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On a viscous fourth-order parabolic equation with boundary degeneracy

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Abstract

A viscous fourth-order parabolic equation with boundary degeneracy is studied. By using the variational method, the existence of a time-discrete fourth-order elliptic equation with homogeneous boundary conditions is solved. Moreover, the existence and uniqueness for the corresponding parabolic problem with nondegenerate coefficient is shown by several asymptotic limit processes. Finally, by applying the regularization method, the existence and uniqueness for the problem with degenerate boundary coefficient is obtained by applying the energy method and a small parameter limit process.

MSC: 35G20; 35G30; 35G61

Keywords: Fourth-order degenerate; Boundary degeneracy; Double degeneracy

1 Introduction

Many physical phenomena can be described by nonlinear fourth-order parabolic equations. The Cahn–Hilliard equation can be used to establish the model for phase transformation theory (see [3]). The degenerate fourth-order parabolic equations can show the motion of a very thin layer of a viscous compressible fluid (see [2, 12], and [8]). Specially, in materials science, the epitaxial growth of nanoscale thin films can be given by nonlinear fourth-order parabolic equations (see [23] and [6]).

For the research of fourth-order parabolic equations, Liu [10] studied a Cahn–Hilliard equation with a zero-mass flux boundary condition, and the global existence of classical solutions with a nondegenerate $m(w)$ and small initial energy was shown. Xu and Zhou [18] considered a nonlinear fourth-order parabolic equation with gradient degeneracy, and the corresponding existence of weak solutions was studied in the sense of distribution. For the nonlinear source problem, the existence and asymptotic behavior of solutions were given by Liang and Zheng in [9]. In the paper, we consider a viscous fourth-order parabolic equation with boundary degeneracy conditions. For the boundary degeneracy problem, there have been some research results about second-order equations. Yin and Wang (see [21] and [22]) gave the existence of weak solutions for a second-order singular diffusion problem, and the corresponding diffusion coefficients were allowed to degenerate on a portion of the boundary. For the boundary degeneracy problem with a gradient flow, Zhan in [24] obtained the existence and stability of solutions.

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In the paper, a viscous fourth-order parabolic equation with boundary degeneracy is considered. If we drop the viscous term, the model can be treated as a thin film equation with a degenerate mobility rate. If the fourth-order diffusion term is replaced by a classic second-order diffusion, it often appeared in the research for pseudo-parabolic equations. For their research works, Xu and Su in [19] considered the initial-boundary value problem for a semi-linear pseudo-parabolic equation, and the corresponding global existence and finite time blow-up of solutions were given by potential well theory. In [7], a pseudo-parabolic equation with a singular potential was shown. Moreover, the papers [20] and [16] studied the related nonlinear parabolic systems with power type source terms and time-fractional pseudo-parabolic problems. For the other references, the readers may refer to [4, 11, 13], and [14].

Our research problem with initial-boundary conditions has the following form:

$$w_t - \gamma\triangle w_t + \triangle\big(\varrho^{\alpha}(x)|\triangle w|^{p-2}\triangle w\big) = 0, \quad (x,t)\in Q_T, \tag{1.1}$$

$$w = \triangle w = 0, \quad (x,t)\in\Gamma, \tag{1.2}$$

$$w(x,0) = w_I(x), \quad x\in\Omega, \tag{1.3}$$

where $\Omega\subset R^N$ is a bounded domain with $N\le 2$, $Q_T = \Omega\times(0,T)$, and $\Gamma = \partial\Omega\times(0,T)$. $\alpha>0$, $p>1$, and $\gamma>0$ are all constants. In physics, the capillarity-driven surface diffusion is from the term $\triangle(\varrho^{\alpha}(x)|\triangle w|^{p-2}\triangle w)$ (see Zangwill [23]). Here the function $\varrho(x)$ is defined by $\varrho = \operatorname{dist}(x,\partial\Omega)$, which can yield the degeneration at $\partial\Omega$. $\gamma>0$ is the viscosity coefficient. We always suppose that the boundary $\partial\Omega$ is smooth enough and simple enough. Besides, for any constant $\sigma\in(0,1)$, the domain Ω satisfies the condition $\int_\Omega \varrho^{-\sigma}\,\mathrm{d}x<\infty$. The term $\gamma\triangle w_t$ denotes the viscous relaxation factor or viscosity.

In order to obtain the existence of weak solutions for (1.1)–(1.3), we need to deal with the degenerate coefficient $\varrho(x)$, and so we introduce the following approximate problem:

$$w_{\varepsilon t} - \gamma\triangle w_{\varepsilon t} + \triangle\big(\varrho_\varepsilon^{\alpha}|\triangle w_\varepsilon|^{p-2}\triangle w_\varepsilon\big) = 0, \quad (x,t)\in Q_T, \tag{1.4}$$

$$w_\varepsilon = \triangle w_\varepsilon = 0, \quad (x,t)\in\Gamma, \tag{1.5}$$

$$w_\varepsilon(x,0) = w_{\varepsilon I}(x), \quad x\in\Omega, \tag{1.6}$$

where $\varrho_\varepsilon = \varrho+\varepsilon$ with $\varepsilon>0$. From the existence of (1.4)–(1.6), we can conclude the existence of (1.1)–(1.3) by a limit process for $\varepsilon\to 0$.

The weak solution of (1.4)–(1.6) is shown in the following definition.

Definition 1 If a function w_ε satisfies the conditions

(i) $w_\varepsilon\in C([0,T];H^1(\Omega))\cap L^\infty(0,T;W_0^{2,p}(\Omega))$, $w_{\varepsilon t}\in L^2(0,T;H^1(\Omega))$, $\varrho_\varepsilon^{\alpha}|\triangle w_\varepsilon|^p\in L^1(Q_T)$ with $W_0^{2,p}(\Omega)\doteq W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)$;

(ii) For each $\varphi\in C_0^\infty(Q_T)$, it has

$$\begin{aligned}&\iint_{Q_T}\frac{\partial w_\varepsilon}{\partial t}\varphi\,\mathrm{d}x\,\mathrm{d}t + \gamma\iint_{Q_T}\nabla w_{\varepsilon t}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\iint_{Q_T}\varrho_\varepsilon^{\alpha}|\triangle w_\varepsilon|^{p-2}\triangle w_\varepsilon\triangle\varphi\,\mathrm{d}x\,\mathrm{d}t = 0,\end{aligned}$$

then it is called a weak solution of (1.4)–(1.6).

