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# On a nonlocal implicit problem under Atangana–Baleanu–Caputo fractional derivative

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# Abstract

In this paper, we study a class of initial value problems for a nonlinear implicit fractional differential equation with nonlocal conditions involving the Atangana–Baleanu–Caputo fractional derivative. The applied fractional operator is based on a nonsingular and nonlocal kernel. Then we derive a formula for the solution through the equivalent fractional functional integral equations to the proposed problem. The existence and uniqueness are obtained by means of Schauder's and Banach's fixed point theorems. Moreover, two types of the continuous dependence of solutions to such equations are discussed. Finally, the paper includes two examples to substantiate the validity of the main results.

MSC: 26A33; 34A08; 34A12; 34B15

**Keywords:** ABC-type fractional differential equation; Nonlocal conditions; Fixed point theorem

# **1** Introduction

Fractional calculus [1, 2] has persistently magnetized the attention of many researchers in the few past decades. Recently, novel fractional derivatives which mix the Riemann– Liouville, Caputo, Hadamard, Hilfer, and generalized fractional derivatives have emerged (see [3–8]). Some interested authors and researchers have realized that innovation for novel fractional derivatives with nonsingular (nonlocal) or singular (local) kernels is an urgent necessity to satisfy the need to model more realistic problems in different fields of applied science.

Caputo and Fabrizio in [9] suggested a novel kind of fractional derivatives where the kernel relies on the exponential function. Some properties of this novel operator were studied by Losada and Nieto in [10]. In [11] the authors proposed interesting new fractional operators called Atangana–Baleanu (AB) fractional operators. One of these operators is called Atangana–Baleanu–Caputo (ABC) fractional derivative, and it is basically a generalization of the Caputo operator. Then, in [12, 13], the authors discussed the discrete versions of those novel operators. Some recent and interesting contributions on fractional differential

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equations (FDEs) and mathematical modeling that incorporate ABC fractional derivatives can be found in the following series of articles [14–26].

The recent investigations of the qualitative analysis of FDEs, e.g., the evolution, impulsive, and functional problems with initial (or boundary) nonlocal conditions, can be found in [27-35] and the references therein.

On the other hand, in the case where a physical procedure is described by IVPs for FDEs, at that point it is desirable that any mistakes made in the estimation of initial data do not impact the solution so much. Mathematically, this is known as continuous dependence of solution of an IVP on the data introduced in the proposed problem. Actually, nonlocal conditions come up when estimations of the function on the limit are associated with values in the domain. It is seen as more reasonable than the classical initial conditions for the forming of some physical phenomena in specific problems of wave spread and thermodynamics. In crossing, we saw that the nonlocal condition  $\sum_{k=1}^{m} \beta_k \varkappa(\tau_k) = \varkappa_0$  that may be applied in physical models yields preferred impact over the initial conditions  $\varkappa(0) = \varkappa_0$ .

In this regard, many interested authors have presented excellent results on the existence and continuous dependence of solution of FDEs with the nonlocal conditions and classical fractional operators. For the recent review of these studies, we refer to [36–44].

Recently, ABC-fractional IVP is one of the studied problems by Thabet et al. [14] which is of type

$$\begin{split} & {}^{ABC} \mathbb{D}^{\varrho}_{a,\theta} \, \varkappa(\theta) = f(\theta, \, \varkappa(\theta)), \quad \theta \in [a, \chi], 0 < \varrho \le 1, \\ & \varkappa(a) = \varkappa_0. \end{split}$$

Through the above discussions, and motivated by [14, 40], in this work, we will prove some new results based on a novel version of fractional operators. More precisely, we consider the following ABC-type nonlocal fractional problem:

$${}^{ABC}\mathbb{D}^{\varrho}_{0,\theta}\,\varkappa(\theta) = f\left(\theta,\,\varkappa(\theta),\,{}^{ABC}\mathbb{D}^{\varrho}_{0,\theta}\,\varkappa(\theta)\right), \quad \theta \in [0,\chi],\tag{1.1}$$

$$\sum_{k=1}^{m} \beta_k \varkappa(\tau_k) = \varkappa_0, \quad \tau_k \in (0, \chi), \tag{1.2}$$

where  $0 < \rho \le 1$ ,  ${}^{ABC}\mathbb{D}_{a^+}^{\rho}$  is the ABC fractional derivative of order  $\rho, f: [0, \chi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with  $f(0, \varkappa(0), {}^{ABC}\mathbb{D}_{0,\theta}^{\rho}\varkappa(0)) = 0, 0 < \tau_1 < \tau_2 < \cdots < \tau_m < \chi, \beta_k$  are real numbers  $(k = 1, 2, \dots, m)$ , and  $\varkappa \in C[0, \chi]$  such that the operator  ${}^{ABC}\mathbb{D}_{0,\theta}^{\rho}$  exists and  ${}^{ABC}\mathbb{D}_{0,\theta}^{\rho}\varkappa \in C[0, \chi]$ .

The main aim of this work is to study the existence, uniqueness of solutions and their continuous dependence on the nonlinear nonlocal problem (1.1)-(1.2) in the frame of ABC fractional derivative by means of Schauder's and Banach's fixed point theorems. To the best of our knowledge in the subject, no one considered the existence and data dependence of the ABC-type fractional problem with nonlocal conditions. Therefore, the acquired results are recent studies and an extension of the development of FDEs involving an ABC fractional derivative. Furthermore, the analysis of the results is restricted to a minimum of hypotheses.

The rest of the paper is arranged as follows. In Sect. 2, we recall some useful preliminaries related to the main outcomes. Section 3 is dedicated to obtaining the solution representation to a given problem. Then the existence and uniqueness results are proved via functional integral equation with the aid of some fixed point approaches. Moreover, we discuss the continuous dependence of solutions for the problem at hand. Illustrative examples are given in Sect. 4. Finally, concluding remarks are mentioned in Sect. 5.

## 2 Background materials and preliminaries

Here, we recall some essential definitions and preliminary facts related to AB fractional operators.

Let  $C([0, \chi], \mathbb{R}) = C[0, \chi]$  be the space of continuous functions  $\upsilon : [0, \chi] \to \mathbb{R}$  with the norm

 $\|\boldsymbol{\upsilon}\| = \max\{|\boldsymbol{\upsilon}(\boldsymbol{\theta})| : \boldsymbol{\theta} \in [0, \chi]\}.$ 

Clearly,  $C[0, \chi]$  is a Banach space with the norm  $\|\cdot\|$ .

**Definition 2.1** ([11]) Let  $\rho \in [0, 1]$  and  $\upsilon \in H^1(0, \chi)$ . Then the AB-Riemann–Liouville and AB-Caputo fractional derivatives are given by

$${}^{ABR}\mathbb{D}^{\varrho}_{a^{+},\theta}\upsilon(\theta) = \frac{\mathfrak{N}(\varrho)}{1-\varrho}\frac{d}{d\theta}\int_{a}^{\theta}\mathbb{E}_{\varrho}\left(\frac{-\varrho}{1-\varrho}(\theta-\sigma)^{\varrho}\right)\upsilon(\sigma)\,d\sigma, \quad \theta > a,$$

and

$${}^{ABC}\mathbb{D}^{\varrho}_{a^{+},\theta}\upsilon(\theta) = \frac{\mathfrak{N}(\varrho)}{1-\varrho}\int_{a}^{\theta}\mathbb{E}_{\varrho}\left(\frac{-\varrho}{1-\varrho}(\theta-\sigma)^{\varrho}\right)\upsilon'(\sigma)\,d\sigma,\quad\theta>a,$$

respectively, where  $\mathbb{E}_{\varrho}$  is called the MLF defined by

$$\mathbb{E}_{\varrho}(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(k\varrho+1)}, \qquad \operatorname{Re}(\varrho) > 0, \quad \theta \in \mathbb{C}.$$

The AB fractional integral is described by

$${}^{AB}\mathbb{I}^{\varrho}_{a^{+},\theta}\upsilon(\theta)=\frac{1-\varrho}{\mathfrak{N}(\varrho)}\upsilon(\theta)+\frac{\varrho}{\mathfrak{N}(\varrho)}I^{\varrho}_{a^{+}}\upsilon(\theta),\quad \theta>a,$$

where  $\mathfrak{N}(\varrho) > 0$  is a normalization function satisfying  $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$  and

$$I^{\varrho}_{a^+}\upsilon(\theta)=\frac{1}{\Gamma(\varrho)}\int_a^\theta (\theta-\sigma)^{\varrho-1}\upsilon(\sigma)\,d\sigma.$$

