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Blow-up for a viscoelastic von Karman equation with strong damping and variable exponent source terms

Sun-Hye Park[1]*

*Correspondence:
sh-park@pusan.ac.kr
[1]Office for Education Accreditation,
Pusan National University, Busan
46241, South Korea

Abstract

In this article, we deal with a strongly damped von Karman equation with variable exponent source and memory effects. We investigate blow-up results of solutions with three levels of initial energy such as non-positive initial energy, certain positive initial energy, and high initial energy. Furthermore, we estimate not only the upper bound but also the lower bound of the blow-up time.

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1 Introduction

In this work, we discuss a viscoelastic von Karman equation with strong damping and variable exponent source terms,

$$w_{tt} + \Delta^2 w - \int_0^t h(t-s)\Delta^2 w(s)\,ds - \Delta w_t$$

$$= \left[w, \chi(w)\right] + |w|^{q(x)-2}w \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$\Delta^2 \chi(w) = -[w, w] \quad \text{in } \Omega \times (0, T), \tag{1.2}$$

$$w = \frac{\partial w}{\partial \nu} = 0, \qquad \chi(w) = \frac{\partial \chi(w)}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.3}$$

$$w(0) = w_0, \qquad w_t(0) = w_1 \quad \text{in } \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $\nu = (\nu_1, \nu_2)$ is the unit normal vector outward to $\partial\Omega$, the differentiable kernel function h defined on $[0, \infty)$ satisfies $h(0) > 0$, $h(t) \geq 0$, $h'(t) \leq 0$, and

$$1 - \int_0^\infty h(s)\,ds := l > 0.$$

The Von Karman bracket $[\cdot, \cdot]$ is given as

$$[y, v] = y_{x_1 x_1} v_{x_2 x_2} + y_{x_2 x_2} v_{x_1 x_1} - 2 y_{x_1 x_2} v_{x_1 x_2},$$

here $x = (x_1, x_2) \in \Omega$. The exponent function $q(\cdot)$ is measurable and verifies

$$\left| q(\overline{x}) - q(x) \right| \leq -\frac{a}{\log |\overline{x} - x|} \quad \text{for all } \overline{x}, x \in \Omega \text{ with } |\overline{x} - x| < \kappa, \tag{1.5}$$

where $a > 0$ and $0 < \kappa < 1$, and

$$2 \leq q_1 := \operatorname{ess} \inf_{x \in \Omega} q(x) \leq q(x) \leq q_2 := \operatorname{ess} \sup_{x \in \Omega} q(x) < \infty.$$

The von Karman equations (1.1)–(1.4) model a nonlinear elastic plate by describing the transversal displacement w and the Airy-stress function $\chi(w)$. Von Karman equation also arises in many applications such as bifurcation theory and shells. The type of von Karman equations

$$w_{tt} + \Delta^2 w - \int_0^t h(t-s)\Delta^2 w(s)\, ds + g(x, w_t) = \left[w, \chi(w) \right] \tag{1.6}$$

associated with different boundary conditions has been intensively treated about existence and stability (see [5, 8, 9, 16–18] and the references therein). When $h = g = 0$, the authors of [8] studied the unique existence of global solution. When $g(x, u_t) = a(x)u_t$ and $h = 0$, Horn and Lasiecka [9] discussed energy decay estimates. When $g = 0$, Park et al. [18] obtained the general decay behavior of solutions.

On the other hand, problems with variable exponent source have been attracting great interest [13–15, 21]. Such problems appear in physical phenomena such as nonlinear elastics [22], electrorheological fluids [20], stationary thermorheological viscous flows of non-Newtonian fluids [2], and image precessing [1]. Recently, Messaoudi et al. [15] considered wave equations with source and damping terms of variable exponent,

$$w_{tt} - \Delta w + a|w_t|^{\gamma(x)-2} w_t = b|w|^{q(x)-2} w.$$

They obtained the local existence of solutions under appropriate conditions on $\gamma(\cdot)$ and $q(\cdot)$ by utilizing the Faedo–Galerkin's technique and the contraction mapping theorem. Furthermore, they showed that the solution with negative initial energy blows up in a finite time. Later, Park and Kang [19] improved and complemented the result of [15] by obtaining a blow-up result of solution with certain positive initial energy for a wave equation of memory type. We also refer to a recent work [4] for a nonlinear diffusion system involving variable exponents dependent on spatial and time variables and cross-diffusion terms. At this point, it is worthwhile to mention that there is little work concerning global nonexistence of solutions for viscoelastic von Karman equations with variable source effect. Particularly, there is no literature concerning blow-up results of solutions with high initial energy for the equations. Thus, in this article, we establish blow-up results of solutions with three levels of initial energy such as non-positive initial energy, certain positive initial energy, and high initial energy. Furthermore, we estimate not only the upper bound

but also the lower bound of the blow-up time. These are inspired by the ideas of [23], where the authors proved blow-up results of solutions with high initial energy and estimated bounds of existence time of solutions for wave equation with logarithmic nonlinear source term.

The outline of this article is here. In Sect. 2, we state some definitions, notations, and auxiliary lemmas. In Sect. 3, we construct blow-up results and obtain bounds of the blow-up time.