Its existence is shown in the following proposition.

Proposition 1 *Let $w_{\varepsilon I} \in W_0^{2,p}(\Omega)$. Problem (1.4)–(1.6) owns a unique weak solution.*

For (1.1)–(1.3), its weak solutions are defined as follows.

Definition 2 If a function w satisfies the conditions
(i) $w \in C([0,T];H^1(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$, $w_t \in L^2(0,T;H^1(\Omega))$, $\varrho^\alpha|\triangle w|^p \in L^1(Q_T)$, $\triangle w \in L^p_{\mathrm{loc}}(Q_T)$;
(ii) For each $\varphi \in C_0^\infty(Q_T))$, it has

$$\iint_{Q_T} \frac{\partial w}{\partial t}\varphi \,\mathrm{d}x\,\mathrm{d}t + \gamma \iint_{Q_T} \nabla w_t \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \varrho^\alpha |\triangle w|^{p-2}\triangle w \triangle \varphi \,\mathrm{d}x\,\mathrm{d}t = 0,$$

then it is called a weak solution of (1.1)–(1.3).

The main existence is as follows.

Theorem 1 *Let $w_I \in W_0^{2,p}(\Omega)$ and $\alpha < p-1$. Problem (1.1)–(1.3) has a unique weak solution.*

In the paper, C, C_j $(j = 1, 2, \ldots)$ represent general constants, and the values may change from line to line. The paper is organized as follows. Section 2 gives the existence, uniqueness, and iterative estimates for the semi-discrete elliptic problem. In Sect. 3, we show the existence and uniqueness for the nondegenerate parabolic problem. The final section establishes the existence and uniqueness for the degenerate problem.

2 Elliptic problem

In this section, we introduce a semi-discrete problem, and some important iterative estimates are established. For the time interval $[0,T]$, we make it into n subintervals with the equal width $h = \frac{T}{n}$. Let $w_i = w(x, ih)$ and $w_0 = w_{\varepsilon I}$ for $i = 1, 2, \ldots, n$. We get the semi-discrete elliptic problem

$$\frac{1}{h}(w_i - w_{i-1}) - \frac{\gamma}{h}(\triangle w_i - \triangle w_{i-1}) + \triangle\big(\varrho_\varepsilon^\alpha |\triangle w_i|^{p-2}\triangle w_i\big) = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$w_i = \triangle w_i = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

We will use the variational method to study the existence of (2.1)–(2.2), and so we define the functional as follows:

$$\begin{aligned} \mathcal{K}[w_i] = {} & \frac{1}{2h}\int_\Omega |w_i|^2 \,\mathrm{d}x + \frac{\gamma}{2h}\int_\Omega |\nabla w_i|^2 \,\mathrm{d}x + \frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_i|^p \,\mathrm{d}x \\ & - \frac{\gamma}{h}\int_\Omega \nabla w_{i-1}\nabla w_i \,\mathrm{d}x - \frac{1}{h}\int_\Omega w_{i-1} w_i \,\mathrm{d}x \end{aligned} \tag{2.3}$$

for $w_i \in W_0^{2,p}(\Omega)$.

The corresponding existence result is shown in the following lemma.

Lemma 1 *For fixed $\varepsilon > 0$ and $w_{i-1} \in W_0^{2,p}(\Omega)$, problem (2.1)–(2.2) has a unique weak solution $w_i \in W_0^{2,p}(\Omega)$ with*

$$\frac{1}{h}\int_\Omega (w_i - w_{i-1})\varphi\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega (\nabla w_i - \nabla w_{i-1})\nabla\varphi\,\mathrm{d}x + \int_\Omega \varrho_\varepsilon^\alpha |\triangle w_i|^{p-2}\triangle w_i \triangle\varphi\,\mathrm{d}x = 0 \tag{2.4}$$

for each $\varphi \in C_0^\infty(\Omega)$. Moreover, it has

$$\frac{1}{h}\int_\Omega |w_i|^2\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega |\nabla w_i|^2\,\mathrm{d}x + 2\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_i|^p\,\mathrm{d}x \le \frac{1}{h}\int_\Omega |w_{i-1}|^2\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega |\nabla w_{i-1}|^2\,\mathrm{d}x, \tag{2.5}$$

$$\int_\Omega |w_j|^2\,\mathrm{d}x + \gamma\int_\Omega |\nabla w_j|^2\,\mathrm{d}x + 2h\sum_{i=1}^{j}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_i|^p\,\mathrm{d}x \le \int_\Omega |w_0|^2\,\mathrm{d}x + \gamma\int_\Omega |\nabla w_0|^2\,\mathrm{d}x, \tag{2.6}$$

$$\frac{1}{h}\int_\Omega (w_i - w_{i-1})^2\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega (\nabla w_i - \nabla w_{i-1})^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_i|^p\,\mathrm{d}x \le \frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_{i-1}|^p\,\mathrm{d}x, \tag{2.7}$$

$$\frac{1}{h}\int_\Omega (w_i - w_{i-1})^2\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega (\nabla w_i - \nabla w_{i-1})^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_i|^p\,\mathrm{d}x \le \frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_0|^p\,\mathrm{d}x, \tag{2.8}$$

$$\sum_{i=1}^{n}\frac{1}{h}\int_\Omega (w_i - w_{i-1})^2\,\mathrm{d}x + \sum_{i=1}^{n}\frac{\gamma}{h}\int_\Omega (\nabla w_i - \nabla w_{i-1})^2\,\mathrm{d}x \le \frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_0|^p\,\mathrm{d}x \tag{2.9}$$

for $i,j = 1,\dots,n$.