**Lemma 2.2** ([11, 45]) Let  $\rho \in (0, 1]$  and  $\upsilon \in H^1(0, \chi)$ , if an ABC fractional derivative exists, then we have

$${}^{ABC}\mathbb{D}^{\varrho}_{a^+,\theta}{}^{AB}\mathbb{I}^{\varrho}_{a^+,\theta}\upsilon(\theta)=\upsilon(\theta)$$

and

$${}^{AB}\mathbb{I}^{\varrho}_{a^+,\theta}{}^{ABC}\mathbb{D}^{\varrho}_{a^+,\theta}\upsilon(\theta)=\upsilon(\theta)-\upsilon(a).$$

**Definition 2.3** ([11]) The relation between the AB-Caputo and AB-Riemann–Liouville operator is

$${}^{ABC}\mathbb{D}^{\varrho}_{a^+,\theta}\upsilon(\theta) = {}^{ABR}\mathbb{D}^{\varrho}_{a^+,\theta}\upsilon(\theta) - \frac{\mathfrak{N}(\varrho)}{1-\varrho}\upsilon(a)\mathbb{E}_{\varrho}\bigg(\frac{-\varrho}{1-\varrho}(\theta-a)^{\varrho}\bigg).$$

**Lemma 2.4** ([15]) For  $n < \varrho \le n + 1$ , for some  $n \in \mathbb{N}_0$  and  $\upsilon(\theta)$  defined on  $[0, \chi]$ , we have

- (i)  ${}^{ABC}\mathbb{D}^{\varrho}_{a^{+},\theta}{}^{AB}\mathbb{I}^{\varrho}_{a^{+},\theta}\upsilon(\theta) = \upsilon(\theta);$
- (ii)  ${}^{AB}\mathbb{I}^{\varrho}_{a^{+},\theta} {}^{ABC}\mathbb{D}^{\varrho}_{a^{+},\theta}\upsilon(\theta) = \upsilon(\theta) \sum_{k=0}^{n} \frac{\upsilon^{(k)}(a)}{k!}(\theta-a)^{k};$ (iii)  ${}^{AB}\mathbb{I}^{\varrho}_{a^{+},\theta} {}^{ABR}\mathbb{D}^{\varrho}_{a^{+},\theta}\upsilon(\theta) = \upsilon(\theta) - \sum_{k=0}^{n-1} \frac{\upsilon^{(k)}(a)}{k!}(\theta-a)^{k}.$

**Lemma 2.5** ([15]) For  $n < \varrho \le n + 1$ ,  ${}^{ABC} \mathbb{D}^{\varrho}_{a^+,\theta}$   $(\theta - a)^k = 0, k = 0, 1, ..., n$ . Moreover,  ${}^{ABC} \mathbb{D}^{\varrho}_{a^+,\theta} \upsilon(\theta) = 0$  if  $\upsilon(\theta)$  is a constant function.

**Lemma 2.6** ([11, 15]) *Let*  $\rho \in (0, 1]$  *and*  $\varpi \in C[0, 1]$  *with*  $\varpi(0) = 0$ *. Then the solution of* 

$$\begin{split} ^{ABC} \mathbb{D}^{\varrho}_{0^+} \upsilon(\theta) &= \varpi(\theta), \quad \theta \in [0,1], \\ \upsilon(0) &= c \end{split}$$

is given by

$$\upsilon(\theta) = c + {}^{AB} \mathbb{I}_{0^+}^{\varrho} \varpi(\theta).$$

**Theorem 2.7** ([46]) Let  $\mathfrak{X}$  be a Banach space and  $\mathfrak{K}$  be a nonempty closed subset of  $\mathfrak{X}$ . If  $\mathfrak{B}: \mathfrak{K} \to \mathfrak{K}$  is a contraction, then there exists a unique fixed point of  $\mathfrak{B}$ .

**Theorem 2.8** ([46]) Let  $\mathfrak{X}$  be a Banach space and  $\mathfrak{K}$  be a convex subset of  $\mathfrak{X}$  and  $Q : \mathfrak{K} \to \mathfrak{K}$  be a compact and continuous map. Then Q has at least one fixed point in  $\mathfrak{K}$ .

## 3 Main results

This section is devoted to obtaining formula of the solution to ABC-type nonlocal problem (1.1)-(1.2). Then we prove the existence and uniqueness of solution for problem (1.1)-(1.2) by means of Schauder's fixed point theorem (Theorem 2.8) and Banach's fixed point theorem (Theorem 2.7). Moreover, we also discuss the continuous dependence of solutions to such equations on arbitrary data.

## 3.1 Solution representation

**Lemma 3.1** Let  $0 < \rho \le 1$  and  $\sum_{k=1}^{m} \beta_k \ne 0$ . Then the solution of ABC-type nonlocal problem (1.1)–(1.2) can be indicated by the fractional integral equation

$$\varkappa(\theta) = \mathcal{A}\left(\varkappa_0 - \sum_{k=1}^m \beta_k{}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\varkappa}(\tau_k)\right) + {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \mathfrak{F}_{\varkappa}(\theta), \tag{3.1}$$

where  $\mathfrak{F}_{\varkappa}$  is the solution of the functional integral equation

$$\mathfrak{F}_{\varkappa}(\theta) = f\left(\theta, \mathcal{A}_{\varkappa_{0}} - \mathcal{A}\sum_{k=1}^{m} \beta_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}) + {}^{AB} \mathbb{I}_{0,\theta}^{\varrho} \mathfrak{F}_{\varkappa}(\theta), \mathfrak{F}_{\varkappa}(\theta)\right)$$
(3.2)

and  $\mathcal{A} := (\sum_{k=1}^{m} \beta_k)^{-1}$ .

*Proof* Set  ${}^{ABC}\mathbb{D}^{\varrho}_{0,\theta} \varkappa(\theta) = \mathfrak{F}_{\varkappa}(\theta)$  in (1.1). Then we get

$$\mathfrak{F}_{\varkappa}(\theta) = f(\theta, \varkappa(\theta), \mathfrak{F}_{\varkappa}(\theta)).$$

Applying  ${}^{AB}\mathbb{I}^{\varrho}_{0,\theta}$  on both sides of (1.1) and using Lemma 2.2, we have

$$\varkappa(\theta) = \varkappa(0) + {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \mathfrak{F}_{\varkappa}(\theta).$$
(3.3)

Putting  $\theta = \tau_k$  into (3.3), we get

$$\varkappa(\tau_k) = \varkappa(0) + {}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\varkappa}(\tau_k).$$
(3.4)

Multiplying  $\beta_k$  and taking the sum to both sides of (3.4), we can write

$$\sum_{k=1}^m \beta_k \varkappa(\tau_k) = \sum_{k=1}^m \beta_k \varkappa(0) + \sum_{k=1}^m \beta_k^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\varkappa}(\tau_k).$$

By nonlocal condition (1.2), we obtain

$$\begin{split} \varkappa_0 &= \sum_{k=1}^m \beta_k \varkappa(\tau_k) \\ &= \sum_{k=1}^m \beta_k \varkappa(0) + \sum_{k=1}^m \beta_k {}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\varkappa}(\tau_k), \end{split}$$

which implies

$$\varkappa(0) = \left(\sum_{k=1}^{m} \beta_{k}\right)^{-1} \left[\varkappa_{0} - \sum_{k=1}^{m} \beta_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k})\right].$$

Since  $\mathcal{A} = (\sum_{k=1}^{m} \beta_k)^{-1}$ , we get

$$\varkappa(\theta) = \mathcal{A}\left(\varkappa_0 - \sum_{k=1}^m \beta_k{}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\varkappa}(\tau_k)\right) + {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \mathfrak{F}_{\varkappa}(\theta).$$

Here,  $\mathfrak{F}_{\varkappa}$  is the solution of equation  $\mathfrak{F}_{\varkappa}(\theta) = f(\theta, \varkappa(\theta), \mathfrak{F}_{\varkappa}(\theta))$ , i.e.,

$$\mathfrak{F}_{\varkappa}(\theta) = f\left(\theta, \mathcal{A}_{\varkappa_{0}} - A\sum_{k=1}^{m}\beta_{k}{}^{AB}\mathbb{I}_{0,\tau_{k}}^{\varrho}\mathfrak{F}_{\varkappa}(\tau_{k}) + {}^{AB}\mathbb{I}_{0,\theta}^{\varrho}\mathfrak{F}_{\varkappa}(\theta), \mathfrak{F}_{\varkappa}(\theta)\right).$$

The proof is completed.