2 Preliminaries

We denote $(y, v) = \int_\Omega y(x)v(x)\,dx$, $\|y\|_2^2 = (y, y)$. Generally, we denote by $\| \cdot \|_Y$ the norm of a space Y. For simplicity, we write $\| \cdot \|_{L^p(\Omega)}$ by $\| \cdot \|_p$. If there is no ambiguity, we will omit the variables t and x. Let B_1 and B_2 be the constants with

$$\|y\|_2 \le B_1 \|\nabla y\|_2 \quad \text{for } y \in H_0^1(\Omega), \qquad \|\nabla y\|_2 \le B_2 \|\Delta y\|_2 \quad \text{for } y \in H_0^2(\Omega). \tag{2.1}$$

For a measurable function $p : \Omega \subset \mathbb{R}^n \to [1, \infty]$, the Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{ y : \Omega \to \mathbb{R} \,\Big|\, y \text{ is measurable in } \Omega, \int_\Omega |\delta y(x)|^{p(x)}\,dx < \infty \text{ for some } \delta > 0 \right\}$$

is a Banach space equipped with Luxembourg-type norm

$$\|y\|_{p(\cdot)} = \inf\left\{ \delta > 0 \,\Big|\, \int_\Omega \left| \frac{y(x)}{\delta} \right|^{p(x)}\,dx \le 1 \right\}.$$

Lemma 2.1 ([3]) *If* $1 < p_1 := \operatorname{ess\,inf}_{x \in \Omega} p(x) \le p(x) \le p_2 := \operatorname{ess\,sup}_{x \in \Omega} p(x) < \infty$, *then*

$$\min\left\{ \|y\|_{p(\cdot)}^{p_1}, \|y\|_{p(\cdot)}^{p_2} \right\} \le \int_\Omega |y(x)|^{p(x)}\,dx \le \max\left\{ \|y\|_{p(\cdot)}^{p_1}, \|y\|_{p(\cdot)}^{p_2} \right\}$$

for any $y \in L^{p(\cdot)}(\Omega)$.

Since $\dim(\Omega) = 2$, the embedding $H_0^2(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ ($2 \le q(x) < \infty$) is continuous and compact [11]. We let B be the constant of the embedding inequality

$$\|y\|_{q(\cdot)} \le B \|\Delta y\|_2 \quad \text{for } y \in H_0^2(\Omega). \tag{2.2}$$

See [6, 7, 11] for more properties of a Lebesgue space with variable exponent.

Lemma 2.2 ([9], Lemma 2.1) *If* $y \in H^2(\Omega)$, *then* $\chi(y) \in W^{2,\infty}(\Omega)$ *and* $\|\chi(y)\|_{W^{2,\infty}(\Omega)} \le a_1 \|y\|_{H^2(\Omega)}^2$.

Lemma 2.3 ([8], p. 270) *If* $y \in H^2(\Omega)$ *and* $v \in W^{2,\infty}(\Omega)$, *then* $[y, v] \in L^2(\Omega)$ *and* $\|[y, v]\|_2 \le a_2 \|y\|_{H^2(\Omega)} \|v\|_{W^{2,\infty}(\Omega)}$.

Lemma 2.4 ([5]) *Let* $y_1, y_2, y_3 \in H^2(\Omega)$. *If at least one of them is an element of* $H_0^2(\Omega)$, *then*

$$\int_\Omega [y_1, y_2] y_3\,dx = \int_\Omega [y_1, y_3] y_2\,dx.$$

By combining the arguments of [10, 19], for every $(w_0, w_1) \in H_0^2(\Omega) \times L^2(\Omega)$, we can get a unique local solution w of problem (1.1)–(1.4) with $w \in C(0, T; H_0^2(\Omega)) \cap C^1(0, T; L^2(\Omega))$ and $w_t \in L^2(0, T; H_0^1(\Omega))$.

3 Blow-up results

In this section, we establish blow-up results of solutions with three levels of initial energy and estimate bounds of blow-up time. For this, we need ab auxiliary lemma.

Lemma 3.1 ([12]) *Let $B(t)$ be a positive, twice differentiable function verifying*

$$B(t)B''(t) - (1 + \theta)\big(B'(t)\big)^2 \geq 0$$

for $t > 0$, where θ is a positive constant. If $B(0) > 0$ and $B'(0) > 0$, then there exists a $T_1 \leq \frac{B(0)}{\theta B'(0)}$ with $\lim_{t \to T_1^-} B(t) = \infty$.

