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Optimal feedback control for fractional evolution equations with nonlinear perturbation of the time-fractional derivative term

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Abstract

We study the optimal feedback control for fractional evolution equations with a nonlinear perturbation of the time-fractional derivative term involving Caputo fractional derivatives with arbitrary kernels. Firstly, we derive a mild solution in terms of the semigroup operator generated by resolvents and a kernel from the general Caputo fractional operators and establish the existence and uniqueness of mild solutions for the feedback control systems. Then, the existence of feasible pairs by applying Filippov's theorem is obtained. In addition, the existence of optimal control pairs for the Lagrange problem has been investigated.

Keywords: Fractional evolution equations; Mild solutions; Existence and uniqueness theorems; Feedback control; Feasible pairs; Optimal control

1 Introduction

Control theory has received considerable attention due to its extensive applications in various areas of science, e.g., ecology, economics, and engineering, particularly in systems with controllability, feedback control, and optimal control [1–5]. Control systems are most often based on the principle of feedback, whereby the signal to be controlled is compared to a desired reference signal and the discrepancy is used to compute a corrective control action.

It is wonderful that the study of fractional control systems has attracted research recently [6–12]. In [7], Wang et al. considered the optimal feedback control of a nonlinear system, given by fractional evolution equations, that has the form

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha} u(t) = \mathscr{A} u(t) + f(t, u(t), v(t)), & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

where ${}^{C}\mathcal{D}^{\alpha}$ is Caputo fractional derivative of order $\alpha \in (0,1)$, $u_0 \in E$, and $\mathscr{A} : D(\mathscr{A}) \to E$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{\mathbb{T}(t)\}_{t\geq 0}$ in a reflexive Banach space E. The control function $v(\cdot)$ takes values in the Polish space $\mathcal{V}$ and $f:[0,T]\times E\times\mathcal{V}\to E$ is a given function satisfying suitable assumptions.

Motivated by the previous work, we are concerned with the optimal feedback control of the semilinear fractional evolution equations with a nonlinear perturbation of the time-fractional derivative term as follows:

$$\begin{cases} {}^{C}\mathcal{D}_0^{\alpha;\omega}(u(t)-g(t,u(t))) = \mathscr{A}u(t)+f(t,u(t),v(t)), & 0<t\leq T, \\ u(0)=u_0, \end{cases} \tag{1}$$

where ${}^{C}\mathcal{D}_0^{\alpha;\omega}$ is the Caputo fractional derivative with arbitrary kernel ω of order $\alpha\in(0,1)$, $\mathscr{A}:D(\mathscr{A})\subseteq E\to E$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{\mathbb{T}(t),t\geq 0\}$ in a reflexive Banach space E, and $u_0\in E$. The control v has a value in a control set $\mathcal{V}[0,T]$ and $f:[0,T]\times E\times\mathcal{V}\to E$ and $g:[0,T]\times E\to E$ will be specified in what follows. It should be noted that the nonlinear perturbation term g in (1) contributes to a more complicated derivation of a mild solution, which requires certain assumptions on the semigroup and operator $\mathscr{A}$. Furthermore, when the evolution operator $\mathscr{A}$ is defined to be the zero operator on the Banach space $E=\mathbb{R}$, our problem (1) can be modified and rewritten as hybrid fractional differential equations. The fractional derivative of an unknown function is hybrid nonlinear, as a dependent variable is used in this class of equations. Moreover, this problem can be reduced to that considered in [7] where the function g is taken to be zero.

The aim of this paper is to derive a representation of the solution for the problem (1) that depends on fractional derivatives with arbitrary kernels. Furthermore, Krasnoselskii's fixed point theorem is used to investigate the existence results for the nonlinear system (1) under the compactness assumption of the operator semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$. We further investigate the existence of optimal feedback controls for the Lagrange problem. Moreover, our results obtained in this work can be applied for further investigation in many practical problems.

The paper is structured as follows. First, we will outline some definitions and lemmas that will be needed later in Sect. 2. In Sect. 3, we provide a mild solution to the nonlinear system (1) employing the semigroup operator with a function ω that prescribes the generalized Caputo derivative. Next, the Krasnoselskii's fixed point theorem is applied to prove the existence and uniqueness results of mild solutions for the problem (1) in Sect. 4. In Sect. 5, the existence of feasible pairs for the system (1) is also demonstrated. Finally, we will investigate the existence result of the optimal control pairs of the system (1).

2 Preliminaries

Throughout this paper, E is a reflexive Banach space and $\|f\|_{L^p}$ is used to denote the $L^p([0,T],E)$-norm of f when $f\in L^p([0,T],E)$ for some p, with $1\leq p<\infty$. Let $O_r(x)$ be the ball of radius $r>0$ centered at x, i.e.,

$$O_r(x)=\{y\in E \mid \|y-x\|\leq r\}.$$

Consider $C([0,T],E)$ as the Banach space of continuous functions from $[0,T]$ to E with the usual supremum norm.

We denote by $\mathcal{V}$ a Polish space; that is, a separable completely metrizable topological space. Let $\mathcal{V}[0,T] = \{v : [0,T] \to \mathcal{V} \mid v(\cdot) \text{ is measurable}\}$. Then, any element $v \in \mathcal{V}[0,T]$ is said to be a control on $[0,T]$.

Suppose H and F are two metric spaces.

Definition 2.1 ([3]) A multifunction $\Gamma : H \to 2^F$ is called pseudocontinuous at $t \in H$ if

$$\bigcap_{\epsilon>0} \overline{\Gamma\big(O_\epsilon(t)\big)} = \Gamma(t).$$

If Γ is pseudocontinuous at each point $t \in H$, then it is called pseudocontinuous on H.

Proposition 2.2 ([3]) *Let $\Gamma : H \to 2^F$ be a multifunction taking closed set values. Then Γ is pseudocontinuous if and only if the graph*

$$\mathcal{G}(\Gamma) \equiv \big\{(t,u) \in H \times F \mid u \in \Gamma(t)\big\}$$

is closed in $H \times F$.

Lemma 2.3 (Mazur's lemma, [3, 13]) *Let $1 < p < \infty$. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in L^p that converges weakly to some f in L^p. Then, for each $n \in \mathbb{N}$, there exist $\alpha_{n,i} \geq 0$ and $\sum_{i\geq 1} \alpha_{n,i} = 1$ such that*

$$\lim_{n\to\infty} \sum_{i\geq 1} \alpha_{n,i} f_{i+n} = f \quad \text{in } L^p.$$

Lemma 2.4 ([3]) *Let H be a Lebesgue measurable set in $\mathbb{R}^n$ and G and F be Polish spaces. Let $\mathcal{V} : H \times F \to 2^G$ be Souslin measurable and $\xi : H \to F$ be measurable. Then, $\Gamma(\cdot) \equiv \mathcal{V}(\cdot, \xi(\cdot)) : H \to 2^G$ is measurable. Moreover, if $\mathcal{V}$ is pseudocontinuous and ξ is continuous, then Γ is Souslin measurable.*

Lemma 2.5 (Filippov's theorem, [14]) *Let $\Gamma : H \to 2^F$ be a measurable closed set valued function and $f : H \times F \to G$ be Souslin measurable. Assume that*

(1) *for each $u \in F$, $f(\cdot, u)$ is measurable;*
(2) *for almost all $t \in H$, $f(t, \cdot)$ is continuous;*
(3) *$y : H \to G$ is a Lebesgue measurable such that*

$$y(t) \in f\big(t, \Gamma(t)\big) \quad \text{a.e. } t \in H.$$

Then, there exists a measurable function $h : H \to G$, satisfying

$$\begin{cases} h(t) \in \Gamma(t), & \text{a.e. } t \in H, \\ y(t) = f(t, h(t)), & \text{a.e. } t \in H. \end{cases}$$

Definition 2.6 (ω-Riemann–Liouville fractional integral, [15]) Let $u \in L^1([a,b])$, $\alpha > 0$, and $\omega \in C^n([a,b])$ be a function such that $\omega'(t) > 0$ for all $t \in [a,b]$. The ω-Riemann–

Liouville fractional integral operator of order α of a function u is defined by

$$\big(\mathcal{I}_a^{\alpha;\omega}u\big)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)u(\tau)\,\mathrm{d}\tau, \tag{2}$$

where Γ is the gamma function.

Definition 2.7 (ω-Riemann–Liouville fractional derivative, [15]) Let $\alpha \in (n-1,n)$, $u \in L^1([a,b])$, and $\omega \in C^1([a,b])$ be a function such that $\omega'(t) > 0$ for all $t \in [a,b]$. The ω-Riemann–Liouville fractional derivative of order α of a function u is defined by

$$\begin{aligned}\big(\mathcal{D}_a^{\alpha;\omega}u\big)(t) &= \left(\frac{1}{\omega'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \big(\mathcal{I}_a^{n-\alpha;\omega}u\big)(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\omega'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \big(\omega(t)-\omega(\tau)\big)^{n-\alpha-1}\omega'(\tau)u(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{3}$$

where $n = 1 + [\alpha]$.

Definition 2.8 (ω-Caputo fractional derivative, [15, 16]) Let $\alpha \in (n-1,n)$, $u \in C^n([a,b])$, and $\omega \in C^1([a,b])$ be a function such that $\omega'(t) > 0$ for all $t \in [a,b]$. The ω-Caputo fractional derivative of a function u of order α is defined by

$$\begin{aligned}\big({}^C\mathcal{D}_a^{\alpha;\omega}u\big)(t) &= \big(\mathcal{I}_a^{n-\alpha;\omega}u^{[n]}\big)(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \big(\omega(t)-\omega(\tau)\big)^{n-\alpha-1}\omega'(\tau)u^{[n]}(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{4}$$

where $u^{[n]}(t) := (\frac{1}{\omega'(t)}\frac{\mathrm{d}}{\mathrm{d}t})^n u(t)$ on $[a,b]$ and $n = [\alpha] + 1$.

Lemma 2.9 ([16]) *Let $u \in C^{n-1}([a,b])$ and $\alpha > 0$. Then, we have*

$$\mathcal{I}_a^{\alpha;\omega}\, {}^C\mathcal{D}_a^{\alpha;\omega}u(t) = u(t) - \sum_{k=0}^{n-1}\frac{u^{[k]}(a^+)}{k!}\big(\omega(t)-\omega(a)\big)^k.$$

Furthermore, we also have

$$\mathcal{I}_a^{\alpha;\omega}\, {}^C\mathcal{D}_a^{\alpha;\omega}u(t) = u(t) - u(a) \quad \textit{for } \alpha \in (0,1).$$

Lemma 2.10 (Gronwall's inequality, [17, 18]) *Let $\alpha > 0$ and $\omega \in C^1([a,b])$ be a function such that $\omega'(t) > 0$ for all $t \in [a,b]$. Suppose that*

(1) *u and v are nonnegative functions, locally integrable on $[a,b]$;*
(2) *$h(t) \geq 0$ is nondecreasing continuous function on $[a,b]$.*

If

$$u(t) \leq v(t) + h(t)\int_a^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)u(\tau)\,\mathrm{d}\tau,$$

then

$$u(t) \le v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[h(t)\Gamma(\alpha)]^k}{\Gamma(n\alpha)} \big(\omega(t) - \omega(\tau)\big)^{k\alpha-1} \omega'(\tau) v(\tau)\, \mathrm{d}\tau$$

for all $t \in [a,b]$.

