Wang and Ding *Boundary Value Problems* (2021) 2021:96
https://doi.org/10.1186/s13661-021-01560-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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On positive solutions of second-order delayed differential system with indefinite weight

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Abstract

In this paper, we study the existence of positive solutions of a second-order delayed differential system, in which the weight functions may change sign. To prove our main results, we apply Krasnosel'skii's fixed point theorems in cones.

Keywords: Change of sign; Delay; Positive solutions; Fixed point theorem

1 Introduction

This paper is mainly concerned with the existence of positive solutions of a second-order two-delay differential system with

$$\begin{cases} x_1''(t) + h_1(t)f_1(x_1(t-\tau_1), x_2(t-\tau_2)) = 0, & 0 < t < 1, \\ x_2''(t) + h_2(t)f_2(x_1(t-\tau_1), x_2(t-\tau_2)) = 0, & 0 < t < 1, \\ x_1(t) = 0, \quad -\tau_1 \le t \le 0, \quad \text{and} \quad x_1(1) = 0, \\ x_2(t) = 0, \quad -\tau_2 \le t \le 0, \quad \text{and} \quad x_2(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \tau_1 < \tau_2 < \frac{1}{2}$ are constants, and the weight functions $h_i(t)$ may change sign (see (H_1)).

Many researchers have been attracted to study the theory, methodology and application of functional differential equations with delays, which have often been put forward as mathematical model to describe various natural phenomena [6, 9]. One of the important aspects of research is that there are many papers devoted to studying nontrivial solutions of boundary-value problems for functional differential equations with delays. For example, J.W. Lee and D. O'Regan established the general existence principle of differential-difference equations with delay type based on the nonlinear alternative (see [10, 11]). Since then, T. Candan [4, 5] applied Krasnosel'skii's fixed point theorem for the sum of a completely continuous and a contraction mapping to prove the existence of positive periodic solutions for the first- (second-) order neutral differential equation. Y. Liu [12] applied the Mönch fixed point theorem to study the existence and uniqueness of solutions for the nonlinear functional differential equations on infinite interval. D. Bai and Y. Xu [1, 2] employed fixed-point theory to show how the parameters effect the number of positive solution for a two-delay singular differential system (1.1). In addition, to the best of our knowledge, many papers are concerned with the existence of positive solutions of differential equations with indefinite weight functions by using various methods, such as the fixed point theorems, the Leray–Schauder degree theory, bifurcation and so on. We refer the reader to [3, 7, 13, 14, 16–18].

Motivated by the related references, in this paper we mainly apply the Krasnosel'skii fixed point theorems in cones to discuss the existence of positive solutions of problem (1.1), if the functions $h_i(t)$ and $f_i(x_1,x_2)$ satisfy the following assumptions:

(H_1) $h_i : [0,1] \to (-\infty,+\infty)$ are continuous, and there exists a $\xi \in (0,1-\tau_2)$ such that

$$\begin{cases} h_i(t) \geq 0, & \text{if } t \in [0,\xi], \\ h_i(t) \equiv 0, & \text{if } t \in [\xi,\xi+\tau_2], \\ h_i(t) \leq 0, & \text{if } t \in [\xi+\tau_2,1]. \end{cases}$$

Furthermore, h_i do not vanish identically on any subinterval of $[0,\xi]$ and $[\xi+\tau_2,1]$.

(H_2) $f_i : \mathbb{R}^{+2} \to \mathbb{R}^+$ are continuous and there exists a $\theta \in (0,1]$ such that

$$f_i(u,v) \geq \theta\phi_i(u,v),$$

$\phi_i(u,v) = \max\{f_i(x_1,x_2) : 0 \leq x_1 \leq u, 0 \leq x_2 \leq v\}$.

(H_3) There exist positive constants k_i, α_i and continuous functions

$$F_i(x_1,x_2) : \mathbb{R}^{+2} \to \mathbb{R}^+$$

satisfying

(i) $F_i(x_1,x_2)$ are strictly increasing functions with respective to (x_1,x_2),

(ii) $F_i(\lambda x_1,\lambda x_2) = \lambda^{\alpha_i}F_i(x_1,x_2)$,

such that

$$k_iF_i(x_1,x_2) \leq f_i(x_1,x_2) \leq F_i(x_1,x_2), \quad (x_1,x_2) \in \mathbb{R}^{+2}.$$

(H_4) There exist σ_i satisfying $\tau_2 < \sigma_i < \xi$ such that

$$\theta^2k_i(\sigma_i-\tau_2)^{\alpha_i}\int_{\sigma_i}^{\xi} G(t,s)h_i^+(s)\,ds \geq \xi^{\alpha_i}\int_{\xi+\tau_2}^{1} G(t,s)h_i^-(s)\,ds.$$

2 Preliminaries

Let

$$E_i = \bigl\{x_i \in C[-\tau_i,1] : x_i(t) = 0, \forall t \in [-\tau_i,0] \text{ and } x_i(1) = 0\bigr\} \quad (i = 1,2)$$

be a Banach space with norm $|x_i(t)|_i = \max_{-\tau_i\leq t\leq 1} x_i(t) = \max_{0\leq t\leq 1} x_i(t)$. Then we can define a Banach space E by $E_1 \times E_2$ with norm $\|x\| = \max\{|x_1|_1, |x_2|_2\}$, for $x = (x_1,x_2) \in E$. Also, define a subcone $K \subset E$ by

$$K = \bigl\{x \in E : x(t) \geq 0, x_i \text{ is concave on } [0,\xi] \text{ and convex on } [\xi,1]\bigr\}.$$

For $\forall x(t) \in K$, it is obvious that

$$|x_i|_i = \max_{0 \le t \le \xi} x_i(t).$$

For any $\gamma > 0$, in the following paragraphs, we set

$$K_\gamma = \{x \in K : \|x\| < \gamma\}$$

and

$$\partial K_\gamma = \{x \in K : \|x\| = \gamma\}.$$

Define an operator T by $Tx(t) = (T_1 x(t), T_2 x(t))$, where

$$T_i x(t) = \begin{cases} \int_0^1 G(t,s)h_i(s)f_i(x_1(s-\tau_1), x_2(s-\tau_2))\,ds, & 0 < t \le 1, \\ 0, & -\tau_i \le t \le 0, \end{cases}$$

where

$$G(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Now solutions of (1.1) can be rewritten as fixed points of T in Banach space E.