Now, we consider the following hypotheses:

(*H*<sub>1</sub>) There exists a constant  $L_1 > 0$  such that

$$\left|f(\theta, x, y) - f\left(\theta, x^*, y^*\right)\right| \le L_1\left(\left|x - x^*\right| + \left|y - y^*\right|\right)$$

for all  $\theta \in [0, \chi]$  and  $x, x^*, y, y^* \in \mathbb{R}$ .

(*H*<sub>2</sub>) There exists a constant  $\kappa > 0$  such that

$$|f(\theta, x, y)| \leq \kappa (1 + |x| + |y|), \quad \forall (\theta, x, y) \in [0, \chi] \times \mathbb{R} \times \mathbb{R}.$$

## 3.2 Existence results

In this subsection, we prove the existence and uniqueness of solution to ABC-type non-local problem (1.1)-(1.2).

The following result is based on Theorem 2.8.

**Theorem 3.2** Assume that  $f : [0, \chi] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$  is continuous. If  $(H_2)$  holds with  $\kappa \neq 1$ , and

$$\eta_1 := \frac{\kappa}{1-\kappa} \left[ \frac{(|\mathcal{A}| \sum_{k=1}^m |\beta_k| + 1)(1-\varrho)}{\mathfrak{N}(\varrho)} + \frac{|\mathcal{A}| \sum_{k=1}^m |\beta_k| \tau_k^{\varrho} + \chi^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right] < 1,$$
(3.5)

then ABC-type nonlocal problem (1.1)–(1.2) has at least one solution  $\varkappa \in C[0, \chi]$ .

*Proof* Define the operator  $\mathcal{T}: C[0, \chi] \to C[0, \chi]$  by

$$(\mathcal{T}\varkappa)(\theta) = \mathcal{A}\varkappa_0 - \mathcal{A}\sum_{k=1}^m \beta_k{}^{AB}\mathbb{I}^{\varrho}_{0,\tau_k}\mathfrak{F}_{\varkappa}(\tau_k) + {}^{AB}\mathbb{I}^{\varrho}_{0,\theta}\mathfrak{F}_{\varkappa}(\theta),$$
(3.6)

where

$$\mathfrak{F}_{\varkappa}(\theta) = f\left(\theta, \mathcal{A}_{\varkappa_{0}} - \mathcal{A}\sum_{k=1}^{m}\beta_{k}{}^{AB}\mathbb{I}^{\varrho}_{0,\tau_{k}}\mathfrak{F}_{\varkappa}(\tau_{k}) + {}^{AB}\mathbb{I}^{\varrho}_{0,\theta}\mathfrak{F}_{\varkappa}(\theta), \mathfrak{F}_{\varkappa}(\theta)\right).$$
(3.7)

The operator  $\mathcal{T}$  is well defined. Indeed, we consider a function  $\varkappa \in C[0, \chi]$ . It is clear that  $\mathcal{T} \varkappa \in C[0, \chi]$ . Also, by equation (3.6), Lemmas 2.4 and 2.5, we have

$$\begin{split} \left({}^{ABC}\mathbb{D}^{\varrho}_{0,\theta}\mathcal{T}\varkappa\right)(\theta) &= \mathcal{A}\varkappa_{0}\left({}^{ABC}\mathbb{D}^{\varrho}_{0,\theta}\mathbf{1}\right)(\theta) - \mathcal{A}\sum_{k=1}^{m}\beta_{k}{}^{AB}\mathbb{I}^{\varrho}_{0,\tau_{k}}\mathfrak{F}_{\varkappa}(\tau_{k})\left({}^{ABC}\mathbb{D}^{\varrho}_{0,\theta}\mathbf{1}\right)(\theta) \\ &+ {}^{ABC}\mathbb{D}^{\varrho}_{0,\theta}{}^{AB}\mathbb{I}^{\varrho}_{0,\theta}\mathfrak{F}_{\varkappa}(\theta) \\ &= \mathfrak{F}_{\varkappa}(\theta). \end{split}$$

Since  $\varkappa \in C[0, \chi]$  and  ${}^{ABC}\mathbb{D}^{\varrho}_{0,\theta} \varkappa(\theta) = \mathfrak{F}_{\varkappa}(\theta)$  in equation (1.1), it follows that

$$\left( {}^{ABC} \mathbb{D}^{\varrho}_{0,\theta} \mathcal{T} \varkappa \right) (\theta) = f \left( \theta, \varkappa(\theta), {}^{ABC} \mathbb{D}^{\varrho}_{0,\theta} \varkappa(\theta) \right)$$

As  $f(\theta, \varkappa(\theta), {}^{ABC}\mathbb{D}^{\varrho}_{0,\theta} \varkappa(\theta))$  is continuous on  $[0, \chi]$ , then  ${}^{ABC}\mathbb{D}^{\varrho}_{0,\theta} \mathcal{T}\varkappa(\theta) \in C[0, \chi]$ . Let  $r \geq \frac{\eta_2}{1-\eta_1}$  and  $\mathcal{B}_r = \{\varkappa \in C[0, \chi] : \|\varkappa\| \leq r\}$ , where  $\mathcal{B}_r$  is a nonempty, closed, convex, and bounded subset of  $C[0, \chi]$  and

$$\eta_2 = |\mathcal{A}_{\mathcal{H}_0}| + \eta_1. \tag{3.8}$$

Now, we show that T fulfills the hypotheses of Theorem 2.8. The proof is presented in numerous steps as follows.

# Step 1. $\mathcal{TB}_r \subseteq \mathcal{B}_r$ . For $\theta \in [0, \chi]$ , we get

$$\begin{split} \mathcal{T}_{\varkappa}(\theta) \Big| &\leq |\mathcal{A}_{\varkappa_{0}}| + |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}|^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \Big| \mathfrak{F}_{\varkappa}(\tau_{k}) \Big| + {}^{AB} \mathbb{I}_{0,\theta}^{\varrho} \Big| \mathfrak{F}_{\varkappa}(\theta) \Big| \\ &\leq |\mathcal{A}_{\varkappa_{0}}| + |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| \bigg[ \frac{1-\varrho}{\mathfrak{N}(\varrho)} \Big| \mathfrak{F}_{\varkappa}(\tau_{k}) \Big| \\ &+ \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\tau_{k}} (\tau_{k} - \sigma)^{\varrho-1} \Big| \mathfrak{F}_{\varkappa}(\sigma) \Big| d\sigma \bigg] \\ &+ \frac{1-\varrho}{\mathfrak{N}(\varrho)} \Big| \mathfrak{F}_{\varkappa}(\theta) \Big| + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\theta} (\theta - \sigma)^{\varrho-1} \Big| \mathfrak{F}_{\varkappa}(\sigma) \Big| d\sigma \end{split}$$

and

$$\begin{split} \left|\mathfrak{F}_{\varkappa}(\theta)\right| &= \left|f\left(\theta,\varkappa(\theta),\mathfrak{F}_{\varkappa}(\theta)\right)\right| \\ &\leq \kappa \left(1+\left|\varkappa(\theta)\right|+\left|\mathfrak{F}_{\varkappa}(\theta)\right|\right). \end{split}$$

Thus

$$\left|\mathfrak{F}_{\varkappa}(\theta)\right| \leq \frac{\kappa (1 + |\varkappa(\theta)|)}{1 - \kappa}.\tag{3.9}$$

It follows from (3.5) and (3.8) that, for each  $\varkappa \in \mathcal{B}_r$ ,

$$\begin{split} \left| \mathcal{T} \varkappa(\theta) \right| &\leq |\mathcal{A} \varkappa_{0}| + |\mathcal{A}| \frac{\kappa (1+r)}{1-\kappa} \sum_{k=1}^{m} |\beta_{k}| \left( \frac{1-\varrho}{\mathfrak{N}(\varrho)} + \frac{\tau_{k}^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right) \\ &\quad + \frac{\kappa (1+r)}{1-\kappa} \left( \frac{1-\varrho}{\mathfrak{N}(\varrho)} + \frac{\theta^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right) \\ &= |\mathcal{A} \varkappa_{0}| + \frac{\kappa}{1-\kappa} \left[ \left( |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| + 1 \right) \frac{(1-\varrho)}{\mathfrak{N}(\varrho)} + \frac{|\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| \tau_{k}^{\varrho} + \theta^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right] \\ &\quad + \frac{\kappa}{1-\kappa} \left[ \left( |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| + 1 \right) \frac{(1-\varrho)}{\mathfrak{N}(\varrho)} + \frac{|\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| \tau_{k}^{\varrho} + \theta^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right] r \\ &\leq \eta_{2} + \eta_{1} r \\ &\leq r. \end{split}$$

Step 2.  ${\mathcal T}$  is continuous.