Moreover, if v *is a nondecreasing function on* $[a,b]$ *then*

$$u(t) \le v(t) E_\alpha\big(h(t)\Gamma(\alpha)\big[\omega(t) - \omega(a)\big]^\alpha\big), \quad \textit{for all } t \in [a,b],$$

where E_α *is the Mittag-Leffler function.*

Definition 2.11 ([15]) Let $u, \omega : [a,\infty) \to \mathbb{R}$ and $\omega(t)$ be a nonnegative increasing function. Then, the Laplace transform of u with respect to ω is given by

$$\mathcal{L}_\omega\big\{u(t)\big\}(s) = \int_a^\infty e^{-s(\omega(t)-\omega(a))} \omega'(t) u(t)\, \mathrm{d}t$$

for all s such that this integral converges.

Lemma 2.12 ([15]) *Let* $\alpha > 0$, *and* u *be a piecewise continuous function on* $[a,t]$, *and of* $\omega(t)$*-exponential order. Then*

$$\mathcal{L}_\omega\big\{\mathcal{I}_a^{\alpha;\omega} u(t)\big\}(s) = s^{-\alpha} \mathcal{L}_\omega\big\{u(t)\big\}.$$

Definition 2.13 ([15]) Let u and v be two functions which are piecewise continuous at each interval $[a,T]$ and of exponential order. The convolution of u and v with respect to ω is defined by

$$(u *_\omega v)(t) = \int_a^t u(\tau) v\big(\omega^{-1}\big(\omega(t) + \omega(a) - \omega(\tau)\big)\big) \omega'(\tau)\, \mathrm{d}\tau.$$

Definition 2.14 ([19, 20]) The Wright-type function ϕ_α is given by

$$\phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}$$

for $z \in \mathbb{C}$ and $0 < \alpha < 1$.

Proposition 2.15 ([19, 20]) *The Wright function* ϕ_α *is an entire function with the following properties*:

(i) $\phi_\alpha(\theta) \ge 0$ *for* $\theta \ge 0$ *and* $\int_0^\infty \phi_\alpha(\theta)\, \mathrm{d}\theta = 1$;
(ii) $\int_0^\infty \phi_\alpha(\theta)\theta^r\, \mathrm{d}\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ *for* $r > -1$;
(iii) $\int_0^\infty \phi_\alpha(\theta) e^{-z\theta}\, \mathrm{d}\theta = E_\alpha(-z)$, $z \in \mathbb{C}$;
(iv) $\alpha \int_0^\infty \theta \phi_\alpha(\theta) e^{-z\theta}\, \mathrm{d}\theta = E_{\alpha,\alpha}(-z)$, $z \in \mathbb{C}$.

Theorem 2.16 (Bochner's theorem) *A measurable function* $\mathcal{Q} : [0,T] \to E$ *is Bochner integrable if* $|\mathcal{Q}|$ *is Lebesgue integrable.*

Theorem 2.17 (Krasnoselskii's fixed point theorem) *Let B is a nonempty convex, closed, and bounded subset of a Banach space E. Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are operators from B to E such that*

(i) *$\mathcal{F}_1 x + \mathcal{F}_2 y \in B$ for every pair $x, y \in B$;*
(ii) *$\mathcal{F}_1$ is a contraction map;*
(iii) *$\mathcal{F}_2$ is completely continuous.*

Then, $x = \mathcal{F}_1 x + \mathcal{F}_2 x$ has a solution on B.

Now, we outline some facts about the semigroups of linear operators which can be found in [21, 22].

The infinitesimal generator of $\{\mathbb{T}(t)\}_{t\geq 0}$ of a strongly continuous semigroup (i.e., C_0-semigroup) $\{\mathbb{T}(t)\}_{t\geq 0}$ is given by

$$\mathscr{A} u = \lim_{t\to 0^+} \frac{\mathbb{T}(t)u - u}{u}, \quad u \in E.$$

We denote the domain of $\mathscr{A}$ by $D(\mathscr{A})$, that is,

$$D(\mathscr{A}) = \left\{ u \in E : \lim_{t\to 0^+} \frac{\mathbb{T}(t)u - u}{u} \text{ exists} \right\}.$$

Lemma 2.18 ([21, 22]) *Let $\{\mathbb{T}(t)\}_{t\geq 0}$ be a C_0-semigroup and let $\mathscr{A}$ be its infinitesimal generator. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{T}(t)u = \mathscr{A}\mathbb{T}(t)u = \mathbb{T}(t)\mathscr{A} u$$

for $u \in D(\mathscr{A})$ and $\mathbb{T}(t)u \in D(\mathscr{A})$.

Throughout this work, we assume that the analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$ has the following properties:

(i) There is a constant $M \geq 1$ satisfying

$$M = \sup_{t\in[0,\infty)} \left\| \mathbb{T}(t) \right\|. \tag{5}$$

(ii) For any $0 < \eta \leq 1$, there exists a positive constant C_η such that

$$\left\| \mathscr{A}^\eta \mathbb{T}(t) \right\| \leq \frac{C_\eta}{t^\eta} \tag{6}$$

for all $t \in [0, T]$.

3 Representation formula of mild solutions based on semigroup theory

Lemma 3.1 *Any solution of the problem (1) satisfies the following integral equation:*

$$\begin{aligned} u(t) = {} & \int_0^\infty \phi_\alpha(\theta)\mathbb{T}\big(\big(\omega(t) - \omega(0)\big)^\alpha \theta\big)\big(u_0 - g(0, u_0)\big)\, \mathrm{d}\theta + g\big(t, u(t)\big) \\ & + \alpha \int_0^t \int_0^\infty \theta\phi_\alpha(\theta)\big(\omega(t) - \omega(\tau)\big)^{\alpha-1}\omega'(\tau) \end{aligned}$$

$$
\begin{aligned}
&\times \mathscr{A}\mathbb{T}\big(\big(\omega(t)-\omega(\tau)\big)^{\alpha}\theta\big)g\big(s,u(\tau)\big)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&+\alpha\int_0^t\int_0^\infty \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau) \\
&\times \mathbb{T}\big(\big(\omega(t)-\omega(\tau)\big)^{\alpha}\theta\big)f\big(s,u(\tau),v(\tau)\big)\,\mathrm{d}\theta\,\mathrm{d}\tau.
\end{aligned}
$$

Proof Applying Definition 2.8 and Lemma 2.9 to the problem (1), it can be rewritten in the form of the integral representation as follows:

$$
\begin{aligned}
u(t) &= u_0 - g(0,u_0) + g(t,u) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \big(\omega(t)-\omega(\eta)\big)^{\alpha-1}\omega'(\eta)\big(\mathscr{A}u(\eta)+f\big(\eta,u(\eta),v(\eta)\big)\big)\,\mathrm{d}\eta.
\end{aligned}
\tag{7}
$$

Taking the generalized Laplace transform on both sides of equation (7), we have that for $s > 0$,

$$
U(s) = \frac{1}{s}\big(u_0 - g(0,u_0)\big) + G(s) + \frac{1}{s^\alpha}\big(\mathscr{A}U(s) + F(s)\big),
$$

where

$$
\begin{aligned}
U(s) &= \int_0^\infty e^{-s(\omega(\eta)-\omega(0))}u(\eta)\omega'(\eta)\,\mathrm{d}\eta, \\
F(s) &= \int_0^\infty e^{-s(\omega(\eta)-\omega(0))}f\big(\eta,u(\eta),v(\eta)\big)\omega'(\eta)\,\mathrm{d}\eta,
\end{aligned}
$$

and

$$
G(s) = \int_0^\infty e^{-s(\omega(\eta)-\omega(0))}g\big(\eta,u(\eta)\big)\omega'(\eta)\,\mathrm{d}\eta.
$$

It follows that

$$
\begin{aligned}
U(s) &= s^{\alpha-1}\big(s^\alpha I - \mathscr{A}\big)^{-1}\big(u_0 - g(0,u_0)\big) + s^\alpha\big(s^\alpha I - \mathscr{A}\big)^{-1}G(s) \\
&\quad + \big(s^\alpha I - \mathscr{A}\big)^{-1}F(s) \\
&= s^{\alpha-1}\int_0^\infty e^{-s^\alpha\tau}\mathbb{T}(\tau)\big(u_0 - g(0,u_0)\big)\,\mathrm{d}\tau + s^\alpha\int_0^\infty e^{-s^\alpha\tau}\mathbb{T}(\tau)G(s)\,\mathrm{d}\tau \\
&\quad + \int_0^\infty e^{-s^\alpha\tau}\mathbb{T}(\tau)F(s)\,\mathrm{d}\tau.
\end{aligned}
$$

Now, we consider the change of variable

$$
\tau = \big(\omega(t)-\omega(0)\big)^\alpha \quad\text{and}\quad \mathrm{d}\tau = \alpha\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)\,\mathrm{d}t.
$$

It follows that

$$
\begin{aligned}
U(s) &= \alpha s^{\alpha-1}\int_0^\infty \big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)e^{-(s(\omega(t)-\omega(0)))^\alpha} \\
&\quad \times \mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\big)\big(u_0 - g(0,u_0)\big)\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha s^{\alpha}\int_0^{\infty}\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\\
&\times\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)G(s)\,\mathrm{d}t\\
&+\alpha\int_0^{\infty}\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\\
&\times\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)F(s)\,\mathrm{d}t\\
={}&\int_0^{\infty}-\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\big)\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)\big(u_0-g(0,u_0)\big)\,\mathrm{d}t\\
&+\alpha s^{\alpha}\int_0^{\infty}\int_0^{\infty}\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\\
&\times\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)e^{-s(\omega(\eta)-\omega(0))}\omega'(\eta)g\big(s,u(s)\big)\,\mathrm{d}\eta\,\mathrm{d}t\\
&+\int_0^{\infty}\int_0^{\infty}\alpha\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\\
&\times\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)e^{-(s(\omega(\eta)-\omega(0)))}\omega'(\eta)f\big(\eta,u(\eta),v(\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}t\\
={}&\int_0^{\infty}-\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\big)\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)\big(u_0-g(0,u_0)\big)\,\mathrm{d}t\\
&+\int_0^{\infty}e^{-s(\omega(\eta)-\omega(0))}\omega'(\eta)g\big(\eta,u(\eta)\big)\,\mathrm{d}\eta\\
&+\int_0^{\infty}\int_0^{\infty}e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\alpha\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)\\
&\times\mathscr{A}\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)e^{-s(\omega(\eta)-\omega(0))}\omega'(\eta)g\big(\eta,u(\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}t\\
&+\int_0^{\infty}\int_0^{\infty}\alpha\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\\
&\times\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)e^{-(s(\omega(\eta)-\omega(0)))}\omega'(\eta)f\big(\eta,u(\eta),v(\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}t.
\end{aligned}
$$

