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# Existence and stability results for nonlocal boundary value problems of fractional order

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**Abstract**

In this paper, we prove the existence and uniqueness of solutions for the nonlocal boundary value problem (BVP) using Caputo fractional derivative (CFD). We derive Green's function and give some estimation for it to derive our main results. The main principles applied to investigate our results are based on the Banach contraction fixed point theorem and Schauder fixed point approach. We dwell in detail on some results concerning the Hyers-Ulam (H-U) type and generalized H-U (g-H-U) type stability also for problem we are considering. We justify our results with an illustrative example.

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**Keywords:** CFD; BVP; Existence and uniqueness; g-H-U stability

## 1 Introduction

Fractional differential equations (FDEs) refer to the generalization of integer-order differential equations. They appear in various scientific and engineering fields, for instance, the mathematical modeling of different dynamical processes in epidemiology [1–4], chemistry, physics [5], psychology [6], biology, signal and image processing, control theory, ecology [7], etc. As a result, the concept of non-integer order differential equations is continuously getting more consequences and relevancy. For instances and details, see [8–11] and the given references therein. After all, even though the BVPs theory for nonlinear and nonclassical differential equations is still in its infancy, many directions of this literature need to be expanded further.

The topic of multi-point nonlocal BVPs has been raised by various researchers (we refer [12–15]). The multi-point boundary constraints arise in various problems of physics, fluid mechanics, and wave propagation (we refer to [16, 17] for interest). For instance, in controllers, the multi-point boundary constraints may be found such that the controllers at the endpoints spread or add energy with related sensors placed at middle-level positions. In the same line, the third-order differential equations in which differentiation of acceleration is involved are called jerk equations, where the time derivative of acceleration occurs (see details in [18, 19]). The said equations are important for engineers and physicists, and they try to plan the vehicles in a way that jerks may be minimal. The third-order differential equations are special cases of FDEs with orders between 2 and 3. The fractional order goes through three, the considered equation possibly corresponds to the jerk equation.

The results related to investigating the existence and uniqueness of solutions for nonlinear multi-point BVPs have been studied by a number of researchers. For example, authors in [20, 21], and references therein have investigated various classes of BVPs of FDEs. Authors [22] have proved the existence of a solution for non-integer order BVP with nonlocal multi-point boundary constraints using Schaefer's and Krasnoselskii's fixed point theorems. On the other hand, stability results are important to be investigated in most cases to demonstrate the authenticity and validity of numerical algorithms, methods, and procedures. In this regard, valuable work has been done in the last many decades. Various concepts of stability have been introduced, including exponential, Mittag-Leffler, and Lyapunov types. The stability aspects are very important from the optimization and numerical point of view. So far, we know, a huge amount of work has been done [23–27]. Another version of stability introduced by Ulam and explained by Hyers attracted the attention of researchers very well. The said version is easy and understandable for approximate solutions. For functional equations, this kind of stability has been investigated very well (see details in a few articles as [28–30]). The mentioned stability has been very well studied for ordinary differential problems. In the last few decades, this aspect has been given more attention and investigated for different classes of initial value problems of FDEs (see [31–34]). However, for simple BVPs, the concerned stability has also been well studied. However, in the case of nonlocal BVPs, it is very rarely investigated, especially for jerk-like problems. In this regard, we refer to [35–38].

Inspired by the above work, we will extend the results of the following problem [39] to fractional order as

$$y''' + \Lambda(t,y) = 0, \quad t \in [c,d], \tag{1.1}$$

$$y(c) = y'(c) = 0, \qquad y(d) = ky(\beta). \tag{1.2}$$

Here $\beta \in (c,d)$, $k \in \mathbb{R}$, $\Lambda \in C([c,d] \times \mathbb{R}, \mathbb{R})$, and $\Lambda(t,0) \neq 0$. In our work, we intend to extend the above (1.1) and (1.2) by taking the Caputo fractional order derivative instead of ordinary (we refer the reader to see [40] for definitions and basic consequences on non-integer order calculus) in the place of the classical operator $y'''$. Here, we prove the existence and uniqueness of the solution for the following non-integer order BVP of FDEs as

$${}^{c}D^{\zeta}_{0+}y(t) + \Lambda(t,y) = 0, \quad t \in [c,d], \tag{1.3}$$

$$y(c) = y'(c) = 0, \qquad y(d) = ky(\beta), \tag{1.4}$$

where $2 < \zeta \leq 3$, by assuming that $\Lambda : [c,d] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping and follows a uniform Lipschitz inequality with respect to $y$ on $[c,d] \times \mathbb{R}$, such that there exists a constant $L > 0$, where for every $(t,y), (t,z) \in [c,d] \times \mathbb{R}$, we have

$$\left|\Lambda(t,y) - \Lambda(t,z)\right| \leq L|y-z|.$$

If $(d-c)^2 \neq k(c-\beta)^2$ with $c \neq \beta$, $c \neq d$ and $(d-c)$ is as small such that

$$\frac{(d-c)^3}{3} + \frac{|k|}{3}\frac{(d-c)^5}{|(d-c)^2 - k(\beta-c)^2|} < \frac{1}{L}.$$

Under the said condition, a unique solution to Problem (1.3) and (1.4) exists. This result will be investigated using fixed point techniques. After that, we will also evaluate the proposed stability analysis for our considered Problem (1.3) and (1.4). Also, H-U and g-H-U stability results are developed for the considered problem using some sophisticated procedure of nonlinear analysis.