Proof Young's inequality can give

$$\begin{aligned}\mathcal{K}[w_i] &\ge \frac{1}{2h}\int_\Omega |w_i|^2\,\mathrm{d}x + \frac{\gamma}{2h}\int_\Omega |\nabla w_i|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_i|^p\,\mathrm{d}x - \frac{\gamma}{2h}\int_\Omega |\nabla w_{i-1}|^2\,\mathrm{d}x \\ &\quad - \frac{\gamma}{2h}\int_\Omega |\nabla w_i|^2\,\mathrm{d}x - \frac{1}{2h}\int_\Omega |w_{i-1}|^2\,\mathrm{d}x - \frac{1}{2h}\int_\Omega |w_i|^2\,\mathrm{d}x \\ &\ge -\frac{\gamma}{2h}\int_\Omega |\nabla w_{i-1}|^2\,\mathrm{d}x - \frac{1}{2h}\int_\Omega |w_{i-1}|^2\,\mathrm{d}x,\end{aligned} \tag{2.10}$$

and thus $\mathcal{K}[w_i]$ is bounded

$$-C \le \inf_{v\in W_0^{2,p}(\Omega)} \mathcal{K}[v] \le \mathcal{K}[0] = 0.$$

It ensures the existence of a subsequence $\{w_{kl}\}_{l=1}^{\infty} \subset W_0^{2,p}(\Omega)$ and a function v such that

$$\mathcal{K}[w_{kl}] \to \inf_{v\in W_0^{2,p}(\Omega)} \mathcal{K}[v], \tag{2.11}$$

as $l \to +\infty$. Using Young's inequality again, we have

$$\begin{aligned}
&\frac{1}{2h}\int_\Omega |w_{kl}|^2\,\mathrm{d}x + \frac{\gamma}{2h}\int_\Omega |\nabla w_{kl}|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha |\triangle w_{kl}|^p\,\mathrm{d}x \\
&\quad\le \bigl|\mathcal{K}[w_{kl}]\bigr| + \left|\frac{\gamma}{h}\int_\Omega \nabla w_{i-1}\nabla w_{kl}\,\mathrm{d}x\right| + \left|\frac{1}{h}\int_\Omega w_{i-1}w_{kl}\,\mathrm{d}x\right| \\
&\quad\le \bigl|\mathcal{K}[w_{kl}]\bigr| + \frac{\gamma}{4h}\int_\Omega |\nabla w_{kl}|^2\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega |\nabla w_{i-1}|^2\,\mathrm{d}x \\
&\qquad + \frac{1}{4h}\int_\Omega |w_{kl}|^2\,\mathrm{d}x + \frac{1}{h}\int_\Omega |w_{i-1}|^2\,\mathrm{d}x.
\end{aligned}$$

Since $\mathcal{K}[w_{kl}]$ is bounded, it has

$$\begin{aligned}
&\frac{1}{4h}\int_\Omega |w_{kl}|^2\,\mathrm{d}x + \frac{\gamma}{4h}\int_\Omega |\nabla w_{kl}|^2\,\mathrm{d}x + \frac{\varepsilon^\alpha}{p}\int_\Omega |\triangle w_{kl}|^p\,\mathrm{d}x \\
&\quad\le \bigl|\mathcal{K}[w_{kl}]\bigr| + \frac{\gamma}{h}\int_\Omega |\nabla w_{i-1}|^2\,\mathrm{d}x + \frac{1}{h}\int_\Omega |w_{i-1}|^2\,\mathrm{d}x \\
&\quad\le C.
\end{aligned}$$

It implies the estimate $\|w_{kl}\|_{W_0^{2,p}(\Omega)} \le C$, and then we can seek a subsequence from $\{w_{kl}\}$ and a function $w_i \in W_0^{2,p}(\Omega)$ so that

$$w_{kl} \rightharpoonup w_i \quad \text{weakly in } W_0^{2,p}(\Omega)$$

as $l \to \infty$.

The weak lower semi-continuity yields

$$\mathcal{K}[w_i] \le \liminf_{l\to\infty} \mathcal{K}[w_{kl}] = \inf_{v\in W_0^{2,p}(\Omega)} \mathcal{K}[v],$$

and then $\mathcal{K}[w_i] = \inf_{v\in W_0^{2,p}(\Omega)} \mathcal{K}[v]$. A standard procedure can show the existence of (2.1)–(2.2) (see [17] or [5]).

For the uniqueness, we suppose that w_{i1} and w_{i2} are two weak solutions, and we choose $w_{i1} - w_{i2}$ as the test function to get

$$\begin{aligned}
&\frac{1}{h}\int_\Omega (w_{i1} - w_{i2})^2\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega (\nabla w_{i1} - \nabla w_{i2})^2\,\mathrm{d}x \\
&\quad = -\int_\Omega \varrho_\varepsilon^\alpha \bigl(|\triangle w_{i1}|^{p-2}\triangle w_{i1} - |\triangle w_{i2}|^{p-2}\triangle w_{i2}\bigr)(\triangle w_{i1} - \triangle w_{i2})\,\mathrm{d}x \le 0.
\end{aligned}$$

Notice that, for arbitrary numbers ζ and η, the inequality

$$\bigl(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta\bigr)(\zeta - \eta) \ge 0 \tag{2.12}$$

holds if $p > 1$. Thus, one has $w_{i1} = w_{i2}$ a.e. in Ω.

To give the proof for the iterative estimates, we take w_i as the test function and apply Young's inequality to find

$$\begin{aligned}&\frac{1}{h}\int_\Omega |w_i|^2\,\mathrm{d}x+\frac{\gamma}{h}\int_\Omega |\nabla w_i|^2\,\mathrm{d}x+\int_\Omega \varrho_\varepsilon^\alpha|\triangle w_i|^p\,\mathrm{d}x\\&\quad=\frac{1}{h}\int_\Omega w_{i-1}w_i\,\mathrm{d}x+\frac{\gamma}{h}\int_\Omega \nabla w_{i-1}\nabla w_i\,\mathrm{d}x\\&\quad\le\frac{1}{2h}\int_\Omega |w_{i-1}|^2\,\mathrm{d}x+\frac{1}{2h}\int_\Omega |w_i|^2\,\mathrm{d}x+\frac{\gamma}{2h}\int_\Omega |\nabla w_{i-1}|^2\,\mathrm{d}x+\frac{\gamma}{2h}\int_\Omega |\nabla w_i|^2\,\mathrm{d}x.\end{aligned}$$

Thus, (2.5) and (2.6) have been shown. Meanwhile, taking $w_i - w_{i-1}$ as the test function, we have

$$\begin{aligned}&\frac{1}{h}\int_\Omega (w_i-w_{i-1})^2\,\mathrm{d}x+\frac{\gamma}{h}\int_\Omega (\nabla w_i-\nabla w_{i-1})^2\,\mathrm{d}x\\&\quad+\int_\Omega \varrho_\varepsilon^\alpha|\triangle w_i|^{p-2}\triangle w_i\triangle(w_i-w_{i-1})\,\mathrm{d}x=0.\end{aligned}$$

