Lin et al. *Boundary Value Problems* (2021) 2021:79
https://doi.org/10.1186/s13661-021-01554-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Multiple solutions for singular semipositone boundary value problems of fourth-order differential systems with parameters

Longfei Lin[1], Yansheng Liu[1*] and Daliang Zhao[1]

*Correspondence:
ysliu@sdnu.edu.cn
[1]School of Mathematics and Statistics, Shandong Normal University, Jinan, P.R. China

Abstract

The aim of this paper is to establish some results about the existence of multiple solutions for the following singular semipositone boundary value problem of fourth-order differential systems with parameters:

$$\begin{cases} u^{(4)}(t) + \beta_1 u''(t) - \alpha_1 u(t) = f_1(t, u(t), v(t)), & 0 < t < 1; \\ v^{(4)}(t) + \beta_2 v''(t) - \alpha_2 v(t) = f_2(t, u(t), v(t)), & 0 < t < 1; \\ u(0) = u(1) = u''(0) = u''(1) = 0; \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases}$$

where $f_1, f_2 \in C[(0,1) \times \mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}]$, $\mathbb{R}_0^+ = (0, +\infty)$. By constructing a special cone and applying fixed point index theory, some new existence results of multiple solutions for the considered system are obtained under some suitable assumptions. Finally, an example is worked out to illustrate the main results.

Keywords: Multiple solutions; Singular semipositone problems; Cone; Fixed point index

1 Introduction

In the recent decades, the topic about the existence of solutions of nonlinear boundary value problems (BVPs for short) has received considerable popularity due to its wide applications in biology, hydrodynamics, physics, chemistry, control theory, and so forth. Some progress has also been made in the study of solutions for various types of equations or systems including differential equation [13, 21, 25, 27], integro-differential equation [2, 19, 27], evolution equations [1, 7], fractional systems [3, 15, 17, 22–24, 30, 31], impulsive systems [14, 18, 28], and delay systems [14]. In consequence, many meaningful results have been obtained in these fields. For more details, please see Lakshmikantham et al. [8], Podlubny [16], and the references therein.

As a branch of research on boundary value problems, singular boundary value problems arise from many fields, such as nuclear physics, biomathematics, mechanics or engineering, and play an extremely important role in both theoretical developments and practical applications [5, 10–12, 26, 29, 32]. Moreover, extensive attention has been drawn to the study of singular semipositone boundary value problems (SBVPs for short) to differential equations or systems recently. For example, in [12] Y. Liu investigated the existence of two positive solutions to the singular semipositone problem

$$\begin{cases} y'' + \lambda f(t,y) = 0, \quad 0 < t < 1; \\ y(0) = y(1) = 0, \end{cases}$$

where $f \in C[J \times \mathbb{R}_0^+, \mathbb{R}]$, $J = (0,1)$, $\mathbb{R}_0^+ = (0,+\infty)$, and the parameter $\lambda > 0$. The nonlinear term f may be singular at $t = 0$, $t = 1$ and $y = 0$. By constructing a special cone, the existence of multiple positive solutions was obtained under some suitable assumptions. In [32], Zhu et al. considered the existence of positive solutions of the two-point boundary value problem for nonlinear singular semipositone systems

$$\begin{cases} x^{(4)} = f(t,x(t),y(t),x''(t),y''(t)), \quad 0 < t < 1; \\ y^{(4)} = g(t,x(t),y(t),x''(t),y''(t)), \quad 0 < t < 1; \\ x(0) = x(1) = x''(0) = x''(1) = 0; \\ y(0) = y(1) = y''(0) = y''(1) = 0, \end{cases}$$

where $f,g \in C[J \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^-, \mathbb{R}]$ may be singular at $t = 0$ or $t = 1$, not singular at $u = 0$, $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{R}^- = (-\infty,0]$. By applying the fixed point theory in cones, the existence results of positive solutions were established.

As we all know, fourth-order boundary value problems have important practical applications in physics and engineering, and, for instance, they are usually used to describe the deformation of an elastic beam supported at the end points [4, 9, 20]. Wang et al. [20] investigated the boundary value problems of a class of fourth-order differential systems with parameters as follows:

$$\begin{cases} u^{(4)}(t) + \beta_1 u''(t) - \alpha_1 u(t) = f_1(t,u(t),v(t)), \quad 0 < t < 1; \\ v^{(4)}(t) + \beta_2 v''(t) - \alpha_2 v(t) = f_2(t,u(t),v(t)), \quad 0 < t < 1; \\ u(0) = u(1) = u''(0) = u''(1) = 0; \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases} \tag{1.1}$$

where $f_1,f_2 \in C[[0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$, and $\beta_i, \alpha_i \in \mathbb{R}$ $(i = 1,2)$ satisfying

$$\beta_i < 2\pi^2, \qquad -\beta_i/4 \le \alpha_i, \qquad \alpha_i/\pi^4 + \beta_i/\pi^2 < 1. \tag{1.2}$$

The existence results of positive solutions were proved by using the fixed point theory under two novel cones being constructed.

Unfortunately, the result obtained in [20] is only the existence of at least one nontrivial positive solution when the nonlinear terms have no singularity. It should be stressed also in [20] that the solutions of BVPs (1.1) are all positive and the nonlinear terms must be nonnegative, which is limited to a certain extent in some cases. Besides, we know that there is always some connection between the nonlinear terms in practical applications, but the description of this connection is rarely mentioned and studied in the present literature. To our best knowledge, there is no paper considering SBVPs (1.1) when $f_1(t,u,v)$ and $f_2(t,u,v)$ are singular at $t = 0$, $t = 1$, and $u = 0$, and also no result is available about the existence of multiple solutions for such boundary value problems.

Motivated by all the above analyses, in this paper we discuss the existence and multiplicity of solutions to SBVPs (1.1) when the parameters $\beta_i, \alpha_i \in \mathbb{R}$ $(i = 1,2)$ satisfy condition (1.2). In addition, $f_1, f_2 \in C[(0,1) \times \mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}]$, namely $f_1(t,u,v)$ and $f_2(t,u,v)$ may be singular at $t = 0$, $t = 1$ and $u = 0$, and f_1, f_2 are semipositone rather than positive with some connection imposed between them. Our approaches are based on the approximation method and the well-known fixed point index theory.

Obviously, what we consider is more different from [20] and [32]. The main features of the present work are as follows. Firstly, $f_1(t,u,v)$ and $f_2(t,u,v)$ may be singular at both $t = 0$, $t = 1$ and $u = 0$, and under some suitable assumptions, the multiple nontrivial solutions for SBVPs (1.1) are established. Secondly, f_1 may be negative for some values of t, u, and v; f_2 is also allowed to change sign. Moreover, f_2 is controlled by f_1. Thirdly, in the obtained solution (u,v), the component u is positive, but the component v is allowed to have different signs, even may be negative.