Taking the scalar product (1.1) by w_t in $L^2(\Omega)$ and using (1.3), we get

$$\frac{d}{dt}\left(\frac{1}{2}\|w_t\|_2^2 + \frac{1}{2}\|\Delta w\|_2^2 - \int_\Omega \frac{|w|^{q(x)}}{q(x)}\,dx\right) - \int_0^t h(t-s)\big(\Delta w(s), \Delta w_t(t)\big)\,ds$$

$$= -\|\nabla w_t\|_2^2 + \big([w, \chi(w)], w_t\big).$$

From Lemma 2.4, (1.2) and (1.3), we have

$$\big([w, \chi(w)], w_t\big) = \frac{1}{2}\left(\frac{d}{dt}[w, w], \chi(w)\right) = -\frac{1}{4}\frac{d}{dt}\big\|\Delta\chi(w)\big\|_2^2.$$

Using this and the relation

$$-\int_0^t h(t-s)\big(\Delta w(s), \Delta w_t(t)\big)\,ds$$

$$= \frac{1}{2}\frac{d}{dt}\left\{(h \circ \Delta w) - \left(\int_0^t h(s)\,ds\right)\|\Delta w\|_2^2\right\} + \frac{h(t)}{2}\|\Delta w\|_2^2 - \frac{1}{2}(h' \circ \Delta w),$$

we get

$$E'(t) = -\|\nabla w_t\|_2^2 - \frac{h(t)}{2}\|\Delta w\|_2^2 + \frac{1}{2}\big(h' \circ \Delta w\big) \leq -\|\nabla w_t\|_2^2 \leq 0, \quad t \in [0, T_*) \tag{3.1}$$

and

$$E(t) + \int_0^t \big\|\nabla w_t(s)\big\|_2^2\,ds \leq E(0), \quad t \in [0, T_*), \tag{3.2}$$

where

$$E(t) = \frac{1}{2}\|w_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)\|\Delta w\|_2^2 + \frac{1}{2}(h \circ \Delta w)$$

$$+ \frac{1}{4}\big\|\Delta\chi(w)\big\|_2^2 - \int_\Omega \frac{|w|^{q(x)}}{q(x)}\,dx, \tag{3.3}$$

here

$$(h \circ \Delta w)(t) = \int_0^t h(t - s) \big\| \Delta w(t) - \Delta w(s) \big\|_2^2 \, ds,$$

and $T_* = \sup\{T : [0, T]$ is the existence interval of the solution to (1.1)–(1.4)$\}$.

3.1 Case of non-positive initial energy

Theorem 3.1 *Let $q_1 \geq 4$ and the kernel function h satisfy*

$$\int_0^\infty h(s) \, ds \leq \frac{q_1(q_1 - 2)}{q_1(q_1 - 2) + 1}. \tag{3.4}$$

Let one of the following hold.

  (i) $E(0) < 0$;
  (ii) $E(0) = 0$ *and* $(w_0, w_1) > \frac{2\|\nabla w_0\|_2^2}{q_1 - 2}$.

Then the solution w of problem (1.1)–(1.4) *blows up in a finite time T_*, that is,*

$$\lim_{t \to T_*^-} \big\| \Delta w(t) \big\|_2 = \infty. \tag{3.5}$$

In addition, T_* satisfies

$$T_* \leq \frac{2\|w_0\|_2^2 + 2\alpha\beta^2}{(q_1 - 2)\{(w_0, w_1) + \alpha\beta\} - 2\|\nabla w_0\|_2^2}, \tag{3.6}$$

where

$$\begin{cases} 0 < \alpha \leq -2E(0) \quad and \quad \beta > \max\{0, -\frac{(w_0, w_1)}{\alpha}, \frac{2\|\nabla w_0\|_2^2 - (q_1 - 2)(w_0, w_1)}{q_1 - 2}\} \quad if\ E(0) < 0; \\ \alpha = 0 \quad if\ E(0) = 0 \quad and \quad (w_0, w_1) > \frac{2\|\nabla w_0\|_2^2}{q_1 - 2}. \end{cases} \tag{3.7}$$

Proof Suppose that w is global. For $0 < T < T_*$, we define a function F on $[0, T]$ by

$$F(t) = \|w\|_2^2 + \int_0^t \big\| \nabla w(s) \big\|_2^2 \, ds + (T - t)\|\nabla w_0\|_2^2 + \alpha(t + \beta)^2, \tag{3.8}$$

where $\alpha \geq 0$ and $\beta > 0$ are the constants satisfying (3.7), then

$$F(t) > 0 \quad \text{for } t \in [0, T], \tag{3.9}$$

$$F'(t) = 2(w, w_t) + 2 \int_0^t \big(\nabla w(s), \nabla w_t(s) \big) \, ds + 2\alpha(t + \beta), \tag{3.10}$$

and

$$F''(t) = 2\|w_t\|_2^2 - 2\left(1 - \int_0^t h(s) \, ds\right)\|\Delta w\|_2^2 + 2\big(w, [w, \chi(w)]\big) + 2 \int_\Omega |w|^{q(x)} \, dx$$

$$- 2 \int_0^t h(t - s)\big(\Delta w(t), \Delta w(t) - \Delta w(s)\big) \, ds + 2\alpha.$$

From Lemma 2.4, (1.2), and (1.3), we have

$$2\big(w, [w, \chi(w)]\big) = 2\big([w, w], \chi(w)\big) = -2\big\|\Delta\chi(w)\big\|_2^2.$$

Thus, we get

$$F''(t) = 2\|w_t\|_2^2 - 2\bigg(1 - \int_0^t h(s)\,ds\bigg)\|\Delta w\|_2^2 - 2\big\|\Delta\chi(w)\big\|_2^2 + 2\int_\Omega |w|^{q(x)}\,dx$$

$$- 2\int_0^t h(t-s)\big(\Delta w(t), \Delta w(t) - \Delta w(s)\big)\,ds + 2\alpha. \tag{3.11}$$