Apply Young's inequality to give

$$\begin{aligned}&\frac{1}{h}\int_\Omega (w_i-w_{i-1})^2\,\mathrm{d}x+\frac{\gamma}{h}\int_\Omega (\nabla w_i-\nabla w_{i-1})^2\,\mathrm{d}x+\int_\Omega \varrho_\varepsilon^\alpha|\triangle w_i|^p\,\mathrm{d}x\\&\quad\le\int_\Omega \varrho_\varepsilon^\alpha|\triangle w_i|^{p-1}|\triangle w_{i-1}|\,\mathrm{d}x\\&\quad\le\frac{p-1}{p}\int_\Omega \varrho_\varepsilon^\alpha|\triangle w_i|^p\,\mathrm{d}x+\frac{1}{p}\int_\Omega \varrho_\varepsilon^\alpha|\triangle w_{i-1}|^p\,\mathrm{d}x.\end{aligned}$$

Therefore, a simple calculation can show assertions (2.7)–(2.9). □

3 Parabolic problem with nondegenerate coefficient

In this section, we would give the proof of Proposition 1 for fixed constant $\varepsilon > 0$. We assume that $w_{\varepsilon I} \to w_I$ in H^1-norm as $\varepsilon \to 0$. For convenience, we use the notation w to represent the weak solutions of (1.4)–(1.6).

For the purpose of existence, we define the following approximate solution:

$$U^{(n)}(x,t)=\sum_{i=1}^{n}\mathbb{S}_i(t)w_i(x)$$

for

$$\mathbb{S}_i(t)=\begin{cases}1, & t\in((i-1)h,ih];\\ 0, & \text{elsewhere}\end{cases}\qquad\text{with } i=1,\dots,n.$$

For $U^{(n)}$, we can establish the uniform estimates as follows.

Lemma 2 *There is uniform constant C such that*

$$\|U^{(n)}\|_{L^\infty(0,T;L^2(\Omega))}\le C, \tag{3.1}$$

$$\left\| \varrho_\varepsilon^\alpha \left| \triangle U^{(n)} \right|^p \right\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{3.2}$$

$$\left\| U^{(n)} \right\|_{L^\infty(0,T;W_0^{2,p}(\Omega))} \le \frac{1}{\varepsilon^{\frac{\alpha}{p}}} C. \tag{3.3}$$

Proof For any time $t \in (0,T]$, there exists some interval $((i-1)h, ih]$ such that $t \in ((i-1)h, ih]$, and then $\|U^{(n)}(x,t)\|^2_{L^2(\Omega)} = \|w_i(x)\|^2_{L^2(\Omega)} \le C$. So we have (3.1). Besides, estimate (2.8) can give

$$\left(\frac{1}{p} \int_\Omega \varrho_\varepsilon^\alpha \left| \triangle U^{(n)} \right|^p \,\mathrm{d}x \right)(t) = \frac{1}{p} \int_\Omega \varrho_\varepsilon^\alpha |\triangle w_i|^p \,\mathrm{d}x \le \frac{1}{p} \int_\Omega \varrho_\varepsilon^\alpha |\triangle w_0|^p \,\mathrm{d}x \le C.$$

It implies (3.2)–(3.3). □

Now we introduce another approximate solution

$$V^{(n)}(x,t) = \sum_{i=1}^n \mathbb{S}_i(t) \big(\Theta_i(t) w_i(x) + \big(1 - \Theta_i(t)\big) w_{i-1}(x) \big) \tag{3.4}$$

with

$$\Theta_i(t) = \begin{cases} \frac{t}{h} - (i-1), & \text{if } t \in ((i-1)h, ih], \\ 0, & \text{otherwise}. \end{cases}$$

For $V^{(n)}$, we establish the estimates as follows.

Lemma 3 *There is a constant C such that*

$$\left\| V_t^{(n)} \right\|_{L^2(0,T;H^1(\Omega))} + \left\| V^{(n)} \right\|_{L^\infty(0,T;W_0^{2,p}(\Omega))} \le C.$$

Proof By using $\frac{\partial V^{(n)}}{\partial t} = \frac{1}{h} \sum_{i=1}^n \mathbb{S}_i (w_i - w_{i-1})$ and (2.8), we have

$$\left\| \frac{\partial \nabla V^{(n)}}{\partial t} \right\|^2_{L^2(Q_T)} = \frac{1}{h^2} \sum_{i=1}^n h \int_\Omega (\nabla w_i - \nabla w_{i-1})^2 \,\mathrm{d}x \le C.$$

For $t \in [0,T]$, there is a positive integer i satisfying $t \in ((i-1)h, ih]$. Thus, (2.7) gives

$$\begin{aligned} &\left(\int_\Omega \left| \triangle V^{(n)} \right|^p \,\mathrm{d}x \right)(t) \\ &\quad = \int_\Omega \left| \big(\Theta_i(t) \triangle w_i(x) + \big(1 - \Theta_i(t)\big) \triangle w_{i-1}(x) \big) \right|^p \,\mathrm{d}x \\ &\quad \le C_1 \int_\Omega \left| \triangle w_i(x) \right|^p \,\mathrm{d}x + C_2 \int_\Omega \left| \triangle w_{i-1}(x) \right|^p \,\mathrm{d}x \\ &\quad \le C. \end{aligned}$$

It shows the estimate in $L^\infty(0,T;W_0^{2,p}(\Omega))$. □

Next we give the proof of Proposition 1.