The rest of the present work is organized as follows. Section 2 contains some preliminaries. In Sect. 3, some transformations are introduced to convert SBVPs (1.1) into the corresponding approximate boundary value problems. The main results will be given and proved in Sect. 4. Finally, in Sect. 5, an example is given to demonstrate the main result.

2 Preliminaries

In view of condition (1.2), as in [9], denote

$$\xi_{i,1} = \frac{-\beta_i + \sqrt{\beta_i^2 + 4\alpha_i}}{2}, \qquad \xi_{i,2} = \frac{-\beta_i - \sqrt{\beta_i^2 + 4\alpha_i}}{2} \quad (i = 1,2),$$

and let $G_{i,j}(t,s)$ $(i,j = 1,2)$ be the Green function of the linear boundary value problem

$$\begin{cases} -u_i''(t) + \xi_{i,j} u_i(t) = 0, & 0 < t < 1; \\ u_i(0) = u_i(1) = 0, & i,j = 1,2. \end{cases}$$

Then, for $h_i \in C[0,1]$, the solution to the following linear boundary value problem

$$\begin{cases} u_i^{(4)}(t) + \beta_i u_i''(t) - \alpha_i u_i(t) = h_i(t), & 0 < t < 1; \\ u_i(0) = u_i(1) = u_i''(0) = u_i''(1) = 0, & i,j = 1,2 \end{cases}$$

can be expressed as

$$u_i(t) = \int_0^1 \int_0^1 G_{i,1}(t,\tau) G_{i,2}(\tau,s) h_i(s)\, ds\, d\tau, \quad t \in [0,1]. \tag{2.1}$$

Lemma 2.1 *The function $G_{i,j}(t,s)$ $(i = 1,2)$ has the following properties*:

(1) $G_{i,j}(t,s) = G_{i,j}(s,t)$ *and* $G_{i,j}(t,s) > 0$ *for* $t,s \in (0,1)$;

(2) $G_{i,j}(t,s) \le C_{i,j} G_{i,j}(s,s)$ *for* $t,s \in [0,1]$, *where* $C_{i,j} > 0$ *is a constant*;

(3) $G_{i,j}(t,s) \geq \delta_{i,j} G_{i,j}(t,t) G_{i,j}(s,s)$ *for* $t,s \in [0,1]$, *where* $\delta_{i,j} > 0$ *is a constant*;
(4) $G_{2,j}(t,s) \leq N_j G_{1,j}(t,s)$ *for* $t,s \in [0,1]$, *where* $N_j > 0$ *is a constant*.

Proof (1)–(3) can be seen from [9]. In addition, by careful calculation and Lemma 2.1 in [9], it is not difficult to prove that $N_j := \sup_{0<t,s<1} \frac{G_{2,j}(t,s)}{G_{1,j}(t,s)} < +\infty$. Immediately, (4) is derived. □

The main tool used here is the following fixed point index theory.

Lemma 2.2 ([6]) *Let* E_1 *be a Banach space and* P *be a cone in* E_1. *Denote* $P_r = \{u \in P : \|u\| < r\}$ *and* $\partial P_r = \{u \in P : \|u\| = r\}$ $(\forall r > 0)$. *Let* $T : P \to P$ *be a complete continuous mapping, then the following conclusions are valid.*
(1) *If* $\mu Tu \neq u$ *for* $u \in \partial P_r$ *and* $\mu \in (0,1]$, *then* $i(T, P_r, P) = 1$;
(2) *If* $\inf_{u \in \partial P_r} \|Tu\| > 0$ *and* $\mu Tu \neq u$ *for* $u \in \partial P_r$ *and* $\mu \geq 1$, *then* $i(T, P_r, P) = 0$.

3 Conversion of boundary value problem (1.1)

In order to overcome the difficulties arising from singularity and semipositone, we convert boundary value problem (1.1) into another form (see (3.4)). For simplicity and convenience, set

$$C_1 = \int_0^1 G_{1,1}(\tau,\tau) G_{1,2}(\tau,\tau)\, d\tau, \qquad M_{i,j} = \max_{t \in [0,1]} G_{i,j}(t,t).$$

Then C_1 and $M_{i,j}$ $(i,j = 1,2)$ are positive numbers.

Now let us list the following assumptions which will be satisfied throughout the paper.

(H1) There exist functions $p \in L^1[J, \mathbb{R}^+]$ such that

$$f_1(t,u,v) + p(t) \geq 0, \quad \forall (t,u,v) \in J \times \mathbb{R}_0^+ \times \mathbb{R}.$$

(H2) $f_1, f_2 \in C[J \times \mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}]$, and there exists $N_3 > 0$ such that

$$\left| f_2(t,u,v) \right| \leq N_3 \cdot \left[f_1(t,u,v) + p(t) \right].$$

In this paper, the basic space is $E := C[0,1] \times C[0,1]$. It is a Banach space endowed with the norm $\|(u,v)\| = \max\{N\|u\|, \|v\|\}$ for $(u,v) \in E$, where $\|u\| = \max_{t\in[0,1]} |u(t)|$, $\|v\| = \max_{t\in[0,1]} |v(t)|$, and $N := N_1 N_2 N_3$. N_1, N_2, and N_3 are defined in Lemma 2.1 and (H2), respectively.

Moreover, let

$$\begin{aligned}
&\pounds\left[u(t)\right] = u^{(4)}(t) + \beta_1 u''(t) - \alpha_1 u(t), \\
&\Im\left[u(t)\right] = u^{(4)}(t) + \beta_2 u''(t) - \alpha_2 u(t), \\
&r_p = \frac{[C_{1,1}C_{1,2}]^2 M_{1,1} M_{1,2}}{\delta_{1,1}\delta_{1,2} C_1} \int_0^1 p(s)\, ds.
\end{aligned} \tag{3.1}$$

Define a function $w : [0,1] \to \mathbb{R}^+$ by

$$w(t) = \int_0^1 \int_0^1 G_{1,1}(t,\tau) G_{1,2}(\tau,s) p(s)\, ds\, d\tau.$$

Applying (3.1) and Lemma 2.1, one can easily obtain that

$$\begin{aligned} w(t) &= \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)p(s)\,ds\,d\tau \\ &\le C_{1,1}C_{1,2}M_{1,2}G_{1,1}(t,t)\int_0^1 p(s)\,ds = r_p\frac{\delta_{1,1}\delta_{1,2}C_1}{C_{1,1}C_{1,2}M_{1,1}}G_{1,1}(t,t), \quad t\in[0,1]. \end{aligned} \tag{3.2}$$

This together with (2.1) guarantees that $w(t)$ is the positive solution of the following boundary value problem:

$$\begin{cases} \pounds[w(t)] = p(t), \quad 0<t<1; \\ w(0) = w(1) = w''(0) = w''(1) = 0. \end{cases} \tag{3.3}$$

Now we are in a position to convert SBVPs (1.1) into an approximate boundary value problem. For this matter, it will be carried out in two steps.