Lemma 2.1 *Assume that* (H_1)–(H_4) *hold. Then the operator* T *is positive and* $T : K \to K$ *is completely continuous.*

Proof First of all, we show that the operator T_i is positive, namely, for any $x(t) \in K$, we have

$$\int_0^1 G(t,s)h_i(s)f_i\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \ge \int_0^{\sigma_i} G(t,s)h_i^+(s)f_i\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds.$$

For any $x(t) \in K$, we can obtain

$$x_i(t) \ge q(t)x_i(\xi), \quad t \in [0,\xi] \quad \text{and} \quad x_i(t) \le q(t)x_i(\xi), \quad t \in [\xi,1],$$

where $q(t) : [0,1] \to [0,1]$ defined by $q(t) = \min\{\frac{t}{\xi}, \frac{1-t}{1-\xi}\}$. Hence, we have

$$\min_{s \in [\sigma_1,\xi]} q(s-\tau_i) = \frac{\sigma_1 - \tau_i}{\xi}, \qquad \max_{s \in [\xi+\tau_2,1]} q(s-\tau_i) = \frac{1-\xi-\tau_2+\tau_i}{1-\xi}.$$

Then, from (H_1)–(H_4), it follows that

$$\begin{aligned} &\int_0^1 G(t,s)h_1(s)f_1\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds - \int_0^{\sigma_1} G(t,s)h_1^+(s)f_1\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\ &\quad = \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)f_1\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \end{aligned}$$

$$
\begin{aligned}
&- \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)f_1\bigl(x_1(s-\tau_1),x_2(s-\tau_2)\bigr)\,ds \\
&\geq \theta \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\phi_1\bigl(x_1(s-\tau_1),x_2(s-\tau_2)\bigr)\,ds \\
&\quad - \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\phi_1\bigl(x_1(s-\tau_1),x_2(s-\tau_2)\bigr)\,ds \\
&\geq \theta \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\phi_1\bigl(q(s-\tau_1)x_1(\xi),q(s-\tau_2)x_2(\xi)\bigr)\,ds \\
&\quad - \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\phi_1\bigl(q(s-\tau_1)x_1(\xi),q(s-\tau_2)x_2(\xi)\bigr)\,ds \\
&\geq \theta \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)f_1\bigl(q(s-\tau_1)x_1(\xi),q(s-\tau_2)x_2(\xi)\bigr)\,ds \\
&\quad - \frac{1}{\theta}\int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)f_1\bigl(q(s-\tau_1)x_1(\xi),q(s-\tau_2)x_2(\xi)\bigr)\,ds \\
&= \frac{1}{\theta}\biggl[\theta^2 \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)f_1\bigl(q(s-\tau_1)x_1(\xi),q(s-\tau_2)x_2(\xi)\bigr)\,ds \\
&\quad - \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)f_1\bigl(q(s-\tau_1)x_1(\xi),q(s-\tau_2)x_2(\xi)\bigr)\,ds\biggr] \\
&\geq \frac{1}{\theta}\biggl[\theta^2 \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)k_1F_1\bigl(q(s-\tau_1)x_1(\xi),q(s-\tau_2)x_2(\xi)\bigr)\,ds \\
&\quad - \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)F_1\bigl(q(s-\tau_1)x_1(\xi),q(s-\tau_2)x_2(\xi)\bigr)\,ds\biggr] \\
&\geq \frac{1}{\theta}\biggl[\theta^2 \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)k_1F_1\Bigl(\min_{[\sigma_1,\xi]} q(s-\tau_1)x_1(\xi),\min_{[\sigma_1,\xi]} q(s-\tau_2)x_2(\xi)\Bigr)\,ds \\
&\quad - \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)F_1\Bigl(\max_{[\xi+\tau_2,1]} q(s-\tau_1)x_1(\xi),\max_{[\xi+\tau_2,1]} q(s-\tau_2)x_2(\xi)\Bigr)\,ds\biggr] \\
&= \frac{1}{\theta}\biggl[\theta^2 \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\frac{k_1}{\xi^{\alpha_1}}F_1\bigl((\sigma_1-\tau_1)x_1(\xi),(\sigma_1-\tau_2)x_2(\xi)\bigr)\,ds \\
&\quad - \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\frac{1}{(1-\xi)^{\alpha_1}}F_1\bigl((1-\xi-\tau_2+\tau_1)x_1(\xi),(1-\xi)x_2(\xi)\bigr)\,ds\biggr] \\
&\geq \frac{1}{\theta}\biggl[\theta^2 \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\frac{k_1}{\xi^{\alpha_1}}F_1\bigl((\sigma_1-\tau_2)x_1(\xi),(\sigma_1-\tau_2)x_2(\xi)\bigr)\,ds \\
&\quad - \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\frac{1}{(1-\xi)^{\alpha_1}}F_1\bigl((1-\xi)x_1(\xi),(1-\xi)x_2(\xi)\bigr)\,ds\biggr] \\
&= \frac{1}{\theta}\biggl[\theta^2 \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\frac{k_1}{\xi^{\alpha_1}}(\sigma_1-\tau_2)^{\alpha_1}\,ds \\
&\quad - \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\frac{1}{(1-\xi)^{\alpha_1}}(1-\xi)^{\alpha_1}\,ds\biggr]\cdot F_1\bigl(x_1(\xi),x_2(\xi)\bigr) \\
&\geq 0,
\end{aligned}
$$

which implies that the operator T_1 is positive.

In the similar way, we also get

$$\int_0^1 G(t,s)h_2(s)f_2\big(x_1(s-\tau_1),x_2(s-\tau_2)\big)\,ds \ge \int_0^{\sigma_2} G(t,s)h_2^+(s)f_2\big(x_1(s-\tau_1),x_2(s-\tau_2)\big)\,ds,$$

which implies that the operator T_2 is positive.

Secondly, from the conditions (H_1)–(H_3), we can obtain

$$(T_ix)''(t) \le 0, \quad t \in [0,\xi] \quad \text{and} \quad (T_ix)''(t) \ge 0, \quad t \in [\xi,1].$$

Ultimately, from the standard process, we can prove that $T : K \to K$ is completely continuous. □

The proofs of this paper are mainly based on the Krasnosel'skii fixed point theorems in cones such as the following.