The following one-sided stable probability density in [23] is considered:

$$\rho_\alpha(\theta)=\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-\alpha k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\pi\alpha),\quad \theta\in(0,\infty),$$

whose standard Laplace transform is provided by

$$\int_0^{\infty}e^{-s\theta}\rho_\alpha(\theta)\,\mathrm{d}\theta=e^{-s^{\alpha}}\quad\text{where }\alpha\in(0,1).\tag{8}$$

Using (8), we obtain

$$
\begin{aligned}
&\int_0^{\infty}-\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-(s(\omega(t)-\omega(0)))^{\alpha}}\big)\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)\big(u_0-g(0,u_0)\big)\,\mathrm{d}t\\
&\quad=\int_0^{\infty}\int_0^{\infty}\theta\rho_\alpha(\theta)e^{-s(\omega(t)-\omega(0))\theta}\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^{\alpha}\big)\big(u_0-g(0,u_0)\big)\omega'(t)\,\mathrm{d}\theta\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty e^{-s(\omega(t)-\omega(0))}\left(\int_0^\infty \rho_\alpha(\theta)\mathbb{T}\left(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\right)\big(u_0 - g(0,u_0)\big)\,\mathrm{d}\theta\right)\omega'(t)\,\mathrm{d}t,\\
&\int_0^\infty \int_0^\infty e^{-(s(\omega(t)-\omega(0)))^\alpha}\alpha\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)\\
&\quad\times \mathscr{A}\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\big)e^{-s(\omega(\eta)-\omega(0))}\omega'(\eta)g\big(\eta,u(\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}t\\
&= \alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s(\omega(t)-\omega(0)))\theta}\rho_\alpha(\theta)\big(\omega(t)-\omega(0)\big)^{\alpha-1}\\
&\quad\times \mathscr{A}\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\big)e^{-s(\omega(\eta)-\omega(0))}g\big(s,u(\eta)\big)\omega'(\eta)\,\mathrm{d}\theta\,\mathrm{d}s\omega'(t)\,\mathrm{d}t\\
&= \alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s(\omega(t)+\omega(\eta)-2\omega(0)))\theta}\rho_\alpha(\theta)\frac{(\omega(t)-\omega(0))^{\alpha-1}}{\theta^\alpha}\\
&\quad\times \mathscr{A}\mathbb{T}\left(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\right)g\big(s,u(\eta)\big)\omega'(\eta)\,\mathrm{d}\theta\,\mathrm{d}s\omega'(t)\,\mathrm{d}t\\
&= \int_0^\infty\int_t^\infty\int_0^\infty \alpha e^{-s(\omega(\eta)-\omega(0))}\rho_\alpha(\theta)\frac{(\omega(t)-\omega(0))^{\alpha-1}}{\theta^\alpha}\mathscr{A}\mathbb{T}\left(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\right)\\
&\quad\times g\big(\omega^{-1}\big(\omega(\eta)-\omega(t)+\omega(0)\big)\big),u\big(\omega^{-1}\big(\omega(\eta)-\omega(t)+\omega(0)\big)\big)))\omega'(\eta)\omega'(t)\,\mathrm{d}\theta\,\mathrm{d}\eta\,\mathrm{d}t\\
&= \int_0^\infty\int_0^\eta\int_0^\infty \alpha e^{-s(\omega(\eta)-\omega(0))}\rho_\alpha(\theta)\frac{(\omega(t)-\omega(0))^{\alpha-1}}{\theta^\alpha}\mathscr{A}\mathbb{T}\left(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\right)\\
&\quad\times g\big(\omega^{-1}\big(\omega(\eta)-\omega(t)+\omega(0)\big)\big),u\big(\omega^{-1}\big(\omega(\eta)-\omega(t)+\omega(0)\big)\big)))\omega'(\eta)\omega'(t)\,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}\eta\\
&= \int_0^\infty e^{-s(\omega(\eta)-\omega(0))}\left(\int_0^\eta\int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\omega(t)-\omega(\eta))^{\alpha-1}}{\theta^\alpha}\right.\\
&\quad\left.\times \mathscr{A}\mathbb{T}\left(\frac{(\omega(t)-\omega(\eta))^\alpha}{\theta^\alpha}\right)g\big(s,u(\eta)\big)\omega'(\eta)\,\mathrm{d}\theta\,\mathrm{d}\eta\right)\omega'(\eta)\,\mathrm{d}\eta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty \alpha\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)e^{-(s(\omega(t)-\omega(0)))^\alpha}\\
&\quad\times \mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\big)e^{-(s(\omega(\eta)-\omega(0)))}\omega'(\eta)f\big(\eta,u(\eta),v(\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}t\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\big(\omega(t)-\omega(0)\big)^{\alpha-1}\omega'(t)e^{-s(\omega(t)-\omega(0))\theta}\rho_\alpha(\theta)\\
&\quad\times \mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\big)e^{-s(\omega(\eta)-\omega(0))}\omega'(\eta)f\big(\eta,u(\eta),v(\eta)\big)\,\mathrm{d}\theta\,\mathrm{d}\eta\,\mathrm{d}t\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \alpha e^{-s(\omega(t)+\omega(\eta)-2\omega(0))}\frac{(\omega(t)-\omega(0))^{\alpha-1}}{\theta^\alpha}\rho_\alpha(\theta)\mathbb{T}\left(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\right)\\
&\quad\times f\big(\eta,u(\eta),v(\eta)\big)\omega'(\eta)\omega'(t)\,\mathrm{d}\theta\,\mathrm{d}\eta\,\mathrm{d}t\\
&= \int_0^\infty\int_t^\infty\int_0^\infty \alpha e^{-s(\omega(\eta)-\omega(0))}\rho_\alpha(\theta)\frac{(\omega(t)-\omega(0))^{\alpha-1}}{\theta^\alpha}\mathbb{T}\left(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\right)\\
&\quad\times f\big(\omega^{-1}\big(\omega(\eta)-\omega(t)+\omega(0)\big)\big),u\big(\omega^{-1}\big(\omega(\eta)-\omega(t)+\omega(0)\big)\big)))\omega'(\eta)\omega'(t)\,\mathrm{d}\theta\,\mathrm{d}\eta\,\mathrm{d}t\\
&= \int_0^\infty\int_0^\eta\int_0^\infty \alpha e^{-s(\omega(\eta)-\omega(0))}\rho_\alpha(\theta)\frac{(\omega(t)-\omega(0))^{\alpha-1}}{\theta^\alpha}\mathbb{T}\left(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\right)\\
&\quad\times f\big(\omega^{-1}\big(\omega(\eta)-\omega(t)+\omega(0)\big)\big),u\big(\omega^{-1}\big(\omega(\eta)-\omega(t)+\omega(0)\big)\big)))\omega'(\eta)\omega'(t)\,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}\eta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty e^{-s(\omega(\eta)-\omega(0))}\omega'(\eta)\biggl(\int_0^\eta \int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\omega(t)-\omega(\tau))^{\alpha-1}}{\theta^\alpha} \\
&\quad \times \mathbb{T}\biggl(\frac{(\omega(t)-\omega(\tau))^\alpha}{\theta^\alpha}\biggr)f\bigl(\tau,u(\tau),v(\tau)\bigr)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau\biggr)\,\mathrm{d}\eta.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&U(s) \\
&= \int_0^\infty e^{-s(\omega(t)-\omega(0))}\biggl(\int_0^\infty \rho_\alpha(\theta)\mathbb{T}\biggl(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\biggr)u_0\,\mathrm{d}\theta\biggr)\omega'(t)\,\mathrm{d}t \\
&\quad + \int_0^\infty e^{-s(\omega(\eta)-\omega(0))}g\bigl(s,u(\eta)\bigr)\omega'(\eta)\,\mathrm{d}s \\
&\quad + \int_0^\infty e^{-s(\omega(\eta)-\omega(0))}\biggl(\int_0^\eta \int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\omega(t)-\omega(\eta))^{\alpha-1}}{\theta^\alpha} \\
&\quad \times \mathscr{A}\mathbb{T}\biggl(\frac{(\omega(t)-\omega(\eta))^\alpha}{\theta^\alpha}\biggr)g\bigl(s,u(\eta)\bigr)\omega'(\eta)\,\mathrm{d}\theta\,\mathrm{d}\eta\biggr)\omega'(\eta)\,\mathrm{d}\eta \\
&\quad + \int_0^\infty e^{-s(\omega(\eta)-\omega(0))}\biggl(\int_0^\eta \int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\omega(t)-\omega(\eta))^{\alpha-1}}{\theta^\alpha} \\
&\quad \times \mathbb{T}\biggl(\frac{(\omega(t)-\omega(\eta))^\alpha}{\theta^\alpha}\biggr)f\bigl(s,u(\eta),v(\eta)\bigr)\omega'(\eta)\,\mathrm{d}\theta\,\mathrm{d}\eta\biggr)\omega'(\eta)\,\mathrm{d}\eta.
\end{aligned}
$$

Now, we take the inverse Laplace transform to obtain

$$
\begin{aligned}
&u(t) \\
&= \int_0^\infty \rho_\alpha(\theta)\mathbb{T}\biggl(\frac{(\omega(t)-\omega(0))^\alpha}{\theta^\alpha}\biggr)\bigl(u_0-g(0,u_0)\bigr)\,\mathrm{d}\theta + g\bigl(t,u(t)\bigr) \\
&\quad + \alpha\int_0^t \int_0^\infty \rho_\alpha(\theta)\frac{(\omega(t)-\omega(\tau))^{\alpha-1}}{\theta^\alpha}\mathscr{A}\mathbb{T}\biggl(\frac{(\omega(t)-\omega(\tau))^\alpha}{\theta^\alpha}\biggr)g\bigl(\tau,u(\tau)\bigr)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&\quad + \alpha\int_0^t \int_0^\infty \rho_\alpha(\theta)\frac{(\omega(t)-\omega(\tau))^{\alpha-1}}{\theta^\alpha} \\
&\quad \times \mathbb{T}\biggl(\frac{(\omega(t)-\omega(\tau))^\alpha}{\theta^\alpha}f\bigl(\tau,u(\tau),v(\tau)\bigr)\biggr)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&= \int_0^\infty \phi_\alpha(\theta)\mathbb{T}\bigl(\bigl(\omega(t)-\omega(0)\bigr)^\alpha\theta\bigr)\bigl(u_0-g(0,u_0)\bigr)\,\mathrm{d}\theta + g\bigl(t,u(t)\bigr) \\
&\quad + \alpha\int_0^t \int_0^\infty \theta\phi_\alpha(\theta)\bigl(\omega(t)-\omega(\tau)\bigr)^{\alpha-1}\mathscr{A}\mathbb{T}\bigl(\bigl(\omega(t)-\omega(\tau)\bigr)^\alpha\theta\bigr)g\bigl(\tau,u(\tau)\bigr)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&\quad + \alpha\int_0^t \int_0^\infty \theta\phi_\alpha(\theta)\bigl(\omega(t)-\omega(\tau)\bigr)^{\alpha-1} \\
&\quad \times \mathbb{T}\bigl(\bigl(\omega(t)-\omega(\tau)\bigr)^\alpha\theta\bigr)f\bigl(\tau,u(\tau),v(\tau)\bigr)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau,
\end{aligned}
$$

where $\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\rho_\alpha(\theta^{-\frac{1}{\alpha}})$ is the probability density function defined on $(0,\infty)$. □