Firstly, in order to overcome the difficulties arising from semipositone, consider the following singular nonlinear differential boundary value problem:

$$\begin{cases} \pounds[u(t)] = f_1(t,[u(t)-w(t)]^*,v(t)) + p(t), \quad 0<t<1; \\ \Im[v(t)] = f_2(t,[u(t)-w(t)]^*,v(t)), \quad 0<t<1; \\ u(0) = u(1) = u''(0) = u''(1) = 0; \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases} \tag{3.4}$$

where

$$\big[u(t)\big]^* = \begin{cases} u(t), & u(t)\ge 0; \\ 0, & u(t)\le 0. \end{cases}$$

Then we can obtain the following conclusion.

Lemma 3.1 *Assume that* (u,v) *is a solution of BVPs* (3.4) *and* $u(t)\ge w(t)$ *for* $t\in[0,1]$. *Then* (u,v) *is a solution of SBVPs* (1.1).

Proof Let the vector (u,v) be a solution of BVPs (3.4) and $u(t)\ge w(t)$ for $t\in[0,1]$. Then the definition of function $[\cdot]^*$ together with (3.4) guarantees that

$$\begin{cases} \pounds[u(t)] = f_1(t,u(t)-w(t),v(t)) + p(t), \quad 0<t<1; \\ \Im[v(t)] = f_2(t,u(t)-w(t),v(t)), \quad 0<t<1; \\ u(0) = u(1) = u''(0) = u''(1) = 0; \\ v(0) = v(1) = v''(0) = v''(1) = 0. \end{cases} \tag{3.5}$$

Set $u_1 = u-w$, $v_1 = v$. Then $\pounds[u_1(t)] = \pounds[u(t)] - \pounds[w(t)]$ and $\Im[v_1(t)] = \Im[v(t)]$, which implies

$$\begin{aligned} &\pounds\big[u(t)\big] = \pounds\big[u_1(t)\big] + \pounds\big[w(t)\big] = \pounds\big[u_1(t)\big] + p(t), \\ &\Im\big[v(t)\big] = \Im\big[v_1(t)\big], \quad t\in[0,1]. \end{aligned}$$

So, (3.3) together with (3.5) guarantees that

$$\begin{cases} \pounds[u_1(t)] = f_1(t, u_1(t), v_1(t)), & 0 < t < 1; \\ \Im[v_1(t)] = f_2(t, u_1(t), v_1(t)), & 0 < t < 1; \\ u_1(0) = u_1(1) = u_1''(0) = u_1''(1) = 0; \\ v_1(0) = v_1(1) = v_1''(0) = v_1''(1) = 0. \end{cases} \tag{3.6}$$

Namely $(u_1, v_1) = (u - w, v)$ is a solution of SBVPs (1.1). □

Secondly, in order to overcome the singularity associated with SBVPs (1.1), consider the following approximate boundary value problem:

$$\begin{cases} \pounds[u(t)] = f_1^j(t, [u(t) - w(t)]_j^*, v(t)) + p(t), & 0 < t < 1; \\ \Im[v(t)] = f_2^j(t, [u(t) - w(t)]_j^*, v(t)), & 0 < t < 1; \\ u(0) = u(1) = u''(0) = u''(1) = 0; \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases} \tag{3.7}$$

where

$$f_i^j\left(t, [u]_j^*, v)\right] = \begin{cases} f_i(t, u + \frac{1}{j}, v), & u \geq 0; \\ f_i(t, \frac{1}{j}, v), & u < 0 \ (i = 1, 2; j \in \mathbb{N}). \end{cases}$$

In the following, we shall mainly discuss the existence results for BVPs (3.7) by using the fixed point index theory. For this matter, first we define the following mappings:

$$\begin{aligned} T_1^j(u, v)(t) &= \int_0^1 \int_0^1 G_{1,1}(t, \tau) G_{1,2}(\tau, s) \left[f_1^j\left(s, \left[u(s) - w(s)\right]_j^*, v(s)\right) + p(s)\right] ds\, d\tau, \\ T_2^j(u, v)(t) &= \int_0^1 \int_0^1 G_{2,1}(t, \tau) G_{2,2}(\tau, s) f_2^j\left(s, \left[u(s) - w(s)\right]_j^*, v(s)\right) ds\, d\tau, \\ T^j(u, v)(t) &= \left(T_1^j(u, v)(t), T_2^j(u, v)(t)\right), \quad \forall t \in [0, 1], (u, v) \in E, j \in \mathbb{N}. \end{aligned} \tag{3.8}$$

Obviously, it is easy to see that the existence of nontrivial solutions for BVPs (3.7) is equivalent to the existence of the nontrivial fixed point of T^j. Therefore, we just need to find the nontrivial fixed point of T^j in the following work.

For the sake of obtaining the nontrivial fixed point of operator T^j, set

$$P = \left\{(u, v) \in E : u(t) \geq \sigma(t)\|u\| \text{ and } \left|v(t)\right| \leq Nu(t), \forall t \in [0, 1]\right\},$$

where $\sigma(t) = \frac{\delta_{1,1}\delta_{1,2}C_1}{C_{1,1}C_{1,2}M_{1,1}} G_{1,1}(t, t)$ and $N = N_1N_2N_3$. N_1, N_2, and N_3 are defined in Lemma 2.1 and (H2), respectively.

Evidently, P is a nonempty, convex, and closed subset of E. Furthermore, one can prove that P is a cone of Banach space E. For simplicity, denote

$$P_r := \left\{(u, v) \in P : \left\|(u, v)\right\| < r\right\}.$$

Then, by the definition of cone P and the norm $\|(u,v)\|$, one can see that

$$\partial P_r := \{(u,v) \in P : \|(u,v)\| = r\} = \left\{(u,v) \in P : \|u\| = \frac{r}{N}\right\},$$
$$\overline{P_r} := \{(u,v) \in P : \|(u,v)\| \le r\} = \left\{(u,v) \in P : \|u\| \le \frac{r}{N}\right\}.$$

Clearly, for each $r > 0$, P_r is a relatively open and bounded set of P.

4 Main results

In this section, we present the main results of this paper. To do this, first we need to investigate the properties of mapping T^j ($j \in \mathbb{N}$).

