follows:

$$E(m,j) := \frac{1}{2^{j}+1} \sum_{k=0}^{2^{j}} \left(\nu\left(\frac{k}{2^{j}}\right) - \nu_{m,j}\left(\frac{k}{2^{j}}\right) \right)^{2},$$

where ν is the exact and $\nu_{m,j}$ is the approximated solution obtained by approach II at scale *j* for iteration index *m*. As we can see by choosing higher scale *j*, the relevant E(m, j) is de-





creasing exponentially, while the required CPU time is increasing linearly. Also the results of Table 6 show that, for different values of m, the CPU time is nearly fixed. It is interesting that the required CPU time for j = 5 and m = 10 is only 1.2 seconds. Of course this result was predictable because of sparsity of operational connection coefficients matrices.

5.4 Numerical results of approach III

Using approach III, we solve nonlinear boundary value problem (1)–(2) in scale j = 5 for different values of the Troesch parameter λ and compare our findings with the results of some other existing methods. The absolute errors of numerical results evaluated by approach III are shown in Tables 7–8 for different values of the Troesch parameter. In Table 7 absolute errors relevant to approach III for m = 4, 5, the modified homotopy perturbation technique (MHPT) [4], and the Laplace transform and a modified decomposi-





tion technique (LT-MDM) [11] are reported for Troesch parameters $\lambda = 0.5$ and 1. Table 8 contains absolute errors of numerical solutions evaluated by approach III, sinc-Galerkin method based upon double exponential transformation (DESG) [13], and B-SCM [25] for $\lambda = 5$. For having a comparison between the results of presented methods, L_2 -errors of numerical solutions are given in Table 9, where L_2 -error is defined as follows. Suppose that ν and ν_* are the exact and numerical solutions of problem (1), respectively, we put

$$\epsilon_j = \nu(x_j) - \nu_*(x_j), \quad j = 0, 1, \dots, N, x_j = \frac{j}{10},$$

so

$$\|\epsilon\|_{L_2} = \sqrt{\sum_{j=1}^N \epsilon_j^2}$$



Table 2 Approximated solutions utilizing approach II for $\lambda = 0.5$

X	<i>m</i> = 4	<i>m</i> = 5	DM [4]	SCM [8]	Exact
0.1	0.0979671	0.092398	0.0959477	0.095944	0.0951769
0.2	0.196057	0.186122	0.1921352	0.192128	0.1906339
0.3	0.294391	0.281305	0.2888034	0.288793	0.2866534
0.4	0.393094	0.378081	0.3861955	0.386183	0.3835229
0.5	0.492288	0.476585	0.4845585	0.484546	0.4815374
0.6	0.592097	0.576957	0.5841442	0.584132	0.5810020
0.7	0.692647	0.679339	0.6852105	0.685200	0.6822351
0.8	0.794063	0.783875	0.7880234	0.788015	0.7855718
0.9	0.896471	0.890712	0.8928578	0.892853	0.8913670
1.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 3 Approximated solutions utilizing approach II for $\lambda = 1$

x	<i>m</i> = 4	<i>m</i> = 5	VIM [4]	LT-MDM [11]	Exact
0.1	0.0930076	0.0852337	0.1001668	0.0846631	0.0817969
0.2	0.1837690	0.1713204	0.2101338	0.1701750	0.1645308
0.3	0.2736864	0.2591218	0.3045410	0.2573995	0.2491673
0.4	0.3641402	0.3495170	0.4108413	0.3472304	0.3367322
0.5	0.4565536	0.4434094	0.5213734	0.4406094	0.4283471
0.6	0.5523283	0.5417400	0.6373663	0.5385460	0.5252740
0.7	0.6529541	0.6454926	0.7601789	0.6421421	0.6289711
0.8	0.7599845	0.7557054	0.8913449	0.7526227	0.7411683
0.9	0.8750744	0.8734816	1.0326302	0.8713749	0.8639700
1.0	1.0000000	1.0000000	1.1861098	1.0000512	1.0000000

