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Some notes on conformable fractional Sturm–Liouville problems

Wei-Chuan Wang[1*]

*Correspondence:
wangwc72@gmail.com;
wangwc@nqu.edu.tw
[1]Center for General Education,
National Quemoy University, 892,
Kinmen, Taiwan, ROC

Abstract

The conformable fractional eigenvalue problem

$$-D_x^\alpha D_x^\alpha y + q(x)y = \lambda \rho(x)y$$

is considered. We employ an easy and efficient method to derive its eigenvalue asymptotic expansion. On the basis of this result, we also investigate Ambarzumyan problems related to this eigenvalue problem as an application.

Keywords: Conformable fractional Sturm–Liouville problem; Eigenvalue expansion; Prüfer-type substitution; Ambarzumyan problem

1 Introduction

Fractional calculus is a generalization of integer differential calculus, allowing to define integrals and derivatives of real or complex order. Various types of fractional derivatives, such as Riemann–Liouville, Caputo, Grunwald–Letnikov, Riesz and Weyl, have been introduced. All of them are defined via fractional integrals, thus they inherit nonlocal properties from integral. This leads to failure to satisfy some of the basic properties owned by usual derivatives, e.g., the product rule, chain rule, Rolle's theorem, mean value theorem, and composition rule. These inconsistencies come with difficulties and inconvenience in mathematical analysis. For this reason, some scholars introduced the concept of local fractional derivatives [5, 9, 12, 14, 20]. In 2014, Khalil et al. introduced a new local fractional derivative, called the conformable fractional derivative (*CFD*) [13]. Now this subject is under strong development, see [2, 7, 10, 11] and the references therein. Later, Al-Refai and Abdeljawad [6] investigated the conformable fractional Sturm–Liouville problem (CFSLP) and showed that the eigenvalues are real and simple and the eigenfunctions corresponding to distinct eigenvalues are orthogonal. Also, they established a fractional Rayleigh quotient result that can be used to estimate the first eigenvalue.

However, some authors have argued that conformable fractional derivative is not a truly fractional operator [1, 16–19]. And the physical interpretation and some potential applications of this derivative seem not solid. In [8], the authors pointed out that the conformable derivative really results from the ordinary derivative with the change of $x \to \frac{x^\alpha}{\alpha}$

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when applied to differential functions. They also tried to contribute to the physical interpretation when these derivatives are applied to physics and engineering. In another recent paper [21], the authors provided a good account of heredity/nonheredity and locality/nonlocality. They introduced a general conformable fractional derivative (GCFD) and gave the physical and geometrical interpretations of GCFD. The above is done by a new framework named extended Gâteaux derivative and linear extended Gâteaux derivative which are natural extensions of Gâteaux derivative. So, in our opinion, the exploration of the conformable derivative and its generalizations can still be interesting and valuable.

In this note we consider the CFSLP

$$-D_x^\alpha D_x^\alpha y + q(x)y = \lambda \rho(x)y, \quad \text{on } (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where λ is the spectral parameter and D_x^α is the conformable fractional derivative of order α. In 2020, Mortazaasl and Akbarfam [15] developed some spectral theories for the CFSLP with $\rho \equiv 1$ and deduced the asymptotic behavior of eigenvalues and eigenfunctions by analyzing the conformable fractional Volterra integral equations. They also showed that the eigenvalues of the CFSLP with separated boundary conditions are all real and simple, and the nth eigenfunction has precisely $n-1$ nodes in the interval $(0,1)$ for sufficiently large n. Also in [4], uniqueness theorems for the solutions of inverse problems according to the Weyl function, two given spectra, classical spectral data, and mixed data are studied. Ambarzumyan-type problems are investigated for the conformable fractional Sturm–Liouville operator in [3]. Motivated by the idea in [3, 4, 6, 15], we intend to give an alternative proof (an easy method) for eigenvalue asymptotic estimates and acquire an application of Ambarzumyan problems as a consequence. For $q \equiv 0$ and $\rho \equiv 1$, it is known that the eigenfunctions of (1) are $\sin(n\pi x^\alpha)$ and the corresponding eigenvalues are $(\alpha n\pi)^2$ via a direct calculation. Note that, for $q \equiv 0$ and $\alpha = 1$, (1) is a string equation and ρ is called a density function. For general q and ρ, we plan to apply an easy and efficient method, the Prüfer angle approach, to achieve our goal. Throughout this note we assume that the following conditions hold:

(C_1) $\alpha \in (\frac{1}{2}, 1]$, $q \in C[0,1]$;

(C_2) $\rho > 0$ with $\rho \in C^2(0,1)$.