The remaining article is arranged as follows: In Sect. 2, we recall some basic definitions and results. In Sect. 3, we compute fractional Green's function. In Sect. 4, some results about Green's function and its estimation are given. Section 5 is devoted to existing results. In Sect. 6, we elaborate on stability results. Also, we have given an example to demonstrate our results in Sect. 7. The conclusion is given in Sect. 7.

## 2 Elementary results

Here, we recall some basic results of fractional integral and derivative found in [40].

**Definition 2.1** The fractional integral of order $\zeta > 0$, for an absolutely continuous function $y : (0,\infty) \to R$ is defined as

$$I^{\zeta}_{+0}y(t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} y(s)\,ds, \tag{2.1}$$

provided the integral converges at the right sides over $(0,\infty)$.

**Definition 2.2** The Caputo fractional derivative of order $\zeta > 0$, for an absolutely continuous function $y \in C^n[c,d]$ is defined as

$$\begin{aligned}&{}^{c}D^{\zeta}_{0+}y(t) = \frac{1}{\Gamma(n-\zeta)} \int_0^t (t-s)^{n-\zeta-1} \frac{d^n}{dt^n} y(s)\,ds, \quad n-1 < \zeta \le n, t \in [c,d],\\ &\text{where } n = [\zeta]+1,\end{aligned} \tag{2.2}$$

provided that the right side is point wise defined on $(0,\infty)$.

**Lemma 2.1** ([40]) *Let* $\zeta > 0$ *and if* $y$ *is absolutely continuous function, then we have*

$$I^{\zeta}_{+0}\left[{}^{c}D^{\zeta}_{+0}y(t)\right] = y(t) + C_0 + C_1 t + \cdots + C_{n-1} t^{\zeta-n}, \tag{2.3}$$

*for some constants* $C_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$.

## 3 Computation and estimation of Green's function

Now, let us establish Green's function for the following two-point BVP

$${}^{c}D^{\zeta}_{+0}u(t) + h(t) = 0, \quad t \in [c,d], \tag{3.1}$$

$$u(c) = u'(c) = 0, \quad u(d) = 0 \tag{3.2}$$

with $c \neq \beta$, $c \neq d$ and afterwards, supposing that the solution of the following three-point BVP

$${}^{c}D^{\zeta}_{+0}y(t) + h(t) = 0, \quad t \in [c,d], \tag{3.3}$$

$$y(c) = y'(c) = 0, \qquad y(d) = ky(\beta) \tag{3.4}$$

can be stated as follows

$$y(t) = u(t) + \left(\lambda_0 + \lambda_1 t + \lambda_2 t^2\right)u(\beta),$$

where $\lambda_0$, $\lambda_1$, and $\lambda_2$ are constants that will be specified later. We will estimate Green's function for (3.3) and (3.4), respectively.

**Proposition 3.1** *If $h : [c,d] \to \mathbb{R}$ is continuous mapping, then BVP (3.1) and (3.2) has a unique solution given by*

$$u(t) = \int_c^t \left[\frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} - \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\right]h(s)\,ds + \int_t^d \left[\frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}\right]h(s)\,ds,$$

*which can be expressed in compact form as*

$$u(t) = \int_c^d R(t,s)h(s)\,ds,$$

*where*

$$R(t,s) = \begin{cases} \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} - \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}, & c \le s \le t \le d, \\ \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}, & c \le t \le s \le d. \end{cases} \tag{3.5}$$

*Proof* It is well known that Problem (3.1) and (3.2) is similar to solving the integral equation

$$u(t) = c_1 + c_2(t-c) + c_3(t-c)^2 - \frac{1}{\Gamma(\zeta)}\int_c^t (t-s)^{\zeta-1}h(s)\,ds,$$

where $c_1$, $c_2$, and $c_3$ are some real constants. Using boundary conditions given in (1.4), we can obtain

$$\begin{aligned} c_1 &= \frac{c^2}{\Gamma(\zeta)(c-d)^2}\int_c^d (d-s)^{\zeta-1}h(s)\,ds, \\ c_2 &= -\frac{2c}{\Gamma(\zeta)(c-d)^2}\int_c^d (d-s)^{\zeta-1}h(s)\,ds, \\ c_3 &= \frac{1}{\Gamma(\zeta)(c-d)^2}\int_c^d (d-s)^{\zeta-1}h(s)\,ds. \end{aligned}$$

Thus, we get

$$\begin{aligned} u(t) &= \int_c^d \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}h(s)\,ds - \int_c^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}h(s)\,ds \\ &= \int_c^t \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}h(s)\,ds + \int_t^d \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}h(s)\,ds \\ &\quad - \int_c^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}h(s)\,ds \end{aligned}$$

$$= \int_c^t \left[ \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} - \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} \right] h(s)\,ds + \int_t^d \left[ \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} \right] h(s)\,ds. \tag{3.6}$$

□

The unique result exists from the assumption that the completely homogeneous BVP has only the trivial solution. So Proposition 3.1 has been proved.