Let  $\varkappa_n$  be a sequence such that  $\varkappa_n \to \varkappa$  as  $n \to \infty$ . Then

$$\begin{split} \left| \mathcal{T} \varkappa_{n}(\theta) - \mathcal{T} \varkappa(\theta) \right| \\ &\leq |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}|^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \left| \mathfrak{F}_{\varkappa_{n}}(\tau_{k}) - \mathfrak{F}_{\varkappa}(\tau_{k}) \right| \\ &+ {}^{AB} \mathbb{I}_{0,\theta}^{\varrho} \left| \mathfrak{F}_{\varkappa_{n}}(\theta) - \mathfrak{F}_{\varkappa}(\theta) \right| \end{split}$$

$$\leq |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| \left[ \frac{1-\varrho}{\mathfrak{N}(\varrho)} \big| \mathfrak{F}_{\varkappa_{n}}(\tau_{k}) - \mathfrak{F}_{\varkappa}(\tau_{k}) \big| \right. \\ \left. + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\tau_{k}} (\tau_{k} - \sigma)^{\varrho-1} \big| \mathfrak{F}_{\varkappa_{n}}(\sigma) - \mathfrak{F}_{\varkappa}(\sigma) \big| \, d\sigma \right] \\ \left. + \frac{1-\varrho}{\mathfrak{N}(\varrho)} \big| \mathfrak{F}_{\varkappa_{n}}(\theta) - \mathfrak{F}_{\varkappa}(\theta) \big| + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\theta} (\theta - \sigma)^{\varrho-1} \big| \mathfrak{F}_{\varkappa_{n}}(\sigma) - \mathfrak{F}_{\varkappa}(\sigma) \big| \, d\sigma \\ \\ \leq \frac{(|\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| + 1)(1-\varrho)}{\mathfrak{N}(\varrho)} \big\| \mathfrak{F}_{\varkappa_{n}}(\cdot) - \mathfrak{F}_{\varkappa}(\cdot) \big\| \\ \left. + \frac{|\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| \tau_{k}^{\varrho} + \chi^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \big\| \mathfrak{F}_{\varkappa_{n}}(\cdot) - \mathfrak{F}_{\varkappa}(\cdot) \big\|.$$

Since  $\mathfrak{F}_{\varkappa}(\cdot) = f(\cdot, \varkappa(\cdot), \mathfrak{F}_{\varkappa}(\cdot)) \in C[0, \chi]$ , it follows that  $\|\mathcal{T} \varkappa_n(\cdot) - \mathcal{T} \varkappa(\cdot)\| \to 0$  as  $n \to \infty$ , which proves the required result.

Step 3.  $\mathcal{T}$  is compact.

We shall show that  $\mathcal{TB}_r$  is relatively compact. Clearly,  $\mathcal{TB}_r$  is uniformly bounded due to Step 1. It remains to show that  $\mathcal{TB}_r$  is equicontinuous. Let  $\theta_1, \theta_2 \in [0, \chi]$  such that  $0 \leq \theta_1 \leq \theta_2 \leq \chi$ . Then

$$\begin{split} \left| \mathcal{T}_{\varkappa}(\theta_{2}) - \mathcal{T}_{\varkappa}(\theta_{1}) \right| \\ &= \left| {}^{AB} \mathbb{I}^{\varrho}_{0,\theta_{2}} \mathfrak{F}_{\varkappa}(\theta_{2}) - {}^{AB} \mathbb{I}^{\varrho}_{0,\theta_{1}} \mathfrak{F}_{\varkappa}(\theta_{1}) \right| \\ &\leq \left| \frac{1 - \varrho}{\mathfrak{N}(\varrho)} |\mathfrak{F}_{\varkappa}(\theta_{2}) \right| + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\theta_{2}} (\theta_{2} - \sigma)^{\varrho - 1} |\mathfrak{F}_{\varkappa}(\sigma)| \, d\sigma \\ &- \frac{1 - \varrho}{\mathfrak{N}(\varrho)} |\mathfrak{F}_{\varkappa}(\theta_{1})| - \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\theta_{1}} (\theta_{1} - \sigma)^{\varrho - 1} |\mathfrak{F}_{\varkappa}(\sigma)| \, d\sigma | \\ &\leq \frac{1 - \varrho}{\mathfrak{N}(\varrho)} |\mathfrak{F}_{\varkappa}(\theta_{2}) - \mathfrak{F}_{\varkappa}(\theta_{1})| \\ &+ \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\theta_{1}} |(\theta_{1} - \sigma)^{\varrho - 1} - (\theta_{2} - \sigma)^{\varrho - 1}| |\mathfrak{F}_{\varkappa}(\sigma)| \, d\sigma \\ &+ \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{\theta_{1}}^{\theta_{2}} (\theta_{2} - \sigma)^{\varrho - 1} |\mathfrak{F}_{\varkappa}(\sigma)| \, d\sigma . \end{split}$$

It follows from (3.9) that, for each  $\varkappa \in \mathcal{B}_r$ ,

$$\begin{split} \left| \mathcal{T} \varkappa(\theta_{2}) - \mathcal{T} \varkappa(\theta_{1}) \right| &\leq \frac{1-\varrho}{\mathfrak{N}(\varrho)} \left| \mathfrak{F}_{\varkappa}(\theta_{2}) - \mathfrak{F}_{\varkappa}(\theta_{1}) \right| \\ &+ \frac{\kappa(1+r)}{1-\kappa} \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\theta_{1}} \left| (\theta_{1} - \sigma)^{\varrho - 1} - (\theta_{2} - \sigma)^{\varrho - 1} \right| d\sigma \\ &+ \frac{\kappa(1+r)}{1-\kappa} \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{\theta_{1}}^{\theta_{2}} (\theta_{2} - \sigma)^{\varrho - 1} d\sigma \\ &= \frac{1-\varrho}{\mathfrak{N}(\varrho)} \left| \mathfrak{F}_{\varkappa}(\theta_{2}) - \mathfrak{F}_{\varkappa}(\theta_{1}) \right| \\ &+ \frac{\kappa(1+r)}{1-\kappa} \frac{1}{\mathfrak{N}(\varrho)\Gamma(\varrho)} \left[ (\theta_{1}^{\varrho} + (\theta_{2} - \theta_{1})^{\varrho} - \theta_{2}^{\varrho} \right] \\ &+ \frac{\kappa(1+r)}{1-\kappa} \frac{1}{\mathfrak{N}(\varrho)\Gamma(\varrho)} (\theta_{2} - \theta_{1})^{\varrho} \end{split}$$

$$\leq \frac{1-\varrho}{\mathfrak{N}(\varrho)} \left| \mathfrak{F}_{\varkappa}(\theta_2) - \mathfrak{F}_{\varkappa}(\theta_1) \right| + \frac{\kappa (1+r)}{1-\kappa} \frac{2(\theta_2 - \theta_1)^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)}.$$

Since  $\mathfrak{F}_{\varkappa}(\cdot) = f(\cdot, \varkappa(\cdot), \mathfrak{F}_{\varkappa}(\cdot)) \in C[\theta_1, \theta_2]$ , it follows that  $|\mathcal{T}\varkappa(\theta_2) - \mathcal{T}\varkappa(\theta_1)| \to 0$  as  $\theta_2 \to \theta_1$ .

As a result of Steps 1 to 3 together with the Arzela–Ascoli theorem, we arrive at  $\mathcal{T}$  being continuous and compact. According to Theorem 2.8, ABC-type nonlocal problem (1.1)–(1.2) has at least one solution in  $\mathcal{B}_r$ .

The following result is based on Theorem 2.7.