Definition 3.2 A function $u \in C([0,T],E)$ is called a mild solution of the problem (1) if satisfies the following integral equation:

$$\begin{aligned} u(t) &= \mathcal{Q}^{\alpha;\omega}(t,0)\bigl(u_0 - g(0,u_0)\bigr) + g\bigl(t,u(t)\bigr) \\ &\quad + \int_0^t \bigl(\omega(t)-\omega(\tau)\bigr)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\bigl(s,u(\tau)\bigr)\,\mathrm{d}\tau \\ &\quad + \int_0^t \bigl(\omega(t)-\omega(\tau)\bigr)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f\bigl(s,u(\tau),v(\tau)\bigr)\,\mathrm{d}\tau, \end{aligned} \tag{9}$$

where the operators $\mathcal{Q}^{\alpha;\omega}(t,\tau)$ and $\mathcal{R}^{\alpha;\omega}(t,\tau)$ are defined by

$$\mathcal{Q}^{\alpha;\omega}(t,\tau)u = \int_0^\infty \phi_\alpha(\theta)\mathbb{T}\bigl(\bigl(\omega(t)-\omega(\tau)\bigr)^\alpha\theta\bigr)u\,\mathrm{d}\theta$$

and

$$\mathcal{R}^{\alpha;\omega}(t,\tau)u = \alpha\int_0^\infty \theta\phi_\alpha(\theta)\mathbb{T}\bigl(\bigl(\omega(t)-\omega(\tau)\bigr)^\alpha\theta\bigr)u\,\mathrm{d}\theta$$

for $0 \le \tau \le t \le T$.

Lemma 3.3 ([6, 24]) *The operators $\mathcal{Q}^{\alpha;\omega}$ and $\mathcal{R}^{\alpha;\omega}$ satisfy the following properties*:

(i) *The operators $\mathcal{Q}^{\alpha;\omega}(t,s)$ and $\mathcal{R}^{\alpha;\omega}(t,s)$ are bounded, linear and such that*

$$\bigl\|\mathcal{Q}^{\alpha;\omega}(t,\tau)u\bigr\| \le M\|u\| \quad \textit{and} \quad \bigl\|\mathcal{R}^{\alpha;\omega}(t,\tau)u\bigr\| \le \frac{M}{\Gamma(\alpha)}\|u\|$$

for all $u \in E$ and $0 \le \tau \le t$.

(ii) *For any $0 \le \tau \le t$, $\mathcal{Q}^{\alpha;\omega}(t,\tau)$ and $\mathcal{R}^{\alpha;\omega}(t,\tau)$ are strongly continuous on E.*

(iii) *$\mathcal{Q}^{\alpha;\omega}(t,\tau)$ and $\mathcal{R}^{\alpha;\omega}(t,\tau)$ are compact for all $0 < \tau < t$ if $\mathbb{T}(t)$ is a compact operator for every $t > 0$.*

(iv) *If $\mathcal{Q}^{\alpha;\omega}(t,\tau)$ and $\mathcal{R}^{\alpha;\omega}(t,\tau)$ are compact strongly continuous semigroups of bounded linear operators for $0 < \tau < t$, then $\mathcal{Q}^{\alpha;\omega}(t,\tau)$ and $\mathcal{R}^{\alpha;\omega}(t,\tau)$ are continuous in the uniform operator topology.*

(v) *For any $u \in E$, $\beta \in (0,1)$ and $\eta \in (0,1]$, we have*

$$\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)u = \mathscr{A}^{1-\beta}\mathcal{R}^{\alpha;\omega}(t,\tau)\mathscr{A}^\beta u, \quad 0 \le \tau \le t \le T,$$

and

$$\bigl\|\mathscr{A}^\eta\mathcal{R}^{\alpha;\omega}(t,\tau)\bigr\| \le \frac{\alpha C_\eta}{(\omega(t)-\omega(\tau))^{\alpha\eta}}\frac{\Gamma(2-\eta)}{\Gamma(1+\alpha(1-\eta))}, \quad 0 \le \tau < t \le T.$$

4 Existence and uniqueness of a mild solution

In order to demonstrate the main results, we outline the following assumptions:

(A_1) The operator $\mathbb{T}(t)$ is a compact for all $t > 0$,

(A_2) The function $f : [0,T] \times E \times \mathcal{V} \to E$ is a Carathéodory function, that is,

- (F_1) For each $t \in [0,T]$, the function $f(t,\cdot,\cdot) : E \to E$ is continuous,
- (F_2) For each $u \in E$, the function $f(\cdot,u,v) : [0,T] \to E$ is measurable.

(A$_3$) For any $k > 0$, there exist $\alpha p > 1$ and function $m_k \in L^p([0,T],E)$ such that for any $u \in E$ and $v \in \mathcal{V}$ satisfying $\|u\| \le k$,

$$\big\|f(t,u,v)\big\| \le m_k(t), \quad \text{a.e. } t \in [0,T],$$

and there exists $L > 0$ such that

$$\liminf_{k\to\infty} \frac{\|m_k\|_{L^p}}{k} = L.$$

(A$_4$) The function $g : [0,T] \times E \to E$ is continuous and there exist a positive constant $\beta \in (0,1)$ and $M_1, M_2 > 0$ such that

$$\big\|\mathscr{A}^\beta g(t,u) - \mathscr{A}^\beta g(t,w)\big\| \le M_1\|u-w\|$$

and

$$\big\|\mathscr{A}^\beta g(t,u)\big\| \le M_2\big(\|u\|+1\big).$$

(A$_5$) The function f is a locally Lipschitz continuous with respect to $\mathcal{V}$, i.e., for all $t \in [0,T]$ and $u_1, u_2 \in E$, there exists constant $L_f > 0$ such that

$$\big\|f\big(t,u_1(t),v\big) - f\big(t,u_2(t),v\big)\big\| \le L_f\|u_1-u_2\|.$$

The following existence of mild solutions for the problem (1) will be proved by using Krasnoselskii's fixed point theorem.

Theorem 4.1 *Assume* (A$_1$)–(A$_4$) *are true. Then, the problem* (1) *has at least one mild solution provided that*

$$\begin{aligned}&(M+1)M_2\big\|\mathscr{A}^{-\beta}\big\| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}M_2\big(\omega(T)-\omega(0)\big)^{\alpha\beta}\\&\quad + \frac{ML}{\Gamma(\alpha)}\sup_{0\le t\le T}\big|\omega'(t)\big|^{\frac{1}{p}}\left\{\frac{(\omega(T)-\omega(0))^{\frac{\alpha p-1}{p-1}}}{\frac{\alpha p-1}{p-1}}\right\}^{\frac{p-1}{p}} < 1\end{aligned}\tag{10}$$

and

$$\left((M+1)\big\|\mathscr{A}^{-\beta}\big\| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}\big(\omega(T)-\omega(0)\big)^{\alpha\beta}\right)M_1 < 1.\tag{11}$$

Proof For any $k > 0$, we let $B_k = \{u \in C([0,T],E) : \|u(t)\| \le k \text{ for all } t \in [0,T]\}$. Then, B_k is a bounded closed convex subset of $C([0,T],E)$. For each positive k, we define two operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on B_k as follows:

$$\begin{aligned}(\mathcal{F}_1u)(t) &= \mathcal{Q}^{\alpha;\omega}(t,0)\big(u_0 - g(0,u_0)\big) + g\big(t,u(t)\big)\\&\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\end{aligned}$$

and

$$(\mathcal{F}_2 u)(t) = \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau) f\big(\tau,u(\tau),v(\tau)\big)\,\mathrm{d}\tau,$$

for $t \in [0,T]$.

From Lemma 3.3(v), for $t \in [0,T]$, we obtain

$$\begin{aligned}
&\left\| \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau) g\big(\tau,u(\tau)\big)\,\mathrm{d}\tau \right\| \\
&\quad\le \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\big\|\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau) g\big(\tau,u(\tau)\big)\big\|\,\mathrm{d}\tau \\
&\quad= \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\big\|\mathscr{A}^{1-\beta}\mathcal{R}^{\alpha;\omega}(t,\tau)\mathscr{A}^{\beta} g\big(\tau,u(\tau)\big)\big\|\,\mathrm{d}\tau \\
&\quad\le \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)(\omega(t)-\omega(\tau))^{\alpha(1-\beta)}}\big\|\mathscr{A}^{\beta} g\big(\tau,u(\tau)\big)\big\|\,\mathrm{d}\tau \\
&\quad\le \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} M_2\big(1+\|u\|\big)\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha\beta-1}\omega'(\tau)\,\mathrm{d}\tau \\
&\quad= \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} M_2\big(1+\|u\|\big)\frac{(\omega(t)-\omega(0))^{\alpha\beta}}{\alpha\beta} \\
&\quad\le \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} M_2(1+k)\frac{(\omega(T)-\omega(0))^{\alpha\beta}}{\alpha\beta}.
\end{aligned}$$

Thus, $\|(\omega(t)-\omega(\tau))^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g(\tau,u(\tau))\|$ is Lebesgue integrable with respect to $\tau \in [0,t]$ for all $t \in [0,T]$. From Theorem 2.16 (Bochner's theorem), we obtain that $\|(\omega(t)-\omega(\tau))^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g(\tau,u(\tau))\,\mathrm{d}\tau\|$ is Bochner integrable with respect to $\tau \in [0,t]$ for all $t \in [0,T]$.