Lemma 2.2 ([8]) *Let E be a Banach space, and $K \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.3 ([15]) *Let $(X, |\cdot|)$ be a normal linear space; $K_1, K_2 \subset X$ two cones; $K : K_1 \times K_2$; $r, R \in R_+^2$ with $0 < r_i < R_i$ for $i = 1,2$, $K_{r,R} := \{u = (u_1,u_2) \in K : r_i \le |u_i| \le R_i, i = 1,2\}$, and let $T = (T_1,T_2)$ be a compact map. Assume that, for each $i \in \{1,2\}$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) $T_i(u) \not< (\ge) u_i$ *if* $|u_i| = r_i$, *and* $T_i(u) \not> (\le) u_i$ *if* $|u_i| = R_i$;

(b) $T_i(u) \not> (\le) u_i$ *if* $|u_i| = r_i$, *and* $T_i(u) \not< (\ge) u_i$ *if* $|u_i| = R_i$.

Then T has a fixed point u in K with $r_i \le |u_i| \le R_i$.

3 Main results I

Define a function $\delta(t)$ by

$$\delta(t) = \min\left\{\frac{t}{\xi}, \frac{\xi - t}{\xi}\right\}, \quad t \in [0,\xi].$$

Then, for $i = 1,2$, let

$$\gamma_i = \min\left\{\min_{\frac{\sigma_i+\tau_2}{2} \le s \le \sigma_i} \delta(s-\tau_1), \min_{\frac{\sigma_i+\tau_2}{2} \le s \le \sigma_i} \delta(s-\tau_2)\right\},$$

$$A_i = k_i\gamma_i^{\alpha_i} \max_{t\in[0,\xi]} G(t,t) \cdot \int_{\frac{\sigma_i+\tau_2}{2}}^{\sigma_i} G(s,s)h_i^+(s)\,ds,$$

$$B_i = \int_{\tau_1}^{\tau_2} G(s,s)h_i^+(s)\,ds, C_i = \int_{\tau_2}^{\xi} G(s,s)h_i^+(s)\,ds.$$

Theorem 3.1 *Assume that* (H_1)–(H_4) *hold. If there exist two positive constants* r, R *with* $r < R$, *satisfying*

$$\begin{aligned}
&\min\bigl\{A_1F_1(r,0), A_2F_2(0,r)\bigr\} \geq r,\\
&\max\bigl\{B_1F_1(R,0) + C_1F_1(R,R), B_2F_2(R,0) + C_2F_2(R,R)\bigr\} \leq R,
\end{aligned}$$

then problem (1.1) *at least has a positive solution.*

Proof On one hand, for any $x \in \partial K_R$, we have $0 \leq x_1, x_2 \leq R$. Then, by the assumptions, we have

$$\begin{aligned}
|T_1x|_1 &\leq \int_0^1 G(s,s)h_1(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds\\
&= \int_0^\xi G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds\\
&\quad - \int_\xi^1 G(s,s)h_1^-(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds\\
&\leq \int_0^\xi G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds\\
&= \left(\int_{\tau_1}^{\tau_2} + \int_{\tau_2}^{\xi}\right) G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds\\
&\leq \int_{\tau_1}^{\tau_2} G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), 0\bigr)\,ds\\
&\quad + \int_{\tau_2}^{\xi} G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds\\
&\leq \int_{\tau_1}^{\tau_2} G(s,s)h_1^+(s)F_1\bigl(x_1(s-\tau_1), 0\bigr)\,ds\\
&\quad + \int_{\tau_2}^{\xi} G(s,s)h_1^+(s)F_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds\\
&\leq \int_{\tau_1}^{\tau_2} G(s,s)h_1^+(s)F_1(R,0)\,ds\\
&\quad + \int_{\tau_2}^{\xi} G(s,s)h_1^+(s)F_1(R,R)\,ds\\
&\leq R.
\end{aligned}$$

In the similar way, we also have

$$\begin{aligned}
|T_2x|_2 &\leq \int_0^1 G(s,s)h_2(s)f_2\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds\\
&\leq \int_{\tau_1}^{\tau_2} G(s,s)h_2^+(s)F_2(R,0)\,ds + \int_{\tau_2}^{\xi} G(s,s)h_2^+(s)F_2(R,R)\,ds\\
&\leq R.
\end{aligned}$$

So, from the above discussions, we get

$$\|Tx\| = \max\{|T_1x|_1, |T_2x|_2\} \le R = \|x\|, \quad \text{for } x \in \partial K_R. \tag{3.1}$$

On the other hand, for any $x(t) \in K$, from the concave property of $x_i(t)$ it follows that

$$x_i(t) \ge \delta(t)\big|x_i(t)\big|_i, \quad t \in [0,\xi].$$

Then, for any $x(t) \in \partial K_r$, we have the following.

Case I: if $|x_1(t)|_1 = r$, then $\delta(t)r \le x_1(t) \le r$ and $0 \le x_2(t) \le r$. By (H_2), we have

$$\begin{aligned}
\max_{t\in[0,\xi]} T_1x(t) &= \max_{t\in[0,\xi]} \int_0^1 G(t,s)h_1(s)f_1\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\
&\ge \max_{t\in[0,\xi]} \int_0^{\sigma_1} G(t,s)h_1^+(s)f_1\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\
&\ge \max_{t\in[0,\xi]} G(t,t)\cdot \int_0^{\sigma_1} G(s,s)h_1^+(s)f_1\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\
&\ge k_1 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\
&\ge k_1 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1\big(\delta(s-\tau_1)|x_1|_1, \delta(s-\tau_2)|x_2|_2\big)\,ds \\
&\ge k_1 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1\big(\gamma_1|x_1|_1, \gamma_1|x_2|_2\big)\,ds \\
&= k_1\gamma_1^{\alpha_1} \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1\big(|x_1|_1, |x_2|_2\big)\,ds \\
&\ge k_1\gamma_1^{\alpha_1} \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1(r,0)\,ds \\
&= k_1\gamma_1^{\alpha_1} \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)\,ds\cdot F_1(r,0) \\
&\ge r.
\end{aligned}$$