6 Conclusion

In this paper three new and applicable methods are implemented on a class of strongly nonlinear differential equations. The proposed methods are based on some simple quasilinearization technique, the generalized Coiflet scaling functions based homotopy analysis method, and Galerkin and also some interesting approach for converting a boundary value problem to an integral equation. A common property of the presented method was reduc-

x	<i>m</i> = 4	<i>m</i> = 5	FD [25]	B-SCM [25]
0.1	0.0076520	0.0095523	-	-
0.2	0.0110199	0.0109043	0.01075342	0.01002027
0.3	0.0213986	0.0250004	-	-
0.4	0.0314220	0.0328037	0.0332005	0.0309979
0.5	0.0537343	0.0720751	-	-
0.6	0.0114229	0.9600047	-	-
0.7	0.1349407	0.1579514	_	-
0.8	0.2496878	0.2600308	0.2582166	0.2417049
0.9	0.4369226	0.4544960	0.4550603	0.4246183
1.0	1.0000000	1.000000	1.000000	1.0000000

Table 4 Approximated solutions utilizing approach II for $\lambda = 5$

Table 5 Averaged square error of numerical solution and the required CPU time of approach II for $\lambda = 1$ and m = 5

j	E(5,j)	CPU time (second)
3	1.016551 × 10 ⁻⁷	0.337402
4	2.345741×10^{-9}	0.620151
5	1.889065×10^{-12}	0.981400

Table 6 Averaged square error of numerical solution and the required CPU time of approach II for $\lambda = 1$ and j = 4

т	E(m, 4)	CPU time (second)
5	2.345741 × 10 ⁻⁹	1.085459
10	6.081362×10^{-11}	1.119225
15	9.106281 × 10 ⁻¹³	1.153337
20	3.042266×10^{-14}	1.198106

Table 7 Comparison of absolute errors of approximated solutions utilizing approach III for λ = 0.5 and λ = 1

x	M = 4	<i>M</i> = 5	MHPT [4]	LT-MDM [11]
$\lambda = 0.5$				
0.1	3.7×10^{-9}	4.4×10^{-13}	8.2×10^{-4}	7.7×10^{-4}
0.2	2.2×10^{-8}	5.8×10^{-12}	1.6×10^{-3}	1.5×10^{-3}
0.3	1.9 × 10 ⁻⁸	1.6×10^{-12}	2.3×10^{-3}	2.1×10^{-3}
0.4	4.1×10^{-8}	8.8×10^{-12}	2.9×10^{-3}	2.7×10^{-3}
0.5	9.6×10^{-7}	7.0×10^{-12}	3.2×10^{-3}	3.0×10^{-3}
0.6	7.5×10^{-7}	6.1×10^{-12}	3.4×10^{-3}	3.1×10^{-3}
0.7	3.9×10^{-8}	4.5×10^{-12}	3.2×10^{-3}	3.0×10^{-3}
0.8	6.4×10^{-8}	3.7×10^{-12}	2.7×10^{-3}	2.4×10^{-3}
0.9	3.0×10^{-8}	2.9×10^{-13}	1.6×10^{-3}	1.5×10^{-3}
$\lambda = 1$				
0.1	1.2×10^{-8}	1.4×10^{-12}	3.6×10^{-3}	2.9×10^{-3}
0.2	5.3×10^{-86}	8.3×10^{-11}	7.1×10^{-2}	5.9×10^{-3}
0.3	1.8×10^{-9}	2.9×10^{-11}	1.0×10^{-2}	8.2×10^{-3}
0.4	9.7 × 10 ^{−8}	5.0×10^{-11}	1.3×10^{-2}	1.0×10^{-2}
0.5	4.0×10^{-8}	5.7×10^{-11}	1.6×10^{-2}	1.2×10^{-2}
0.6	2.6×10^{-8}	3.3×10^{-11}	1.7×10^{-2}	1.3×10^{-2}
0.7	8.5×10^{-7}	6.1×10^{-11}	1.7×10^{-2}	1.3×10^{-2}
0.8	7.7×10^{-7}	4.3×10^{-12}	1.5×10^{-2}	1.1×10^{-2}
0.9	1.3×10^{-8}	7.4×10^{-12}	9.7×10^{-3}	7.4×10^{-3}

