Lakshmipriya et al. *Boundary Value Problems* (2022) 2022:30
https://doi.org/10.1186/s13661-022-01611-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence and blow-up of weak solutions of a pseudo-parabolic equation with logarithmic nonlinearity

N. Lakshmipriya[1], S. Gnanavel[1], K. Balachandran[2] and Yong-Ki Ma[3*]

*Correspondence:
ykma@kongju.ac.kr
[3]Department of Applied Mathematics, Kongju National University, Chungcheongnam-do 32588, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we prove the existence of weak solutions of a pseudo-parabolic equation with logarithmic nonlinearity in an interval $[0, T)$ by employing the Galerkin approximation method and compactness arguments. We show that the solutions become unbounded at a finite time $T^{\star}$ and find upper and lower bounds for this time.

MSC: 35D30; 35B44; 35K70

Keywords: Weak solutions; Blow-up; Variable exponent spaces; Galerkin method; $p(x)$-Laplacian operator

1 Introduction

Showalter [39] has initiated the study of pseudo-parabolic equations, and subsequently, many authors have contributed to the various type of pseudo-parabolic equations. These equations explain the physical phenomena like unidirectional travel of long waves, aggregation of population, oozing of homogeneous fluids through cracked rocks, etc. Regarding the study of the existence and blow-up of solutions to pseudo-parabolic equations, Xu and Su [44] considered the following type of equation

$$\begin{cases} w_t(x,t) - \Delta w_t - \Delta w = w^p, & (x,t) \in \Omega \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T), \\ w(x,0) = w_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary, $\partial\Omega$. The authors proved the global existence and unboundedness of solutions of (1.1) in finite time using a potential well method, variational methods, and comparison principle. This problem was also studied by Luo [30]; he obtained a lower bound for the blow-up time using a differential inequality technique. Moreover, for any $p > 1$, he established an upper bound for the finite time of blow-up. Xu et al. [45] derived a new theorem to prove blow-up at the finite time and established an upper bound for the time using concavity method. Motivated by [44], Chen and Tian [8] studied (1.1) by considering logarithmic nonlinearity instead of

w^p. In their work, the authors proved the existence of solutions by employing potential well method and derived blow-up at infinity and a condition for finite time blow-up. Han [20] derived a criterion for finite time blow-up of solutions of (1.1) by considering a general nonlinearity $f(w)$ and established an upper bound for blow-up time using the known concavity method.

Sun et al. [41] considered the pseudo-parabolic equation of the form

$$\begin{cases} w_t(x,t) - a\Delta w_t - \Delta w + bw = k(t)|w|^{p-2}w, & (x,t) \in \Omega \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T), \\ w(x,0) = w_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.2}$$

where $k(t) > 0$, $a \geq 0$, $b > \lambda_1$, λ_1 being the principal eigenvalue of $-\Delta$. The authors analyzed the unboundedness of solutions at finite time under super-critical, critical, and sub-critical initial energy levels. Using potential wells, differential inequalities, and concavity method, they found bounds for blow-up time. The problem (1.2) has already been analyzed by Zhu et al. [46] for $a, b = 1$ and $k(t) \equiv 1$; they have established global existence and unboundedness of solutions at finite time. For the solutions of pseudo-parabolic equations with source term depending on gradient term, blow-up properties are analyzed in [31]. Sufficient conditions on the coefficients are introduced in order to specify bounded and blow-up cases, and a lower bound for blow-up time is explicitly found. Blow- up phenomena of the equation in which the source term depends only on the solution are probed, and upper and lower bounds and a blow-up criterion under specific conditions were obtained in [36].

Meyvaci [32] studied the asymptotic behavior of solutions of a pseudo-parabolic equation and in [33] generalized the study by incorporating a bounded function involving gradient term and observed the conditions under which the solution does not blow-up. Also, the author studied the finite time unboundedness of solutions and obtained lower and upper bounds for the time. Lian et al. [26] considered an initial boundary value problem of pseudo-parabolic equation with singular potential and derived global existence, asymptotic behavior, and blow-up of solutions with initial energy. Moreover, they estimated an upper bound of the blow-up time. For a nonlocal source, Wang and Xu [43] investigated a semilinear pseudo-parabolic equation for all the three initial energy levels. For subcritical and critical initial energy cases, the authors obtained results on existence, uniqueness, asymptotic behavior, and blow-up of solutions. Also, they proved that the solutions blow-up for super-critical initial energy.

Di et al. [10] studied the Dirichlet problem of the following equation

$$\begin{cases} w_t - \nu\Delta w_t - \operatorname{div}(|\nabla w|^{m(x)-2}\nabla w) = |w|^{p(x)-2}w, & (x,t) \in \Omega \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T), \\ w(x,0) = w_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.3}$$

where $p(x)$ and $m(x)$ are continuous variable exponents, and $\nu > 0$. They have stated a theorem on the existence of solution to (1.3) and proved the unboundedness of solutions in finite time. They established an upper bound for the time using Kaplan's first eigenvalue method while the lower bound is acquired by a differential inequality. Liao et al. [29] improved the results of [10] by answering some unsolved questions therein. They used

Galerkin's approximation technique to show the global existence of solutions for $p^+ \leq 2$ and negative initial energy and presented results on nonextinction of these solutions. In [28], Liao then analyzed the case of positive initial energy and proved the nonexistence of a global solution.