Proof of Proposition 1 Lemma 2 can ensure the existence of a subsequence of $U^{(n)}$ (we always take the same notation) and two functions $w \in L^\infty(0,T;W_0^{2,p}(\Omega))$ and $\nu \in L^{\frac{p}{p-1}}(Q_T)$ such that

$$U^{(n)} \stackrel{*}{\rightharpoonup} w \quad \text{weakly* in } L^\infty\big(0,T;W_0^{2,p}(\Omega)\big),$$
$$\varrho_\varepsilon^\alpha \big|\triangle U^{(n)}\big|^{p-2}\triangle U^{(n)} \rightharpoonup \varrho_\varepsilon^\alpha \nu \quad \text{weakly in } L^{\frac{p}{p-1}}(Q_T),$$

as $n \to \infty$. Besides, from Lemma 3, we can find a subsequence of $V^{(n)}$ and a function ϖ such that

$$\frac{\partial V^{(n)}}{\partial t} \rightharpoonup \frac{\partial \varpi}{\partial t} \quad \text{weakly in } L^2\big(0,T;H^1(\Omega)\big),$$
$$V^{(n)} \stackrel{*}{\rightharpoonup} \varpi \quad \text{weakly* in } L^\infty\big(0,T;W_0^{2,p}(\Omega)\big),$$
$$V^{(n)} \to \varpi \quad \text{strongly in } L^2\big(0,T;H^1(\Omega)\big),$$
$$V^{(n)} \to \varpi \quad \text{a.e. in } Q_T.$$

On the other hand, for any $\varphi \in C_0^\infty(Q_T)$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \big|\big(\nabla U^{(n)} - \nabla V^{(n)}\big)\big|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \left|\sum_{i=1}^n \mathbb{S}_i(t)\big(1-\Theta_i(t)\big)(\nabla w_k - \nabla w_{k-1})\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad \le \sum_{i=1}^n \int_{(i-1)h}^{ih} \int_\Omega \big|(\nabla w_i - \nabla w_{i-1})\big|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le CTh \to 0
\end{aligned}$$

as $n \to \infty$ (i.e. $h \to 0$). It implies $w = \varpi$ a.e. in Q_T and

$$U^{(n)} \to w \quad \text{strongly in } L^2\big(0,T;H^1(\Omega)\big),$$
$$U^{(n)} \to w \quad \text{a.e. in } Q_T.$$

If we perform the limit $n \to \infty$ in the expression

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial V^{(n)}}{\partial t}\varphi \,\mathrm{d}x\,\mathrm{d}t + \gamma \iint_{Q_T} \nabla V_t^{(n)} \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_{Q_T} \varrho_\varepsilon^\alpha \big|\triangle U^{(n)}\big|^{p-2}\triangle U^{(n)} \triangle\varphi \,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned} \tag{3.5}$$

then we have

$$\iint_{Q_T} \frac{\partial w}{\partial t}\varphi \,\mathrm{d}x\,\mathrm{d}t + \gamma \iint_{Q_T} \nabla w_t \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \varrho_\varepsilon^\alpha \nu \triangle\varphi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.6}$$

for any $\varphi \in C_0^\infty(Q_T)$.

The next job is to prove $v = |\triangle w|^{p-2}\triangle w$. For each test function $\psi \in C_0^\infty(0,T)$, we define $\varphi = \psi w$ as the multiplier in (3.6) to get

$$-\frac{1}{2}\iint_{Q_T} w^2 \frac{\mathrm{d}\psi}{\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}t - \frac{\gamma}{2}\iint_{Q_T} |\nabla w|^2 \frac{\mathrm{d}\psi}{\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \psi \varrho_\varepsilon^\alpha v \triangle w\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.7}$$

In (2.4), we use $\psi(t)w_i$ as the test function to give

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \psi w_i^2\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega \psi \nabla w_i^2\,\mathrm{d}x + \int_\Omega \psi \varrho_\varepsilon^\alpha |\triangle w_i|^p\,\mathrm{d}x \\
&\quad = \frac{1}{h}\int_\Omega \psi w_{i-1} w_i\,\mathrm{d}x + \frac{\gamma}{h}\int_\Omega \psi \nabla w_{i-1}\nabla w_i\,\mathrm{d}x \\
&\quad \le \frac{1}{2h}\int_\Omega \psi w_i^2\,\mathrm{d}x + \frac{1}{2h}\int_\Omega \psi w_{i-1}^2\,\mathrm{d}x + \frac{\gamma}{2h}\int_\Omega \psi \nabla w_i^2\,\mathrm{d}x + \frac{\gamma}{2h}\int_\Omega \psi \nabla w_{i-1}^2\,\mathrm{d}x.
\end{aligned}$$

That becomes

$$\begin{aligned}
&\frac{1}{2h}\int_\Omega \psi w_i^2\,\mathrm{d}x - \frac{1}{2h}\int_\Omega \psi w_{i-1}^2\,\mathrm{d}x + \frac{\gamma}{2h}\int_\Omega \psi \nabla w_i^2\,\mathrm{d}x - \frac{\gamma}{2h}\int_\Omega \psi \nabla w_{i-1}^2\,\mathrm{d}x \\
&\quad + \int_\Omega \psi \varrho_\varepsilon^\alpha |\triangle w_i|^p\,\mathrm{d}x \le 0.
\end{aligned}$$

By introducing the notation $\tilde{U}^{(n)}(x,t) = \sum_{i=1}^n \mathbb{S}_i(t) w_{i-1}(x)$, we have

$$\begin{aligned}
&\frac{1}{2h}\iint_{Q_T} \psi \bigl|U^{(n)}\bigr|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2h}\iint_{Q_T} \psi \bigl|\tilde{U}^{(n)}\bigr|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2h}\iint_{Q_T} \psi \bigl|\nabla U^{(n)}\bigr|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \frac{\gamma}{2h}\iint_{Q_T} \psi \bigl|\nabla \tilde{U}^{(n)}\bigr|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \psi \varrho_\varepsilon^\alpha \bigl|\triangle U^{(n)}\bigr|^p\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le 0.
\end{aligned} \tag{3.8}$$

For any function $\varphi_1 \in C_0^\infty(Q_T)$, we can seek two constants t_1 and t_2 with $0 < t_1 < t_2 < T$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\triangle\varphi_1 \subset (t_1,t_2)\times\Omega$. Meanwhile, we redefine ψ as $\psi \equiv 1$ on (t_1,t_2) and $\psi \equiv 0$ on $[0,h)\cup(T-h,T]$ for small h ($h < t_1$ and $T-h > t_2$). Now a direct computation gives

$$\begin{aligned}
&\int_0^T \psi \bigl|\tilde{U}^{(n)}\bigr|^2\,\mathrm{d}t \\
&\quad = \int_0^h \psi \bigl|\tilde{U}^{(n)}\bigr|^2\,\mathrm{d}t + \int_h^T \psi \bigl|\tilde{U}^{(n)}\bigr|^2\,\mathrm{d}t \\
&\quad = \int_h^T \psi(t) \bigl|\tilde{U}^{(n)}(x,t)\bigr|^2\,\mathrm{d}t \\
&\quad \xlongequal{t=\tau+h} \int_0^{T-h} \psi(\tau+h) \bigl|\tilde{U}^{(n)}(x,\tau+h)\bigr|^2\,\mathrm{d}\tau \\
&\quad = \int_0^{T-h} \psi(t+h) \bigl|U^{(n)}(x,t)\bigr|^2\,\mathrm{d}t.
\end{aligned}$$