ing the nonlinear equation to a system of algebraic equations. Comparison between our results and the findings of some other existing methods for solving this kind of problems shows high accuracy and efficiency of the methods. It is considerable that the results of

x	<i>m</i> = 4	<i>m</i> = 5	B-SCM [25]	DESG [13]
0.1	3.6×10^{-8}	6.1×10^{-11}	-	_
0.2	1.4×10^{-8}	8.3×10^{-11}	7.3×10^{-3}	2.0×10^{-9}
0.3	1.6×10^{-7}	2.7×10^{-11}	_	-
0.4	2.7×10^{-7}	2.1×10^{-11}	2.2×10^{-3}	2.0×10^{-9}
0.5	3.5×10^{-8}	6.5×10^{-11}	-	-
0.6	6.6×10^{-8}	7.4×10^{-11}	-	-
0.7	1.1×10^{-7}	2.8×10^{-11}	-	-
0.8	5.9×10^{-7}	1.7×10^{-11}	1.4×10^{-2}	1.6×10^{-7}

Table 8 Comparison of absolute errors of approximated solutions utilizing approach III for $\lambda = 5$

Table 9	$\ \epsilon\ _{l_{2}}$	attained k	y pro	posed	approa	aches
---------	------------------------	------------	-------	-------	--------	-------

		$\lambda = 0.5$	$\lambda = 1$	$\lambda = 5$
Approach I		1.52 × 10 ⁻⁹	2.39×10^{-9}	6.18×10^{-8}
Approach II	<i>m</i> = 4	4.24×10^{-12}	1.99×10^{-11}	6.76×10^{-11}
	<i>m</i> = 5	4.05×10^{-15}	3.61×10^{-14}	5.48×10^{-14}
Approach III	<i>m</i> = 4	2.26×10^{-12}	3.09×10^{-11}	7.11 × 10 ⁻¹¹
	<i>m</i> = 5	2.66×10^{-14}	1.92×10^{-13}	6.66×10^{-12}

approach I for iteration index n = 1 are as accurate as some methods for large iteration index. Also because of some significant properties of generalized Coiflet scaling functions, such as orthogonality, having compact support, and vanishing moments, the operational matrices in approaches II and III are so sparse, and consequently relevant required computational time and memory is so low.

7 Future remarks

The presented methods in this study are attractive and can be extended for similar highorder nonlinear differential equations. Also, they could applied on the fractional-order differential and integro-differential equations with little additional work. Also the mentioned approaches could be implemented for solving nonlinear dynamic systems, including differential equations. Further research along these lines is under progress and will be reported in due time.