**Proposition 3.2** *Let $h : [c,d] \to \mathbb{R}$ be a continuous mapping, if $k(c-\beta)^2 \neq (c-d)^2$ and $c \neq \beta$, $c \neq d$, then BVP (3.3) and (3.4) has a unique solution given by*

$$y(t) = u(t) + \frac{k(c-t)^2}{(c-d)^2 - k(c-\beta)^2} u(\beta).$$

*The solution can be written further as*

$$y(t) = \int_c^d G(t,s)h(s)\,ds,$$

*where*

$$G(t,s) = R(t,s) + \frac{k(c-t)^2}{(c-d)^2 - k(c-\beta)^2} R(\beta,s). \tag{3.7}$$

*Proof* Let

$$y(t) = u(t) + \left(\lambda_0 + \lambda_1 t + \lambda_2 t^2\right) u(\beta),$$

where $\lambda_0$, $\lambda_1$, $\lambda_2$ are constants that will be determined using boundary conditions given in (3.4) and

$$u(t) = \int_c^d R(t,s)h(s)\,ds.$$

Therefore, to compute $\lambda_0$, $\lambda_1$, $\lambda_2$, we proceed as

$$\begin{aligned}
y(c) &= u(c) + \left(\lambda_0 + \lambda_1 c + \lambda_2 c^2\right)u(\beta) = \left(\lambda_0 + \lambda_1 c + \lambda_2 c^2\right)u(\beta),\\
y'(c) &= u'(c) + (\lambda_1 + 2\lambda_2 c)u(\beta) = (\lambda_1 + 2\lambda_2 c)u(\beta),\\
y(d) &= u(d) + \left(\lambda_0 + \lambda_1 d + \lambda_2 d^2\right)u(\beta) = \left(\lambda_0 + \lambda_1 d + \lambda_2 d^2\right)u(\beta),\\
y(\beta) &= u(\beta) + \left(\lambda_0 + \lambda_1 \beta + \lambda_2 \beta^2\right)u(\beta) = u(\beta)\left(\lambda_0 + \lambda_1 \beta + \lambda_2 \beta^2 + 1\right).
\end{aligned}$$

We get

$$\begin{aligned}
&\left(\lambda_0 + \lambda_1 c + \lambda_2 c^2\right)u(\beta) = 0,\\
&(\lambda_1 + 2\lambda_2 c)u(\beta) = 0,\\
&\left(\lambda_0 + \lambda_1 d + \lambda_2 d^2\right)u(\beta) = ku(\beta)\left(\lambda_0 + \lambda_1 \beta + \lambda_2 \beta^2 + 1\right),
\end{aligned}$$

or

$$\lambda_0 + \lambda_1 c + \lambda_2 c^2 = 0,$$
$$(\lambda_1 + 2\lambda_2 c) = 0,$$
$$(1-k)\lambda_0 + (d - k\beta)\lambda_1 + \left(d^2 - k\beta^2\right)\lambda_2 = k.$$

Solving the system, we get the corresponding values as

$$\lambda_0 = \frac{c^2 k}{(c-d)^2 - k(c-\beta)^2}, \qquad \lambda_1 = \frac{-2ck}{(c-d)^2 - k(c-\beta)^2},$$
$$\lambda_2 = \frac{k}{(c-d)^2 - k(c-\beta)^2}.$$

Therefore, the final solution becomes

$$\begin{aligned} y(t) &= u(t) + \left(\frac{c^2 k}{(c-d)^2 - k(c-\beta)^2} - \frac{2ckt}{(c-d)^2 - k(c-\beta)^2} + \frac{kt^2}{(c-d)^2 - k(c-\beta)^2}\right)u(\beta) \\ &= u(t) + \frac{k(c-t)^2}{(c-d)^2 - k(c-\beta)^2}u(\beta). \end{aligned}$$

Now we derive the proof of the uniqueness. Let $z$ be also a solution to (3.3) and (3.4), that is

$${}^{c}D^{\zeta}_{+0} z(t) + h(t) = 0, \quad t \in [c,d], \tag{3.8}$$
$$z(c) = z'(d) = 0, \qquad z(d) = kz(\beta). \tag{3.9}$$

Let $\Omega(t) = z(t) - y(t)$, $t \in [c,d]$. Due to linearity property of the Caputo non-integer order derivative, we have

$${}^{c}D^{\zeta}_{+0}\Omega(t) = {}^{c}D^{\zeta}_{+0} z(t) - {}^{c}D^{\zeta}_{+0} y(t) = -h(t) + h(t) = 0, \quad t \in [c,d].$$