Using the inequality $(a\xi_1 + b\xi_2 + c\xi_3)^2 \le (a^2 + b^2 + c^2)(\xi_1^2 + \xi_2^2 + \xi_3^2)$ and (3.8), we get

$$\big(F'(t)\big)^2 \le 4F(t)\bigg(\|w_t\|_2^2 + \int_0^t \big\|\nabla w_t(s)\big\|_2^2\,ds + \alpha\bigg). \tag{3.12}$$

Using (3.11), (3.12), (3.3), (3.2), and Young's inequality, one finds

$$F(t)F''(t) - \frac{q_1+2}{4}\big(F'(t)\big)^2$$

$$\ge F(t)\bigg\{-q_1\|w_t\|_2^2 - 2\bigg(1 - \int_0^t h(s)\,ds\bigg)\|\Delta w\|_2^2 - 2\big\|\Delta\chi(w)\big\|_2^2 + 2\int_\Omega |w|^{q(x)}\,dx$$

$$- 2\int_0^t h(t-s)\big(\Delta w(t), \Delta w(t) - \Delta w(s)\big)\,ds - (q_1+2)\int_0^t \big\|\nabla w_t(s)\big\|_2^2\,ds - q_1\alpha\bigg\}$$

$$= F(t)\bigg\{-2q_1 E(t) + (q_1-2)\bigg(1 - \int_0^t h(s)\,ds\bigg)\|\Delta w\|_2^2 + q_1(h \circ \Delta w)$$

$$- 2\int_0^t h(t-s)\big(\Delta w(t), \Delta w(t) - \Delta w(s)\big)\,ds + \bigg(\frac{q_1}{2} - 2\bigg)\big\|\Delta\chi(w)\big\|_2^2$$

$$- (q_1+2)\int_0^t \big\|\nabla w_t(s)\big\|_2^2\,ds - 2q_1\int_\Omega \frac{|w|^{q(x)}}{q(x)}\,dx + 2\int_\Omega |w|^{q(x)}\,dx - q_1\alpha\bigg\}$$

$$\ge F(t)\bigg\{-2q_1 E(0) + (q_1-2)\bigg(1 - \int_0^t h(s)\,ds\bigg)\|\Delta w\|_2^2 + q_1(h \circ \Delta w)$$

$$- 2\int_0^t h(t-s)\big(\Delta w(t), \Delta w(t) - \Delta w(s)\big)\,ds + \bigg(\frac{q_1}{2} - 2\bigg)\big\|\Delta\chi(w)\big\|_2^2$$

$$+ (q_1-2)\int_0^t \big\|\nabla w_t(s)\big\|_2^2\,ds - 2q_1\int_\Omega \frac{|u|^{q(x)}}{q(x)}\,dx + 2\int_\Omega |w|^{q(x)}\,dx - q_1\alpha\bigg\}. \tag{3.13}$$

Using the relations

$$-2q_1\int_\Omega \frac{|w|^{q(x)}}{q(x)}\,dx \ge -2\int_\Omega |w|^{q(x)}\,dx$$

and

$$-2\int_0^t h(t-s)\big(\Delta w(t), \Delta w(t) - \Delta w(s)\big)\,ds \ge -\epsilon(h \circ \Delta w) - \frac{1}{\epsilon}\bigg(\int_0^t h(s)\,ds\bigg)\|\Delta w\|_2^2$$

for $\epsilon > 0$, we infer

$$F(t)F''(t) - \frac{q_1 + 2}{4}\left(F'(t)\right)^2 \geq F(t)G(t), \tag{3.14}$$

where

$$G(t) = -2q_1E(0) - q_1\alpha + (q_1 - \epsilon)(h \circ \Delta w)$$
$$+ \left\{(q_1 - 2) - \left(q_1 - 2 + \frac{1}{\epsilon}\right)\int_0^t h(s)\,ds\right\}\|\Delta w\|_2^2. \tag{3.15}$$

Taking $\epsilon = q_1$ in (3.15), and using (3.4) and (3.7), we find

$$G(t) \geq -2q_1E(0) + \left\{(q_1 - 2) - \left(q_1 - 2 + \frac{1}{q_1}\right)\int_0^t h(s)\,ds\right\}\|\Delta w\|^2 - q_1\alpha \geq 0.$$

From the condition (3.7), it is clear that

$$F'(0) = 2(w_0, w_1) + 2\alpha\beta > 0.$$

Thus, applying Lemma 3.1, we get the existence of T_* satisfying

$$T_* \leq \frac{4F(0)}{(q_1 - 2)F'(0)} = \frac{2(\|w_0\|_2^2 + T\|\nabla w_0\|_2^2 + \alpha\beta^2)}{(q_1 - 2)\{(w_0, w_1) + \alpha\beta\}} \tag{3.16}$$

and

$$\lim_{t \to T_*^-} F(t) = \infty,$$

which gives

$$\lim_{t \to T_*^-}\left\|\Delta w(t)\right\|_2 = \infty.$$

Moreover, using (3.16) and the relation $0 < T < T_*$, we see

$$T_* \leq \frac{2(\|w_0\|_2^2 + T_*\|\nabla w_0\|_2^2 + \alpha\beta^2)}{(q_1 - 2)\{(w_0, w_1) + \alpha\beta\}}.$$