The following is the first result.

Theorem 1.1 (Eigenvalue asymptotics)

(i) *For the potential type of* (1)–(2), $\rho \equiv 1$, *the Dirichlet eigenvalue* λ_n *satisfies*

$$\sqrt{\lambda_n} = \alpha n\pi + \frac{1}{2n\pi} \int_0^1 x^{\alpha-1} q(x)\,dx + o\left(\frac{1}{n}\right) \tag{3}$$

for sufficiently large n.

(ii) *For the density type of* (1)–(2), $q \equiv 0$, *the eigenvalue* λ_n *satisfies*

$$\sqrt{\lambda_n} \int_0^1 x^{\alpha-1} \rho^{1/2}(x)\,dx = n\pi + O\left(\frac{1}{n}\right) \tag{4}$$

for sufficiently large n.

In virtue of Theorem 1.1 and the fractional Rayleigh quotient result [6], we turn to study the Ambarzumyan problems related to (1). The following is the second result.

Theorem 1.2 (Ambarzumyan problems) *Consider the potential type of* (1), $\rho \equiv 1$.

(i) *If the eigenvalues of* (1)–(2) *are* $\lambda_n = (\alpha n\pi)^2$, $n \in \mathbb{N}$, *and the function q satisfies*

$$\int_0^1 x^{\alpha-1}q(x)\cos(2\pi x^\alpha)\,dx = 0, \tag{5}$$

then $q \equiv 0$ *on* $(0,1)$.

(ii) *If the Neumann eigenvalues of* (1) *coupled with* $D_x^\alpha y(0) = D_x^\alpha y(1) = 0$ *are* $\mu_n = [\alpha(n-1)\pi]^2$, *then* $q \equiv 0$ *on* $(0,1)$.

2 Some preliminaries and proofs of the main results

In this section, we first recall the elementary definitions and properties of conformable fractional calculus for the reader's convenience.

Definition 2.1 (cf. [2, 13]) Let $0 < \alpha \le 1$ and $f : [0,\infty) \to \mathbb{R}$.

(i) The conformable fractional derivative of f of order α at $x > 0$ is defined by

$$D_x^\alpha f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon},$$

and the conformable fractional derivative at 0 is defined as $D_x^\alpha f(0) = \lim_{x \to 0^+} D_x^\alpha f(x)$. Note that if f is differentiable, then

$$D_x^\alpha f(x) = x^{1-\alpha} f'(x), \tag{6}$$

where $' = \frac{d}{dx}$ is the ordinary derivative with respect to x. If $D_x^\alpha f(x_0)$ exists, one can say that f is α-differentiable at x_0.

(ii) The conformable fractional integral of f of order α is defined by

$$I_\alpha f(x) = \int_0^x t^{\alpha-1} f(t)\,dt \quad \text{for } x > 0.$$

Proposition 2.2 (cf. [2, 13, 15])

(i) *Let* $f : [0,\infty) \to \mathbb{R}$ *be any continuous function. Then, for all $x > 0$, we have*

$$D_x^\alpha I_\alpha f(x) = f(x). \tag{7}$$

(ii) *Let* $f : (0,b) \to \mathbb{R}$ *be differentiable. Then, for $x > 0$, we have*

$$I_\alpha D_x^\alpha f(x) = f(x) - f(0). \tag{8}$$

(iii) *For all* $p \in \mathbb{R}$, $D_x^\alpha(x^p) = px^{p-\alpha}$.