Similarly, by Lemma 3.3(i), we also obtain

$$\begin{aligned}
&\left\| \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau) f\big(\tau,u(\tau),v(\tau)\big)\,\mathrm{d}\tau \right\| \\
&\quad\le \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\big\|\mathcal{R}^{\alpha;\omega}(t,\tau) f\big(\tau,u(\tau),v(\tau)\big)\big\|\,\mathrm{d}\tau \\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\big\|f\big(\tau,u(\tau),v(\tau)\big)\big\|\,\mathrm{d}\tau \\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\big\|f\big(\tau,u(\tau),v(\tau)\big)\big\|\,\mathrm{d}\tau \\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)m_k(\tau)\,\mathrm{d}\tau \\
&\quad\le \frac{M\|m_k\|_{L^p}}{\Gamma(\alpha)}\left\{\int_0^t \Big[\big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\Big]^{\frac{p}{p-1}}\,\mathrm{d}\tau\right\}^{\frac{p-1}{p}} \\
&\quad\le \frac{M\|m_k\|_{L^p}}{\Gamma(\alpha)}\left\{\sup_{0\le t\le T}\big|\omega'(t)\big|^{\frac{1}{p-1}}\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\frac{(\alpha-1)p}{p-1}}\omega'(\tau)\,\mathrm{d}\tau\right\}^{\frac{p-1}{p}}
\end{aligned}$$

$$= \frac{M\|m_k\|_{L^p}}{\Gamma(\alpha)} \sup_{0\le t\le T} \left|\omega'(t)\right|^{\frac{1}{p}} \left\{ \frac{(\omega(t)-\omega(0))^{\frac{\alpha p-1}{p-1}}}{1+\frac{(\alpha-1)p}{p-1}} \right\}^{\frac{p-1}{p}}$$
$$\le \frac{M\|m_k\|_{L^p}}{\Gamma(\alpha)} \sup_{0\le t\le T} \left|\omega'(t)\right|^{\frac{1}{p}} \left\{ \frac{(\omega(T)-\omega(0))^{\frac{\alpha p-1}{p-1}}}{1+\frac{(\alpha-1)p}{p-1}} \right\}^{\frac{p-1}{p}}.$$

Thus, $\|(\omega(t)-\omega(\tau))^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f(\tau,u(\tau),v(\tau))\|$ is Lebesgue integrable with respect to $\tau\in[0,t]$ for all $t\in[0,T]$. From Theorem 2.16 (Bochner's theorem), it follows that $\|(\omega(t)-\omega(\tau))^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f(\tau,u(\tau),v(\tau))\|$ is Bochner integrable with respect to $\tau\in[0,t]$ for all $t\in[0,T]$.

The proof will be separated into three parts.

Step 1: $\mathcal{F}_1u+\mathcal{F}_2w\in B_k$ whenever $u,w\in B_k$.

We assume that for each $k>0$, there exist $u_k,w_k\in B_k$ such that

$$\left\|(\mathcal{F}_1u_k)(t)+(\mathcal{F}_2w_k)(t)\right\|>k, \quad \text{for } t\in[0,T].$$

According to (A_3) and Lemma 3.3(i), it follows that

$$\begin{aligned}
k &< \left\|(\mathcal{F}_1u_k)(t)+(\mathcal{F}_2w_k)(t)\right\| \\
&\le \left\|\mathcal{Q}^{\alpha;\omega}(t,0)\big(u_0-g(0,u_0)\big)\right\| + \left\|g\big(t,u_k(t)\big)\right\| \\
&\quad + \left\|\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(\tau,u_k(\tau)\big)\,\mathrm{d}\tau\right\| \\
&\quad + \left\|\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f\big(\tau,w_k(\tau),v(\tau)\big)\,\mathrm{d}\tau\right\| \\
&\le M\big(\|u_0\| + \left\|\mathscr{A}^{-\beta}\mathscr{A}^{\beta}g(0,u_0)\right\|\big) + \left\|\mathscr{A}^{-\beta}\mathscr{A}^{\beta}g(t,u_k)\right\| \\
&\quad + \left\|\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(\tau,u_k(\tau)\big)\,\mathrm{d}\tau\right\| \\
&\quad + \left\|\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f\big(\tau,w_k(\tau),v(\tau)\big)\,\mathrm{d}\tau\right\| \\
&\le M\big(\|u_0\| + \left\|\mathscr{A}^{-\beta}\right\|M_2(k+1)\big) + \left\|\mathscr{A}^{-\beta}\right\|M_2(k+1) \\
&\quad + \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}M_2(1+k)\frac{(\omega(T)-\omega(0))^{\alpha\beta}}{\alpha\beta} \\
&\quad + \frac{M\|m_k\|_{L^p}}{\Gamma(\alpha)} \sup_{0\le t\le T} \left|\omega'(t)\right|^{\frac{1}{p}} \left\{ \frac{(\omega(T)-\omega(0))^{\frac{\alpha p-1}{p-1}}}{\frac{\alpha p-1}{p-1}} \right\}^{\frac{p-1}{p}}.
\end{aligned}$$

Multiplying to both sides by $\frac{1}{k}$ and taking the limit inferior as $k\to\infty$, we get

$$\begin{aligned}
1 &\le M\left(\liminf_{k\to\infty}\frac{\|u_0\|}{k} + \left\|\mathscr{A}^{-\beta}\right\|M_2\right) + \left\|\mathscr{A}^{-\beta}\right\|M_2 \\
&\quad + \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}M_2\frac{(\omega(T)-\omega(0))^{\alpha\beta}}{\alpha\beta} \\
&\quad + \liminf_{k\to\infty}\frac{M\|m_k\|_{L^p}}{k\Gamma(\alpha)} \sup_{0\le t\le T} \left|\omega'(t)\right|^{\frac{1}{p}} \left\{ \frac{(\omega(T)-\omega(0))^{\frac{\alpha p-1}{p-1}}}{\frac{\alpha p-1}{p-1}} \right\}^{\frac{p-1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= (M+1)M_2\left\|\mathscr{A}^{-\beta}\right\| \\
&\quad + \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}M_2\frac{(\omega(T)-\omega(0))^{\alpha\beta}}{\alpha\beta} \\
&\quad + \frac{ML}{\Gamma(\alpha)}\sup_{0\le t\le T}\left|\omega'(t)\right|^{\frac{1}{p}}\left\{\frac{(\omega(T)-\omega(0))^{\frac{\alpha p-1}{p-1}}}{1+\frac{(\alpha-1)p}{p-1}}\right\}^{\frac{p-1}{p}} < 1,
\end{aligned}
$$

which is contradiction.

Step 2: $\mathcal{F}_1$ is a contraction on B_k.

For arbitrary $u, w \in B_k$, we have

$$
\begin{aligned}
&\left\|(\mathcal{F}_1 u)(t)-(\mathcal{F}_1 w)(t)\right\| \\
&\quad\le \left\|\mathcal{Q}^{\alpha;\omega}(t,0)\big(g\big(0,u(0)\big)-g\big(0,w(0)\big)\big)\right\| + \left\|g\big(t,u(t)\big)-g\big(t,w(t)\big)\right\| \\
&\qquad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\left\|\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(\tau,u(\tau)\big)-\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(\tau,w(\tau)\big)\right\|\,\mathrm{d}\tau \\
&\quad\le M\left\|\mathscr{A}^{-\beta}\big(\mathscr{A}^{\beta}g\big(0,u(0)\big)-\mathscr{A}^{\beta}g\big(0,w(0)\big)\big)\right\| \\
&\qquad + \left\|\mathscr{A}^{-\beta}\big(\mathscr{A}^{\beta}g\big(t,u(t)\big)-\mathscr{A}^{\beta}g\big(t,w(t)\big)\big)\right\| \\
&\qquad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\left\|\mathscr{A}^{1-\beta}\mathcal{R}^{\alpha;\omega}(t,\tau)\mathscr{A}^{\beta}\big(g\big(\tau,u(\tau)\big)-g\big(\tau,w(\tau)\big)\big)\right\|\,\mathrm{d}\tau \\
&\quad\le M\left\|\mathscr{A}^{-\beta}\right\|M_1\|u-w\| + \left\|\mathscr{A}^{-\beta}\right\|M_1\|u-w\| \\
&\qquad + \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}M_1\frac{(\omega(T)-\omega(0))^{\alpha\beta}}{\alpha\beta}\|u-w\| \\
&\quad = \left((M+1)\left\|\mathscr{A}^{-\beta}\right\| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}\big(\omega(T)-\omega(0)\big)^{\alpha\beta}\right)M_1\|u-w\|
\end{aligned}
$$

for $t \in [0,T]$. According to (11) of Theorem 4.1, we obtain that $\mathcal{F}_1$ is a contraction.

Step 3: $\mathcal{F}_2$ is a completely continuous operator.

Firstly, we claim that $\mathcal{F}_2$ is continuous on B_k. Let $\{u_n\} \subset B_k$ be such that $u_n \to u \in B_k$ as $n \to \infty$. For $t \in [0,T]$, by Assumptions (A_2) and (A_3), we have

$$
f\big(t,u_n(t),v(t)\big) \to f\big(t,u(t),v(t)\big) \quad \text{as } n \to \infty
$$

and

$$
\left\|f\big(t,u_n(t),v(t)\big)-f\big(t,u(t),v(t)\big)\right\| \le 2m_k(t) \quad \text{for all } n \in \mathbb{N}.
$$

Using the Lebesgue dominated convergence theorem, for any $t \in [0,T]$, we obtain

$$
\begin{aligned}
&\left\|(\mathcal{F}_2 u_n)(t)-(\mathcal{F}_2 u)(t)\right\| \\
&\quad\le \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\left\|\mathcal{R}^{\alpha;\omega}(t,\tau)\big[f\big(t,u_n(\tau),v(\tau)\big)-f\big(t,u(\tau),v(\tau)\big)\big]\right\|\,\mathrm{d}\tau \\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\left\|f\big(t,u_n(\tau),v(\tau)\big)-f\big(t,u(\tau),v(\tau)\big)\right\|\,\mathrm{d}\tau \to 0
\end{aligned}
$$

as $n \to \infty$. This implies that $\|(\mathcal{F}_2 u_n)(t) - (\mathcal{F}_2 u)(t)\|_C \to 0$ as $n \to \infty$. Hence $\mathcal{F}_2$ is continuous.

Next, we prove the equicontinuity of $\mathcal{F}_2(B_k)$. For any $u \in B_k$, we have for $0 \le t_1 < t_2 \le T$,

$$\begin{aligned}
&\big\|(\mathcal{F}_2 u)(t_2) - (\mathcal{F}_2 u)(t_1)\big\| \\
&\quad\le \bigg\| \int_0^{t_2} \big(\omega(t_2) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_2,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \\
&\qquad - \int_0^{t_1} \big(\omega(t_1) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_1,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \bigg\| \\
&\quad= \bigg\| \int_0^{t_1} \big(\omega(t_2) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_2,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \\
&\qquad + \int_{t_1}^{t_2} \big(\omega(t_2) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_2,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \\
&\qquad + \int_0^{t_1} \big(\omega(t_1) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_2,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \\
&\qquad - \int_0^{t_1} \big(\omega(t_1) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_2,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \\
&\qquad - \int_0^{t_1} \big(\omega(t_1) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_1,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \bigg\| \\
&\quad\le \bigg\| \int_{t_1}^{t_2} \big(\omega(t_2) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_2,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \bigg\| \\
&\qquad + \bigg\| \int_0^{t_1} \Big[\big(\omega(t_2) - \omega(\tau)\big)^{\alpha-1} - \big(\omega(t_1) - \omega(\tau)\big)^{\alpha-1}\Big] \omega'(\tau) \mathcal{R}^{\alpha;\omega}(t_2,\tau) f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \bigg\| \\
&\qquad + \bigg\| \int_0^{t_1} \big(\omega(t_1) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \big[\mathcal{R}^{\alpha;\omega}(t_2,\tau) - \mathcal{R}^{\alpha;\omega}(t_1,\tau)\big] f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \bigg\| \\
&\quad=: I_1 + I_2 + I_3.
\end{aligned}$$