Therefore, $\Omega(t) = c_1 + c_2 t + c_3 t^2$, where $c_1$, $c_2$, and $c_3$ are constants that will be computed later. We have

$$\Omega(c) = z(c) - y(c),$$
$$\Omega'(c) = z'(c) - y'(c),$$
$$\Omega(d) = z(d) - y(d) = kz(\beta) - ky(\beta) = k\big(z(\beta) - y(\beta)\big) = k\Omega(\beta),$$

or

$$\Omega(c) = c_1 + c_2 c + c_3 c^2,$$
$$\Omega'(c) = c_2 + 2c_3 c,$$
$$\Omega(d) = c_1 + c_2 d + c_3 d^2 = k\left(c_1 + c_2\beta + c_3\beta^2\right) = k\Omega(\beta).$$

We get the following homogeneous system

$$\begin{cases} c_1 + c_2 a + c_3 c^2 = 0, \\ c_2 + 2c_3 c = 0, \\ c_1(1-k) + c_2(d-k\beta) + c_3\left(d^2 - k\beta^2\right) = 0, \end{cases}$$

with determinant

$$\begin{vmatrix} 1 & c & c^2 \\ 0 & 1 & 2c \\ 1-k & d-k\beta & d^2-k\beta^2 \end{vmatrix} = (c-d)^2 - k(c-\beta)^2 \neq 0.$$

Therefore, the homogeneous system contains only the trivial solution, and hence $\Omega(t) \equiv 0$, $t \in [c,d]$ or $y(t) \equiv z(t)$, $t \in [c,d]$. Thus, the proof is completed. □

## 4 Green's function estimations

**Proposition 4.1** *Let $R(t,s)$ be Green's function given in Proposition 3.1, then*

$$\int_c^d \left|R(t,s)\right| ds \le 2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)}, \quad \text{for } t \in [c,d].$$

*Proof* We deduce the proof as

$$\begin{aligned}
\int_c^d \left|R(t,s)\right| ds &\le \int_c^t \left|R(t,s)\right| ds + \int_t^d \left|R(t,s)\right| ds \\
&= \int_c^t \left| \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} - \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} \right| ds + \int_t^d \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}\, ds \\
&\le \int_c^t \left( \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} + \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} \right) ds + \int_t^d \frac{(c-t)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}\, ds \\
&= \left( \frac{(c-t)^2(d-s)^\zeta}{\Gamma(\zeta+1)(c-d)^2} + \frac{(t-s)^\zeta}{\Gamma(\zeta+1)} \right)\Big|_c^t + \left( \frac{(c-t)^2(d-s)^\zeta}{\Gamma(\zeta+1)(c-d)^2} \right)\Big|_t^d \\
&= \frac{(c-t)^2(d-t)^\zeta}{\Gamma(\zeta+1)(c-d)^2} - \frac{(c-t)^2(d-c)^\zeta}{\Gamma(\zeta+1)(d-c)^2} - \frac{(t-c)^\zeta}{\Gamma(\zeta+1)} - \frac{(c-t)^2(d-t)^\zeta}{\Gamma(\zeta+1)(c-d)^2} \\
&= -\frac{(c-t)^2(d-c)^\zeta}{\Gamma(\zeta+1)(c-d)^2} - \frac{(t-c)^\zeta}{\Gamma(\zeta+1)} \\
&\le \frac{(d-c)^2(d-c)^\zeta}{\Gamma(\zeta+1)(c-d)^2} + \frac{(d-c)^\zeta}{\Gamma(\zeta+1)} = \frac{(d-c)^\zeta}{\Gamma(\zeta+1)} + \frac{(d-c)^\zeta}{\Gamma(\zeta+1)} = 2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)}.
\end{aligned}$$

□

The next result is also important in our study.

**Proposition 4.2** *Green's function $G(t,s)$ given in Proposition 3.2 satisfies the following inequality*

$$\int_c^d \left|G(t,s)\right| ds \le 2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)} + 2\frac{|k|}{\Gamma(\zeta+1)} \frac{(d-c)^{\zeta+2}}{|(d-c)^2 - k(\beta-c)^2|},$$

*for $t \in [c,d]$.*

*Proof* Here we derive the proof as

$$\begin{aligned}\int_c^d \left|G(t,s)\right| ds &= \int_c^d \left|R(t,s) + \frac{k(c-t)^2}{(c-d)^2 - k(c-\beta)^2} R(\beta,s)\right| ds \\ &\le \int_c^d \left|R(t,s)\right| ds + \left|\frac{k(c-t)^2}{(c-d)^2 - k(c-\beta)^2}\right| \int_c^d \left|R(\beta,s)\right| ds \\ &\le 2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)} + \frac{|k(c-t)^2|}{|(c-d)^2 - k(c-\beta)^2|} 2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)} \\ &\le 2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)} + 2\frac{|k|}{\Gamma(\zeta+1)} \frac{(d-c)^{\zeta+2}}{|(d-c)^2 - k(\beta-c)^2|}.\end{aligned}$$