(iv) *Let* $f, g : (0,\infty) \to \mathbb{R}$ *be α-differentiable. Then*

$$D_x^\alpha(fg) = (D_x^\alpha f)g + f(D_x^\alpha g) \tag{9}$$

and

$$D_x^\alpha\left(\frac{f}{g}\right) = \frac{(D_x^\alpha f)g - f(D_x^\alpha g)}{g^2} \quad \text{with } g \neq 0. \tag{10}$$

(v) (*α-chain rule*) *Let $f, g : (0, \infty) \to \mathbb{R}$ be α-differentiable and $h(x) = f(g(x))$. Then $h(x)$ is α-differentiable, and for all x with $x \neq 0$ and $g(x) \neq 0$, we have*

$$D_x^\alpha h(x) = D_x^\alpha f\big(g(x)\big) \cdot D_x^\alpha g(x) \cdot g(x)^{\alpha-1}. \tag{11}$$

(vi) (*α-integration by parts*) *Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b x^{\alpha-1} f(x) D_x^\alpha g(x)\, dx = f(x)g(x)\big|_a^b - \int_a^b x^{\alpha-1} g(x) D_x^\alpha f(x)\, dx. \tag{12}$$

Proof of Theorem 1.1 (i) For $\rho \equiv 1$, define

$$y(x) = r(x)\sin\big(\sqrt{\lambda}\theta(x)\big) \quad \text{and} \quad D_x^\alpha y(x) = \sqrt{\lambda} r(x)\cos\big(\sqrt{\lambda}\theta(x)\big). \tag{13}$$

Taking

$$D_x^\alpha\left(\frac{D_x^\alpha y(x)}{y(x)}\right) = D_x^\alpha\left(\frac{\sqrt{\lambda}\cos(\sqrt{\lambda}\theta(x))}{\sin(\sqrt{\lambda}\theta(x))}\right)$$

and applying (1), one can obtain the phase equation

$$\theta'(x) = x^{\alpha-1}\left[1 - \frac{q(x)}{\lambda}\sin^2\big(\sqrt{\lambda}\theta(x)\big)\right]. \tag{14}$$

With each eigenvalue λ_n of (1), one can associate the Prüfer angle $\theta_n(x) \equiv \theta(x; \lambda_n)$ via (13) if one also specifies the initial condition

$$\theta_n(0) = 0, \quad n = 1, 2, 3, \ldots.$$

In particular, $\theta_n(1) = n\pi$. Integrating (14) over $[0, x]$ with $\lambda = \lambda_n$, one can obtain

$$\theta_n(x) = \frac{1}{\alpha}x^\alpha - \frac{1}{\lambda_n}\int_0^x t^{\alpha-1}q(t)\sin^2\big(\sqrt{\lambda_n}\theta_n(t)\big)\, dt. \tag{15}$$

Besides, integrating (14) over $[0, 1]$ with $\lambda = \lambda_n$, one has

$$\frac{n\pi}{\sqrt{\lambda_n}} = \frac{1}{\alpha} - \frac{1}{\lambda_n}\int_0^1 x^{\alpha-1}q(x)\sin^2\big(\sqrt{\lambda_n}\theta_n(x)\big)\, dx. \tag{16}$$

That is,

$$\sqrt{\lambda_n} = \alpha n\pi + \frac{\alpha}{\sqrt{\lambda_n}}\int_0^1 x^{\alpha-1}q(x)\sin^2\big(\sqrt{\lambda_n}\theta_n(x)\big)\, dx$$

$$= \alpha n\pi + \frac{\alpha}{2\sqrt{\lambda_n}} \int_0^1 x^{\alpha-1} q(x)\, dx - \frac{\alpha}{2\sqrt{\lambda_n}} \int_0^1 x^{\alpha-1} q(x) \cos\left(2\sqrt{\lambda_n}\theta_n(x)\right) dx \qquad (17)$$

$$= \alpha n\pi + O\left(\frac{1}{n}\right) \qquad (18)$$

for sufficiently large n. Substituting (18) into (15), for sufficiently large n one can obtain

$$\sqrt{\lambda_n}\theta_n(x) = n\pi x^\alpha + O\left(\frac{1}{n}\right). \qquad (19)$$

Then the second integral term in (17) becomes

$$\int_0^1 x^{\alpha-1} q(x) \cos\left(2\sqrt{\lambda_n}\theta_n(x)\right) dx = \int_0^1 x^{\alpha-1} q(x) \cos\left(2n\pi x^\alpha\right) dx + o(1)$$

$$= \frac{1}{\alpha} \int_0^1 q\left(s^{1/\alpha}\right) \cos(2n\pi s)\, ds + o(1)$$

$$= o(1) \qquad (20)$$

for sufficiently large n by the Riemann–Lebesgue lemma. Substituting (18) and (20) into (17), the Dirichlet eigenvalue asymptotic expansion (3) is valid.