By Lemma 3.3, we obtain that

$$I_1 \le \frac{M\|m_k\|_{L^p}}{\Gamma(\alpha)} \sup_{0\le t\le T} \big|\omega'(t)\big|^{\frac{1}{p}} \left\{ \frac{(\omega(t_2) - \omega(t_1))^{1+\frac{(\alpha-1)p}{p-1}}}{1 + \frac{(\alpha-1)p}{p-1}} \right\}^{\frac{p-1}{p}}$$

and

$$\begin{aligned}
I_2 &\le \frac{M\|m_k\|_{L^p}}{\Gamma(\alpha)} \\
&\quad\times \sup_{0\le t\le T} \big|\omega'(t)\big|^{\frac{1}{p}} \left\{ \frac{(\omega(t_1))^{1+\frac{(\alpha-1)p}{p-1}} - (\omega(t_2))^{1+\frac{(\alpha-1)p}{p-1}} - (\omega(t_2) - \omega(t_1))^{1+\frac{(\alpha-1)p}{p-1}}}{1 + \frac{(\alpha-1)p}{p-1}} \right\}^{\frac{p-1}{p}}
\end{aligned}$$

and hence $I_1 \to 0$ and $I_2 \to 0$ as $t_2 \to t_1$. For $t_1 = 0$ and $0 < t_2 \le T$, it easy to see that $I_4 = 0$. Thus, for any $\varepsilon \in (0, t_1)$, we have

$$\begin{aligned}
I_3 &\le \bigg\| \int_0^{t_1-\varepsilon} \big(\omega(t_1) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \big[\mathcal{R}^{\alpha;\omega}(t_2,\tau) - \mathcal{R}^{\alpha;\omega}(t_1,\tau)\big] f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \bigg\| \\
&\quad + \bigg\| \int_{t_1-\varepsilon}^{t_1} \big(\omega(t_1) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \big[\mathcal{R}^{\alpha;\omega}(t_2,\tau) - \mathcal{R}^{\alpha;\omega}(t_1,\tau)\big] f\big(t, u(\tau), v(\tau)\big)\,\mathrm{d}\tau \bigg\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M\|m_k\|_{L^p}}{\Gamma(\alpha)} \\
&\quad \times \sup_{0\leq t\leq T} \left|\omega'(t)\right|^{\frac{1}{p}} \left\{ \frac{(\omega(t_1)-\omega(0))^{1+\frac{(\alpha-1)p}{p-1}} - (\omega(t_1)-\omega(t_1-\varepsilon))^{1+\frac{(\alpha-1)p}{p-1}}}{1+\frac{(\alpha-1)p}{p-1}} \right\}^{\frac{p-1}{p}} \\
&\quad \times \sup_{0\leq s<t_1-\varepsilon} \left\| \mathcal{R}^{\alpha;\omega}(t_2,\tau) - \mathcal{R}^{\alpha;\omega}(t_1,\tau) \right\| \\
&\quad + \frac{2M\|m_k\|_{L^p}}{\Gamma(\alpha)} \left\{ \frac{(\omega(t_1)-\omega(t_1-\varepsilon))^{1+\frac{(\alpha-1)p}{p-1}}}{1+\frac{(\alpha-1)p}{p-1}} \right\}^{\frac{p-1}{p}}.
\end{aligned}
$$

Therefore $I_3 \to 0$ as $t_2 \to t_1$ and $\varepsilon \to 0$ by Lemma 3.3, (iii) and (iv). It follows that

$$
\left\|(\mathcal{F}_2 u)(t_2) - (\mathcal{F}_2 u)(t_1)\right\| \to 0 \quad \text{independently of } u \in B_k \text{ as } t_2 \to t_1,
$$

which means that $\mathcal{F}_2(B_k)$ is equicontinuous.

Now, we will prove that $N(t) = \{(\mathcal{F}_2 u)(t) : u \in B_k\}$ is relatively compact in E for all $t \in [0,T]$. Notice that $N(0)$ is relatively compact in E. Fix $t \in (0,T]$, then, for every $\varepsilon > 0$ and $\delta > 0$, we define an operator $\mathcal{F}_2^{\varepsilon,\delta}$ on B_k as

$$
\begin{aligned}
&\left(\mathcal{F}_2^{\varepsilon,\delta} u\right)(t) \\
&\quad = \alpha \int_0^{t-\varepsilon} \int_\delta^\infty \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1} \\
&\qquad \times \mathbb{T}\Big(\big(\omega(t)-\omega(0)\big)^\alpha \theta\Big) f\big(\tau, u(\tau), v(\tau)\big)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&\quad = \alpha \int_0^{t-\varepsilon} \int_\delta^\infty \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1} \\
&\qquad \times \mathbb{T}\Big(\big(\omega(t)-\omega(0)\big)^\alpha \theta + \varepsilon^\alpha\delta - \varepsilon^\alpha\delta\Big) f\big(\tau, u(\tau), v(\tau)\big)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&\quad = \alpha \int_0^{t-\varepsilon} \int_\delta^\infty \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1} \\
&\qquad \times \Big[\mathbb{T}\big(\varepsilon^\alpha\delta\big)\mathbb{T}\Big(\big(\omega(t)-\omega(0)\big)^\alpha \theta - \varepsilon^\alpha\delta\Big)\Big] f\big(\tau, u(\tau), v(\tau)\big)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&\quad = \alpha\mathbb{T}\big(\varepsilon^\alpha\delta\big) \int_0^{t-\varepsilon} \int_\delta^\infty \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1} \\
&\qquad \times \mathbb{T}\Big(\big(\omega(t)-\omega(0)\big)^\alpha \theta - \varepsilon^\alpha\delta\Big) f\big(\tau, u(\tau), v(\tau)\big)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau,
\end{aligned}
$$

where $u \in B_k$.

By the compactness of $\mathbb{T}(\varepsilon^\alpha\delta)$ for $\varepsilon^\alpha\delta > 0$, it follows that the set $N_{\varepsilon,\delta}(t) = \{(\mathcal{F}_2^{\varepsilon,\delta} u)(t) : u \in B_k\}$ is relatively compact in E for all $\varepsilon > 0$ and $\delta > 0$. Furthermore, for any $u \in B_k$, we have

$$
\begin{aligned}
&\left\|(\mathcal{F}_2 u)(t) - \left(\mathcal{F}_2^{\varepsilon,\delta} u\right)(t)\right\| \\
&\quad = \alpha \left\| \int_0^t \int_0^\delta \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathbb{T}\Big(\big(\omega(t)-\omega(0)\big)^\alpha\theta\Big) f\big(\tau, u(\tau), v(\tau)\big)\,\mathrm{d}\theta\,\mathrm{d}\tau \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_\delta^\infty \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\theta\big)f\big(\tau,u(\tau),v(\tau)\big)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&+\int_0^{t-\varepsilon}\int_\delta^\infty \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1} \\
&\times \mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\theta\big)f\big(\tau,u(\tau),v(\tau)\big)\omega'(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau\bigg\| \\
\le{}& \alpha\bigg\|\int_0^t\int_0^\delta \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\theta\big)f\big(\tau,u(\tau),v(\tau)\big)\,\mathrm{d}\theta\,\mathrm{d}\tau\bigg\| \\
&+\alpha\bigg\|\int_{t-\varepsilon}^t\int_\delta^\infty \theta\phi_\alpha(\theta)\big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau) \\
&\times \mathbb{T}\big(\big(\omega(t)-\omega(0)\big)^\alpha\theta\big)f\big(\tau,u(\tau),v(\tau)\big)\,\mathrm{d}\theta\,\mathrm{d}\tau\bigg\| \\
\le{}& \alpha M\|m_k\|_{L^p}\left\{\sup_{0\le s\le t}\big|\omega'(\tau)\big|^{\frac{1}{p-1}}\int_0^t\big(\omega(t)-\omega(\tau)\big)^{\frac{(\alpha-1)p}{p-1}}\omega'(\tau)\,\mathrm{d}\tau\right\}^{\frac{p-1}{p}}\left(\int_0^\delta\theta\phi_\alpha(\theta)\,\mathrm{d}\theta\right) \\
&+\alpha M\|m_k\|_{L^p}\left\{\sup_{t-\varepsilon\le s\le t}\big|\omega'(t)\big|^{\frac{1}{p-1}}\int_{t-\varepsilon}^t\big(\omega(t)-\omega(\tau)\big)^{\frac{(\alpha-1)p}{p-1}}\omega'(\tau)\,\mathrm{d}\tau\right\}^{\frac{p-1}{p}} \\
&\times\left(\int_0^\delta\theta\phi_\alpha(\theta)\,\mathrm{d}\theta\right) \\
={}& \alpha M\|m_k\|_{L^p}\sup_{0\le s\le t}\big|\omega'(\tau)\big|^{\frac{1}{p}}\left\{\frac{(\omega(t)-\omega(0))^{1+\frac{(\alpha-1)p}{p-1}}}{1+\frac{(\alpha-1)p}{p-1}}\right\}^{\frac{p-1}{p}}\left(\int_0^\delta\theta\phi_\alpha(\theta)\,\mathrm{d}\theta\right) \\
&+\alpha M\|m_k\|_{L^p}\sup_{t-\varepsilon\le s\le t}\big|\omega'(t)\big|^{\frac{1}{p}}\left\{\frac{(\omega(t)-\omega(t-\varepsilon))^{1+\frac{(\alpha-1)p}{p-1}}}{1+\frac{(\alpha-1)p}{p-1}}\right\}^{\frac{p-1}{p}} \\
&\times\left(\int_0^\delta\theta\phi_\alpha(\theta)\,\mathrm{d}\theta\right) \\
\to{}& 0 \quad \text{as } \varepsilon,\delta\to 0^+.
\end{aligned}
$$

Hence, there are relatively compact sets arbitrary close to the set $N(t)$ for $t > 0$. Therefore, we conclude that $N(t)$ is relatively compact in E. It follows that the set $\mathcal{F}_2(B_k)$ is relatively compact in $C([0,T],E)$ by Arzelá–Ascoli theorem. This implies that $\mathcal{F}_2$ is a completely continuous by the continuity of $\mathcal{F}_2$ and relatively compactness of $\mathcal{F}_2(B_k)$. Hence, Krasnoselskii's fixed point theorem implies that $\mathcal{F}_1 + \mathcal{F}_2$ has a fixed point u^* in B_k, which is a mild solution of (1). □

Next, we prove a uniqueness result by means of the Banach contraction theorem.