□

## 5 Existence results for the solution

For further correspondence in this work, we define the following:

$$G(t,s) = R(t,s) + \frac{k(c-t)^2}{(c-d)^2 - k(c-\beta)^2} R(\beta,s).$$

For further analysis, we also denote the Banach space by $X = C[c,d]$ under the norm $\|y\| = \sup_{t\in[c,d]} |y(t)|$. Hence we define the operator $T : X \to X$ by

$$Ty(t) = \int_c^d G(t,s)\Lambda\bigl(s,y(s)\bigr)\, ds, \quad t \in [c,d]. \tag{5.1}$$

The following hypothesis needed also to be held:

($H_1$) Let $\Lambda : [c,d] \times \mathbb{R} \to \mathbb{R}$ is continuous and follows the uniform Lipschitz inequality for $y$ on $[c,d] \times \mathbb{R}$, such that there exists a constant $L$, for all $(t,y),(t,z) \in [c,d] \times \mathbb{R}$,

$$\bigl|\Lambda(t,y) - \Lambda(t,z)\bigr| \le L|y-z|.$$

**Theorem 5.1** *Under the hypothesis* ($H_1$) *and if the condition* $(d-c)^2 \neq k(c-\beta)^2$, *with* $c \neq \beta$ *holds and* $d-c$ *is sufficiently small, such that*

$$2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)} + 2\frac{|k|}{\Gamma(\zeta+1)} \frac{(d-c)^{\zeta+2}}{|(d-c)^2 - k(\beta-c)^2|} < \frac{1}{L}, \tag{5.2}$$

*then there exists a unique solution to* (1.3) *and* (1.4).

*Proof* Note that $y$ is a solution to (1.3) and (1.4) if and only if it is a solution to (3.3) and (3.4) with $h(t)$ replaced by $\Lambda(t,y(t))$. However, (3.3) and (3.4) have a unique solution given by

$$y(t) = \int_c^d G(t,s)\Lambda\bigl(s,y(s)\bigr)\, ds, \tag{5.3}$$

where $G(t,s)$ is specified in Proposition 3.2. Define the operator $T : X \to X$ by

$$Ty(t) = \int_c^d G(t,s)\Lambda\bigl(s,y(s)\bigr)\, ds, \quad t \in [c,d]. \tag{5.4}$$

We will apply the Banach fixed point theorem to determine whether the operator $T$ has a unique fixed point. Assume that $y, z \in X$, then for $t \in [c, d]$, we have

$$\begin{aligned}
\left|Ty(t) - Tz(t)\right| &= \left|\int_c^d G(t,s)\big(\Lambda\big(s,y(s)\big) - \Lambda\big(s,z(s)\big)\big)\,ds\right| \\
&\le \int_c^d \left|G(t,s)\right|\left|\Lambda\big(s,y(s)\big) - \Lambda\big(s,z(s)\big)\right|\,ds \\
&\le \int_c^d \left|G(t,s)\right|L\left|y(s) - z(s)\right|\,ds \\
&\le L\int_c^d \left|G(t,s)\right|\,ds\|y - z\| \\
&\le L\left[2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)} + 2\frac{|k|}{\Gamma(\zeta+1)}\frac{(d-c)^{\zeta+2}}{|(d-c)^2 - k(\beta-c)^2|}\right]\|x - y\|,
\end{aligned}$$

where we have utilized Proposition 4.1. It agrees that

$$\|Ty - Tz\| \le \Omega L\|y - z\|,$$

where

$$\Omega = \left[2\frac{(d-c)^\zeta}{\Gamma(\zeta+1)} + 2\frac{|k|}{\Gamma(\zeta+1)}\frac{(d-c)^{\zeta+2}}{|(d-c)^2 - k(\beta-c)^2|}\right]. \tag{5.5}$$

Hence, we deduce that $T$ is a contraction mapping on $X$, and from the Banach contraction mapping theorem, we receive the required result. □

For next result, the following hypothesis need to be hold:

$(H_2)$ For $(i = 1, 2)$, let there exists $\varphi_i : [c, d] \to R$ and $y \in X$, such that

$$\left|\Lambda\big(t,y(t)\big)\right| \le \varphi_1(t) + \varphi_2(t)|y|, \quad t \in [c,d].$$

Further putting $\varphi_1^* = \sup_{t\in[c,d]}|\varphi_1(t)|$ and $\varphi_2^* = \sup_{t\in[c,d]}|\varphi_2(t)|$.

**Theorem 5.2** *Under the hypothesis $(H_2)$, if $T : B \to B$ is a completely continuous operator, then $T$ has at least one fixed point in $B_r$.*

*Proof* Let $X$ be a Banach space and $B_r \subset X$ be a bounded closed convex subset.