　(ii) For $q \equiv 0$, introduce another Prüfer-type substitution

$$\lambda^{1/2} \rho^{1/2}(x) y(x) = R(x) \sin\left(\phi(x)\right) \quad \text{and} \quad D_x^\alpha y(x) = R(x) \cos\left(\phi(x)\right). \qquad (21)$$

Taking

$$D_x^\alpha\left(\frac{D_x^\alpha y(x)}{\lambda^{1/2} \rho^{1/2}(x) y(x)}\right) = D_x^\alpha\left(\frac{\cos(\phi(x))}{\sin(\phi(x))}\right)$$

and applying (1), one can obtain

$$\phi'(x) = \lambda^{1/2} \rho^{1/2}(x) x^{\alpha-1} + \frac{1}{4}\left(\frac{\rho'(x)}{\rho(x)}\right) \sin 2\phi(x). \qquad (22)$$

With each eigenvalue λ_n of (1), one can associate the Prüfer angle $\phi_n(x) \equiv \phi(x; \lambda_n)$ via (21) if one also specifies the initial condition

$$\phi_n(0) = 0, \quad n = 1, 2, 3, \ldots.$$

In particular, $\phi_n(1) = n\pi$. Now integrating both sides of (22) with $\lambda = \lambda_n$, one can obtain

$$n\pi = \lambda_n^{1/2} \int_0^1 x^{\alpha-1} \rho^{1/2}(x)\, dx + \frac{1}{4} \int_0^1 \left(\frac{\rho'(x)}{\rho(x)}\right) \sin 2\phi_n(x)\, dx. \qquad (23)$$

Note that if $\phi_n'(x) = 0$ is valid in some subinterval of $(0, 1)$, $\sin 2\phi_n(x)$ will be constant in this subinterval. This implies that the function $\rho(x)$ depends on λ_n in this subinterval from (22). This will contradict our original problem. Hence, the points satisfying $\phi_n'(x) = 0$ shall be

isolated. Let $f(x) = \rho^{-1/2}(x)$, and then $-2\rho^{1/2}f' = (\frac{\rho'}{\rho})$. Note that, for sufficiently large n, by (22), one has

$$
\begin{aligned}
\frac{1}{4}\left(\frac{\rho'}{\rho}\right)\sin 2\phi_n &= \frac{1}{4}\left(-2\rho^{1/2}f'\right)\sin 2\phi_n \frac{\phi_n'}{\lambda_n^{1/2}\rho^{1/2}x^{\alpha-1} + \frac{1}{4}(-2\rho^{1/2}f')\sin 2\phi_n} \\
&= -\left(\frac{1}{2}\lambda_n^{-1/2}x^{1-\alpha}f'\sin 2\phi_n\right)\frac{\phi_n'}{1 - \frac{1}{2}\lambda_n^{-1/2}x^{1-\alpha}f'\sin 2\phi_n} \\
&= -\sum_{k=0}^{\infty}\left(\frac{1}{2}\lambda_n^{-1/2}x^{1-\alpha}f'\sin 2\phi_n\right)^{k+1}\phi_n' \\
&= -\frac{\lambda_n^{-1/2}}{2}x^{1-\alpha}f'\sin 2\phi_n \cdot \phi_n' + O\left(\frac{1}{\lambda_n^{1/2}}\right).
\end{aligned}
$$

Then, by (23), for sufficiently large n, one gets

$$
\begin{aligned}
n\pi &= \lambda_n^{1/2}\int_0^1 x^{\alpha-1}\rho^{1/2}(x)\,dx - \frac{\lambda_n^{-1/2}}{2}\int_0^1 x^{1-\alpha}f'(x)\sin 2\phi_n \cdot \phi_n'\,dx + O\left(\frac{1}{\lambda_n^{1/2}}\right) \\
&= \lambda_n^{1/2}\int_0^1 x^{\alpha-1}\rho^{1/2}(x)\,dx - \frac{\lambda_n^{-1/2}}{2}\left(f'(1) - \int_0^1 \frac{d}{dx}\left[x^{1-\alpha}f'(x)\right]\cos 2\phi_n\,dx\right) \\
&\quad + O\left(\frac{1}{\lambda_n^{1/2}}\right) \\
&= \lambda_n^{1/2}\int_0^1 x^{\alpha-1}\rho^{1/2}(x)\,dx + O\left(\frac{1}{\lambda_n^{1/2}}\right).
\end{aligned} \tag{24}
$$