**Theorem 3.3** Assume that  $f : [0, \chi] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$  is continuous. If  $(H_1)$  holds with  $L_1 \neq 1$ , then ABC-type nonlocal problem (1.1)–(1.2) has a unique solution  $\varkappa \in C[0, \chi]$  provided that

$$\Upsilon := \frac{L_1}{1 - L_1} \left[ \frac{(|\mathcal{A}| \sum_{k=1}^m |\beta_k| + 1)(1 - \varrho)}{\mathfrak{N}(\varrho)} + \frac{|\mathcal{A}| \sum_{k=1}^m |\beta_k| \tau_k^{\varrho} + \chi^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right] < 1.$$
(3.10)

*Proof* We shall use Theorem 2.7 to prove that  $\mathcal{T}$  defined by (3.6) has a fixed point. Let  $\varkappa, \varkappa^* \in C[0, \chi]$  and  $\theta \in [0, \chi]$ . Then

$$\left| (\mathcal{T}_{\varkappa})(\theta) - (\mathcal{T}_{\varkappa}^{*})(\theta) \right| \leq |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}|^{AB} \mathbb{I}^{\varrho}_{0,\tau_{k}} \left| \mathfrak{F}_{\varkappa}(\tau_{k}) - \mathfrak{F}_{\varkappa}^{*}(\tau_{k}) \right|$$
$$+ {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \left| \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\varkappa}^{*}(\theta) \right|.$$
(3.11)

On the other hand, we have, for each  $\theta \in [0, \chi]$ ,

$$\begin{split} \left| \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\varkappa^{\ast}}(\theta) \right| &= \left| f\left(\theta, \varkappa(\theta), \mathfrak{F}_{\varkappa}(\theta)\right) - f\left(\theta, \varkappa^{\ast}(\theta), \mathfrak{F}_{\varkappa^{\ast}}(\theta)\right) \right| \\ &\leq L_1 \left( \left| \varkappa(\theta) - \varkappa^{\ast}(\theta) \right| + \left| \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\varkappa^{\ast}}(\theta) \right| \right). \end{split}$$

Thus

$$\left|\mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\varkappa^{*}}(\theta)\right| \leq \frac{L_{1}}{1 - L_{1}} \left|\varkappa(\theta) - \varkappa^{*}(\theta)\right|.$$
(3.12)

By replacing (3.12) in (3.11), we get

$$\begin{split} \left| (\mathcal{T}\varkappa)(\theta) - \left(\mathcal{T}\varkappa^*\right)(\theta) \right| &\leq \frac{|\mathcal{A}|L_1}{1 - L_1} \sum_{k=1}^m |\beta_k|^{AB} \mathbb{I}_{0,\tau_k}^{\varrho} \left| \varkappa(\tau_k) - \varkappa^*(\tau_k) \right| \\ &\quad + \frac{L_1}{1 - L_1} {}^{AB} \mathbb{I}_{0,\theta}^{\varrho} \left| \varkappa(\theta) - \varkappa^*(\theta) \right| \\ &= \frac{|\mathcal{A}|L_1}{1 - L_1} \sum_{k=1}^m |\beta_k| \left[ \frac{1 - \varrho}{\mathfrak{N}(\varrho)} \left| \varkappa(\tau_k) - \varkappa^*(\tau_k) \right| \right. \\ &\quad + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_0^{\tau_k} (\tau_k - \sigma)^{\varrho - 1} \left| \varkappa(\sigma) - \varkappa^*(\sigma) \right| d\sigma \right] \\ &\quad + \frac{L_1}{1 - L_1} \left[ \frac{1 - \varrho}{\mathfrak{N}(\varrho)} \left| \varkappa(\theta) - \varkappa^*(\theta) \right| \right] \end{split}$$

$$\begin{aligned} &+ \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\theta} (\theta - \sigma)^{\varrho - 1} \Big| \varkappa(\sigma) - \varkappa^{*}(\sigma) \Big| d\sigma \Big] \\ &\leq \frac{L_{1}}{1 - L_{1}} \bigg[ \frac{(|\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| + 1)(1 - \varrho)}{\mathfrak{N}(\varrho)} \\ &+ \frac{|\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| \tau_{k}^{\varrho} + \chi^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \bigg] \big\| \varkappa - \varkappa^{*} \big\|. \end{aligned}$$

Consequently, by (3.10),  $\mathcal{T}$  is a contraction. As a consequence of Theorem 2.7, we conclude that  $\mathcal{T}$  has a fixed point which is a solution of problem (1.1)–(1.2).

# 3.3 Continuous dependence

This portion is devoted to discussing the continuous dependence of the solution for ABC-type nonlocal problem (1.1)-(1.2).

**Definition 3.4** The solution  $\varkappa \in C[0, \chi]$  of ABC-type nonlocal problem (1.1)–(1.2) is called continuously dependent on  $\varkappa_0$  if, for every  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that  $|\varkappa_0 - \widetilde{\varkappa}_0| < \delta$  implies  $\|\varkappa - \widetilde{\varkappa}\| < \epsilon$ , where  $\widetilde{\varkappa}$  is the solution of equation (1.1) with the nonlocal condition

$$\sum_{k=1}^{m} \beta_k \varkappa(\tau_k) = \widetilde{\varkappa}_0, \quad \tau_k \in (0, \chi).$$
(3.13)

**Theorem 3.5** Assume that the hypotheses of Theorem 3.3 are fulfilled. Then the solution of ABC-type nonlocal problem (1.1)-(1.2) depends continuously on  $\varkappa_0$ .

*Proof* In view of Lemma 3.1, the solution of ABC-type nonlocal problem (1.1)-(1.2) is

$$\varkappa(\theta) = \mathcal{A}\left(\varkappa_0 - \sum_{k=1}^m \beta_k{}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\varkappa}(\tau_k)\right) + {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \mathfrak{F}_{\varkappa}(\theta), \tag{3.14}$$

and the solution of ABC-type nonlocal problem (1.1)-(3.13) is

$$\widetilde{\varkappa}(\theta) = \mathcal{A}\left(\widetilde{\varkappa}_{0} - \sum_{k=1}^{m} \beta_{k}{}^{AB}\mathbb{I}^{\varrho}_{0,\tau_{k}}\mathfrak{F}_{\widetilde{\varkappa}}(\tau_{k})\right) + {}^{AB}\mathbb{I}^{\varrho}_{0,\theta}\mathfrak{F}_{\widetilde{\varkappa}}(\theta),$$
(3.15)

where  $\mathfrak{F}_{\varkappa}$  and  $\mathfrak{F}_{\widetilde{\varkappa}}$  are the solutions of

$$\begin{split} \mathfrak{F}_{\varkappa}(\theta) &= f\left(\theta, \mathcal{A}_{\varkappa_{0}} - \mathcal{A}\sum_{k=1}^{m}\beta_{k}{}^{AB}\mathbb{I}^{\varrho}_{0,\tau_{k}}\mathfrak{F}_{\varkappa}(\tau_{k}) + {}^{AB}\mathbb{I}^{\varrho}_{0,\theta}\mathfrak{F}_{\varkappa}(\theta), \mathfrak{F}_{\varkappa}(\theta)\right) \\ &= f\left(\theta, \varkappa(\theta), \mathfrak{F}_{\varkappa}(\theta)\right) \end{split}$$

and

$$\begin{split} \mathfrak{F}_{\widetilde{\varkappa}}(\theta) &= f\left(\theta, \mathcal{A}\widetilde{\varkappa}_{0} - \mathcal{A}\sum_{k=1}^{m}\beta_{k}{}^{AB}\mathbb{I}^{\varrho}_{0,\tau_{k}}\mathfrak{F}_{\widetilde{\varkappa}}(\tau_{k}) + {}^{AB}\mathbb{I}^{\varrho}_{0,\theta}\mathfrak{F}_{\widetilde{\varkappa}}(\theta), \mathfrak{F}_{\widetilde{\varkappa}}(\theta)\right) \\ &= f\left(\theta, \widetilde{\varkappa}(\theta), \mathfrak{F}_{\widetilde{\varkappa}}(\theta)\right). \end{split}$$

Hence,

$$\left|\varkappa(\theta) - \widetilde{\varkappa}(\theta)\right| \leq |\mathcal{A}| \left( |\varkappa_{0} - \widetilde{\varkappa}_{0}| + \sum_{k=1}^{m} |\beta_{k}|^{AB} \mathbb{I}^{\varrho}_{0,\tau_{k}} \left| \mathfrak{F}_{\varkappa}(\tau_{k}) - \mathfrak{F}_{\widetilde{\varkappa}}(\tau_{k}) \right| \right) + {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \left| \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\widetilde{\varkappa}}(\theta) \right|.$$

$$(3.16)$$

However, we have from  $(H_1)$  that

$$\begin{split} \left| \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\widetilde{\varkappa}}(\theta) \right| &\leq \left| f\left(\theta, \varkappa(\theta), \mathfrak{F}_{\varkappa}(\theta)\right) - f\left(\theta, \widetilde{\varkappa}(\theta), \mathfrak{F}_{\widetilde{\varkappa}}(\theta)\right) \right| \\ &\leq L_1 \left| \varkappa(\theta) - \widetilde{\varkappa}(\theta) \right| + L_1 \left| \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\widetilde{\varkappa}}(\theta) \right|. \end{split}$$