Theorem 4.2 *Assume that* (A_3)–(A_5) *are satisfied and condition* (10) *of Theorem* 4.1 *holds. Then, the problem* (1) *has a unique mild solution if*

$$
\begin{aligned}
&(M+1)M_1\big\|\mathscr{A}^{-\beta}\big\| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}M_1\big(\omega(T)-\omega(0)\big)^{\alpha\beta} \\
&\quad+\frac{M}{\Gamma(\alpha+1)}L_f\big(\omega(T)-\omega(0)\big)^\alpha < 1. \qquad (12)
\end{aligned}
$$

Proof For $u \in B_k$, we define the operator $\mathcal{G}$ on B_k by

$$\begin{aligned}(\mathcal{G}u)(t) &= \mathcal{Q}^{\alpha;\omega}(t,0)\big(u_0 - g(0,u_0)\big) + g\big(t,u(t)\big) \\ &\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(s,u(\tau)\big)\,\mathrm{d}\tau \\ &\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f\big(s,u(\tau),v(\tau)\big)\,\mathrm{d}\tau.\end{aligned}$$

Notice that it is enough to show the uniqueness of a fixed point of $\mathcal{G}$ on B_k. According to (10), we know that $\mathcal{G}$ is an operator from B_k into itself.

For any $u, u^* \in B_k$ and $t \in [0,T]$, according to (A_3)–(A_5), we have

$$\begin{aligned}&\big\|(\mathcal{G}u)(t) - \big(\mathcal{G}u^*\big)(t)\big\| \\ &\quad\le \big\|\mathcal{Q}^{\alpha;\omega}(t,0)\big(g\big(0,u(0)\big) - g\big(0,u^*(0)\big)\big)\big\| + \big\|g\big(t,u(t)\big) - g\big(t,u^*(t)\big)\big\| \\ &\qquad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau) \\ &\qquad \times \big\|\mathscr{A}^{1-\beta}\mathcal{R}^{\alpha;\omega}(t,\tau)\big[\mathscr{A}^{\beta}g\big(s,u(\tau)\big) - \mathscr{A}^{\beta}g\big(s,u^*(\tau)\big)\big]\big\|\,\mathrm{d}\tau \\ &\qquad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau) \\ &\qquad \times \big\|\mathcal{R}^{\alpha;\omega}(t,\tau)\big[f\big(s,u(\tau),v(\tau)\big) - f\big(s,u^*(\tau),v(\tau)\big)\big]\big\|\,\mathrm{d}\tau \\ &\quad\le M\big\|\mathscr{A}^{-\beta}\big(\mathscr{A}^{\beta}g\big(0,u(0)\big) - \mathscr{A}^{\beta}g\big(0,u^*(0)\big)\big)\big\| \\ &\qquad + \big\|\mathscr{A}^{-\beta}\big(\mathscr{A}^{\beta}g\big(t,u(t)\big) - \mathscr{A}^{\beta}g\big(t,u^*(t)\big)\big)\big\| \\ &\qquad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\frac{\alpha C_{1-\beta}}{(\omega(t)-\omega(\tau))^{\alpha(1-\beta)}}\frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} \\ &\qquad \times \big\|\mathscr{A}^{\beta}g\big(s,u(\tau)\big) - \mathscr{A}^{\beta}g\big(s,u^*(\tau)\big)\big\|\,\mathrm{d}\tau \\ &\qquad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau) \\ &\qquad \times \frac{M}{\Gamma(\alpha)}\big\|f\big(s,u(\tau),v(\tau)\big) - f\big(s,u^*(\tau),v(\tau)\big)\big\|\,\mathrm{d}\tau \\ &\quad\le M\big\|\mathscr{A}^{-\beta}\big\|M_1\big\|u-u^*\big\| + \big\|\mathscr{A}^{-\beta}\big\|M_1\big\|u-u^*\big\| \\ &\qquad + \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}M_1\frac{(\omega(T)-\omega(0))^{\alpha\beta}}{\alpha\beta}\big\|u-u^*\big\| \\ &\qquad + \frac{M}{\Gamma(\alpha+1)}L_f\big(\omega(T)-\omega(0)\big)^{\alpha}\big\|u-u^*\big\| \\ &\quad= \bigg((M+1)M_1\big\|\mathscr{A}^{-\beta}\big\| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}M_1\big(\omega(T)-\omega(0)\big)^{\alpha\beta} \\ &\qquad + \frac{M}{\Gamma(\alpha+1)}L_f\big(\omega(T)-\omega(0)\big)^{\alpha}\bigg)\big\|u-u^*\big\|.\end{aligned}$$

This implies that $\mathcal{G}$ is a contraction map satisfying (12). Hence the uniqueness of a fixed point of the map $\mathcal{G}$ on B_k follows from the Banach contraction principle. □

5 Existence of feasible pairs for fractional evolution equations

In this section, we present the existence of feasible pairs for system (1). To establish our results, we introduce the following hypotheses:

(U_1) The set $f(t,u,\Gamma(t,u))$ satisfies

$$\bigcap_{\delta>0} \overline{\mathrm{co}} f\big(t, O_\delta(u), \Gamma\big(O_\delta(t,u)\big)\big) = f\big(t,u,\Gamma(t,u)\big), \quad \text{a.e. } t \in [0,T].$$

(U_2) $\Gamma : [0,T] \times E \to 2^U$ is pseudocontinuous.

Definition 5.1 (Feasible pairs) A pair (u,v) is called feasible if v satisfies (9) and

$$v(t) \in \Gamma\big(t,u(t)\big), \quad \text{a.e. } t \in [0,T].$$

Let $[\tau,\nu] \subseteq [0,T]$, and set

$$H[\tau,\nu] = \big\{(u,v) \in C\big([\tau,\nu],E\big) \times \mathcal{V}[\tau,\nu] \mid (u,v) \text{ is feasible}\big\},$$
$$H[0,T] = \big\{(u,v) \in C\big([0,T],E\big) \times \mathcal{V}[0,T] \mid (u,v) \text{ is feasible}\big\}.$$

Lemma 5.2 *Operators $\mathscr{Q}_j : L^p([0,T],E) \to C([0,T]E)$, $j = 1,2$ for some $\frac{1}{p} < \alpha < 1$, $p > 1$, given by*

$$(\mathscr{Q}_1 h)(\cdot) = \int_0^\cdot \big(\omega(\cdot) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{Q}^{\alpha;\omega}(\cdot,\tau) h(\tau)\,\mathrm{d}\tau$$

and

$$(\mathscr{Q}_2 h)(\cdot) = \int_0^\cdot \big(\omega(\cdot) - \omega(\tau)\big)^{\alpha-1} \omega'(\tau) \mathcal{R}^{\alpha;\omega}(\cdot,\tau) h(\tau)\,\mathrm{d}\tau$$

are also compact for $h \in L^p([0,T],E)$.

Proof Assume that $\{h^n\} \subseteq L^p([0,T],E)$ is bounded. We define

$$\mathscr{E}_n^j(t) = \big(\mathscr{Q}_j h^n\big)(t), \quad t \in [0,T].$$

We can verify that, for any $t \in [0,T]$ and $\frac{1}{p} < \alpha < 1$, $\|\mathscr{E}_n^j(t)\|$ is bounded. By Lemma 3.3, it is not difficult to verify that $\mathscr{E}_n^j(t)$ is compact in E and also equicontinuous. Due to Ascoli–Arzela theorem, $\{\mathscr{E}_n^j(t)\}$ is relatively compact in $C([0,T],E)$. Obviously, $\mathscr{Q}_j$ is a continuous linear operator. Therefore, $\mathscr{Q}_j$ is a compact operator for $j = 1,2$. □

We need to investigate the following result in order to solve our optimal feedback control problem.

Theorem 5.3 *Under the assumptions of Theorem 4.2 and Conditions* (U_1) *and* (U_2), *for any $u_0 \in E$ and $\frac{1}{p} < \alpha < 1$ for some $p > 1$, the set $H[0,T]$ is nonempty, that is, $H[0,T] \neq \emptyset$.*

Proof For any $k \geq 0$, we define

$$v_k(t) = \sum_{j=0}^{k-1} v^j \chi_{[t_j,t_{j+1})}(t), \quad t \in [0,T],$$

where $t_j = \frac{j}{k}T$ for $0 \leq j \leq k-1$ and $\chi_{[t_j,t_{j+1})}$ is the characteristic function of the interval $[t_j, t_{j+1})$. The sequence $\{v^j\}$ is constructed as follows.

Firstly, we take $v_0 \in \Gamma(0, u_0)$. From Theorem 4.2, there exists a unique $u_k(\cdot)$, which is defined as

$$\begin{aligned} u_k(t) &= \mathcal{Q}^{\alpha;\omega}(t,0)\big(u_0 - g(0,u_0)\big) + g\big(t,u_k(t)\big) \\ &\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(\tau,u_k(\tau)\big)\,\mathrm{d}\tau \\ &\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f\big(\tau,u_k(\tau),v_0(\tau)\big)\,\mathrm{d}\tau, \quad t \in \left[0,\frac{T}{k}\right]. \end{aligned}$$

Then, we take $v_1 \in \Gamma(\frac{T}{k}, u_k(\frac{T}{k}))$. Repeating this this procedure, we obtain u_k on $[\frac{T}{k}, \frac{2T}{k}]$.

To conclude, we construct the following integral equation by induction:

$$\begin{cases} u_k(t) = \mathcal{Q}^{\alpha;\omega}(t,0)(u_0 - g(0,u_0)) + g(t,u_k(t)) \\ \qquad + \int_0^t (\omega(t)-\omega(\tau))^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g(\tau,u_k(\tau))\,\mathrm{d}\tau \\ \qquad + \int_0^t (\omega(t)-\omega(\tau))^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f(\tau,u_k(\tau),v_0(\tau))\,\mathrm{d}\tau, \quad t \in [0,\frac{T}{k}], \\ v_k(\cdot) \in \Gamma(\frac{jT}{k}, u_k(\frac{jT}{k})), \\ t \in [\frac{jT}{k}, \frac{(j+1)T}{k}), \quad 0 \leq j \leq k-1. \end{cases} \tag{13}$$

From Lemma 3.3 and Gronwall's inequality, we can find $L > 0$ such that

$$\|u_k(t)\| \leq L, \quad t \in [0,T].$$

By (A_3), it follows that

$$\|f\big(t,u_k(t),v_k(t)\big)\| \leq m_k(t), \quad \text{a.e. } t \in [0,T].$$

By Lemma 5.2, there exists a subsequence of $\{u_k\}$, denoted by $\{u_k\}$ again, such that

$$u_k \to \overline{u} \quad \text{in } C\big([0,T],E\big) \tag{14}$$

for some $\overline{u} \in C([0,T],E)$, and

$$f\big(\cdot,u_k(\cdot),v_k(\cdot)\big) \to \overline{f}(\cdot) \quad \text{in } L^p\big([0,T],E\big) \tag{15}$$

for some $\overline{f} \in L^p([0,T],E)$.