*Step 1*: First, we are going to show that operator $T$ is continuous.

Let $\{y_n\}$ be a sequence, such that $B_r = \{y \in X : \|y\| \le r\}$, then, for $t \in [c, d]$, we have

$$\begin{aligned}
\|Ty_n - Ty\| &\le \sup_{t\in[c,d]}\int_c^d \left|G(t,s)\right|\left|\Lambda\big(s,y_n(s)\big) - \Lambda\big(s,\bar{y}(s)\big)\right|\,ds, \\
&\le 2\left[\frac{(d-c)^\zeta}{\Gamma(\zeta+1)} + \frac{|k|}{\Gamma(\zeta+1)}\frac{(d-c)^{\zeta+1}}{(d-c)^2 - (\beta-c)^2}\right]\|y_n - y\| \\
&= \Omega L\|y_n - y\|,
\end{aligned}$$

where $\Omega$ is given in (5.5).

$$y_n \to y \quad \text{as } n \to \infty,$$

then we have

$$\|Ty_n - Ty\| \to 0, \quad \text{as } n \to \infty.$$

Hence $T$ is continuous.

*Step 2*: Next, to show that $T$ is bounded means maps bounded sets to bounded sets on $X$. Let $y \in B_r$, then for $t \in [c,d]$, we have

$$\begin{aligned}\|Ty(t)\| &\leq \sup_{t\in[c,d]} \int_c^d \bigl|G(t,s)\|\Lambda\bigl(s,y(s)\bigr)\bigr|\,ds \\ &\leq \Omega \operatorname*{Sup}_{t\in[c,d]} \int_c^d \bigl[\bigl|\varphi_1(s)\bigr| + \bigl|\varphi_2(s)\bigr||y|\bigr]\,ds \\ &\leq \Omega \int_c^d |\bigl[\varphi_1^*| + |\varphi_2^*|\bigr]\,ds \\ &= \Omega\bigl(\varphi_1^* + \varphi_2^* r\bigr)(d-c) = \epsilon^*.\end{aligned}$$

Hence one has

$$\|Ty\| \leq \epsilon^*.$$

Thus, $T$ is a bounded operator.

*Step 3*: Now, we are going to show that $T$ is equi-continuous. Let $t_1$ and $t_2 \in [c,d]$, then

$$\begin{aligned}&\bigl\|Ty(t_2) - Ty(t_1)\bigr\| \\ &\quad\leq \int_c^d \bigl|G(t_2,s) - G(t_1,s)\bigr|\bigl|\Lambda\bigl(s,y(s)\bigr)\bigr|\,ds \\ &\quad\leq \int_c^{t_2} \biggl[\frac{(c-t_2)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} - \frac{(t_2-s)^{\zeta-1}}{\Gamma(\zeta)}\biggr]\bigl[\varphi_1^* + \varphi_2^* r\bigr]\,ds \\ &\qquad + \int_{t_2}^d \frac{(c-t_2)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}\bigl[\varphi_1^* + \varphi_2^* r\bigr]\,ds \\ &\qquad + \frac{k(c-t_2)^2}{(c-d)^2 - k(c-\beta)^2}\int_c^d \bigl|R(\beta,s)\bigr|\bigl|\varphi_1^* + \varphi_2^* r\bigr|\,ds \\ &\qquad - \int_c^{t_1} \biggl[\frac{(c-t_1)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} - \frac{(t_1-s)^{\zeta-1}}{\Gamma(\zeta)}\biggr]\bigl[\varphi_1^* + \varphi_2^* r\bigr]\,ds \\ &\qquad - \int_{t_1}^d [\frac{(c-t_1)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}\bigl[\varphi_1^* + \varphi_2^* r\bigr]\,ds \\ &\qquad - \frac{k(c-t_1)^2}{(c-d)^1 - k(c-\beta)^2}\int_c^d \bigl|R(\beta,s)\bigr|\bigl|\varphi_1^* + \varphi_2^* r\bigr|\,ds \\ &\quad= \bigl[\varphi_1^* + \varphi_2^* r\bigr]\biggl(\int_c^{t_2} \frac{(c-t_2)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2} - \int_c^{t_1} \frac{(c-t_1)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}\biggr)\end{aligned}$$

$$
\begin{aligned}
&+ \left[\varphi_1^* + \varphi_2^* r\right]\left(\int_c^{t_1} \frac{(t_1 - s)^{\zeta-1}}{\Gamma(\zeta)} - \int_c^{t_2} \frac{(t_2 - s)^{\zeta-1}}{\Gamma(\zeta)}\right) \\
&+ \left[\varphi_1^* + \varphi_2^* r\right]\left(\int_{t_2}^{d} \frac{(c-t_2)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}\,ds - \int_{t_1}^{d} \frac{(c-t_1)^2(d-s)^{\zeta-1}}{\Gamma(\zeta)(c-d)^2}\right) \\
&+ \frac{k}{(c-d)^2 - k(c-\beta)^2}\left[(c-t_2)^2 - (c-t_1)^2\right]\int_c^d \left|R(\beta,s)\right|\left[\varphi_1^* + \varphi_2^* r\right]ds.
\end{aligned}
$$