Thus

$$\left|\mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\widetilde{\varkappa}}(\theta)\right| \leq \frac{L_1}{1 - L_1} \left|\varkappa(\theta) - \widetilde{\varkappa}(\theta)\right|. \tag{3.17}$$

By replacing (3.17) in (3.16), we get

$$\begin{split} \varkappa(\theta) - \widetilde{\varkappa}(\theta) \Big| &\leq |\mathcal{A}| \left( |\varkappa_0 - \widetilde{\varkappa}_0| + \frac{L_1}{1 - L_1} \sum_{k=1}^m |\beta_k|^{AB} \mathbb{I}^{\theta}_{0,\tau_k} | \varkappa(\theta) - \widetilde{\varkappa}(\theta) | \right) \\ &+ \frac{L_1}{1 - L_1}^{AB} \mathbb{I}^{\theta}_{0,\theta} | \varkappa(\theta) - \widetilde{\varkappa}(\theta) | \\ &\leq |\mathcal{A}| |\varkappa_0 - \widetilde{\varkappa}_0| + \frac{L_1}{1 - L_1} |\mathcal{A}| \sum_{k=1}^m |\beta_k| \frac{1 - \varrho}{\mathfrak{N}(\varrho)} || \varkappa - \widetilde{\varkappa} || \\ &+ \frac{L_1}{1 - L_1} |\mathcal{A}| \sum_{k=1}^m |\beta_k| \frac{\tau_k^{\theta}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} || \varkappa - \widetilde{\varkappa} || \\ &+ \frac{L_1}{1 - L_1} \frac{1 - \varrho}{\mathfrak{N}(\varrho)} || \varkappa - \widetilde{\varkappa} || + \frac{L_1}{1 - L_1} \frac{\theta^{\theta}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} || \varkappa - \widetilde{\varkappa} || \\ &\leq |\mathcal{A}| |\varkappa_0 - \widetilde{\varkappa}_0| + \frac{L_1}{1 - L_1} \frac{(1 - \varrho)(|\mathcal{A}| \sum_{k=1}^m |\beta_k| + 1)}{\mathfrak{N}(\varrho)} || \varkappa - \widetilde{\varkappa} || \\ &\leq |\mathcal{A}| |\varkappa_0 - \widetilde{\varkappa}_0| + \frac{L_1}{1 - L_1} \frac{(1 - \varrho)(|\mathcal{A}| \sum_{k=1}^m |\beta_k| + 1)}{\mathfrak{N}(\varrho)} || \varkappa - \widetilde{\varkappa} || \\ &\leq |\mathcal{A}| |\varkappa_0 - \widetilde{\varkappa}_0| + \frac{L_1}{1 - L_1} \frac{|\mathcal{A}| \sum_{k=1}^m |\beta_k| \tau_k^{\theta} + \theta^{\theta}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} || \varkappa - \widetilde{\varkappa} || \\ &\leq |\mathcal{A}| |\varkappa_0 - \widetilde{\varkappa}_0| + \frac{L_1}{1 - L_1} \Big[ \frac{(|\mathcal{A}| \sum_{k=1}^m |\beta_k| + 1)(1 - \varrho)}{\mathfrak{N}(\varrho)} \\ &+ \frac{|\mathcal{A}| \sum_{k=1}^m |\beta_k| \tau_k^{\theta} + \chi^{\theta}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \Big] || \varkappa - \varkappa^* ||. \end{split}$$

Since  $\Upsilon < 1$ , we get

$$\|\varkappa - \widetilde{\varkappa}\| \leq \frac{|\mathcal{A}|}{1 - \Upsilon} |\varkappa_0 - \widetilde{\varkappa}_0| < \frac{|\mathcal{A}|}{1 - \Upsilon} \delta = \epsilon.$$

**Definition 3.6** The solution  $\varkappa \in C[0, \chi]$  of ABC-type nonlocal problem (1.1)–(1.2) is called continuously dependent on the coefficients  $\sum_{k=1}^{m} \beta_k$  if, for every  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that  $\sum_{k=1}^{m} |\beta_k - \widetilde{\beta_k}| < \delta$  implies  $||\varkappa - \widetilde{\varkappa}|| < \epsilon$ , where  $\widetilde{\varkappa}$  is the solution of

equation (1.1) with the nonlocal condition

$$\sum_{k=1}^{m} \widetilde{\beta}_{k} \varkappa(\tau_{k}) = \varkappa_{0}, \quad \tau_{k} \in (0, \chi).$$
(3.18)

**Theorem 3.7** Assume that the hypotheses of Theorem 3.3 hold. Then the solution of ABCtype nonlocal problem (1.1)–(1.2) depends continuously on the coefficients  $\sum_{k=1}^{m} \beta_k$ , provided that

$$\left|\frac{1}{\sum_{k=1}^{m} \beta_{k} \sum_{k=1}^{m} \widetilde{\beta_{k}}}\right| \leq 1,$$

$$\Theta_{1} := L_{1}\left(1 + |\widetilde{\mathcal{A}}| \sum_{k=1}^{m} |\widetilde{\beta_{k}}| \frac{\tau_{k}^{\varrho}}{\mathfrak{N}(\varrho)\Gamma(\varrho)}\right) < 1,$$
(3.19)

and

$$\Theta_{2} := \left( |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| + 1 \right) \left( \frac{1-\varrho}{\mathfrak{N}(\varrho)} + \frac{|\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| \tau_{k}^{\varrho} + \chi^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right) \frac{L_{1}}{1-L_{1}} < 1,$$

where  $\widetilde{\mathcal{A}} = \frac{1}{\sum_{k=1}^{m} \widetilde{\beta_k}}$  and  $\sum_{k=1}^{m} \widetilde{\beta_k} \neq 0$ .

*Proof* In view of Lemma 3.1, the solution of ABC-type nonlocal problem (1.1)-(1.2) is

$$\varkappa(\theta) = \mathcal{A}\left(\varkappa_0 - \sum_{k=1}^m \beta_k{}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\varkappa}(\tau_k)\right) + {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \mathfrak{F}_{\varkappa}(\theta),$$

and the solution of ABC-type nonlocal problem (1.1)-(3.18) is

$$\widetilde{\varkappa}(\theta) = \widetilde{\mathcal{A}}\left(\varkappa_0 - \sum_{k=1}^m \widetilde{\beta_k}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\widetilde{\varkappa}}(\tau_k)\right) + {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \mathfrak{F}_{\widetilde{\varkappa}}(\theta),$$

where  $\mathfrak{F}_\varkappa$  and  $\mathfrak{F}_{\widetilde\varkappa}$  are the solutions of

$$\begin{split} \mathfrak{F}_{\varkappa}(\theta) &= f\left(\theta, \mathcal{A}_{\varkappa_{0}} - A\sum_{k=1}^{m}\beta_{k}^{AB}\mathbb{I}_{0,\tau_{k}}^{\varrho}\mathfrak{F}_{\varkappa}(\tau_{k}) + {}^{AB}\mathbb{I}_{0,\theta}^{\varrho}\mathfrak{F}_{\varkappa}(\theta), \mathfrak{F}_{\varkappa}(\theta)\right),\\ \mathfrak{F}_{\varkappa}(\theta) &= f\left(\theta, \widetilde{\mathcal{A}}_{\varkappa_{0}} - \widetilde{\mathcal{A}}\sum_{k=1}^{m}\widetilde{\beta_{k}}^{AB}\mathbb{I}_{0,\tau_{k}}^{\varrho}\mathfrak{F}_{\varkappa}(\tau_{k}) + {}^{AB}\mathbb{I}_{0,\theta}^{\varrho}\mathfrak{F}_{\varkappa}(\theta), \mathfrak{F}_{\varkappa}(\theta)\right). \end{split}$$

Hence,

$$\varkappa(\theta) - \widetilde{\varkappa}(\theta) = \varkappa_{0}(\mathcal{A} - \widetilde{\mathcal{A}}) - \mathcal{A} \sum_{k=1}^{m} \beta_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}) 
+ \widetilde{\mathcal{A}} \sum_{k=1}^{m} \widetilde{\beta_{k}}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}) + {}^{AB} \mathbb{I}_{0,\theta}^{\varrho} \big( \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\varkappa}(\theta) \big) 
= \varkappa_{0}(\mathcal{A} - \widetilde{\mathcal{A}}) - \mathcal{J} + \widetilde{\mathcal{J}} + {}^{AB} \mathbb{I}_{0,\theta}^{\varrho} \big( \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\varkappa}(\theta) \big).$$
(3.20)