Lemma 4.1 *Assume that* (H1) *and* (H2) *hold. Then, for any* $j \in \mathbb{N}$, $T^j : P \to P$ *is completely continuous and* $T^j(P) \subset P$.

Proof For $(u,v) \in P$, by virtue of Lemma 2.1, one can easily get that

$$\begin{aligned} T_1^j(u,v)(t) &= \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\big[f_1^j\big(s,[u(s)-w(s)]_j^*, v(s)\big) + p(s)\big]\,ds\,d\tau \\ &\ge \frac{\delta_{1,1}\delta_{1,2}C_1}{C_{1,1}C_{1,2}M_{1,1}} G_{1,1}(t,t)\big\|T_1^j(u,v)\big\| = \sigma(t)\big\|T_1^j(u,v)\big\|, \quad \forall t \in [0,1], j \in \mathbb{N}. \end{aligned}$$

Moreover, (H2) together with Lemma 2.1 implies that

$$\begin{aligned} \big|T_2^j(u,v)(t)\big| &= \left|\int_0^1 \int_0^1 G_{2,1}(t,\tau)G_{2,2}(\tau,s)f_2^j\big(s,[u(s)-w(s)]_j^*, v(s)\big)\,ds\,d\tau\right| \\ &\le N_3 \int_0^1 \int_0^1 G_{2,1}(t,\tau)G_{2,2}(\tau,s)f_1^j\big(s,[u(s)-w(s)]_j^*, v(s)\big)\,ds\,d\tau \\ &\le N_1N_2N_3 \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)f_1^j\big(s,[u(s)-w(s)]_j^*, v(s)\big)\,ds\,d\tau \\ &= N\big|T_1^j(u,v)(t)\big|, \quad \forall t \in [0,1], j \in \mathbb{N}. \end{aligned}$$

Therefore, $T^j(u,v) \in P$, namely $T^j(P) \subset P$. In addition, notice that f_1, f_2, and $G_{i,j}$ are continuous, one can deduce that T^j is completely continuous for each $j \in \mathbb{N}$ by using normal methods such as Ascoli–Arzela theorem, etc. □

For convenience of expression, for each $R_1 > r_1 > Nr_p$, take

$$\Lambda_{[r_1,R_1](t)} = \left(\left(\frac{r_1}{N} - r_p\right)\sigma(t), \frac{R_1}{N} + 1\right] \times [-R_1, R_1],$$

where r_p is defined in (3.1).

At the same time, define a functional $\digamma : L^1(J) \to \mathbb{R}^+$ by

$$\digamma(y) = \max_{t\in[0,1]} \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)y(s)\,ds\,d\tau, \quad \text{for } y \in L^1(J). \tag{4.1}$$

Next, let us list the following assumptions which will be used in what follows.

(H1′) For each $R_1 > r_1 > Nr_p$, there exists $\Psi_{r_1,R_1} \in L^1(J)$ such that

$$0 \leq f_1(t,u,v) + p(t) \leq \Psi_{r_1,R_1}(t), \quad \text{for } \forall (t,u,v) \in J \times \Lambda_{[r_1,R_1](t)}. \tag{4.2}$$

(H3) There exist $R > r > Nr_p$ and function Φ_r such that
(1) $f_1(t,u,v) + p(t) \geq \Phi_r(t)$, $\forall (t,u,v) \in J \times \Lambda_{[r,r](t)}$;
(2) $\digamma(\Psi_{R,R}) < \frac{R}{N}$, $\digamma(\Phi_r) > \frac{r}{N}$.

Now we are in a position to give the following two lemmas to calculate the fixed point index of T^j ($j \in \mathbb{N}$) in P_r.

Lemma 4.2 *Assume that* (H1′) *and* (H2)–(H3) *hold. Then the following conclusions are valid*:
(i) *For any* $j \in \mathbb{N}$, $i(T^j, P_r, P) = 0$;
(ii) *For any* $j \in \mathbb{N}$, $i(T^j, P_R, P) = 1$.

Proof (i) For the sake of obtaining the desired result, we firstly prove that

$$\inf_{(u,v)\in\partial P_r} \|T^j(u,v)\| > 0 \quad \text{and} \quad (u,v) \neq \mu T^j(u,v), \quad \forall (u,v) \in \partial P_r, \mu \geq 1 \text{ and } j \in \mathbb{N}. \tag{4.3}$$

In fact, if it is not true, then there exist $\mu_0 \geq 1$ and $(u_0,v_0) \in \partial P_r$ such that $(u_0,v_0) = \mu_0 T^j(u_0,v_0)$. By (3.1), (3.2), and the definition of cone P, one can obtain that

$$u_0(t) \geq \sigma(t)\|u_0\| = \frac{r}{N}\sigma(t), \qquad w(t) \leq r_p\sigma(t), \quad t \in [0,1].$$

That is,

$$u_0(t) - w(t) \geq \frac{r}{N}\sigma(t) - r_p\sigma(t) = \left(\frac{r}{N} - r_p\right)\sigma(t) \geq 0.$$

Moreover, by the definition of function $[\cdot]_j^*$, we have

$$\begin{aligned}
&[u_0(t) - w(t)]_j^* = u_0(t) - w(t) + \frac{1}{j} \leq u_0(t) + \frac{1}{j} \leq \frac{r}{N} + 1, \\
&|v_0(t)| \leq Nu_0(t) \leq r, \quad \forall t \in [0,1], j \in \mathbb{N},
\end{aligned} \tag{4.4}$$

which means

$$\left([u_0(t) - w(t)]_j^*, v_0(t)\right) \in \Lambda_{[r,r](t)}, \quad \forall t \in [0,1], j \in \mathbb{N}.$$

Hence, applying $(u_0,v_0) = \mu_0 T^j(u_0,v_0)$ and (H3), we obtain immediately that

$$\begin{aligned}
u_0(t) &= \mu_0 T_1^j(u_0,v_0)(t) \geq T_1^j(u_0,v_0)(t) \\
&= \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\left[f_1^j\left(t,[u_0(t) - w(t)]_j^*, v_0(t)\right) + p(t)\right] ds\, d\tau \\
&\geq \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\Phi_r(s)\, ds\, d\tau.
\end{aligned} \tag{4.5}$$

Taking the maximum for both sides of (4.5) in $[0,1]$, we get

$$\|u_0\| \geq \mathcal{F}(\Phi_r) > \frac{r}{N}, \quad j \in \mathbb{N}.$$

This is in contradiction with $(u_0, v_0) \in \partial P_r$. Besides, it is clear that $\inf_{(u,v)\in\partial P_r} \|T^j(u,v)\| > 0$ by (4.5), and then (4.3) holds.