Case II: if $|x_1(t)|_1 < r$, then $0 \le x_1(t) < r$ and $\delta(t)r \le x_2(t) \le r$. By ($H_2$), we have

$$\begin{aligned}
\max_{t\in[0,\xi]} T_2x(t) &= \max_{t\in[0,\xi]} \int_0^1 G(t,s)h_2(s)f_2\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\
&\ge \max_{t\in[0,\xi]} \int_0^{\sigma_2} G(t,s)h_2^+(s)f_2\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\
&\ge \max_{t\in[0,\xi]} G(t,t)\cdot \int_0^{\sigma_2} G(s,s)h_2^+(s)f_2\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\
&\ge k_2 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_2+\tau_2}{2}}^{\sigma_2} G(s,s)h_2^+(s)F_2\big(x_1(s-\tau_1), x_2(s-\tau_2)\big)\,ds \\
&\ge k_2 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_2+\tau_2}{2}}^{\sigma_2} G(s,s)h_1^+(s)F_2\big(\delta(s-\tau_1)|x_1|_1, \delta(s-\tau_2)|x_2|_2\big)\,ds \\
&\ge k_2 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_2+\tau_2}{2}}^{\sigma_2} G(s,s)h_1^+(s)F_2\big(\gamma_2|x_1|_1, \gamma_2|x_2|_2\big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&= k_2\gamma_2^{\alpha_2} \max_{t\in[0,\xi]} G(t,t) \cdot \int_{\frac{\sigma_2+\tau_2}{2}}^{\sigma_2} G(s,s)h_2^+(s)F_2\bigl(|x_1|_1,|x_2|_2\bigr)\,ds \\
&\geq k_2\gamma_2^{\alpha_2} \max_{t\in[0,\xi]} G(t,t) \cdot \int_{\frac{\sigma_2+\tau_2}{2}}^{\sigma_2} G(s,s)h_2^+(s)F_2(0,r)\,ds \\
&= k_2\gamma_2^{\alpha_2} \max_{t\in[0,\xi]} G(t,t) \cdot \int_{\frac{\sigma_2+\tau_2}{2}}^{\sigma_2} G(s,s)h_2^+(s)\,ds \cdot F_2(0,r) \\
&\geq r.
\end{aligned}
$$

So, from the above discussions, we get

$$
\|Tx\| = \max\bigl\{|T_1x|_1, |T_2x|_2\bigr\} \geq r = \|x\|, \quad \text{for } x \in \partial K_r. \tag{3.2}
$$

Therefore, from (3.1) and (3.2) and Lemma 2.2, the operator T has a fixed point in $K \cap (\overline{\Omega}_R \setminus \Omega_r)$. □

As by a similar proof to Theorem 3.1, we also get the following.

Theorem 3.2 *Assume that* (H$_1$)–(H$_4$) *hold. If there exist two positive constants* r, R *with* $r < R$, *satisfying*

$$
\begin{aligned}
&\min\bigl\{A_1F_1(R,0), A_2F_2(0,R)\bigr\} \geq R, \\
&\max\bigl\{B_1F_1(r,0) + C_1F_1(r,r), B_2F_2(r,0) + C_2F_2(r,r)\bigr\} \leq r,
\end{aligned}
$$

then problem (1.1) *at least has a positive solution.*

4 Main results II

Theorem 4.1 *Assume that* (H$_1$)–(H$_4$) *hold. If there exist four positive constants* r_1, r_2, R_1, R_2 *with* $r_1 < R_1$, $r_2 < R_2$, *satisfying*

$$
\begin{aligned}
&A_1F_1(r_1,r_2) > r_1, \qquad A_2F_2(r_1,r_2) > r_2, \\
&B_1F_1(R_1,0) + C_1F_1(R_1,R_2) < R_1, \\
&B_2F_2(R_1,0) + C_2F_2(R_1,R_2) < R_2,
\end{aligned}
$$

then problem (1.1) *at least has a positive solution.*

Proof of Theorem 4.1 Let

$$
K_{r,R} := \bigl\{x = (x_1,x_2) \in K : r_i \leq |x_i|_i \leq R_i\bigr\} \quad (i = 1,2).
$$

Then, for any $x(t) \in K_{r,R}$, we have

$$
x_i(t) \geq \delta(t)\bigl|x_i(t)\bigr|_i, \quad t \in [0,\xi],
$$

where $\delta(t) = \min\{\frac{t}{\xi}, \frac{\xi-t}{\xi}\}$, $t \in [0,\xi]$.

On the one hand, for any $x(t) \in K_{r,R}$, we show that $T_1(x) \nless x_1$ if $|x_1|_1 = r_1$ and $r_2 \leq |x_2|_2 \leq R_2$. On the contrary, if $T_1(x) < x_1$, then we have

$$\begin{aligned}
r_1 &\geq \max_{t\in[0,\xi]} x_1(t) \geq \max_{t\in[0,\xi]} T_1 x(t) \\
&= \max_{t\in[0,\xi]} \int_0^1 G(t,s)h_1(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&\geq \max_{t\in[0,\xi]} \int_0^{\sigma_1} G(t,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&\geq \max_{t\in[0,\xi]} G(t,t)\cdot \int_0^{\sigma_1} G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&\geq k_1 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&\geq k_1 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1\bigl(\delta(s-\tau_1)|x_1|_1, \delta(s-\tau_2)|x_2|_2\bigr)\,ds \\
&\geq k_1 \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1\bigl(\gamma_1|x_1|_1, \gamma_1|x_2|_2\bigr)\,ds \\
&= k_1\gamma_1^{\alpha_1} \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1\bigl(|x_1|_1, |x_2|_2\bigr)\,ds \\
&\geq k_1\gamma_1^{\alpha_1} \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)F_1(r_1, r_2)\,ds \\
&= k_1\gamma_1^{\alpha_1} \max_{t\in[0,\xi]} G(t,t)\cdot \int_{\frac{\sigma_1+\tau_2}{2}}^{\sigma_1} G(s,s)h_1^+(s)\,ds\cdot F_1(r_1, r_2) \\
&> r_1,
\end{aligned}$$

which implies a contradiction. In the similar way, for any $x(t) \in K_{r,R}$, we also can obtain $T_2(x) \nless x_2$ if $|x_2|_2 = r_2$ and $r_1 \leq |x_1|_1 \leq R_1$.