This gives (3.6) under the condition β given in (3.7).   □

3.2 Case of certain positive initial energy
We set

$$\overline{B} = \max\left\{1, \frac{B}{\sqrt{l}}\right\}, \qquad \eta_1 = \left(\frac{1}{\overline{B}}\right)^{\frac{q_1}{q_1 - 2}}, \qquad E_1 = \frac{(q_1 - 2)\eta_1^2}{2q_1}, \tag{3.17}$$

where B is the embedding constant given in (2.2), and define a function g by

$$g(\eta) = \frac{1}{2}\eta^2 - \frac{\overline{B}^{q_1}}{q_1}\eta^{q_1}. \tag{3.18}$$

Then one knows

    (i) $g(0) = 0$ and $\lim_{\eta \to +\infty} g(\eta) = -\infty$,

    (ii) g is increasing on $(0, \eta_1)$ and decreasing on (η_1, ∞),

    (iii) g has the maximum value $g(\eta_1) = E_1$.

Lemma 3.2 *Let w be the solution of problem* (1.1)–(1.4). *Assume that*

$$E(0) < E_1 \quad and \quad \eta_1 < \sqrt{l}\|\Delta w_0\|_2 \leq \frac{1}{B}. \tag{3.19}$$

Then there exists a constant $\eta_* > \eta_1$ such that

$$l\|\Delta w\|_2^2 \geq \eta_*^2 \quad for\ 0 \leq t < T_*. \tag{3.20}$$

Proof From (3.3), Lemma 2.1, (2.2) and (3.17), we have

$$
\begin{aligned}
E(t) &\geq \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)\|\Delta w\|_2^2 - \int_\Omega \frac{|w|^{q(x)}}{q(x)}\,dx \\
&\geq \frac{l}{2}\|\Delta w\|_2^2 - \frac{1}{q_1}\max\left\{\|w\|_{q(\cdot)}^{q_1}, \|w\|_{q(\cdot)}^{q_2}\right\} \\
&\geq \frac{l}{2}\|\Delta w\|_2^2 - \frac{1}{q_1}\max\left\{B^{q_1}\|\Delta w\|_2^{q_1}, B^{q_2}\|\Delta w\|_2^{q_2}\right\}\} \\
&\geq \frac{1}{2}\left(\sqrt{l}\|\Delta w\|_2\right)^2 - \frac{1}{q_1}\max\left\{\overline{B}^{q_1}\left(\sqrt{l}\|\Delta w\|_2\right)^{q_1}, \overline{B}^{q_2}\left(\sqrt{l}\|\Delta w\|_2\right)^{q_2}\right\} \\
&= f\left(\sqrt{l}\|\Delta w\|_2\right),
\end{aligned}
\tag{3.21}
$$

where

$$f(\eta) = \frac{1}{2}\eta^2 - \frac{1}{q_1}\max\left\{\overline{B}^{q_1}\eta^{q_1}, \overline{B}^{q_2}\eta^{q_2}\right\}.$$

It is easily seen that

$$f(\eta) = g(\eta) \quad for\ 0 \leq \eta \leq \frac{1}{B}. \tag{3.22}$$

Since $E(0) < E_1$, there exists $\eta_* > \eta_1$ such that

$$E(0) = g(\eta_*). \tag{3.23}$$

From (3.23), (3.21), and (3.19), we observe

$$g(\eta_*) = E(0) \geq f\left(\sqrt{l}\|\Delta w_0\|_2\right) = g\left(\sqrt{l}\|\Delta w_0\|_2\right).$$

Since g is decreasing on (η_1, ∞), we see that

$$\eta_* \leq \sqrt{l}\|\Delta w_0\|_2.$$

From (3.19), we also know

$$\eta_1 < \eta_* \leq \frac{1}{B}. \tag{3.24}$$

We will show (3.20) by contradiction. Suppose that there exists $t_0 \in [0, T_*)$ such that

$$\sqrt{l}\left\| \Delta w(t_0) \right\|_2 < \eta_*.$$

Because the solution w is continuous in t, there exists $t_1 > 0$ such that

$$\eta_1 < \sqrt{l}\left\| \Delta w(t_1) \right\|_2 < \eta_*. \tag{3.25}$$

Noting that g is decreasing on (η_1, ∞) and using (3.23), (3.25), (3.22), (3.24), (3.21), (3.1), we get

$$E(0) = g(\eta_*) < g\big(\sqrt{l}\left\| \Delta w(t_1) \right\|_2\big) = f\big(\sqrt{l}\left\| \Delta w(t_1) \right\|_2\big) \leq E(t_1) \leq E(0).$$