(ii) Next, we claim that

$$(u,v) \neq \mu T^j(u,v), \quad \forall (u,v) \in \partial P_R, \mu \in (0,1], \text{and } j \in \mathbb{N}. \tag{4.6}$$

Suppose on the contrary that there exist $\mu_\lambda \in (0,1]$ and $(u_\lambda, v_\lambda) \in \partial P_R$ such that $(u_\lambda, v_\lambda) = \mu_\lambda T^j(u_\lambda, v_\lambda)$. Using a similar process of the proof as (i), we immediately get that

$$\begin{aligned} &\left|v_\lambda(t)\right| \leq N u_\lambda(t) \leq R, \\ &u_\lambda(t) - w(t) \geq \left(\frac{R}{N} - r_p\right)\sigma(t) \geq 0, \\ &\left[u_0(t) - w(t)\right]_j^* = u_0(t) - w(t) + \frac{1}{j} \leq u_0(t) + \frac{1}{j} \leq \frac{R}{N} + 1, \end{aligned} \tag{4.7}$$

which indicates

$$\left(\left[u_0(t) - w(t)\right]_j^*, v_0(t)\right) \in \Lambda_{[R,R](t)}, \quad \forall t \in [0,1], j \in \mathbb{N}.$$

In addition, $(u_\lambda, v_\lambda) = \mu_\lambda T^j(u_\lambda, v_\lambda)$ together with (4.2), (4.7), and (H3) deduces that

$$\begin{aligned} u_\lambda(t) &= \mu_\lambda T_1^j(u_\lambda, v_\lambda)(t) \leq T_1^j(u_\lambda, v_\lambda)(t) \\ &= \int_0^1 \int_0^1 G_{1,1}(t,\tau) G_{1,2}(\tau,s) \left[f_1^j\left(s, \left[u_\lambda(s) - w(s)\right]_j^*, v_\lambda(s)\right) + p(s)\right] ds\, d\tau \\ &\leq \int_0^1 \int_0^1 G_{1,1}(t,\tau) G_{1,2}(\tau,s) \Psi_{R,R}(s)\, ds\, d\tau, \end{aligned}$$

which implies

$$\|u_\lambda\| \leq \mathcal{F}(\Psi_{R,R}) < \frac{R}{N}, \quad j \in \mathbb{N}.$$

This is in contradiction with $(u_\lambda, v_\lambda) \in \partial P_R$. Therefore, (4.6) holds. To sum up, the proof is complete. □

Lemma 4.3 *Assume that* (H1′) *and* (H2)–(H3) *hold. In addition, suppose that*:

(H4) *There exists an interval* $[\alpha, \beta] \subset J$ *such that*

$$\lim_{\substack{|v| \leq Nu \\ u \to +\infty}} \min_{t \in [\alpha,\beta]} \frac{f_1(t,u,v)}{u} = +\infty.$$

Then there exists a constant $R^* > R$ *such that* $i(T^j, P_{R^*}, P) = 0$ *for each* $j \in \mathbb{N}$.

Proof First, choose a positive number Υ satisfying

$$\Upsilon > 2\left(\min_{t\in[\alpha,\beta]}\sigma(t)\cdot\max_{t\in[0,1]}\int_\alpha^\beta\int_\alpha^\beta G_{1,1}(t,\tau)G_{1,2}(\tau,s)\,ds\,d\tau\right)^{-1}. \tag{4.8}$$

Then, by (H4), it is easy to see that there exists $\ell > \frac{R}{N}$ such that

$$\frac{f_1(t,u,v)}{u} \geq \Upsilon, \quad \forall t\in[\alpha,\beta], u\geq \ell, |v|\leq Nu. \tag{4.9}$$

Let R^* be a positive number satisfying $R^* > \frac{2N\ell}{\min_{t\in[\alpha,\beta]}\sigma(t)}$. Then

$$\frac{2R}{N} < 2\ell < \frac{R^*}{N}. \tag{4.10}$$

Next we show

$$(u,v)\neq \mu T^j(u,v), \quad \forall (u,v)\in\partial P_{R^*}, \mu\geq 1, \text{and } j\in\mathbb{N}. \tag{4.11}$$

In fact, if it is not true, then there exist $\mu_0\geq 1$ and $(u_0,v_0)\in\partial P_{R^*}$ such that $(u_0,v_0) = \mu_0 T^j(u_0,v_0)$. Therefore, for any $t\in[\alpha,\beta]$, by (3.1), (3.2), and (4.10), one can easily get that

$$\begin{aligned} u_0(t)-w(t) &\geq u_0(t)-r_p\sigma(t)\geq u_0(t)-\frac{R}{N}\sigma(t)\\ &\geq u_0(t)-\frac{R}{N}\cdot\frac{N}{R^*}u_0(t) = u_0(t)-\frac{R}{R^*}u_0(t)\\ &\geq \frac{u_0(t)}{2}\geq\frac{R^*}{2N}\cdot\min_{t\in[\alpha,\beta]}\sigma(t) > \ell > 0. \end{aligned} \tag{4.12}$$

Hence, from (4.9) and (4.12), we have

$$\begin{aligned} u_0(t) &= \mu_0 T_1^j(u_0,v_0)(t)\geq T_1^j(u_0,v_0)(t)\\ &= \int_0^1\int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\big[f_1^j\big(t,\big[u_0(s)-w(s)\big]_j^*,v_0(s)\big)+p(s)\big]\,ds\,d\tau\\ &= \int_0^1\int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\left[f_1\left(t,u_0(s)-w(s)+\frac{1}{j},v_0(s)\right)+p(s)\right]ds\,d\tau\\ &\geq \int_0^1\int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\cdot f_1\left(t,u_0(s)-w(s)+\frac{1}{j},v_0(s)\right)ds\,d\tau\\ &\geq \Upsilon\cdot\int_\alpha^\beta\int_\alpha^\beta G_{1,1}(t,\tau)G_{1,2}(\tau,s)\cdot\left(u_0(s)-w(s)+\frac{1}{j}\right)ds\,d\tau\\ &\geq \frac{\Upsilon R^*}{2N}\min_{t\in[\alpha,\beta]}\sigma(t)\int_\alpha^\beta\int_\alpha^\beta G_{1,1}(t,\tau)G_{1,2}(\tau,s)\,ds\,d\tau. \end{aligned} \tag{4.13}$$

Consequently, by (4.8) and (4.13), we immediately obtain that

$$\|u_0\|\geq\frac{\Upsilon R^*}{2N}\min_{t\in[\alpha,\beta]}\sigma(t)\cdot\left(\max_{t\in[\alpha,\beta]}\int_\alpha^\beta\int_\alpha^\beta G_{1,1}(t,\tau)G_{1,2}(\tau,s)\,ds\,d\tau\right) > \frac{R^*}{N}.$$

This is in contradiction with $(u_0, v_0) \in \partial P_{R^*}$. Moreover, in view of (4.13) we know that $\inf_{(u,v)\in\partial P_{R^*}} \|T^j(u,v)\| > 0$. So, by Lemma 2.2, the conclusion of this lemma follows. □

Now, we are in a position to prove the main theorem of the present paper.