On the one hand, for any $x(t) \in K_{r,R}$, we show that $T_1(x) \ngtr x_1$ if $|x_1|_1 = R_1$ and $r_2 \leq |x_2|_2 \leq R_2$. On the contrary, if $T_1(x) > x_1$, then we have

$$\begin{aligned}
x_1(t) < T_1x(t) &= \int_0^1 G(t,s)h_1(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&= \int_0^{\xi} G(t,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&\quad - \int_{\xi}^1 G(t,s)h_1^-(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&\leq \int_0^{\xi} G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&= \left(\int_0^{\tau_1} + \int_{\tau_1}^{\tau_2} + \int_{\tau_2}^{\xi}\right) G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds \\
&= \int_0^{\tau_1} G(s,s)h_1^+(s)f_1(0,0)\,ds + \int_{\tau_1}^{\tau_2} G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), 0\bigr)\,ds \\
&\quad + \int_{\tau_2}^{\xi} G(s,s)h_1^+(s)f_1\bigl(x_1(s-\tau_1), x_2(s-\tau_2)\bigr)\,ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\tau_1}^{\tau_2} G(s,s)h_1^+(s)F_1\bigl(x_1(s-\tau_1),0\bigr)\,ds \\
&\quad + \int_{\tau_2}^{\xi} G(s,s)h_1^+(s)F_1\bigl(x_1(s-\tau_1),x_2(s-\tau_2)\bigr)\,ds \\
&\leq \int_{\tau_1}^{\tau_2} G(s,s)h_1^+(s)F_1(R_1,0)\,ds + \int_{\tau_2}^{\xi} G(s,s)h_1^+(s)F_1(R_1,R_2)\,ds \\
&< R_1,
\end{aligned}$$

which implies a contradiction. In the similar way, for any $x(t) \in K_{r,R}$, we also can obtain $T_2(x) \not\geq x_2$ if $|x_2|_2 = R_2$ and $r_1 \leq |x_1|_1 \leq R_1$.

Therefore, from (a) of Lemma 2.3, the operator T has a fixed point x in K with $r_i \leq |x_i| \leq R_i$. □

As in a similar proof to Theorem 4.1, we also get the following.

Theorem 4.2 *Assume that* (H_1)–(H_4) *hold. If there exist four positive constants* r_1, r_2, R_1, R_2 *with* $r_1 < R_1$, $r_2 < R_2$, *satisfying*

$$\begin{aligned}
&A_1F_1(R_1,r_2) > R_1, \qquad A_2F_2(r_1,R_2) > R_2, \\
&B_1F_1(r_1,0) + C_1F_1(r_1,R_2) < r_1, \\
&B_2F_2(R_1,0) + C_2F_2(R_1,r_2) < r_2,
\end{aligned}$$

then problem (1.1) *at least has a positive solution.*

5 Examples

Example 5.1 Now we consider the following problem:

$$\begin{cases}
x_1''(t) + h_1(t)f_1(x_1(t-\tau_1),x_2(t-\tau_2)), & t > 0, \\
x_2''(t) + h_2(t)f_2(x_1(t-\tau_1),x_2(t-\tau_2)), & t > 0, \\
x_1(t) = 0, \quad -\tau_1 \leq t \leq 0, & \text{and} \quad x_1(1) = 0, \\
x_2(t) = 0, \quad -\tau_2 \leq t \leq 0, & \text{and} \quad x_2(1) = 0,
\end{cases} \tag{5.1}$$

where $\tau_1 = \frac{1}{10}$, $\tau_2 = \frac{1}{5}$,

$$h_1(t) = h_2(t) = \begin{cases}
400(\frac{1}{2} - t), & t \in [0, \frac{1}{2}], \\
0, & t \in [\frac{1}{2}, \frac{7}{10}], \\
-\frac{5}{3}t + \frac{7}{6}, & t \in [\frac{7}{10}, 1],
\end{cases}$$

$$f_1 = \frac{(x_1 + x_2)^{\frac{1}{2}}}{\frac{3}{2} + \frac{1}{4}\sin^2(x_1 + x_2)},$$

$$f_2 = \frac{(x_1 + x_2)^{\frac{1}{3}}}{\frac{4}{3} + \frac{1}{3\pi}\arctan(x_1^2 x_2^4)}.$$

It is obvious that (H_1) and (H_2) hold. Furthermore, we also have

$$\frac{1}{2}F_1(x_1,x_2) = \frac{1}{2}(x_1+x_2)^{\frac{1}{2}} \le f_1 \le (x_1+x_2)^{\frac{1}{2}} = F_1(x_1,x_2),$$
$$\frac{1}{2}F_2(x_1,x_2) = \frac{1}{2}(x_1+x_2)^{\frac{1}{3}} \le f_2 \le (x_1+x_2)^{\frac{1}{3}} = F_2(x_1,x_2),$$

which shows that (H_3) holds. Now, we prove that (H_4) holds.

Firstly, we show that there exist σ_i satisfying $\frac{1}{10} < \sigma_i < \frac{1}{5}$ such that

$$\theta^2 k_1(\sigma_1-\tau_2)^{\frac{1}{2}} \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\,ds \ge \xi^{\frac{1}{2}} \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\,ds \tag{5.2}$$

and

$$\theta^2 k_2(\sigma_2-\tau_2)^{\frac{1}{3}} \int_{\sigma_2}^{\xi} G(t,s)h_2^+(s)\,ds \ge \xi^{\frac{1}{3}} \int_{\xi+\tau_2}^{1} G(t,s)h_2^-(s)\,ds. \tag{5.3}$$

For fixed $\theta = \frac{1}{2}$, $\sigma_1 = \sigma_2 = \frac{2}{5}$, we set

$$M(t) = \theta^2 \frac{1}{2}(\sigma_1-\tau_2)^{\frac{1}{2}} \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\,ds = \frac{50}{\sqrt{5}} \int_{\frac{2}{5}}^{\frac{1}{2}} G(t,s)\left(\frac{1}{2}-s\right)ds,$$
$$N(t) = \xi^{\frac{1}{2}} \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\,ds = \frac{1}{\sqrt{2}} \int_{\frac{7}{10}}^{1} G(t,s)\left(\frac{5}{3}s-\frac{7}{6}\right)ds, \quad t \in [0,1].$$

For $t \in [0, \frac{2}{5}]$, we have

$$M(t) = \frac{50t}{\sqrt{5}} \int_{\frac{2}{5}}^{\frac{1}{2}} (1-s)\left(\frac{1}{2}-s\right)ds = \frac{17}{120\sqrt{5}}t,$$
$$N(t) = \frac{t}{\sqrt{2}} \int_{\frac{7}{10}}^{1} (1-s)\left(\frac{5}{3}s-\frac{7}{6}\right)ds = \frac{3}{400\sqrt{2}}t.$$

It is obvious that $M(t) \ge N(t)$, $\forall t \in [0, \frac{2}{5}]$.