This is a contradiction.   □

Theorem 3.2 *Let the conditions of Lemma* 3.2 *are valid. If* $E(0) = \gamma E_1$, *where* $0 < \gamma < 1$, *and*

$$\int_0^\infty h(s)\,ds \leq \frac{q_1 - 2}{q_1 - 2 + \frac{1}{(1-\gamma)^2 q_1 + 2\gamma(1-\gamma)}}, \tag{3.26}$$

the solution w to problem (1.1)–(1.4) *blows up in a finite time T_*. Moreover, T_* satisfies*

$$T_* \leq \frac{2\|w_0\|_2^2 + 2\alpha\beta^2}{(q_1 - 2)\{(w_0, w_1) + \alpha\beta\} - 2\|\nabla w_0\|_2^2},$$

where

$$0 < \alpha \leq \frac{\gamma\lambda(q_1 - 2)}{q_1}, \quad and$$

$$\beta > \max\left\{ 0, -\frac{(w_0, w_1)}{\alpha}, \frac{2\|w_0\|_2^2 - (q_1 - 2)(w_0, w_1)}{q_1 - 2} \right\}, \tag{3.27}$$

here $0 < \lambda < \eta_*^2 - \eta_1^2$.

Proof Let F be the function given in (3.8) with (3.27). Then (3.9), (3.10), (3.11), (3.14), and (3.15) are valid. Taking $\epsilon = (1 - \gamma)q_1 + 2\gamma$ in (3.15), we have

$$G(t) \geq -2q_1 E(0) - q_1\alpha + \gamma(q_1 - 2)(h \circ \Delta w)$$
$$+ \left\{ (q_1 - 2) - \left(q_1 - 2 + \frac{1}{(1-\gamma)q_1 + 2\gamma} \right) \int_0^t h(s)\,ds \right\} \|\Delta w\|_2^2. \tag{3.28}$$

The condition (3.26) implies

$$(q_1 - 2) - \left(q_1 - 2 + \frac{1}{(1-\gamma)q_1 + 2\gamma} \right) \int_0^t h(s)\,ds \geq \gamma(q_1 - 2)\left(1 - \int_0^\infty h(s)\,ds \right).$$

Since w is continuous in t, Lemma 3.2 guarantees the existence of $\lambda > 0$ with

$$\eta_1^2 + \lambda < \eta_*^2 \le l\|\Delta w\|_2^2 \quad \text{for all } t \in [0, T].$$

Adapting these too, noting the definition of E_1 given in (3.17), and using (3.27), we have

$$
\begin{aligned}
G(t) &\ge -2q_1 E(0) - q_1 \alpha + \gamma(q_1 - 2)l\|\Delta w\|_2^2 \\
&\ge -2q_1 \gamma E_1 - q_1 \alpha + \gamma(q_1 - 2)\left(\eta_1^2 + \lambda\right) \\
&= \gamma(q_1 - 2)\lambda - q_1 \alpha \ge 0.
\end{aligned}
$$

By the same argument of Theorem 3.1, we complete the proof. $\qquad\square$

3.3 Case of high initial energy

Lemma 3.3 *If $q_1 \ge 4$ and h satisfies* (3.4), *it fulfills*

$$\left((w, w_t) - \frac{2q_1 + kB_2^2}{2kQ}E(t)\right) \ge e^{rt}\left((w_0, w_1) - \frac{2q_1 + kB_2^2}{2kQ}E(0)\right) \quad \text{for } t \in [0, T_*), \quad (3.29)$$

here

$$
\begin{aligned}
Q &= (q_1 - 2) - \left(q_1 - 2 + \frac{1}{q_1}\right)(1 - l), \\
k &= \min\left\{\frac{q_1 + 2}{q_1 - 2}, \frac{1}{B_1^2 B_2^2}\right\}, \qquad r = \frac{2kq_1 Q}{2q_1 + kB_2^2}.
\end{aligned}
\tag{3.30}
$$

Proof Using (1.1)–(1.4) and Young's inequality, we get

$$
\begin{aligned}
\frac{d}{dt}(w, w_t) &= \|w_t\|_2^2 - \left(1 - \int_0^t h(s)\,ds\right)\|\Delta w\|_2^2 - (\nabla w, \nabla w_t) - \left\|\Delta \chi(w)\right\|_2^2 \\
&\quad + \int_0^t h(t - s)\left(\Delta w(s) - \Delta w(t), \Delta w(t)\right)ds + \int_\Omega |w|^{q(x)}\,dx \\
&\ge \|w_t\|_2^2 - \left\{\left(1 - \int_0^t h(s)\,ds\right) + \frac{1}{2q_1}\int_0^t h(s)\,ds\right\}\|\Delta w\|_2^2 - \left\|\Delta \chi(w)\right\|_2^2 \\
&\quad - \frac{\delta}{2}\|\nabla w\|_2^2 - \frac{1}{2\delta}\|\nabla w_t\|_2^2 - \frac{q_1}{2}(h \circ \Delta w) + \int_\Omega |w|^{q(x)}\,dx
\end{aligned}
\tag{3.31}
$$

for $\delta > 0$. From (3.3), we observe

$$
\begin{aligned}
\int_\Omega |w|^{q(x)}\,dx &\ge -q_1 E(t) + \frac{q_1}{2}\|w_t\|_2^2 + \frac{q_1}{2}\left(1 - \int_0^t h(s)\,ds\right)\|\Delta w\|_2^2 \\
&\quad + \frac{q_1}{2}(h \circ \Delta w) + \frac{q_1}{4}\left\|\Delta \chi(w)\right\|_2^2.
\end{aligned}
$$