Since

$$\mathcal{J} - \widetilde{\mathcal{J}} = \mathcal{A} \sum_{k=1}^{m} \beta_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}) - \widetilde{\mathcal{A}} \sum_{k=1}^{m} \widetilde{\beta}_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k})$$

$$= \mathcal{A} \sum_{k=1}^{m} \beta_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}) - \mathcal{A} \sum_{k=1}^{m} \beta_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k})$$

$$+ \mathcal{A} \sum_{k=1}^{m} \beta_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}) - \mathcal{A} \sum_{k=1}^{m} \widetilde{\beta}_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k})$$

$$+ \mathcal{A} \sum_{k=1}^{m} \widetilde{\beta}_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}) - \widetilde{\mathcal{A}} \sum_{k=1}^{m} \widetilde{\beta}_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k})$$

$$= \mathcal{A} \sum_{k=1}^{m} \beta_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} (\mathfrak{F}_{\varkappa}(\tau_{k}) - \mathfrak{F}_{\varkappa}(\tau_{k}))$$

$$+ \mathcal{A} \sum_{k=1}^{m} (\beta_{k} - \widetilde{\beta}_{k})^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}) + (\mathcal{A} - \widetilde{\mathcal{A}}) \sum_{k=1}^{m} \widetilde{\beta}_{k}^{AB} \mathbb{I}_{0,\tau_{k}}^{\varrho} \mathfrak{F}_{\varkappa}(\tau_{k}). \quad (3.21)$$

Substituting from (3.21) in (3.20), we get

$$\begin{aligned} \varkappa(\theta) - \widetilde{\varkappa}(\theta) &= \varkappa_0 (\mathcal{A} - \widetilde{\mathcal{A}}) - \mathcal{A} \sum_{k=1}^m \beta_k{}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \big( \mathfrak{F}_{\varkappa}(\tau_k) - \mathfrak{F}_{\widetilde{\varkappa}}(\tau_k) \big) \\ &- \mathcal{A} \sum_{k=1}^m (\beta_k - \widetilde{\beta_k}){}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\widetilde{\varkappa}}(\tau_k) - (\mathcal{A} - \widetilde{\mathcal{A}}) \sum_{k=1}^m \widetilde{\beta_k}{}^{AB} \mathbb{I}^{\varrho}_{0,\tau_k} \mathfrak{F}_{\widetilde{\varkappa}}(\tau_k) \\ &+ {}^{AB} \mathbb{I}^{\varrho}_{0,\theta} \big( \mathfrak{F}_{\varkappa}(\theta) - \mathfrak{F}_{\widetilde{\varkappa}}(\theta) \big). \end{aligned}$$

Then

$$\begin{split} \|\varkappa - \widetilde{\varkappa}\| \\ &\leq |\varkappa_{0}||\mathcal{A} - \widetilde{\mathcal{A}}| + |\mathcal{A}| \left| \sum_{k=1}^{m} \beta_{k} \right| \left( \frac{1-\varrho}{\mathfrak{N}(\varrho)} \|\mathfrak{F}_{\varkappa} - \mathfrak{F}_{\widetilde{\varkappa}}\| \right) \\ &\quad + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\tau_{k}} (\tau_{k} - \sigma)^{\varrho-1} \|\mathfrak{F}_{\varkappa} - \mathfrak{F}_{\widetilde{\varkappa}}\| d\sigma \right) \\ &\quad + |\mathcal{A}| \left| \sum_{k=1}^{m} (\beta_{k} - \widetilde{\beta}_{k}) \right| \left( \frac{1-\varrho}{\mathfrak{N}(\varrho)} \|\mathfrak{F}_{\widetilde{\varkappa}}\| + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\tau_{k}} (\tau_{k} - \sigma)^{\varrho-1} \|\mathfrak{F}_{\widetilde{\varkappa}}\| d\sigma \right) \\ &\quad + |\mathcal{A} - \widetilde{\mathcal{A}}| \left| \sum_{k=1}^{m} \widetilde{\beta}_{k} \right| \left( \frac{1-\varrho}{\mathfrak{N}(\varrho)} \|\mathfrak{F}_{\widetilde{\varkappa}}\| + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\tau_{k}} (\tau_{k} - \sigma)^{\varrho-1} \|\mathfrak{F}_{\widetilde{\varkappa}}\| d\sigma \right) \\ &\quad + \frac{1-\varrho}{\mathfrak{N}(\varrho)} \|\mathfrak{F}_{\varkappa} - \mathfrak{F}_{\widetilde{\varkappa}}\| + \frac{\varrho}{\mathfrak{N}(\varrho)} \frac{1}{\Gamma(\varrho)} \int_{0}^{\theta} (\theta - \sigma)^{\varrho-1} \|\mathfrak{F}_{\varkappa} - \mathfrak{F}_{\widetilde{\varkappa}}\| d\sigma \\ &\leq |\varkappa_{0}||\mathcal{A} - \widetilde{\mathcal{A}}| + \left( |\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| + 1 \right) \left( \frac{1-\varrho}{\mathfrak{N}(\varrho)} + \frac{|\mathcal{A}| \sum_{k=1}^{m} |\beta_{k}| \tau_{k}^{\varrho} + \theta^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right) \|\mathfrak{F}_{\varkappa} - \mathfrak{F}_{\widetilde{\varkappa}}\| \\ &\quad + \left( |\mathcal{A}| \left| \sum_{k=1}^{m} (\beta_{k} - \widetilde{\beta}_{k} \right| \right) + |\mathcal{A} - \widetilde{\mathcal{A}}| \left| \sum_{k=1}^{m} \widetilde{\beta}_{k} \right| \right) \left( \frac{1-\varrho}{\mathfrak{N}(\varrho)} + \frac{\tau_{k}^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right) \|\mathfrak{F}_{\widetilde{\varkappa}}\|. \tag{3.22}$$

Now, we have from (3.19) that

$$|\mathcal{A} - \widetilde{\mathcal{A}}| = \left|\frac{1}{\sum_{k=1}^{m} \beta_k} - \frac{1}{\sum_{k=1}^{m} \widetilde{\beta_k}}\right| = \left|\frac{\sum_{k=1}^{m} \widetilde{\beta_k} - \beta_k}{\sum_{k=1}^{m} \beta_k \sum_{k=1}^{m} \widetilde{\beta_k}}\right| \le \sum_{k=1}^{m} |\beta_k - \widetilde{\beta_k}| \le \delta.$$
(3.23)

By  $(H_1)$ , we obtain

$$\|\mathfrak{F}_{\varkappa} - \mathfrak{F}_{\widetilde{\varkappa}}\| \le \frac{L_1}{1 - L_1} \|\varkappa - \widetilde{\varkappa}\|$$
(3.24)

and

$$\begin{split} \|\mathfrak{F}_{\widetilde{\varkappa}}\| &\leq \frac{L_{1}\|\widetilde{\varkappa}\| + \|\mathfrak{F}_{0}\|}{1 - L_{1}} \\ &\leq \frac{L_{1}|\widetilde{\mathcal{A}}_{\varkappa_{0}}| + \|\mathfrak{F}_{0}\|}{1 - L_{1}} + \frac{L_{1}|\widetilde{\mathcal{A}}\sum_{k=1}^{m}\widetilde{\beta}_{k}\frac{\tau_{k}^{\varrho}}{\mathfrak{N}(\varrho)\Gamma(\varrho)}|}{1 - L_{1}}\|\mathfrak{F}_{\widetilde{\varkappa}}\|, \end{split}$$

which gives

$$\|\mathfrak{F}_{\widetilde{\varkappa}}\| \leq \frac{L_1|\widetilde{\mathcal{A}}||\varkappa_0| + \|\mathfrak{F}_0\|}{1 - \Theta_1},\tag{3.25}$$

where  $\|\mathfrak{F}_0\| = \max_{\theta \in [0,\chi]} |f(\theta, 0, 0)|$ . By replacing (3.23), (3.24), and (3.25) in (3.22), we obtain