For $t \in [\frac{2}{5}, \frac{1}{2}]$, we have

$$\begin{aligned} M(t) &= \frac{50}{\sqrt{5}}\left[\int_{\frac{2}{5}}^{t} (1-t)s\left(\frac{1}{2}-s\right)ds + \int_{t}^{\frac{1}{2}} t(1-s)\left(\frac{1}{2}-s\right)ds\right] \\ &= \frac{50}{\sqrt{5}}\left(\frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{737}{6000}t - \frac{7}{375}\right), \\ N(t) &= \frac{t}{\sqrt{2}} \int_{\frac{7}{10}}^{1} (1-s)\left(\frac{5}{3}s-\frac{7}{6}\right)ds = \frac{3}{400\sqrt{2}}t. \end{aligned}$$

Via some computations, we have

$$M(t)_{\min} = M\left(\frac{1}{2}\right) = \frac{13}{240\sqrt{5}} > \frac{3}{800\sqrt{2}} = N\left(\frac{1}{2}\right) = N(t)_{\max},$$

which implies that $M(t) > N(t)$, $\forall t \in [\frac{2}{5}, \frac{1}{2}]$.

For $t \in [\frac{1}{2}, \frac{7}{10}]$, we have

$$M(t) = \frac{50(1-t)}{\sqrt{5}} \int_{\frac{2}{5}}^{\frac{1}{2}} s\left(\frac{1}{2} - s\right) ds = \frac{13}{120\sqrt{5}}(1-t),$$
$$N(t) = \frac{t}{\sqrt{2}} \int_{\frac{7}{10}}^{1} (1-s)\left(\frac{5}{3}s - \frac{7}{6}\right) ds = \frac{3}{400\sqrt{2}}t.$$

Via some computations, we have

$$M(t)_{\min} = M\left(\frac{7}{10}\right) = \frac{13}{400\sqrt{5}} > \frac{21}{4000\sqrt{2}} = N\left(\frac{7}{10}\right) = N(t)_{\max},$$

which implies that $M(t) > N(t)$, $\forall t \in [\frac{1}{2}, \frac{7}{10}]$.

For $t \in [\frac{7}{10}, 1]$, we have

$$M(t) = \frac{50(1-t)}{\sqrt{5}} \int_{\frac{2}{5}}^{\frac{1}{2}} s\left(\frac{1}{2} - s\right) ds = \frac{13}{120\sqrt{5}}(1-t),$$
$$\begin{aligned} N(t) &= \frac{1}{\sqrt{2}}\left[\int_{\frac{7}{10}}^{t} (1-t)s\left(\frac{5}{3}s - \frac{7}{6}\right) ds + \int_{t}^{1} t(1-s)\left(\frac{5}{3}s - \frac{7}{6}\right) ds\right] \\ &= \frac{1}{\sqrt{2}}\left(-\frac{5}{18}t^3 + \frac{7}{12}t^2 - \frac{481}{1200}t + \frac{343}{3600}\right). \end{aligned}$$

Denote $g(t) = M(t) - N(t)$ and $g(1) = 0$. Since

$$g'(t) = \frac{5}{6\sqrt{2}}t^2 - \frac{7}{6\sqrt{2}}t + \frac{481}{1200\sqrt{2}} - \frac{13}{120\sqrt{5}} < 0, \quad t \in \left[\frac{7}{10}, 1\right],$$

$g(t)$ is strictly decreasing on $[\frac{7}{10}, 1]$. Then $g(t) \geq 0$, $\forall t \in [\frac{7}{10}, 1]$, which implies that $M(t) \geq N(t)$, $\forall t \in [\frac{7}{10}, 1]$.

So from these discussions, we have $M(t) \geq N(t)$, $\forall t \in [0, 1]$, which means that (5.2) holds. Hence, in the similar way, we also see that the inequality (5.3) is true.

Secondly, let

$$\begin{aligned} \Gamma(\rho) &= \min\bigl\{A_1F_1(\rho, 0), A_2F_2(0, \rho)\bigr\} = \min\bigl\{A_1\rho^{\frac{1}{2}}, A_2\rho^{\frac{1}{3}}\bigr\}, \\ \Lambda(\rho) &= \max\bigl\{B_1F_1(\rho, 0) + C_1F_1(\rho, \rho), B_2F_2(\rho, 0) + C_2F_2(\rho, \rho)\bigr\} \\ &= \max\bigl\{B_1\rho^{\frac{1}{2}} + C_1(2\rho)^{\frac{1}{2}}, B_2\rho^{\frac{1}{3}} + C_2(2\rho)^{\frac{1}{3}}\bigr\}. \end{aligned}$$

It is obvious that there exist a sufficiently small constant $r > 0$ and a sufficiently large constant $R > 0$ such that

$$\min\bigl\{A_1r^{\frac{1}{2}}, A_2r^{\frac{1}{3}}\bigr\} > r,$$
$$\max\bigl\{B_1R^{\frac{1}{2}} + C_1(2R)^{\frac{1}{2}}, B_2R^{\frac{1}{3}} + C_2(2R)^{\frac{1}{3}}\bigr\} < R.$$

Then, by Theorem 3.1, problem (5.1) has a positive solution.

Example 5.2 Now we consider the following problem:

$$\begin{cases} x_1''(t) + h_1(t)f_1(x_1(t-\tau_1), x_2(t-\tau_2)), & t > 0, \\ x_2''(t) + h_2(t)f_2(x_1(t-\tau_1), x_2(t-\tau_2)), & t > 0, \\ x_1(t) = 0, \quad -\tau_1 \le t \le 0, \quad \text{and} \quad x_1(1) = 0, \\ x_2(t) = 0, \quad -\tau_2 \le t \le 0, \quad \text{and} \quad x_2(1) = 0, \end{cases} \tag{5.4}$$

where $\tau_1 = \frac{1}{10}$, $\tau_2 = \frac{1}{5}$,

$$h_1(t) = h_2(t) = \begin{cases} \frac{1}{2} - t, & t \in [0, \frac{1}{2}], \\ 0, & t \in [\frac{1}{2}, \frac{7}{10}], \\ 10^{-4}(-\frac{5}{3}t + \frac{7}{6}), & t \in [\frac{7}{10}, 1], \end{cases}$$

$$f_1 = \frac{(x_1 + x_2)^2}{1 + |\sin(x_1 x_2)|},$$

$$f_2 = \frac{(x_1 + x_2)^3}{1 + |\sin(x_1 x_2)|}.$$

It is obvious that (H_1) and (H_2) hold. Moreover, we have

$$\frac{1}{2}F_1(x_1, x_2) = \frac{1}{2}(x_1 + x_2)^2 \le f_1 \le (x_1 + x_2)^2 = F_1(x_1, x_2), \quad k_1 > 0,$$
$$\frac{1}{2}F_2(x_1, x_2) = \frac{1}{2}(x_1 + x_2)^3 \le f_2 \le (x_1 + x_2)^3 = F_2(x_1, x_2), \quad k_2 > 0,$$

which shows that (H_3) holds. Now, we prove that (H_4) holds.