Theorem 4.4 *Under assumptions* (H1′) *and* (H2)–(H4), *SBVPs* (1.1) *admits at least two nontrivial solutions.*

Proof This proof will be carried out in four steps.

Claim 1 *System* (3.7) *has at least two nontrivial solutions.*

In fact, applying Lemmas 4.2–4.3 and the additivity of the fixed point index, one can get for any $j \in \mathbb{N}$ that

$$\begin{aligned} &i\bigl(T^j, P_{R^*} \setminus \overline{P_R}, P\bigr) = i\bigl(T^j, P_{R^*}, P\bigr) - i\bigl(T^j - P_R, P\bigr) = 0 - 1 = -1, \\ &i\bigl(T^j, P_R \setminus \overline{P_r}, P\bigr) = i\bigl(T^j, P_R, P\bigr) - i\bigl(T^j - P_r, P\bigr) = 1 - 0 = 1. \end{aligned}$$

So, there exist $(u_j, v_j) \in P_{R^*} \setminus \overline{P_R}$ and $(U_j, V_j) \in P_R \setminus \overline{P_r}$ satisfying

$$(u_j, v_j) = T^j(u_j, v_j) \quad \text{and} \quad (U_j, V_j) = T^j(U_j, V_j), \quad j \in \mathbb{N}.$$

Namely, system (3.7) has at least two nontrivial solutions satisfying

$$r < N\|u_j\| < R < N\|U_j\| < R^*. \tag{4.14}$$

Claim 2 $\{(u_j, v_j)\}_{j\in\mathbb{N}}$ *and* $\{(U_j, V_j)\}_{j\in\mathbb{N}}$ *are bounded equicontinuous families on* $[0, 1]$.

Notice that the boundedness is obvious. To prove the equicontinuity, let us prove that $\{u_j\}_{j\in\mathbb{N}}$ are equicontinuous on $[0, 1]$ first. Since

$$u_j(t) - w(t) \geq \left(\frac{r}{N} - r_p\right)\sigma(t) \geq 0, \tag{4.15}$$

applying (4.2) and (4.15), we get that, for any $0 < t_1 < t_2 < 1$ and $j \in \mathbb{N}$,

$$\begin{aligned} \bigl|u_j(t_1) - u_j(t_2)\bigr| &= \bigl|T_1^j(u_j, v_j)(t_1) - T_1^j(u_j, v_j)(t_2)\bigr| \\ &= \int_0^1 \int_0^1 \bigl|G_{1,1}(t_1, \tau) - G_{1,1}(t_2, \tau)\bigr| \cdot G_{1,2}(\tau, s) \\ &\quad \cdot \bigl[f_1\bigl(s, \bigl[u_j(s) - w(s)\bigr]_j^*, v_j(s)\bigr) + p(s)\bigr]\, ds\, d\tau \\ &\leq C_{1,2} \max_{t\in[0,1]} G_{1,2}(t, t) \int_0^1 \int_0^1 \bigl|G_{1,1}(t_1, \tau) - G_{1,1}(t_2, \tau)\bigr| \\ &\quad \cdot \Psi_{r,R^*}(s)\, ds\, d\tau \\ &\leq C_{1,2} \max_{t\in[0,1]} G_{1,2}(t, t) \cdot \int_0^1 \Psi_{r,R^*}(s)\, ds \\ &\quad \cdot \int_0^1 \bigl|G_{1,1}(t_1, \tau) - G_{1,1}(t_2, \tau)\bigr|\, d\tau. \end{aligned} \tag{4.16}$$

So, by (H1′), (4.16), and the continuity of $G_{1,1}$, one can easily see that the equicontinuity of $\{u_j\}_{j\in\mathbb{N}}$ holds. From a process similar to the above, we get that the equicontinuity of $\{v_j\}_{j\in\mathbb{N}}$ holds by applying condition (H2). Therefore, $\{(u_j,v_j)\}_{j\in\mathbb{N}}$ is an equicontinuous family on $t\in[0,1]$. Very similarly, $\{(U_j,V_j)\}_{j\in\mathbb{N}}$ is also an equicontinuous family on $[0,1]$.

To sum up, $\{(u_j,v_j)\}_{j\in\mathbb{N}}$ and $\{(U_j,V_j)\}_{j\in\mathbb{N}}$ are the bounded equicontinuous families on $[0,1]$. By the Arzela–Ascoli theorem, there exist subsequences of them such that

$$\begin{aligned}
&(u_{j_n},v_{j_n})\to(u_0,v_0)\quad \text{as } n\to+\infty \text{ in } E,\\
&(U_{j_n},V_{j_n})\to(U_0,V_0)\quad \text{as } n\to+\infty \text{ in } E.
\end{aligned}\tag{4.17}$$

Claim 3 (u_0,v_0) *and* (U_0,V_0) *are nontrivial solutions of BVPs* (3.4).

Since (u_{j_n},v_{j_n}) satisfies the integral equations

$$\begin{aligned}
u_{j_n}(t)&=\int_0^1\int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\bigl[f_1^j\bigl(s,\bigl[u_{j_n}(s)-w(s)\bigr]_j^*,v_{j_n}(s)\bigr)+p(s)\bigr]\,ds\,d\tau,\\
v_{j_n}(t)&=\int_0^1\int_0^1 G_{2,1}(t,\tau)G_{2,2}(\tau,s)f_2^j\bigl(s,\bigl[u_{j_n}(s)-w(s)\bigr]_j^*,v_{j_n}(s)\bigr)\,ds\,d\tau.
\end{aligned}\tag{4.18}$$

From (H1′) and the well-known Lebesgue dominated convergence theorem, one can get that

$$\begin{aligned}
u_0(t)&=\int_0^1\int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\bigl[f_1\bigl(s,\bigl[u_0(s)-w(s)\bigr]^*,v_0(s)\bigr)+p(s)\bigr]\,ds\,d\tau,\\
v_0(t)&=\int_0^1\int_0^1 G_{2,1}(t,\tau)G_{2,2}(\tau,s)f_2\bigl(s,\bigl[u_0(s)-w(s)\bigr]^*,v_0(s)\bigr)\,ds\,d\tau.
\end{aligned}\tag{4.19}$$

Therefore, (u_0,v_0) is a nontrivial solution of BVPs (3.4). Similarly, we also get that (U_0,V_0) is a nontrivial solution of BVPs (3.4). In addition, it is obvious that $u_0(t)-w(t)\geq 0$ and $U_0(t)-w(t)\geq 0$. Then, from Lemma 3.1, we know that $(u_0-w(t),v_0)$ and $(U_0(t)-w(t),V_0(t))$ are the nontrivial solution of SBVPs (1.1).