On further simplification, one has

$$
\begin{aligned}
\left\| Ty(t_2) - Ty(t_1)\right\| \le{}& \frac{[\varphi_1^* + \varphi_2^* r]}{\Gamma(\zeta+1)(c-d)^2}\left[(c-t_2)^2\left[(d-c)^\zeta - (d-t_2)^\zeta\right]\right. \\
&+ (c-t_1)^2\left[(d-t_1)^\zeta - (d-c)^\zeta\right]\Big] \\
&+ \frac{[\varphi_1^* + \varphi_2^* r]}{\Gamma(\zeta+1)}\left[(t_1-c)^c - (t_2-c)^c)\right] \\
&+ \frac{[\varphi_1^* + \varphi_2^* r]}{\Gamma(\zeta+1)(c-d)^2}\left[(c-t_2)^2(d-t_2)^\zeta - (c-t_1)^2\left[(d-t_1)^\zeta\right]\right. \\
&+ \frac{k[\varphi_1^* + \varphi_2^* r]}{(c-d)^2 - k(c-\beta)^2}\left[(c-t_2)^2 - (c-t_1)^2\right]\int_c^d \left|R(\beta,s)\right|ds.
\end{aligned}
$$

We see that as $t_1 \to t_2$, then the right-hand side tends to 0. As $T$ is bounded and continuous on $B_r$ therefore is uniformly continuous.

Hence we claim that

$$
\left\| Ty(t_2) - Ty(t_1)\right\| \to 0 \quad \text{as } t_1 \to t_2.
$$

Thus, using Arzelá Arcoli theorem, one can say that operator $T$ is relatively compact, bounded, and uniformly continuous. Thus, $T$ is the completely continuous operator. Hence, $T$ has at least one fixed point. Therefore, the considered problem has at least one solution. □

## 6 Stability results

Here we describe some stability results. The said stability results are based on the H-U concept.

*Remark* 6.1 We consider a mapping $\phi$ independent of $y$, such that $|\phi(t)| \le \epsilon$, for every, $t \in [c,d]$.

**Theorem 6.1** *The solution to the following perturbed problem*

$$
{}^cD^\zeta y(t) = \Lambda\left(t, y(t)\right) + \phi(t), \quad t \in [c,d], 2 < \zeta \le 3 \tag{6.1}
$$

*satisfies the following relation*

$$
\left| y(t) - \int_c^d G(t,s)\Lambda\left(s, y(s)\right)ds\right| \le \Omega\epsilon, \tag{6.2}
$$

*where* $\Omega$ *has been given above in* (5.5).

*Proof* In view of Lemma 2.1, the solution is given by

$$y(t) = \int_c^d G(t,s)\Lambda\big(s,y(s)\big)\,ds + \int_c^d G(t,s)\phi(s)\,ds. \tag{6.3}$$

Further, we have for $t \in [c,d]$

$$\begin{aligned}\left|y(t) - \int_c^d G(t,s)\Lambda\big(s,y(s)\big)\,ds\right| &\le \int_c^d \big|G(t,s)\|\phi(s)\big|\,ds \\ &\le 2\left|\frac{(d-c)^2}{\Gamma(\zeta+1)} + \frac{|k|}{\Gamma(\zeta+1)}\frac{(d-c)^{\zeta+2}}{[(d-c)^2 - k(\beta-c)^2]}\right|\epsilon \\ &= \Omega\epsilon. \end{aligned}$$

□

**Theorem 6.2** *In view of Theorem 6.1 and assumption ($H_1$), the solution of Problem (1.3) and (1.4) is H-U stable and in the same line will be g-H-U stable if and only if the condition $\Omega < \frac{1}{L}$ holds.*

*Proof* Let $y$ be any solution and $x$ be the unique solution of Problem (1.3) and (1.4), such that $x, y \in X$, then consider

$$\begin{aligned}\|x-y\| &= \sup_{t\in[c,d]}\left|x(t) - \int_c^d G(t,s)\Lambda\big(s,y(s)\big)\,ds\right| \\ &\le \sup_{t\in[c,d]}\left|x(t) - \int_c^d G(t,s)\Lambda\big(s,x(s)\big)\,ds\right| \\ &\quad + \sup_{t\in[c,d]}\left|\int_c^d G(t,s)\Lambda\big(s,x(s)\big)\,ds - \int_c^d G(t,s)\Lambda\big(s,y(s)\big)\,ds\right| \\ &\le \Omega\epsilon + L\Omega\|x-y\|.\end{aligned}$$

Hence

$$\|x-y\| \le \frac{\Omega}{1-L\Omega}\epsilon,$$

where $C_{L,\Omega} = \frac{\Omega}{1-L\Omega}$.