Firstly, we show that there exist σ_i satisfying $\frac{1}{10} < \sigma_i < \frac{1}{5}$ such that

$$\theta^2 k_1(\sigma_1 - \tau_2)^2 \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\,ds \ge \xi^2 \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\,ds \tag{5.5}$$

and

$$\theta^2 k_2(\sigma_2 - \tau_2)^3 \int_{\sigma_2}^{\xi} G(t,s)h_2^+(s)\,ds \ge \xi^3 \int_{\xi+\tau_2}^{1} G(t,s)h_2^-(s)\,ds. \tag{5.6}$$

For fixed $\theta = \frac{1}{2}$, $\sigma_1 = \sigma_2 = \frac{2}{5}$, we set

$$m(t) = \theta^2 \frac{1}{2}(\sigma_1 - \tau_2)^2 \int_{\sigma_1}^{\xi} G(t,s)h_1^+(s)\,ds = \frac{1}{200}\int_{\frac{2}{5}}^{\frac{1}{2}} G(t,s)\left(\frac{1}{2} - s\right)ds,$$
$$n(t) = \xi^2 \int_{\xi+\tau_2}^{1} G(t,s)h_1^-(s)\,ds = \frac{1}{4 \times 10^4}\int_{\frac{7}{10}}^{1} G(t,s)\left(\frac{5}{3}s - \frac{7}{6}\right)ds, \quad t \in [0,1].$$

For $t \in [0, \frac{2}{5}]$, we have

$$m(t) = \frac{t}{200}\int_{\frac{2}{5}}^{\frac{1}{2}} (1-s)\left(\frac{1}{2} - s\right)ds = \frac{17}{12 \times 10^5}t,$$

$$n(t) = \frac{t}{4 \times 10^4} \int_{\frac{7}{10}}^{1} (1-s)\left(\frac{5}{3}s - \frac{7}{6}\right) ds = \frac{3}{16 \times 10^6} t.$$

It is obvious that $m(t) \geq n(t)$, $\forall t \in [0, \frac{2}{5}]$.

For $t \in [\frac{2}{5}, \frac{1}{2}]$, we have

$$\begin{aligned} m(t) &= \frac{1}{200} \left[\int_{\frac{2}{5}}^{t} (1-t)s\left(\frac{1}{2} - s\right) ds + \int_{t}^{\frac{1}{2}} t(1-s)\left(\frac{1}{2} - s\right) ds \right] \\ &= \frac{1}{200} \left(\frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{737}{6000}t - \frac{7}{375} \right), \\ n(t) &= \frac{t}{4 \times 10^4} \int_{\frac{7}{10}}^{1} (1-s)\left(\frac{5}{3}s - \frac{7}{6}\right) ds = \frac{3}{16 \times 10^6} t. \end{aligned}$$

Via some computations, we have

$$m(t)_{\min} = m\left(\frac{1}{2}\right) = \frac{13}{24 \times 10^5} > \frac{3}{32 \times 10^6} = n\left(\frac{1}{2}\right) = n(t)_{\max},$$

which implies that $m(t) > n(t)$, $\forall t \in [\frac{2}{5}, \frac{1}{2}]$.

For $t \in [\frac{1}{2}, \frac{7}{10}]$, we have

$$\begin{aligned} m(t) &= \frac{1-t}{200} \int_{\frac{2}{5}}^{\frac{1}{2}} s\left(\frac{1}{2} - s\right) ds = \frac{13}{12 \times 10^5}(1-t), \\ n(t) &= \frac{t}{4 \times 10^4} \int_{\frac{7}{10}}^{1} (1-s)\left(\frac{5}{3}s - \frac{7}{6}\right) ds = \frac{3}{16 \times 10^6} t. \end{aligned}$$

Via some computations, we have

$$m(t)_{\min} = m\left(\frac{7}{10}\right) = \frac{13}{4 \times 10^6} > \frac{21}{16 \times 10^7} = n\left(\frac{7}{10}\right) = n(t)_{\max},$$

which implies that $m(t) > n(t)$, $\forall t \in [\frac{1}{2}, \frac{7}{10}]$.

For $t \in [\frac{7}{10}, 1]$, we have

$$\begin{aligned} m(t) &= \frac{1-t}{200} \int_{\frac{2}{5}}^{\frac{1}{2}} s\left(\frac{1}{2} - s\right) ds = \frac{13}{12 \times 10^5}(1-t), \\ n(t) &= \frac{1}{4 \times 10^4} \left[\int_{\frac{7}{10}}^{t} (1-t)s\left(\frac{5}{3}s - \frac{7}{6}\right) ds + \int_{t}^{1} t(1-s)\left(\frac{5}{3}s - \frac{7}{6}\right) ds \right] \\ &= \frac{1}{4 \times 10^4} \left(-\frac{5}{18}t^3 + \frac{7}{12}t^2 - \frac{481}{1200}t + \frac{343}{3600} \right). \end{aligned}$$

Let $\omega(t) = m(t) - n(t)$ satisfying $\omega(1) = 0$. Since

$$\omega'(t) = \frac{5}{24 \times 10^4}t^2 - \frac{7}{24 \times 10^4}t - \frac{13}{16 \times 10^6} < 0, \quad t \in \left[\frac{7}{10}, 1\right],$$

it means that $\omega(t)$ is strictly decreasing on $[\frac{7}{10}, 1]$. Then $\omega(t) \geq 0$, $\forall t \in [\frac{7}{10}, 1]$, namely, $m(t) \geq n(t)$, $\forall t \in [\frac{7}{10}, 1]$.

So from these discussions, we have $m(t) \geq n(t)$, $\forall t \in [0,1]$, which means that (5.5) holds. Hence, by a similar method, we also can see that the inequality (5.6) is true.