Similarly, one has

$$\int_0^T \psi\left|\nabla\tilde{U}^{(n)}\right|^2 \mathrm{d}t = \int_0^{T-h} \psi(t+h)\left|\nabla U^{(n)}(x,t)\right|^2 \mathrm{d}t.$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{2h}\iint_{Q_T} \psi\left|U^{(n)}\right|^2 \mathrm{d}x\,\mathrm{d}t - \frac{1}{2h}\iint_{Q_T} \psi\left|\tilde{U}^{(n)}\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{\gamma}{2h}\iint_{Q_T} \psi\left|\nabla U^{(n)}\right|^2 \mathrm{d}x\,\mathrm{d}t - \frac{\gamma}{2h}\iint_{Q_T} \psi\left|\nabla\tilde{U}^{(n)}\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2h}\int_{T-h}^T \int_\Omega \psi(t)\left|U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t + \frac{1}{2h}\int_0^{T-h} \int_\Omega \psi(t)\left|U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{2h}\int_0^{T-h} \int_\Omega \psi(t+h)\left|U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{\gamma}{2h}\int_{T-h}^T \int_\Omega \psi(t)\left|\nabla U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2h}\int_0^{T-h} \int_\Omega \psi(t)\left|\nabla U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{\gamma}{2h}\int_0^{T-h} \int_\Omega \psi(t+h)\left|\nabla U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&= -\frac{1}{2}\int_0^{T-h} \int_\Omega \frac{\psi(t+h)-\psi(t)}{h}\left|U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{\gamma}{2}\int_0^{T-h} \int_\Omega \frac{\psi(t+h)-\psi(t)}{h}\left|\nabla U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

(3.8) implies

$$\begin{aligned}
&-\frac{1}{2}\int_0^{T-h} \int_\Omega \frac{\psi(t+h)-\psi(t)}{h}\left|U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{\gamma}{2}\int_0^{T-h} \int_\Omega \frac{\psi(t+h)-\psi(t)}{h}\left|\nabla U^{(n)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \psi\varrho_\varepsilon^\alpha\left|\triangle U^{(n)}\right|^p \mathrm{d}x\,\mathrm{d}t \\
&\leq 0.
\end{aligned} \tag{3.9}$$

If we choose $\zeta = \triangle U^{(n)}$ and $\eta = \triangle(w-\lambda\varphi_1))$ with $\lambda > 0$ and $\varphi_1 \in C_0^\infty(Q_T)$ in (2.12), then we have

$$\begin{aligned}
&\iint_{Q_T} \psi\varrho_\varepsilon^\alpha\left|\triangle U^{(n)}\right|^p \mathrm{d}x\,\mathrm{d}t \\
&\geq -\iint_{Q_T} \psi\varrho_\varepsilon^\alpha\left|\triangle(w-\lambda\varphi_1)\right|^p \mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_{Q_T} \psi\varrho_\varepsilon^\alpha\left|\triangle U^{(n)}\right|^{p-2}\triangle U^{(n)}\triangle(w-\lambda\varphi_1)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_{Q_T} \psi\varrho_\varepsilon^\alpha\left|\triangle(w-\lambda\varphi_1)\right|^{p-2}\triangle(w-\lambda\varphi_1)\triangle U^{(n)}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.10}$$

Use (3.9) and (3.10) to get

$$
\begin{aligned}
&-\frac{1}{2}\int_0^{T-h}\int_\Omega \frac{\psi(t+h)-\psi(t)}{h}\left|U^{(n)}(x,t)\right|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\frac{\gamma}{2}\int_0^{T-h}\int_\Omega \frac{\psi(t+h)-\psi(t)}{h}\left|\nabla U^{(n)}(x,t)\right|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \left|\triangle(w-\lambda\varphi_1)\right|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \left|\triangle U^{(n)}\right|^{p-2}\triangle U^{(n)}\triangle(w-\lambda\varphi_1)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \left|\triangle(w-\lambda\varphi_1)\right|^{p-2}\triangle(w-\lambda\varphi_1)\triangle U^{(n)}\,\mathrm{d}x\,\mathrm{d}t \\
&\le 0.
\end{aligned}
\tag{3.11}
$$

By letting $n\to\infty$ (i.e. $h\to 0$), we have

$$
\begin{aligned}
&-\frac{1}{2}\int_0^{T}\int_\Omega w^2\frac{\mathrm{d}\psi}{\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}t-\frac{\gamma}{2}\int_0^{T}\int_\Omega |\nabla w|^2\frac{\mathrm{d}\psi}{\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \left|\triangle(w-\lambda\varphi_1)\right|^p \,\mathrm{d}x\,\mathrm{d}t+\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \nu\triangle(w-\lambda\varphi_1)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \left|\triangle(w-\lambda\varphi_1)\right|^{p-2}\triangle(w-\lambda\varphi_1)\triangle w\,\mathrm{d}x\,\mathrm{d}t \\
&\le 0.
\end{aligned}
\tag{3.12}
$$

Apply (3.7) and (3.12) to get

$$
\begin{aligned}
&-\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \left|\triangle(w-\lambda\varphi_1)\right|^p \,\mathrm{d}x\,\mathrm{d}t-\lambda\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \nu\triangle\varphi_1\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\iint_{Q_T} \psi\varrho_\varepsilon^\alpha \left|\triangle(w-\lambda\varphi_1)\right|^{p-2}\triangle(w-\lambda\varphi_1)\triangle w\,\mathrm{d}x\,\mathrm{d}t \\
&\le 0.
\end{aligned}
$$

Therefore, we have

$$
\iint_{Q_T} \psi\varrho_\varepsilon^\alpha\left[\left|\triangle(w-\lambda\varphi_1)\right|^{p-2}\triangle(w-\lambda\varphi_1)-\nu\right]\triangle\varphi_1\,\mathrm{d}x\,\mathrm{d}t\le 0.
$$

We pass to the limit $\lambda\to 0$ to get

$$
\iint_{Q_T} \psi\varrho_\varepsilon^\alpha\left[|\triangle w|^{p-2}\triangle w-\nu\right]\triangle\varphi_1\,\mathrm{d}x\,\mathrm{d}t\le 0.
$$