Applying Lemma 5.2 and (13), we get

$$\begin{aligned}\overline{u}(t) &= \mathcal{Q}^{\alpha;\omega}(t,0)\big(u_0 - g(0,u_0)\big) + g\big(t,u(t)\big) \\ &\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(t,u(\tau),v(\tau)\big)\,\mathrm{d}\tau \\ &\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)\overline{f}(\tau)\,\mathrm{d}\tau, \quad t\in[0,T].\end{aligned}$$

From (14), for any $\delta > 0$, there exists a $k_0 > 0$ such that

$$u_k(t) \in O_\delta\big(\overline{u}(t)\big), \quad t\in[0,T], k\geq k_0. \tag{16}$$

In contrast, by the definition of $v_k(\cdot)$ and for sufficiently large k, we have

$$v_k(t) \in \Gamma\big(t_j,u_k(t_j)\big) \subset \Gamma\big(O_\delta\big(t,\overline{u}(t)\big)\big) \tag{17}$$

for all $t\in[\frac{jT}{k},\frac{(j+1)T}{k})$, $0\leq j\leq k-1$.

Next, by (5) and Lemma 2.3 (Mazur's lemma), we let $\alpha_{ij}\geq 0$ and $\sum_{j\geq 0}\alpha_{ij}=1$ be such that

$$\phi_l(\cdot) = \sum_{i\geq 1}\alpha_{ij}f\big(\cdot,u_{i+l}(\cdot),v_{i+l}(\cdot)\big) \to \overline{f}(\cdot) \quad \text{in } L^p\big([0,T],E\big).$$

Hence, there exists a subsequence of $\{\phi_l\}$, denoted by $\{\phi_l\}$ again, such that

$$\phi_l(t) \to \overline{f}(t) \quad \text{in } E, \text{a.e. } t\in[0,T].$$

From (16) and (17), it follows that for sufficiently large l, we have

$$\phi_l(t) \in \overline{\mathrm{co}}f\big(t,O_\delta\big(\overline{u}(t)\big)\big),\Gamma\big(O_\delta\big(t,\overline{u}(t)\big)\big), \quad \text{a.e. } t\in[0,T].$$

Therefore, for any $\delta>0$, we have

$$\overline{f}(t) \in \overline{\mathrm{co}}f\big(t,O_\delta\big(\overline{u}(t)\big)\big),\Gamma\big(O_\delta\big(t,\overline{u}(t)\big)\big), \quad \text{a.e. } t\in[0,T].$$

By Assumption (U_1), we have

$$\overline{f}(t) \in \overline{\mathrm{co}}f\big(t,\overline{u}(t),\Gamma\big(t,\overline{u}(t)\big)\big), \quad \text{a.e. } t\in[0,T].$$

From (U_2) and Corollary 2.4, we have that $\Gamma(\cdot,\overline{u}(\cdot))$ is Souslin measurable. Applying Lemma 2.5 (Fillippov's theorem), we can find a function $\overline{u}\in\mathcal{V}[0,T]$ such that

$$\overline{u}(t) \in \Gamma\big(t,\overline{u}(t)\big), \quad t\in[0,T]$$

and

$$\overline{f}(t) = \overline{f}\big(t,\overline{u}(t),\overline{v}(t)\big), \quad t\in[0,T].$$

This means that $(\overline{u},\overline{v})$ is a feasible pair in $[0,T]$. □

6 Existence of optimal feedback control pairs

In this section, we consider the Lagrange problem (P) for the optimal feedback control as follows: find a pair $(u^0, v^0) \in H[0,T]$ such that

$$J\left(u^0, v^0\right) \leq J(u,v), \quad \text{for all } (u,v) \in H[0,T],$$

where

$$J(u,v) = \int_0^b L\big(t, u(t), y(t)\big)\,\mathrm{d}t. \tag{P}$$

Then, we give some assumptions about L.

(L_1) The functional $L : [0,T] \times E \times \mathcal{V} \to \mathbb{R} \cup \{\infty\}$ is Borel measurable in (t,u,v).

(L_2) The functional $L(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $E \times \mathcal{V}$ for a.e. $t \in [0,T]$ and there is a constant $M_1 > 0$ such that

$$L(t,u,v) \geq -M_1, \quad (t,u,v) \in [0,T] \times E \times \mathcal{V}.$$

For any $(t,u) \in [0,T] \times E$, we denote the set

$$\Sigma(t,u) = \left\{\left(z^0, z\right) \in \mathbb{R} \times E \mid z^0 \geq L(t,u,v), z = f(t,u,v), u \in \Gamma(t,u)\right\}.$$

To investigate the existence of optimal control pairs for problem (P), we assume that

(C) The map $\Sigma(t,\cdot) : E \to 2^{R \times E}$ has the Cesari property for a.e. $t \in [0,T]$, that is,

$$\bigcap_{\delta > 0} \overline{\mathrm{co}}\, \Sigma\big(t, O_\delta(u)\big) = \Sigma(t,u)$$

for all $u \in E$.

Theorem 6.1 *Under the assumptions of Theorem 5.3 and Conditions* (L_1)–(L_2) *and* (C), *the exists at least one optimal control pair for the Lagrange problem* (P).

Proof It is clear that the statement of Theorem 6.1 is true if $\inf\{J(u,v) \mid (u,v) \in H[0,T]\} = +\infty$. Hence, it suffices to prove the statement when $\inf\{J(u,v) \mid (u,v) \in H[0,T]\} = m < +\infty$. By Assumptions ($\mathrm{L}_1$)–($\mathrm{L}_2$), we have

$$J(u) \geq m \geq -M_1 > -\infty.$$

Then, there exists a sequence $\{u^n, v^n\} \subset H[0,T]$ such that

$$J\left(u^n, v^n\right) \to m.$$

We denote

$$J\left(u^n, v^n\right) = \int_0^T L\big(t, u^n(t), v^n(t)\big)\,\mathrm{d}t$$

and

$$\varliminf_{n\to+\infty} J\big(u^n, v^n\big) = m.$$

From the nonlinearity of f and boundedness of $\{u^n\}$, we obtain that

$$\big\{f\big(\cdot, u^n(\cdot), v^n(\cdot)\big)\big\}$$

is bounded in $L^p([0,T],E)$.

Without loss of generality, we can assume that

$$f^n(\cdot) = f\big(\cdot, u^n(\cdot), v^n(\cdot)\big) \xrightarrow{w} f(\cdot) \quad \text{in } L^p\big([0,T],E\big)$$

for some $f(\cdot) \in L^p([0,T],E)$. By Lemma 3.1, we obtain

$$\begin{aligned}
u^n(t) &= \mathcal{Q}^{\alpha;\omega}(t,0)\big(u_0 - g(0,u_0)\big) + g\big(t,u^n(t)\big) \\
&\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(\tau,u^n(\tau)\big)\,\mathrm{d}\tau \\
&\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)\Gamma\big(\tau,u^n(\tau),v^n(\tau)\big)\,\mathrm{d}\tau \\
&\to \overline{u}(t) \\
&= \mathcal{Q}^{\alpha;\omega}(t,0)\big(u_0 - g(0,u_0)\big) + g\big(t,u(t)\big) \\
&\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(t,u(\tau),v(\tau)\big)\,\mathrm{d}\tau \\
&\quad + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)\overline{f}(\tau)\,\mathrm{d}\tau
\end{aligned}$$

uniformly in $t \in [0,T]$, i.e., $u^n(\cdot) \to \overline{u}(\cdot)$ in $C([0,T],E)$.

Applying Lemma 2.3 (Mazur's lemma), we let $\alpha_{k,l} \geq 0$, $\sum_{k\geq 1}\alpha_{k,l} = 1$ be such that

$$\phi_l(\cdot) = \sum_{k\geq 1}\alpha_{k,l} f\big(\cdot, u_{k+l}(\cdot), v_{k+l}(\cdot)\big) \to \overline{f}(\cdot) \quad \text{in } L^p\big([0,T],E\big).$$

Set

$$\phi_l^0(\cdot) = \sum_{k\geq 1}\alpha_{k,l} L\big(\cdot, u_{k+l}(\cdot), v_{k+l}(\cdot)\big)$$

and

$$L^0(t) = \varliminf_{l\to+\infty} \phi_l^0(t) \geq -M_1, \quad \text{a.e. } t \in [0,T].$$

For any $\delta > 0$ and sufficiently large l, we have

$$\big(\phi_l(t), \phi_l^0(t)\big) \in \Sigma\big(t, O_\delta\big(\overline{u}(t)\big)\big).$$

By Assumption (C), we get

$$\big(L^0(t),\overline{f}(t)\big) \in \Sigma\big(t,\overline{u}(t)\big), \quad \text{a.e. } t \in [0,T].$$

Hence, we have

$$\begin{cases} L^0(t) \geq L(t,\overline{u}(t),y), & t \in [0,T], \\ \overline{f}(t) = f(t,\overline{u}(t),y), & t \in [0,T], \\ u \in \Gamma(t,\overline{u}(t)). \end{cases}$$

Then, by Lemma 2.5 (Fillippov's theorem), we can find a measurable function $\overline{u}(\cdot)$ of $\Gamma(\cdot,\overline{u}(\cdot))$ satisfying

$$\begin{cases} L^0(t) \geq L(t,\overline{u}(t),\overline{v}(t)), & t \in [0,T], \\ \overline{f}(t) = f(t,\overline{u}(t),\overline{v}(t)), & \text{a.e. } t \in [0,T]. \end{cases}$$

Moreover, we have

$$\begin{aligned} \overline{u}(t) = {} & \mathcal{Q}^{\alpha;\omega}(t,0)\big(u_0 - g(0,u_0)\big) + g\big(t,\overline{u}(t)\big) \\ & + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathscr{A}\mathcal{R}^{\alpha;\omega}(t,\tau)g\big(\tau,\overline{u}(\tau)\big)\,\mathrm{d}\tau \\ & + \int_0^t \big(\omega(t)-\omega(\tau)\big)^{\alpha-1}\omega'(\tau)\mathcal{R}^{\alpha;\omega}(t,\tau)f\big(\tau,\overline{u}(\tau),\overline{v}(\tau)\big)\,\mathrm{d}\tau, \quad t \in [0,T] \end{aligned}$$

and $(\overline{u},\overline{v}) \in H[0,T]$.

By Fatou's lemma, it follows that

$$\int_0^T L^0(t)\,\mathrm{d}t = \int_0^T \underline{\lim}_{l\to+\infty} \phi_l^0(t)\,\mathrm{d}t \leq \underline{\lim}_{l\to+\infty} \int_0^T \phi_l^0(t)\,\mathrm{d}t,$$

i.e.,

$$J(\overline{u},\overline{v}) = \int_0^T L\big(t,\overline{u}(t),\overline{v}(t)\big)\,\mathrm{d}t = \inf_{(u,v)\in H[0,T]} J(u,v) = m.$$

Thus, $(\overline{u},\overline{v})$ is an optimal pair. □

7 Conclusion

In this paper, we have investigated the existence and uniqueness results concerning mild solutions for fractional evolution equations with nonlinear perturbation of the time-fractional derivative term involving Caputo fractional derivatives with arbitrary kernels. Further, we extended our result to study the existence of optimal control pairs for the Lagrange problem by applying the Filippov's theorem.

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Authors' contributions
The main idea of this paper was proposed and mainly proved by PSN, while AS performed some proofs and provided some examples. All authors read and approved the final manuscript.

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