Secondly, let

$$\begin{aligned}\Gamma(\rho) &= \min\{A_1F_1(\rho,0), A_2F_2(0,\rho)\} = \min\{A_1\rho^2, A_2\rho^3\},\\ \Lambda(\rho) &= \max\{B_1F_1(\rho,0) + C_1F_1(\rho,\rho), B_2F_2(\rho,0) + C_2F_2(\rho,\rho)\}\\ &= \max\{B_1\rho^2 + C_1(2\rho)^2, B_2\rho^3 + C_2(2\rho)^3\}.\end{aligned}$$

It is obvious that there exist a sufficiently small constant $r > 0$ and a sufficiently large constant $R > 0$ such that

$$\begin{aligned}&\min\{A_1R^2, A_2R^3\} > R,\\ &\max\{B_1r^2 + C_1(2r)^2, B_2r^3 + C_2(2r)^3\} < r.\end{aligned}$$

Then, by Theorem 3.2, problem (5.4) has a positive solution.

Example 5.3 Now we consider the following problem:

$$\begin{cases} x_1''(t) + h_1(t)f_1(x_1(t-\tau_1), x_2(t-\tau_2)), & t > 0,\\ x_2''(t) + h_2(t)f_2(x_1(t-\tau_1), x_2(t-\tau_2)), & t > 0,\\ x_1(t) = 0, \quad -\tau_1 \leq t \leq 0, \quad \text{and} \quad x_1(1) = 0,\\ x_2(t) = 0, \quad -\tau_2 \leq t \leq 0, \quad \text{and} \quad x_2(1) = 0,\end{cases} \tag{5.7}$$

where $\tau_1 = \frac{1}{10}$, $\tau_2 = \frac{1}{5}$,

$$\begin{aligned}&h_1(t) = h_2(t) = \begin{cases} 400(\frac{1}{2} - t), & t \in [0, \frac{1}{2}],\\ 0, & t \in [\frac{1}{2}, \frac{7}{10}],\\ -\frac{5}{3}t + \frac{7}{6}, & t \in [\frac{7}{10}, 1],\end{cases}\\ &f_1 = (x_1 + x_2)^{\frac{1}{2}}, \qquad f_2 = (x_1 + x_2)^{\frac{1}{3}}.\end{aligned}$$

Choosing $\theta = \frac{1}{2}$, $k_1 = k_2 = \frac{1}{2}$, $\sigma_1 = \sigma_2 = \frac{2}{5}$. Then, from Example 5.1, it is obvious that (H_1)–(H_4) hold.

Now, we show that there exist $0 < r_i < R_i$ such that

$$k_1\gamma_1^{\frac{1}{2}} \max_{t\in[0,\frac{1}{2}]} G(t,t) \cdot \int_{\frac{\sigma_1+\frac{1}{5}}{2}}^{\sigma_1} G(s,s)h_1^+(s)\,ds \cdot (r_1 + r_2)^{\frac{1}{2}} > r_1, \tag{5.8}$$

$$k_2\gamma_2^{\frac{1}{3}} \max_{t\in[0,\frac{1}{2}]} G(t,t) \cdot \int_{\frac{\sigma_2+\frac{1}{5}}{2}}^{\sigma_2} G(s,s)h_2^+(s)\,ds \cdot (r_1 + r_2)^{\frac{1}{3}} > r_2, \tag{5.9}$$

$$\int_{\frac{1}{10}}^{\frac{1}{5}} G(s,s)h_1^+(s)\,ds \cdot R_1^{\frac{1}{2}} + \int_{\frac{1}{5}}^{\frac{1}{2}} G(s,s)h_1^+(s)\,ds \cdot (R_1 + R_2)^{\frac{1}{2}} < R_1, \tag{5.10}$$

$$\int_{\frac{1}{10}}^{\frac{1}{5}} G(s,s)h_2^+(s)\,ds \cdot R_1^{\frac{1}{3}} + \int_{\frac{1}{5}}^{\frac{1}{2}} G(s,s)h_2^+(s)\,ds \cdot (R_1 + R_2)^{\frac{1}{3}} < R_2. \tag{5.11}$$

Choosing $r_1 = 0.001$, $r_2 = 0.01$, $R_1 = 70$, $R_2 = 140$, we obtain

$$\frac{50}{\sqrt{5}}\int_{\frac{3}{10}}^{\frac{2}{5}} s(1-s)\left(\frac{1}{2}-s\right) ds\cdot(0.001+0.01)^{\frac{1}{2}} = \frac{0.16875}{\sqrt{5}}\sqrt{0.011} > 0.001,$$

$$\frac{50}{\sqrt[3]{5}}\int_{\frac{3}{10}}^{\frac{2}{5}} s(1-s)\left(\frac{1}{2}-s\right) ds\cdot(0.001+0.01)^{\frac{1}{3}} = \frac{0.16875}{\sqrt[3]{5}}\sqrt[3]{0.011} > 0.01,$$

$$\begin{aligned}&400\left[\int_{\frac{1}{10}}^{\frac{1}{5}} s(1-s)\left(\frac{1}{2}-s\right) ds\cdot 70^{\frac{1}{2}} + \int_{\frac{1}{5}}^{\frac{1}{2}} s(1-s)\left(\frac{1}{2}-s\right) ds\cdot(70+140)^{\frac{1}{2}}\right]\\ &\quad = 1.75\sqrt{70} + 3.69\sqrt{210} < 70,\\ &400\left[\int_{\frac{1}{10}}^{\frac{1}{5}} s(1-s)\left(\frac{1}{2}-s\right) ds\cdot 70^{\frac{1}{3}} + \int_{\frac{1}{5}}^{\frac{1}{2}} s(1-s)\left(\frac{1}{2}-s\right) ds\cdot(70+140)^{\frac{1}{3}}\right]\\ &\quad = 1.75\sqrt[3]{70} + 3.69\sqrt[3]{210} < 140,\end{aligned}$$

which implies that the inequalities (5.8)–(5.11) are true.

Therefore, from Theorem 4.1, one concludes that problem (5.7) has a positive solution.

Acknowledgements
We would like to thank the referee for his/her valuable comments and helpful suggestions which have led to an improvement of the presentation of this paper.

Funding
The authors were supported by the Fundamental Research Funds for the Central Universities (B200202003).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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Received: 9 November 2020 Accepted: 1 September 2021 Published online: 23 November 2021

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