$$\begin{split} \|\varkappa - \widetilde{\varkappa}\| &\leq |\varkappa_{0}|\delta + \Theta_{2}\|\varkappa - \widetilde{\varkappa}\| + \left(|\mathcal{A}| \left| \sum_{k=1}^{m} (\beta_{k} - \widetilde{\beta_{k}}) \right| + |\mathcal{A} - \widetilde{\mathcal{A}}| \left| \sum_{k=1}^{m} \widetilde{\beta_{k}} \right| \right) \\ &\times \left( \frac{1 - \varrho}{\mathfrak{N}(\varrho)} + \frac{\tau_{k}^{\varrho}}{\mathfrak{N}(\varrho) \Gamma(\varrho)} \right) \frac{(L_{1}|\widetilde{\mathcal{A}}||\varkappa_{0}| + \|\mathfrak{F}_{0}\|)}{(1 - \Theta_{1})} \\ &= |\varkappa_{0}|\delta + \Theta_{2}\|\varkappa - \widetilde{\varkappa}\| + \Theta_{3}, \end{split}$$

which implies

$$\|\varkappa - \widetilde{\varkappa}\| \leq \frac{|\varkappa_0|\delta + \Theta_3}{1 - \Theta_2} = \epsilon,$$

where

$$\Theta_{3} := \left( \left| \mathcal{A} \right| \left| \sum_{k=1}^{m} (\beta_{k} - \widetilde{\beta}_{k}) \right| + \left| \mathcal{A} - \widetilde{\mathcal{A}} \right| \left| \sum_{k=1}^{m} \widetilde{\beta}_{k} \right| \right) \\ \times \left( \frac{1 - \varrho}{\mathfrak{N}(\varrho)} + \frac{\tau_{k}^{\varrho}}{\mathfrak{N}(\varrho)\Gamma(\varrho)} \right) \frac{(L_{1}|\widetilde{\mathcal{A}}||\varkappa_{0}| + \|\mathfrak{F}_{0}\|)}{(1 - \Theta_{1})}.$$

# 4 Examples

*Example* 4.1 Let us consider the following ABC-type nonlocal problem:

$${}^{ABC}\mathbb{D}_{0^+}^{\frac{1}{3}}\varkappa(\theta) = \frac{\theta^2}{8} \left(1 + \frac{|\varkappa(\theta)| + |{}^{ABC}\mathbb{D}_{0^+}^{\frac{1}{3}}\varkappa(\theta)|}{1 + |\varkappa(\theta)| + |{}^{ABC}\mathbb{D}_{0^+}^{\frac{1}{3}}\varkappa(\theta)|}\right), \quad \theta \in \left[0, \frac{1}{2}\right]$$
(4.1)

with nonlocal conditions

$$\frac{1}{8}\varkappa\left(\frac{1}{6}\right) + \frac{3}{8}\varkappa\left(\frac{1}{4}\right) + \frac{1}{2}\varkappa\left(\frac{1}{3}\right) = 1,$$
(4.2)

where  $0 < (\tau_1 = \frac{1}{6}, \tau_2 = \frac{1}{4}, \tau_3 = \frac{1}{3}) < \frac{1}{2}, (\beta_1 = \frac{1}{8}, \beta_2 = \frac{3}{8}, \beta_3 = \frac{1}{2}) > 0$  (*k* = 1, 2, 3) (*m* = 3),  $\rho = \frac{1}{3}$ . Notice that (4.1)–(4.2) is a particular case of (1.1)–(1.2).

Set  $f(\theta, \varkappa, \nu) = \frac{\theta^2}{8} (1 + \frac{\varkappa}{(1+\varkappa+\nu)} + \frac{\nu}{(1+\varkappa+\nu)})$  for  $(\theta, \varkappa, \nu) \in [0, \frac{1}{2}] \times \mathbb{R}^2$ . Clearly, the function  $f(0, \varkappa(0), \nu(0)) = 0$ . Let  $\varkappa, \varkappa^*, \nu, \nu^* \in \mathbb{R}$  and  $\theta \in [0, \frac{1}{2}]$ . Then we have

$$\begin{split} \left| f(\theta,\varkappa,\nu) \right| &\leq \frac{\theta^2}{8} \left( 1 + \frac{|\varkappa|}{(1+|\varkappa|+|\nu|)} + \frac{|\nu|}{(1+|\varkappa|+|\nu|)} \right) \\ &\leq \frac{1}{8} \left( 1 + |\varkappa|+|\nu| \right). \end{split}$$

Hence condition (*H*<sub>2</sub>) is satisfied with  $\kappa = \frac{1}{8}$ . By choosing  $\mathfrak{N}(\frac{1}{3}) = 1$ , we can find that  $\eta_1 \approx 0.23 < 1$ . It follows from Theorem 3.2 that ABC-type nonlocal problem (4.3)–(4.4) has a solution on  $[0, \frac{1}{2}]$ .

Example 4.2 Consider the ABC-type nonlocal problem

$${}^{ABC}\mathbb{D}_{0^+}^{\frac{1}{2}}\varkappa(\theta) = \frac{\theta}{(8+e^{\theta})(1+|\varkappa(\theta)|+|{}^{ABC}\mathbb{D}_{0^+}^{\frac{1}{2}}\varkappa(\theta)|)}, \quad \theta \in [0,1]$$

$$(4.3)$$

with nonlocal conditions

$$\frac{1}{4}\varkappa\left(\frac{1}{3}\right) + \frac{3}{4}\varkappa\left(\frac{1}{2}\right) = \varkappa_0 \in \mathbb{R},\tag{4.4}$$

where  $0 < (\tau_1 = \frac{1}{3}, \tau_2 = \frac{1}{2}) < 1$ ,  $(\beta_1 = \frac{1}{4}, \beta_2 = \frac{3}{4}) > 0$  (k = 1, 2; m = 2), and  $\rho = \frac{1}{2}$ . Notice that (4.3)–(4.4) is a particular case of (1.1)–(1.2).

Set  $f(\theta, \varkappa, \nu) = \frac{\theta}{8+e^{\theta}} \frac{1}{(1+\varkappa+\nu)}$  for  $(\theta, \varkappa, \nu) \in [0,1] \times \mathbb{R}^2$ . Clearly, the function  $f(0, \varkappa(0), \nu(0)) = 0$ . Let  $\varkappa, \varkappa^*, \nu, \nu^* \in \mathbb{R}$  and  $\theta \in [0,1]$ . Then we have

$$\begin{split} \left| f(\theta, \varkappa, \nu) - f(\theta, \varkappa^*, \nu^*) \right| &= \frac{\theta}{8 + e^{\theta}} \left| \frac{1}{(1 + \varkappa + \nu)} - \frac{1}{(1 + \varkappa^* + \nu^*)} \right| \\ &\leq \frac{\theta}{8 + e^{\theta}} \left( \frac{|\varkappa - \varkappa^*| + |\nu - \nu^*|}{(1 + \varkappa + \nu)(1 + \varkappa^* + \nu^*)} \right) \\ &\leq \frac{1}{8} \left[ \left| \varkappa - \varkappa^* \right| + \left| \nu - \nu^* \right| \right]. \end{split}$$

Hence condition (*H*<sub>1</sub>) is satisfied with  $L_1 = \frac{1}{8}$ . By choosing  $\mathfrak{N}(\frac{1}{2}) = 1$ , we can find that  $\Upsilon \approx 0.28$ . Also, we have  $\frac{1}{\mathcal{A}} = 1 \neq 0$ . It follows from Theorem 3.3 that problem (4.3)–(4.4) has a unique solution on [0, 1].

By Theorems 3.5 and 3.7, the solution of problem (4.3)–(4.4) depends continuously on the coefficients  $\varkappa_0$  and  $\sum_{k=1}^{m} \beta_k$ .

## 5 Concluding remarks

We can conclude that the main outcomes of this manuscript have been effectively accomplished. The existence and uniqueness of solutions for the nonlocal Cauchy problem for a nonlinear implicit FDE involving the ABC fractional derivative have been proved through some fixed point techniques (Theorems 2.8, 2.7) and some outcomes related to AB operators. Then, as an application, the continuous dependence of solution to such equations on arbitrary data involved therein was discussed. This paper adds and contributes to growth FDEs, particularly in the case of nonlocal implicit FDEs involving a novel fractional derivative presented recently by Atangana and Baleanu [11]. There are some works that carried out reported studies on the existence and continuous dependence of solutions of classical FDEs, and one of the destinations of this paper is to contribute with the goal that it can have a more prominent degree of studies identified with FDEs involving generalized fractional operators.

As a future direction, the studied problem would be interesting if it were studied on generalized fractional operators of variable order recently introduced by Yang and Machado [8] and its generalization by Sousa and Oliveira [47].

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#### Availability of data and materials

Not applicable.

## Declarations

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

The authors declare that the work was realized with equal contribution. All authors read and approved the final manuscript.

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