Claim 4 $(u_0,v_0)\neq(U_0,V_0)$.

Since $(u_0,v_0)\subset P_{R^*}\setminus\overline{P_R}$ and $(U_0,V_0)\subset P_R\setminus\overline{P_r}$, we only need to prove that BVPs (3.4) has no solutions on ∂P_R. Suppose on the contrary that there exists $(\tilde{u},\tilde{v})\in\partial P_R$ satisfying BVPs (3.4). Then

$$\begin{aligned}
\tilde{u}(t)&=\int_0^1\int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\bigl[f_1\bigl(t,\bigl[\tilde{u}(t)-w(t)\bigr]^*,\tilde{v}(t)\bigr)+p(t)\bigr]\,ds\,d\tau\\
&\leq\int_0^1\int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)\Psi_{R,R}(s)\,ds\,d\tau.
\end{aligned}\tag{4.20}$$

Taking the maximum on both sides of (4.20) in $[0,1]$, one can easily obtain that

$$\|\tilde{u}\|\leq\mathcal{F}\bigl(\Psi_{R,R}(s)\bigr)<\frac{R}{N}.$$

This is in contradiction with $(\tilde{u},\tilde{v}) \in \partial P_R$. To sum up, the conclusion of this theorem follows. □

5 An example

In this section, an illustrative example is worked out to show the effectiveness of the obtained result.

Example 5.1 Consider the following boundary value problem of fourth-order differential systems:

$$\begin{cases} u^{(4)}(t) + u''(t) - \pi^2 u(t) = \frac{v}{\sqrt{t(1-t)}}(u^2 + \frac{1}{u}) - \kappa \cos(\frac{\pi t}{2}), & 0 < t < 1; \\ v^{(4)}(t) + \frac{1}{2}v''(t) - \frac{\pi^2}{2}v(t) = \frac{v\cos(t)}{N_1 N_2\sqrt{t(1-t)}}(u^2 + \frac{\sin(t)}{u}), & 0 < t < 1; \\ u(0) = u(1) = u''(0) = u''(1) = 0; \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases} \tag{5.1}$$

where $\kappa = \frac{2[C_{1,1}C_{1,2}]^2 M_{1,1}M_{1,2}}{100\pi\delta_{1,1}\delta_{1,2}C_1}$, $u \in \mathbb{R}_0^+$, and $\frac{1}{2} \le |v| \le 1$.

Conclusion: SBVPs (5.1) has at least two nontrivial solutions.

Proof SBVPs (5.1) can be regarded as the form of SBVPs (1.1), where

$$\alpha_1 = \pi^2, \qquad \beta_1 = 1, \qquad \alpha_2 = \frac{\pi^2}{2}, \qquad \beta_2 = \frac{1}{2},$$
$$f_1(t,u,v) = \frac{v}{\sqrt{t(1-t)}}\left(u^2 + \frac{1}{u}\right) - \kappa\cos(t),$$

and

$$f_2(t,u,v) = \frac{v\cos(t)}{N_1 N_2\sqrt{t(1-t)}}\left(u^2 + \frac{\sin(t)}{u}\right).$$

Then

$$\xi_{1,1} = \frac{-1 + \sqrt{1+4\pi^2}}{2}, \qquad \xi_{1,2} = \frac{-1 - \sqrt{1+4\pi^2}}{2}.$$

Clearly, α_i and β_i $(i = 1,2)$ satisfy condition (1.2). Moreover, by careful calculation and Lemma 2.1 in [9], one can obtain that

$$G_{1,1}(t,s) = \begin{cases} \frac{\sinh w_{1,1}t \sinh w_{1,1}(1-s)}{w_{1,1}\sinh w_{1,1}}, & 0 \le t \le s \le 1; \\ \frac{\sinh w_{1,1}s \sinh w_{1,1}(1-t)}{w_{1,1}\sinh w_{1,1}}, & 0 \le s \le t \le 1, \end{cases}$$

$$G_{1,2}(t,s) = \begin{cases} \frac{\sin w_{1,2}t \sin w_{1,2}(1-s)}{w_{1,2}\sin w_{1,2}}, & 0 \le t \le s \le 1; \\ \frac{\sin w_{1,2}s \sin w_{1,2}(1-t)}{w_{1,2}\sin w_{1,2}}, & 0 \le s \le t \le 1, \end{cases}$$

where $w_{1,i} = \sqrt{|\xi_{1,i}|}$ $(i = 1,2)$.

Take $p(t) = \kappa \cos(\frac{\pi t}{2})$, and simple calculation implies that (H1′) holds. Moreover, (H2) holds by choosing $N_3 = \frac{1}{N_1 N_2}$. For convenience, let

$$\Delta_1 =: 2C_{1,1}C_{1,2} \int_0^1 G_{1,1}(\tau,\tau)\,d\tau \int_0^1 G_{1,2}(s,s)\,ds,$$
$$\Delta_2 =: \delta_{1,1}\delta_{1,2}C_1 \max_{t\in[0,1]} G_{1,1}(t,t) \int_0^1 G_{1,2}(s,s)\,ds.$$

Obviously, it is easy to get that $r_p = \frac{1}{100}$ from (3.1) and $N = 1$. Moreover, choose

$$r = \frac{\sqrt{1+4\Delta_2}+1}{4} > Nr_p = \frac{1}{100}, \qquad \Phi_r(t) = \frac{1}{2(r+1)\sqrt{t(1-t)}},$$

and

$$R > \max\bigl\{(\Delta_1)^{\frac{3}{2}}, Nr\bigr\}, \qquad \Psi_{R,R}(t) = \frac{2(R+1)^2}{\sqrt{t(1-t)}}.$$

Then careful calculation indicates that (H3) holds. From $[\alpha,\beta] \subset (0,1)$, it follows that

$$\lim_{\substack{|v|\le Nu\\ u\to+\infty}} \min_{t\in[\alpha,\beta]} \frac{f_1(t,u,v)}{u} = \lim_{\substack{|v|\le Nu\\ u\to+\infty}} \min_{t\in[\alpha,\beta]} \frac{\frac{v}{\sqrt{t(1-t)}}(u^2+\frac{1}{u})) - \kappa\cos(\frac{\pi t}{2})}{u} = +\infty,$$

which implies that condition (H4) holds. Consequently, SBVPs (5.1) has at least two nontrivial solutions by Theorem 4.4. □

Acknowledgements
The authors are thankful to the editor and anonymous referees for their valuable comments and suggestions.