Applying this to (3.31) and using (3.1), we have

$$\frac{d}{dt}(w, w_t) \ge \frac{q_1 + 2}{2}\|w_t\|_2^2 + \frac{1}{2\delta}E'(t) - q_1 E(t)$$

$$+ \left[\left\{ (q_1 - 2) - \left(q_1 - 2 + \frac{1}{q_1} \right) \int_0^t h(s)\, ds \right\} B_2^{-2} - \delta \right] \frac{\|\nabla w\|_2^2}{2}$$

$$\geq \frac{q_1 + 2}{2} \|w_t\|_2^2 + \frac{1}{2\delta} E'(t) - q_1 E(t)$$

$$+ \frac{1}{2B_1^2 B_2^2} \left[\left\{ (q_1 - 2) - \left(q_1 - 2 + \frac{1}{q_1} \right) \int_0^t h(s)\, ds \right\} - B_2^2 \delta \right] \|w\|_2^2.$$

Recalling (3.30), we have

$$\frac{d}{dt} \left((w, w_t) - \frac{1}{2\delta} E(t) \right)$$

$$\geq \frac{q_1 + 2}{2} \|w_t\|_2^2 + \frac{Q - B_2^2 \delta}{2B_1^2 B_2^2} \|w\|_2^2 - q_1 E(t)$$

$$\geq \frac{k(q_1 - 2)}{2} \|w_t\|_2^2 + \frac{k(Q - B_2^2 \delta)}{2} \|w\|_2^2 - q_1 E(t)$$

$$\geq \frac{k(Q - B_2^2 \delta)}{2} \left(\|w_t\|_2^2 + \|w\|_2^2 \right) - q_1 E(t)$$

$$\geq k(Q - B_2^2 \delta) \left((w, w_t) - \frac{q_1}{k(Q - B_2^2 \delta)} E(t) \right) \quad \text{for } 0 < \delta < \frac{Q}{B_2^2}.$$

Taking

$$\delta = \frac{kQ}{2q_1 + kB_2^2},$$

we find

$$\frac{d}{dt} \left((w, w_t) - \frac{2q_1 + kB_2^2}{2kQ} E(t) \right) \geq \frac{2kq_1 Q}{2q_1 + kB_2^2} \left((w, w_t) - \frac{2q_1 + kB_2^2}{2kQ} E(t) \right).$$

This completes the proof.　　　　　　　　　　　　　　　　　　　　　　　　　　　　□

Theorem 3.3 *Let $q_1 \geq 4$ and h satisfy* (3.4). *If $0 < E(0) < \frac{2kQ}{2q_1 + kB_2^2} (w_0, w_1)$, then the solution w blows up in a finite time T_*. Moreover, if $E(0) < \frac{Q \|u_0\|_2^2}{2q_1 B_1^2 B_2^2}$, then T_* satisfies*

$$T_* \leq \frac{2\|w_0\|_2^2 + 2\alpha\beta^2}{(q_1 - 2)\{(w_0, w_1) + \alpha\beta\} - 2\|\nabla w_0\|_2^2},$$

where

$$0 < \alpha \leq \frac{-2q_1 B_1^2 B_2^2 E(0) + Q \|w_0\|_2^2}{q_1 B_1^2 B_2^2} \quad \text{and}$$

$$\beta > \max \left\{ 0, \frac{-(q_1 - 2)(w_0, w_1) + 2\|\nabla w_0\|_2^2}{(q_1 - 2)\alpha} \right\}.$$

$$(3.32)$$

Proof Suppose that w is global. Then, using (3.2), we get

$$\|w\|_2 \le \|w_0\|_2 + \int_0^t \left\|w_t(s)\right\|_2 ds \le \|w_0\|_2 + B_1 t^{\frac{1}{2}} \left(\int_0^t \left\|\nabla w_t(s)\right\|_2^2 ds\right)^{\frac{1}{2}}$$

$$\le \|w_0\|_2 + B_2 \left(t\left(E(0) - E(t)\right)\right)^{\frac{1}{2}}, \quad t \ge 0. \tag{3.33}$$

In the case $E(t) \ge 0$ for all $t \ge 0$, from (3.33), we see

$$\|w\|_2^2 \le 2\|w_0\|_2^2 + 2B_2^2 E(0)t, \quad t \ge 0. \tag{3.34}$$

Applying Lemma 3.3, we also have

$$\|w\|_2^2 = \|w_0\|_2^2 + 2\int_0^t \left(w(s), w_t(s)\right) ds$$

$$\ge \|w_0\|_2^2 + 2\left\{\int_0^t e^{rs}\left((w_0, w_1) - \frac{2q_1 + kB_2^2}{2kQ}E(0)\right) ds + \int_0^t \frac{2q_1 + kB_2^2}{2kQ}E(s)\, ds\right\}$$

$$\ge \|w_0\|_2^2 + 2\left((w_0, w_1) - \frac{2q_1 + kB_2^2}{2kQ}E(0)\right)\frac{e^{rt} - 1}{r} \quad \text{for all } t \ge 0. \tag{3.35}$$

But this contradicts (3.34) for t appropriately large. In the case $E(t_1) < 0$ for some $t_1 > 0$, there exists the first $t_2 > 0$ with $0 < t_2 < t_1$ satisfying $E(t_2) = 0$, $E(t) > 0$ for $0 \le t < t_2$, and $E(t_0) < 0$ for some $t_0 > t_2$. Taking $w(t_0)$ as a new initial datum, by Theorem 3.1, the solution w blows up after the time t_0. This also is a contradiction. Consequently, $T_* < \infty$.