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References

- Lang, J., Walter, A.: A finite element method adaptive in space and time for nonlinear reaction-diffusion systems. Impact Comput. Sci. Eng. 4(4), 269–314 (1992)
- 2. Weibel, E.S.: On the confinement of a plasma by magnetostatic fields. Phys. Fluids 2(1), 52–56 (1959)
- Chiou, J.P., Na, T.Y.: On the solution of Troesch's nonlinear two-point boundary value problem using an initial value method. J. Comput. Phys. 19, 311–316 (1975)
- 4. Feng, X., Mei, L., He, G.: An efficient algorithm for solving Troesch's problem. Appl. Math. Comput. 189, 500–507 (2007)
- 5. Roberts, S.M., Shipman, J.S.: On the closed form solution of Troesch's problem. J. Comput. Phys. 21, 291–304 (1976)
- Troesch, B.A.: Intrinsic difficulties in the numerical solution of a boundary value problem. Internal Report NN-142, TRW Inc., Redondo Beach, California (1960)
- Yang, C., Hou, J.: Chebyshev wavelets method for solving Bratu's problem. Bound. Value Probl. 2013, 142 (2013). http://www.boundaryvalueproblems.com/content/2013/1/142
- El Hajaji, A., Hilal, K., Jalila, E.G., Mhamed, E.M.: New numerical scheme for solving Troesch' problem. Math. Theory Model. 4(2), 65–72 (2014)
- Secer, A., Kurulay, M., Bayram, M., Akinlar, M.A.: An efficient computer application of the sinc-Galerkin approximation for nonlinear boundary value problems. Bound. Value Probl. 2012, 117 (2012). http://www.boundaryvalueproblems.com/content/2012/1/117
- Caglar, H., Caglar, N., Ozer, M.: B-spline solution and the chaotic dynamics of Troesch's problem. Acta Phys. Pol. A 125(2), 554–559 (2014)
- 11. Khuri, S.A.: A numerical algorithm for solving the Troesch's problem. Int. J. Comput. Math. 80(4), 493–498 (2003)
- 12. Inc, M., Akgul, A.: The reproducing kernel Hilbert space method for solving Troesch's problem. J. Assoc. Arab Univ. Basic Appl. Sci. 14, 19–27 (2013)
- Nabati, M., Jalavand, M.: Solution of Troesch's problem through double exponential Sinc–Galerkin method. Comput. Methods Differ. Equ. 5(2), 141–157 (2017)
- 14. Amin, R., Shah, K., Asif, M., Imran, K., Ullah, F.: An efficient algorithm for numerical solution of fractional integro-differential equations via Haar wavelet. J. Comput. Appl. Math. **381**, 113028 (2021)
- Amin, R., Shah, K., Asif, M., Imran, K.: A computational algorithm for the numerical solution of fractional order delay differential equations. Appl. Math. Comput. 402, 125863 (2021)
- Nosrati Sahlan, M.: Four computational approaches for solving a class of boundary value problems arising in chemical reactor industry. Appl. Math. Comput. 355, 253–268 (2019)
- 17. Shah, K.: Using a numerical method by omitting discretization of data to study numerical solutions for boundary value problems of fractional order differential equations. Math. Methods Appl. Sci. 42, 6944–6959 (2019)
- 18. Chui, C.: An Introduction to Wavelets. Academic Press, New York (1992)
- 19. Chui, C.: Wavelets: A Mathematical Tool for Signal Analysis. SIAM, Philadelpia (1997)
- Liu, X.J., Zhou, Y.H., Zhang, L., Wang, J.Z.: Wavelet solutions of Burgers equation with high Reynolds numbers. Sci. China, Technol. Sci. 57(7), 1285–1292 (2014)
- Wang, J.Z., Zhou, Y.H., Gao, H.J.: Computation of the Laplace inverse transform by application of the wavelet theory. Commun. Numer. Methods Eng. 19(12), 959–975 (2003)
- Zhang, L., Liu, X.J., Zhou, Y.H., Wang, J.Z.: Influence of vanishing moments on the accuracy of a modified wavelet Galerkin method for nonlinear boundary value problems. In: 11th International Conference of Numerical Analysis and Applied Mathematics 2013, vol. 1558, pp. 942–945. AIP, New York (2013)
- 23. Wang, J.Z.: Generalized theory and arithmetic of orthogonal wavelets and applications to researches of mechanics including piezoelectric smart structures. PhD thesis, Lanzhou University (2001)
- Zhou, Y.H., Wang, J.Z.: Generalized Gaussian integral method for the calculation of scaling function transforms of wavelets and its applications. Acta Math. Sci. Ser. A Chin. Ed. 19(3), 293–300 (1999)
- 25. Khuri, S.A., Sayfy, A.: Troesch's problem: a B-spline collocation approach. Math. Comput. Model. 54, 1907–1918 (2011)
- Haq, F., Shah, K., Al-Mdallal, Q., Jarad, F.: Application of a hybrid method for systems of fractional order partial differential equations arising in the model of the one-dimensional Keller–Segel equation. Eur. Phys. J. Plus 134, 461 (2019)
- Gul, H., Alrabaiah, H., Ali, S., Shah, K., Muhammad, S.: Computation of solution to fractional order partial reaction diffusion equations. J. Adv. Res. 25, 31–38 (2020)
- 28. Daubechies, I.: Ten Lectures on Wavelets. SIAM, Philadelpia (1992)
- 29. Strang, G., Nguyen, T.: Wavelets and Filter Banks. Wellesley-Cambridge, Cambridge (1997)
- Telesca, L., Hloupis, G., Nikolintaga, I., Vallianatos, F.: Temporal patterns in southern Aegean seismicity revealed by the multiresolution wavelet analysis. Commun. Nonlinear Sci. Numer. Simul. 12, 1418–1426 (2007)