Therefore, one can obtain the eigenvalue asymptotic

$$
\sqrt{\lambda_n}\int_0^1 x^{\alpha-1}\rho^{1/2}(x)\,dx = n\pi + O\left(\frac{1}{n}\right). \tag{25}
$$

$\square$

Remark 2.3 Assume that $\rho \equiv 1$. For the same issue for (1) coupling with the Neumann boundary conditions $D_x^\alpha y(0) = D_x^\alpha y(1) = 0$, one can derive the similar expansion by the following arguments. From the phase equation (14), let $\lambda = \mu_n$ with $\theta_n(0) = \frac{\pi}{2}$ and $\theta_n(1) = (n - \frac{1}{2})\pi$. Applying the same calculation, one can obtain

$$
\sqrt{\mu_n} = \alpha(n-1)\pi + \frac{1}{2(n-1)\pi}\int_0^1 x^{\alpha-1}q(x)\,dx + o\left(\frac{1}{n}\right) \tag{26}
$$

for sufficiently large n.

Before proving Theorem 1.2, we have to quote the fractional Rayleigh quotient result from [6].

Proposition 2.4 ([6, Lemma 12]) *For the fractional eigenvalue problem* (1) *it holds that*

$$
\lambda_1 = \min_{y\neq 0} \frac{\int_0^1 x^{\alpha-1}(D_x^\alpha y(x))^2\,dx - \int_0^1 x^{\alpha-1}q(x)(y(x))^2\,dx}{\int_0^1 x^{\alpha-1}(y(x))^2\,dx}. \tag{27}
$$

Now it suffices to give the proof of Theorem 1.2.

Proof of Theorem 1.2 (i) By the eigenvalue estimates (3), one has

$$\int_0^1 x^{\alpha-1} q(x)\, dx = 0. \tag{28}$$

Next, under the assumption, we will show that $\sin(\pi x^\alpha)$ is the first eigenfunction and then apply Proposition 2.4. Now $\sin(\pi x^\alpha)$ satisfies (2) and

$$(\alpha\pi)^2 = \lambda_1 \le \frac{\int_0^1 x^{\alpha-1} (D_x^\alpha \sin(\pi x^\alpha))^2\, dx - \int_0^1 x^{\alpha-1} q(x) \sin^2(\pi x^\alpha)\, dx}{\int_0^1 x^{\alpha-1} \sin^2(\pi x^\alpha)\, dx}, \tag{29}$$

where

$$\int_0^1 x^{\alpha-1} \left(D_x^\alpha \sin\left(\pi x^\alpha\right)\right)^2 dx = \int_0^1 x^{\alpha-1}(\alpha\pi)^2 \cos^2\left(\pi x^\alpha\right) dx = \frac{\alpha\pi^2}{2},$$

$$\int_0^1 x^{\alpha-1} q(x) \sin^2\left(\pi x^\alpha\right) dx = \frac{1}{2}\left(\int_0^1 x^{\alpha-1} q(x)\, dx - \int_0^1 x^{\alpha-1} q(x) \cos\left(2\pi x^\alpha\right) dx\right) = 0$$

and

$$\int_0^1 x^{\alpha-1} \sin^2\left(\pi x^\alpha\right) dx = \frac{1}{2}\int_0^1 x^{\alpha-1}\left(1 - \cos\left(2\pi x^\alpha\right)\right) dx = \frac{1}{2\alpha}$$

by (5) and (28). The above shows that $\sin(\pi x^\alpha)$ achieves the minimum value and is thus the first eigenfunction. Substituting this into (1), we obtain $q \equiv 0$ on $(0,1)$.