Let F be the function given in (3.8) with (3.32). Then (3.9), (3.10), (3.11), (3.14), and (3.15) are also valid. Taking $\epsilon = q_1$ in (3.15) and using (3.35), we have

$$F(t)F''(t) - \frac{q_1 + 2}{4}\left(F'(t)\right)^2 \ge F(t)\left(-2q_1 E(0) - q_1\alpha + Q\|\Delta w\|_2^2\right)$$

$$\ge F(t)\left(-2q_1 E(0) - q_1\alpha + \frac{Q}{B_1^2 B_2^2}\|w_0\|_2^2\right).$$

From (3.32), we observe

$$F(t)F''(t) - \frac{q_1 + 2}{4}\left(F'(t)\right)^2 \ge 0.$$

By the same argument of Theorem 3.1, we complete the proof. $\qquad\square$

Theorem 3.4 *Let the conditions of one of Theorem 3.1–Theorem 3.3 are satisfied. Then the blow-up time T_* verifies*

$$\int_{D(0)}^{\infty} \frac{1}{2y + d_1 y^3 + d_2 y^{q_1-1} + d_3 y^{q_2-1}}\, dy \le T_*, \tag{3.36}$$

where $D(0) = \|w_1\|_2^2 + \|\Delta w_0\|_2^2$ and $d_i > 0$ $(i = 1, 2, 3)$ are certain constants.

Proof We let

$$D(t) = \|w_t\|_2^2 + \left(1 - \int_0^t h(s)\, ds\right)\|\Delta w\|_2^2 + (h \circ \Delta w). \tag{3.37}$$

From (3.5), it is observed

$$\lim_{t \to T_*^-} D(t) \geq \lim_{t \to T_*^-} \left(\|w_t\|_2^2 + l\|\Delta w\|_2^2 \right) = \infty. \tag{3.38}$$

Using (1.1)–(1.4), one gets

$$D'(t) = -2\|\nabla w_t\|_2^2 - h(t)\|\Delta w\|_2^2 + (h' \circ \Delta w) + 2(w_t, [w, \chi(w)]) + 2\int_\Omega w_t |w|^{q(x)-2} w \, dx$$

$$\leq 2\|w_t\|_2^2 + \left\| [w, \chi(w)] \right\|_2^2 + \int_\Omega |w|^{2(q(x)-1)} \, dx. \tag{3.39}$$

From Lemma 2.2 and Lemma 2.3, we see

$$\left\| [w, \chi(w)] \right\|_2^2 \leq \left(a_2 \|w\|_{H^2(\Omega)} \|\chi(w)\|_{W^{2,\infty}(\Omega)} \right)^2$$

$$\leq \left(a_1 a_2 \|w\|_{H^2(\Omega)}^3 \right)^2$$

$$\leq b_1 \|\Delta w\|_2^6 \tag{3.40}$$

for some $b_1 > 0$. The last term of (3.39) is estimated as

$$\int_\Omega |w|^{2(q(x)-1)} \, dx \leq \int_{|w|<1} |w|^{2(q_1-1)} \, dx + \int_{|w|\geq 1} |w|^{2(q_2-1)} \, dx$$

$$\leq \|w\|_{2(q_1-1)}^{2(q_1-1)} + \|w\|_{2(q_2-1)}^{2(q_2-1)}$$

$$\leq b_2 \|\Delta w\|_2^{2(q_1-1)} + b_3 \|\Delta w\|_2^{2(q_2-1)} \tag{3.41}$$

for some $b_2, b_3 > 0$. From (3.39), (3.40), (3.41), and (3.37), we arrive at

$$D'(t) \leq 2\|w_t\|_2^2 + b_1 \|\Delta w\|_2^6 + b_2 \|\Delta w\|_2^{2(q_1-1)} + b_3 \|\Delta w\|_2^{2(q_2-1)}$$

$$\leq 2D(t) + d_1 \left(D(t) \right)^3 + d_2 \left(D(t) \right)^{q_1-1} + d_3 \left(D(t) \right)^{q_2-1}$$

for some $d_1, d_2, d_3 > 0$. Using the integration of substitution and (3.38), we get (3.36). □

4 Conclusion

In this paper, the author considered a viscoelastic von Karman equation with strong damping and variable exponent source terms. We showed that the solutions with three levels of initial energy such as non-positive initial energy, certain positive initial energy, and high initial energy blow up in a finite time. Moreover, we estimated not only the upper bound but also the lower bound of the blow-up time.

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