Forth, if we have a non-decreasing function $\psi : [c,d] \to R^+$, then

$$\|x-y\| \le C_{L,\Omega}\psi(\epsilon), \quad \text{where } \psi(\epsilon) = \epsilon,$$

where $\psi(0) = 0$. So, the solution to the considered problem is also g-H-U stable. □

## 7 Illustrative example

Here to support our findings, we present the following example.

*Example* 7.1 Consider the following nonlocal nonlinear BVP of FDEs as

$$\begin{cases} {}^cD_{+0}^{\frac{27}{10}}y(t) + 2 + t^2 + \frac{y^2}{2(y^2+1)} = 0, & t \in [0,1], \\ y(0) = y'(0) = 0, & y(1) = \frac{3}{4}y\left(\frac{1}{2}\right). \end{cases} \tag{7.1}$$

We see from (7.1) that

$$\Lambda(t,y) = 2 + t^2 + \frac{y^2}{2(y^2+1)}, \qquad \Lambda(t,0) = 2 + t^2 \neq 0,$$

and

$$\begin{aligned}
\left|\Lambda(t,y_1) - \Lambda(t,y_2)\right| &= \left|\frac{y_1^2}{2(y_1^2+1)} - \frac{y_2^2}{2(y_2^2+1)}\right| \\
&= \left|\frac{y_1^2(y_2^2+1) - y_2^2(y_1^2+1)}{2(y_1^2+1)(y_2^2+1)}\right| \\
&= \left|\frac{(y_1^2 - y_2^2)}{2(y_1^2+1)(y_2^2+1)}\right| \\
&= \left|\frac{(y_1 - y_2)(y_1 + y_2)}{2(y_1^2+1)(y_2^2+1)}\right| \\
&= \frac{1}{2}|y_1 - y_2|\left|\frac{y_1 + y_2}{(y_1^2+1)(y_2^2+1)}\right| \\
&= \frac{1}{2}|y_1 - y_2|\left|\frac{y_1}{(y_1^2+1)(y_2^2+1)} + \frac{y_2}{(y_1^2+1)(y_2^2+1)}\right| \\
&< \frac{1}{2}|y_1 - y_2|\left|\frac{y_1}{y_1^2+1} + \frac{y_2}{y_2^2+1}\right| \\
&< \frac{1}{2}|y_1 - y_2|(2) \\
&= |y_1 - y_2|.
\end{aligned}$$

So, $\Lambda$ is a Lipschitz with respect to $y$ on $[0,1] \times \mathbb{R}$ with the Lipschitz constant $L = 1$.

Since $(d-c)^2 = 1 \neq \frac{3}{16} = k(c-\beta)^2$ and

$$\begin{aligned}
2\frac{(d-c)^{\frac{27}{10}}}{\Gamma(\frac{27}{10}+1)} + 2\frac{|k|}{\Gamma(\frac{27}{10}+1)}\frac{(d-c)^{\frac{27}{10}+2}}{|(d-c)^2 - k(\beta-c)^2|} &= \frac{2}{\Gamma(\frac{37}{10})} + \frac{24}{13\Gamma(\frac{37}{10})} \\
&= \frac{50}{13\Gamma(\frac{37}{10})} \cong 0.922195 < \frac{1}{L} = 1.
\end{aligned}$$

Now in view of Theorem 5.1, Problem (7.1) has a unique solution. The graph of the solution $y(t)$ is displayed in Fig. 1. Note that the solution has been obtained here by the generalized differential transform method, which is a very effective tool to give semi-analytical solutions for FDEs (see for details [41]). Also, the condition of H-U stability and g-H-U is obvious. The graphical presentation of the approximate solution is given in Fig. 1.

## 8 Conclusion

We have given some sufficient conditions that demonstrate the existence and uniqueness of the solution for a non-integer order three-point nonlocal BVP of FDEs. Some pertinent results regarding H-U and g-H-U stability have been incorporated. Thanks to the fixed point approach and nonlinear functional analysis, these findings have been established. For validation of our results, an interesting example has also been given. From the mentioned discussion and results, we conclude that fixed point theory is a powerful tool

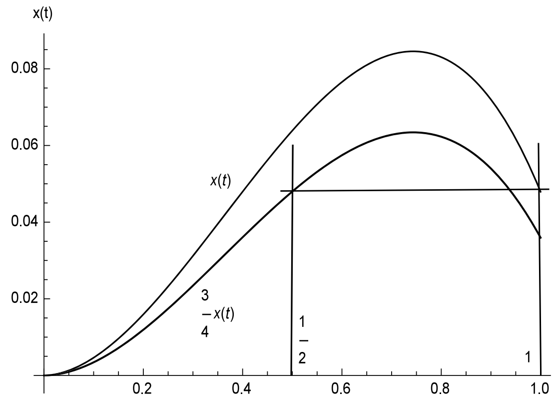


**Figure 1** Graphical presentation of approximate solution of Problem (7.1)

to deal with nonlinear problems of FDEs corresponding to different initial and boundary conditions.

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**Competing interests**
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