(ii) By (26),

$$\int_0^1 x^{\alpha-1} q(x)\, dx = 0.$$

Let $y_1(x)$ be the first Neumann eigenfunction with $\mu_1 = 0$. One may assume, without loss of generality, that $y_1(x) > 0$ on $[0,1]$. Taking into account the relation

$$D_x^\alpha\left(\frac{D_x^\alpha y_1(x)}{y_1(x)}\right) = q(x) - \left(\frac{D_x^\alpha y_1(x)}{y_1(x)}\right)^2,$$

one can obtain

$$0 = \int_0^1 x^{\alpha-1} q(x)\, dx = \left[\frac{D_x^\alpha y_1(x)}{y_1(x)}\right]_0^1 + \int_0^1 x^{\alpha-1}\left(\frac{D_x^\alpha y_1(x)}{y_1(x)}\right)^2 dx$$

$$= \int_0^1 x^{\alpha-1}\left(\frac{D_x^\alpha y_1(x)}{y_1(x)}\right)^2 dx.$$

Then $D_x^\alpha y_1(x) = 0$ on $(0,1)$, and hence $y_1(x)$ is a constant on $(0,1)$. This implies $q \equiv 0$ on $(0,1)$. □

Declarations

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References

 1. Abdelhakim, A.A., Machado, J.A.T.: A critical analysis of the conformable derivative. Nonlinear Dyn. **95**, 3063–3073 (2019)
 2. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
 3. Adalar, I.: Ambarzumyan-type theorem for conformable Sturm–Liouville problem. In: 7th International Congress on Fundamental and Applied Sciences, Prishtina, Kosovo, 6 October (2020)
 4. Adalar, I., Ozkan, A.S.: Inverse problems for a conformable fractional Sturm–Liouville operator. J. Inverse Ill-Posed Probl. **28**(6), 775–782 (2020)
 5. Adda, F.B., Cresson, J.: About non-differentiable functions. J. Math. Anal. Appl. **263**(2), 721–737 (2001)
 6. Al-Refai, M., Abdeljawad, T.: Fundamental results of conformable Sturm–Liouville eigenvalue problems. Complexity **2017**, Article ID 3720471 (2017)
 7. Anderson, D.R., Avery, R.I.: Fractional-order boundary value problem with Sturm–Liouville boundary conditions. Electron. J. Differ. Equ. **2015**, 29 (2015)
 8. Anderson, D.R., Camrud, E., Ulness, D.J.: On the nature of the conformable derivative and its applications to physics. J. Fract. Calc. Appl. **10**(2), 92–135 (2019)
 9. Babakhani, A., Daftardar-Gejji, V.: On calculus of local fractional derivatives. J. Math. Anal. Appl. **270**(1), 66–79 (2002)
10. Batarfi, H., Losada, J., Nieto, J.J., Shammakh, W.: Three-point boundary value problems for conformable fractional differential equations. J. Funct. Spaces **2015**, Article ID 706383 (2015)
11. Bayour, B., Torres, D.F.M.: Existence of solution to a local fractional nonlinear differential equation. J. Comput. Appl. Math. **321**, 127–133 (2017)
12. Chen, Y., Yan, Y., Zhang, K.: On the local fractional derivative. J. Math. Anal. Appl. **362**(1), 17–33 (2010)
13. Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
14. Kolwankar, K.M., Gangal, A.D.: Fractional differentiability of nowhere differentiable functions and dimensions, chaos: an interdisciplinary. J. Nonlinear Sci. **6**(4), 505–513 (1996)
15. Mortazaasl, H., Akbarfam, A.J.: Trace formula and inverse nodal problem for a conformable fractional Sturm–Liouville problem. Inverse Probl. Sci. Eng. **28**(4), 524–555 (2020)
16. Ortigueira, M.D., Martynyuk, V., Fedula, M., Machado, J.T.: The failure of certain fractional calculus operators in two physical models. Fract. Calc. Appl. Anal. **22**(2), 255–270 (2019)
17. Ortigueira, M.D., Tenreiro Machado, J.A.: What is a fractional derivative? J. Comput. Phys. **293**, 4–13 (2015)
18. Tarasov, V.E.: No violation of the Leibniz rule. No fractional derivative. Commun. Nonlinear Sci. Numer. Simul. **18**(11), 2945–2948 (2013)
19. Tarasov, V.E.: No nonlocality. No fractional derivative. Commun. Nonlinear Sci. Numer. Simul. **62**, 157–163 (2018)
20. Xiaojun, Y., Baleanu, D.: Fractal heat conduction problem solved by local fractional variation iteration method. Therm. Sci. **17**(2), 625 (2013)
21. Zhao, D., Luo, M.: General conformable fractional derivative and its physical interpretation. Calcolo **54**, 903–917 (2017)