Funding
Y. Liu was supported by NNSF of P.R. China (62073202) and the Natural Science Foundation of Shandong Province (ZR2020MA007). D. Zhao was supported by a project of Shandong Province Higher Educational Science and Technology Program of China under the grant J18KA233.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Publisher's Note

Received: 13 April 2021 Accepted: 23 August 2021 Published online: 15 September 2021

References
1. Bishop, S.A., Ayoola, E.O., Oghonyon, G.J.: Existence of mild solution of impulsive quantum stochastic differential equation with nonlocal conditions. Anal. Math. Phys. **7**, 255–265 (2017)
2. Chen, Y., Qin, B.: Solutions for *n*th-order boundary value problems of impulsive singular nonlinear integro-differential equations in Banach spaces. Bound. Value Probl. **2013**(1), 128 (2013)
3. Cheng, W., Xu, J., O'Regan, D.: Positive solutions for a nonlinear discrete fractional boundary value problem with a *p*-Laplacian operator. J. Appl. Anal. Comput. **9**(5), 1959–1972 (2019)
4. De, C.C., Fabry, C., Munyamarere, F.: Nonresonance conditions for fourth-order nonlinear boundary value problems. Int. J. Math. Math. Sci. **17**(4), 725–740 (1994)

5. Fan, J., Li, L.: Existence of positive solutions for p-Laplacian dynamic equations with derivative on time scales. J. Appl. Math. **2013**, 736583 (2013)
6. Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cones. Academic Press, New York (1988)
7. Indhumathi, P., Leelamani, A.: Existence and uniqueness results for mild solutions of random impulsive abstract neutral partial differential equation over real axis. Appl. Math. J. Chin. Univ. **33**(1), 71–87 (2018)
8. Lakshmikantham, V., Leela, S., Vasundhara Devi, J.: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
9. Li, Y.: Positive solutions of fourth-order boundary value problems with two parameters. J. Math. Anal. Appl. **281**(2), 477–484 (2003)
10. Liu, B., Liu, Y.: Positive solutions of a two-point boundary value problem for singular fractional differential equations in Banach space. J. Funct. Spaces **2013**(3), 721–730 (2013)
11. Liu, L., Zhang, X., Wu, Y.: On existence of positive solutions of a two-point boundary value problem for a nonlinear singular semipositone system. Appl. Math. Comput. **192**(1), 223–232 (2007)
12. Liu, Y.: Twin solutions to singular semipositone problems. J. Math. Anal. Appl. **286**(1), 248–260 (2003)
13. Liu, Y., O'Regan, D.: Multiplicity results for a class of fourth order semipositone m-point boundary value problems. Appl. Anal. **91**(5/6), 911–921 (2012)
14. Liu, Z., Jia, M., Ume, J.S., Kang, S.M.: Positive solutions for a higher-order nonlinear neutral delay differential equation. Abstr. Appl. Anal. **2011**, 328956 (2011)
15. Mao, J., Zhao, D.: Multiple positive solutions for nonlinear fractional differential equations with integral boundary value conditions and a parameter. J. Funct. Spaces **2019**, 2787569 (2019)
16. Podlubny, I.: Fractional Differential Equations. Mathematics in Science and Engineering. Academic Press, New York (1999)
17. Qi, T., Liu, Y., Cui, Y.: Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions. J. Funct. Spaces **2017**, 6703860 (2017)
18. Rong, D., Zhang, Q.: Solutions and nonnegative solutions for a weighted variable exponent impulsive integro-differential system with multi-point and integral mixed boundary value problems. Bound. Value Probl. **2013**(1), 161 (2013)
19. Thaiprayoon, C., Samana, D., Tariboon, J.: Multi-point boundary value problem for first order impulsive integro-differential equations with multi-point jump conditions. Bound. Value Probl. **2012**(1), 38 (2012)
20. Wang, Q., Lu, Y.: Positive solutions for a nonlinear system of fourth-order ordinary differential equations. Electron. J. Differ. Equ. **2020**, 45 (2020)
21. Wang, W., Shen, J.: Positive periodic solutions for neutral functional differential equations. Appl. Math. Lett. **102**, 106154 (2020)
22. Wang, Y., Liu, Y., Cui, Y.: Infinitely many solutions for impulsive fractional boundary value problem with p-Laplacian. Bound. Value Probl. **2018**(1), 94 (2018)
23. Xu, J., Jiang, J., O'Regan, D.: Positive solutions for a class of p-Laplacian Hadamard fractional-order three-point boundary value problems. Mathematics **8**(3), 308 (2020)
24. Xu, J., Wei, Z., O'Regan, D.: Infinitely many solutions for fractional Schrodinger–Maxwell equations. J. Appl. Anal. Comput. **9**(3), 1165–1182 (2019)
25. Xu, W., Ding, W.: Existence of positive solutions for second-order impulsive differential equations with delay and three-point boundary value problem. Adv. Differ. Equ. **2016**(1), 239 (2016)
26. Yan, B.: Positive solutions for the singular nonlocal boundary value problems involving nonlinear integral conditions. Bound. Value Probl. **2014**(1), 38 (2014)
27. Zhang, H., Liu, L., Wu, Y.: A unique positive solution for nth-order nonlinear impulsive singular integro-differential equations on unbounded domains in Banach spaces. Appl. Math. Comput. **203**(2), 649–659 (2008)
28. Zhang, S., Sun, J.: Existence of mild solutions for the impulsive semilinear nonlocal problem with random effects. Adv. Differ. Equ. **2014**(1), 194 (2014)
29. Zhao, D., Liu, Y.: Eigenvalues of a class of singular boundary value problems of impulsive differential equations in Banach spaces. J. Funct. Spaces **2014**, 720494 (2014)
30. Zhao, D., Liu, Y.: Positive solutions for a class of fractional differential coupled system with integral boundary value conditions. J. Nonlinear Sci. Appl. **9**, 2922–2942 (2016)
31. Zhao, D., Liu, Y.: Twin solutions to semipositone boundary value problems for fractional differential equations with coupled integral boundary conditions. J. Nonlinear Sci. Appl. **10**, 3544–3565 (2017)
32. Zhu, F., Liu, L., Wu, Y.: Positive solutions for systems of a nonlinear fourth-order singular semipositone boundary value problems. Appl. Math. Comput. **216**(2), 448–457 (2010)