Finally, the arbitrariness of φ_1 and ψ implies $v = |\triangle w|^{p-2}\triangle w$ a.e. in Q_T. Thus, (3.6) becomes

$$\iint_{Q_T} \frac{\partial w}{\partial t}\varphi \,\mathrm{d}x\,\mathrm{d}t + \gamma \iint_{Q_T} \nabla w_t \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \varrho_\varepsilon^\alpha |\triangle w|^{p-2}\triangle w \triangle\varphi \,\mathrm{d}x\,\mathrm{d}t$$
$$= 0. \tag{3.13}$$

For other estimates in Proposition 1, we may apply J. Simon's lemma (see [15]) and Sobolev's embedding theorem (see [1] and [5]), and so we omit the details. The uniqueness can be shown as the corresponding proof of Lemma 1. □

4 Existence for degenerate coefficient

For the solutions obtained in Proposition 1, we would use the notation w_ε. In this section, we want to gain necessary uniform estimations with respect to ε so that the limit $\varepsilon \to 0$ can be passed well.

For uniform estimates, we have the lemma.

Lemma 4 *There are a constant C and a constant $\theta > 1$ (close to 1) such that*

$$\|w_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le C, \tag{4.1}$$

$$\left\|\varrho_\varepsilon^{\frac{\alpha}{p}} \triangle w_\varepsilon\right\|_{L^p(Q_T)} \le C, \tag{4.2}$$

$$\left\|\varrho_\varepsilon^\alpha |\triangle w_\varepsilon|^{p-2}\triangle w_\varepsilon\right\|_{L^{\frac{p}{p-1}}(Q_T)} \le C, \tag{4.3}$$

$$\|w_{\varepsilon t}\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{4.4}$$

$$\|\triangle w_\varepsilon\|_{L^\theta(Q_T)} \le C. \tag{4.5}$$

Proof Define the characteristic function

$$\mathbb{S}_{[0,t]}(t) = \begin{cases} 1, & t \in [0,t]; \\ 0, & \text{otherwise} \end{cases}$$

and apply $\varphi = w_\varepsilon \mathbb{S}_{[0,t]}(t)$ as the test function in (3.13) to give

$$\frac{1}{2}\int_\Omega w_\varepsilon^2(x,t)\,\mathrm{d}x + \frac{\gamma}{2}\int_\Omega \nabla w_\varepsilon^2(x,t)\,\mathrm{d}x + \iint_{Q_t} \varrho_\varepsilon^\alpha |\triangle w_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{1}{2}\int_\Omega w_{0\varepsilon}^2(x)\,\mathrm{d}x + \frac{\gamma}{2}\int_\Omega \nabla w_{0\varepsilon}^2(x)\,\mathrm{d}x. \tag{4.6}$$

It implies (4.1) and (4.2). Since ϱ_ε is bounded, (4.3) can be shown. From (3), the limit

$$\nabla w_{\varepsilon t}^{(n)} \rightharpoonup \nabla w_{\varepsilon t} \quad \text{weakly in } L^2(Q_T) \text{ as } n \to \infty$$

can give the estimate

$$\|\nabla w_{\varepsilon t}\|_{L^2(Q_T)} \le \liminf_{n\to\infty} \left\|\nabla w_{\varepsilon t}^{(n)}\right\|_{L^2(Q_T)} \le C,$$

where we have used the weak lower semi-continuity for L^2-norm, and the constant C depends on the W_0^2-norm of the initial function.

On the other hand, the condition $\alpha < p-1$ implies $\frac{\alpha}{p-1} < 1$, and thus we can seek a constant $\alpha_1 \in (\frac{\alpha}{p-1}, 1)$. Besides, we can determine the constant $\theta \in (1, \min\{p - \frac{\alpha}{\alpha_1}, \frac{1}{\alpha_1}\})$. Moreover, the above constants satisfy the conditions $\alpha_1\theta < 1$ and $\frac{\alpha}{\alpha_1} + \theta < p$. Now we have the estimate

$$\begin{aligned}
&\iint_{Q_T} |\triangle w_\varepsilon|^\theta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \iint_{\{\varrho_\varepsilon^{\alpha_1}|\triangle w_\varepsilon|\le 1\}} |\triangle w_\varepsilon|^\theta \,\mathrm{d}x\,\mathrm{d}t + \iint_{\{\varrho_\varepsilon^{\alpha_1}|\triangle w_\varepsilon|> 1\}} |\triangle w_\varepsilon|^\theta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \iint_{Q_T} \varrho_\varepsilon^{-\alpha_1\theta} \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \varrho_\varepsilon^{\alpha}|\triangle w_\varepsilon|^{\frac{\alpha}{\alpha_1}+\theta} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \iint_{Q_T} \varrho^{-\alpha_1\theta} \,\mathrm{d}x\,\mathrm{d}t + C\iint_{Q_T} \varrho_\varepsilon^{\alpha}\bigl(1+|\triangle w_\varepsilon|^p\bigr) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le C,
\end{aligned}$$

where we have applied (4.6). It yields (4.5). □

Proof of Theorem 1 Lemma 4 allows us to find a subsequence of w_ε and two functions w, ν' so that

$$w_{\varepsilon t} \rightharpoonup w_t \quad \text{weakly in } L^2\bigl(0,T;H^1(\Omega)\bigr), \tag{4.7}$$

$$w_\varepsilon \overset{*}{\rightharpoonup} w \quad \text{weakly* in } L^\infty\bigl(0,T;H^1(\Omega)\bigr), \tag{4.8}$$

$$\triangle w_\varepsilon \rightharpoonup \triangle w \quad \text{weakly in } L^p_{\mathrm{loc}}(Q_T), \tag{4.9}$$

$$w_\varepsilon \to w \quad \text{strongly in } L^2\bigl(0,T;H^1(\Omega)\bigr), \tag{4.10}$$

$$w_\varepsilon \to w \quad \text{a. e. in } Q_T, \tag{4.11}$$

$$\varrho_\varepsilon^\alpha|\triangle w_\varepsilon|^{p-2}\triangle w_\varepsilon \rightharpoonup \nu' \quad \text{weakly in } L^{\frac{p}{p-1}}(Q_T) \tag{4.12}$$

as $\varepsilon \to 0$. It can ensure

$$\begin{aligned}
&w_t \in L^2\bigl(0,T;H^1(\Omega)\bigr), \qquad w \in L^\infty\bigl(0,T;H^1(\Omega)\bigr), \\
&\triangle w \in L^p_{\mathrm{loc}}(Q_T), \qquad \nu' \in L^{\frac{p}{p-1}}(Q_T).
\end{aligned}$$

If we perform the limit $\varepsilon \to \infty$ in (3.13), then we have

$$\iint_{Q_T} \frac{\partial w}{\partial t}\varphi \,\mathrm{d}x\,\mathrm{d}t + \gamma\iint_{Q_T} \nabla w_t \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \nu'\triangle\varphi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{4.13}$$

for any $\varphi \in C_0^\infty(Q_T)$. We need to prove

$$\iint_{Q_T} \nu'\triangle\varphi \,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} \varrho^\alpha|\triangle w|^{p-2}\triangle w\triangle\varphi \,\mathrm{d}x\,\mathrm{d}t. \tag{4.14}$$

To show this, for each $\phi \in C_0^\infty(Q_T)$, we can find a small positive constant β such that $\operatorname{supp}\phi, \operatorname{supp}\triangle\phi \subset\subset \Omega_\beta \times (\beta, T-\beta)$, where $\Omega_\beta = \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \beta\}$. Now we can rewrite (4.13) and (3.13) as

$$\iint_{Q_{\beta T}} \frac{\partial w}{\partial t}\phi \,\mathrm{d}x\,\mathrm{d}t + \gamma \iint_{Q_{\beta T}} \nabla w_t \nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_{\beta T}} \nu' \triangle\phi \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.15}$$

$$\begin{aligned} &\iint_{Q_{\beta T}} \frac{\partial w_\varepsilon}{\partial t}\phi \,\mathrm{d}x\,\mathrm{d}t + \gamma \iint_{Q_{\beta T}} \nabla w_{\varepsilon t} \nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_{\beta T}} \varrho_\varepsilon^\alpha |\triangle w_\varepsilon|^{p-2}\triangle w_\varepsilon \triangle\phi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = 0 \end{aligned} \tag{4.16}$$

with $Q_{\beta T} = \Omega_\beta \times (\beta, T-\beta)$.

For any $\psi \in C^\infty(0,T)$ with $\operatorname{supp}\psi \subset (\beta, T-\beta)$, we choose $\phi = \psi w$ as the test function (may need some approximate procedure here and below) in (4.13) to obtain

$$\begin{aligned} &-\frac{1}{2}\iint_{Q_{\beta T}} w^2 \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}t - \frac{\gamma}{2}\iint_{Q_{\beta T}} |\nabla w|^2 \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \iint_{Q_{\beta T}} \psi\nu' \triangle w \,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned} \tag{4.17}$$

On the other hand, we take $\phi = \psi w_\varepsilon$ as the multiplier in (4.13) to get

$$\begin{aligned} &-\frac{1}{2}\iint_{Q_{\beta T}} w_\varepsilon^2 \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}t - \frac{\gamma}{2}\iint_{Q_{\beta T}} |\nabla w_\varepsilon|^2 \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \iint_{Q_{\beta T}} \psi\varrho_\varepsilon^\alpha |\triangle w_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned} \tag{4.18}$$

For $\lambda > 0$, we set $\zeta = \triangle w_\varepsilon$ and $\eta = \triangle(w - \lambda\phi)$ in (2.12) and use (4.18) to find

$$\begin{aligned} &-\frac{1}{2}\iint_{Q_{\beta T}} w_\varepsilon^2 \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}t - \frac{\gamma}{2}\iint_{Q_{\beta T}} |\nabla w_\varepsilon|^2 \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \iint_{Q_{\beta T}} \psi\varrho_\varepsilon^\alpha \big|\triangle(w-\lambda\phi)\big|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \iint_{Q_{\beta T}} \psi\varrho_\varepsilon^\alpha |\triangle w_\varepsilon|^{p-2}\triangle w_\varepsilon \triangle(w-\lambda\phi) \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \iint_{Q_{\beta T}} \psi\varrho_\varepsilon^\alpha \big|\triangle(w-\lambda\phi)\big|^{p-2}\triangle(w-\lambda\phi)\triangle w_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le 0. \end{aligned}$$

By letting $\varepsilon \to 0$, we find

$$\begin{aligned} &-\frac{1}{2}\iint_{Q_{\beta T}} w^2 \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}t - \frac{\gamma}{2}\iint_{Q_{\beta T}} |\nabla w|^2 \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \iint_{Q_{\beta T}} \psi\varrho^\alpha \big|\triangle(w-\lambda\phi)\big|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \iint_{Q_{\beta T}} \psi\nu' \triangle(w-\lambda\phi) \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\iint_{Q_{\beta T}} \psi \varrho^{\alpha} \left|\triangle(w-\lambda\phi)\right|^{p-2} \triangle(w-\lambda\phi)\triangle w \,\mathrm{d}x\,\mathrm{d}t \\
&\leq 0.
\end{aligned}
$$

Apply (4.15) to have

$$
\iint_{Q_{\beta T}} \psi \left(\varrho^{\alpha} \left|\triangle(w-\lambda\phi)\right|^{p-2} \triangle(w-\lambda\phi) - \nu'\right)\triangle\phi \,\mathrm{d}x\,\mathrm{d}t \leq 0.
$$

By passing to the limit $\lambda \to 0^{+}$, we get

$$
\begin{aligned}
&\iint_{Q_T} \psi \left(\varrho^{\alpha} |\triangle w|^{p-2} \triangle w - \nu'\right)\triangle\phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \iint_{Q_{\beta T}} \psi \left(\varrho^{\alpha} |\triangle w|^{p-2} \triangle w - \nu'\right)\triangle\phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq 0.
\end{aligned}
$$

For negative λ, we can have the same result with an opposite inequality sign. Therefore, we can show (4.14) from the arbitrariness of ϕ and ψ.

Finally, a standard process can give the other estimates of the theorem and the uniqueness of weak solutions. Now we have completed the proof of Theorem 1. □

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Ethics approval and consent to participate
Our research is in pure mathematics direction and the research subjects do not include any humans or animals. Thus the research was carried out in accordance with the Declaration of Helsinki.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BL, CS, and YW introduced the main idea and were the major contributors in writing the manuscript. XL and ZZ participated in applying the method for solving this problem. All authors read and approved the final manuscript.

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