All the above discussed investigations motivated us to work on the problem (1.6). The equation we consider is not only a pseudo-parabolic one but also involves logarithmic nonlinearity. Equations involving logarithmic nonlinearity are widely applied in nuclear physics, geophysics, and optics [2, 4, 16]. They appear naturally in inflation cosmology, supersymmetric field theories, and quantum mechanics [1, 13]. Looking at the very close history of problems having logarithmic nonlinearity, Chen [7] considered the following problem

$$\begin{cases} w_t - \Delta w = w \log |w|, & x \in \Omega, t > 0, \\ w(x,t) = 0, & x \in \partial\Omega, t > 0, \\ w(x,0) = w_0(x). & x \in \overline{\Omega}. \end{cases} \tag{1.4}$$

By setting a family of potential wells, he derived the global existence of solutions. In addition, decay estimates for these solutions are obtained, and the solutions for getting suitable conditions for blow-up at infinity are analyzed. Later, Han [23] improved the results of [7]. He established the criterion for the existence of global weak solutions and demonstrated the unboundedness of solutions in finite time by the concavity method. In [8], Chen and Tian studied a pseudo-parabolic problem with the same source term and boundary condition. They established the existence of global solutions, blow-up at infinity, and asymptotic behaviour of solutions under particular assumptions. In these works, the authors concluded that the presence of polynomial nonlinearity is important for the solutions to blow-up at a finite time.

However, then the studies took a turn and scientists used a more powerful logarithmic nonlinearity in their works, which made the solutions blow-up in finite time. A p-Laplacian parabolic equation with the source term $|w|^{p-2}w \log |w|$ was studied in [21, 35]. Using the potential well method and logarithmic Sobolev inequality, the authors obtained the existence and nonexistence of global solutions. They provided sufficient conditions for the finite time blow-up of solutions. Nhan and Truong [34] established existence and finite time blow-up results for the generalized equation

$$\begin{cases} w_t - \Delta w_t - \operatorname{div}(|\nabla w|^{p-2}\nabla w) = w^{q-2} \log |w|, & x \in \Omega, t > 0, \\ w(x,t) = 0, & x \in \partial\Omega, t > 0, \\ w(x,0) = w_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.5}$$

when $p = q$. Cao and Liu [6] also introduced a family of potential wells to prove the global existence of solutions of (1.5) for $q = p$, and the logarithmic Sobolev inequality is used to show the blow-up of solutions at infinity. In addition, they established some decay and growth estimates and analyzed the behavior of solutions. Then, the problem (1.5) for $p < q$ was examined in [9, 12, 22]. He et al. [22] obtained finite time blow-up and decay results for weak solutions by setting a family of potential wells and using the concavity method under the condition $2 < p < q < p(1 + \frac{2}{n})$. Ding and Zhou [9] considered more general assumptions on p and q and classified the ranges of p and q into cases under which global existence of weak solutions, finite time blow-up, and blow-up at infinity are explicitly determined for sub-critical and critical initial energy. The case of super-critical initial energy was discussed in detail by Dai et al. [12]. Lian and Xu [27] examined an initial boundary value problem of nonlinear wave equation with weak and strong damping terms and logarithmic term at three different initial energy levels. They proved the local existence of weak solution using contraction mapping principle and global existence, decay and infinite time blow-up using potential well method. The global well-posedness of a Kirchhoff-type wave system with logarithmic nonlinearities and weak damping was investigated by Wang et al. [42]. They obtained several results and sufficient conditions for the existence and unboundedness of solutions at different initial energy levels using potential well method and concavity method. In [25], the authors studied a semilinear wave equation with logarithmic nonlinearity and arrived at results on the existence and blow-up of solutions using potential well method. Weak solutions and blow-up of different partial differential equations are discussed in [3, 17–19, 38].

Based on the above-mentioned works and motivated by [10, 22], we are excited to study the existence and blow-up of weak solutions of the following pseudo-parabolic equation with logarithmic nonlinearity

$$\begin{cases} w_t - \Delta w_t - \operatorname{div}(|\nabla w|^{p(x)-2}\nabla w) & \\ \quad = |w|^{s(x)-2}w + |w|^{h-2}w\log|w|, & (x,t)\in\Omega\times(0,\infty), \\ w(x,t)=0, & (x,t)\in\partial\Omega\times[0,\infty), \\ w(x,0)=w_0(x), & x\in\overline{\Omega}, \end{cases} \tag{1.6}$$

where $\Omega\subset\mathbb{R}^n (n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The model considered in (1.6) is used to describe the non-stationary process in semiconductors in the presence of sources; the first two terms represent the free-electron density rate and logarithmic and polynomial nonlinearity stands for the source of free-electron current [24]. The motivation of this work is to address the existence and finite time blow-up of solutions of the non-stationary process in semiconductors in the presence of logarithmic and polynomial sources.

The log-Hölder continuous variable exponents $p(x)$, $s(x)$ and the constant h satisfy the following hypotheses

- $$2\leq p_-\leq p(x)\leq p_+ < s_-\leq s(x)\leq s_+ < h < \infty, \tag{1.7}$$

- $$p^*(x) > 2, \qquad p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & p(x) < n, \\ \infty, & p(x)\geq n, \end{cases} \tag{1.8}$$

- $$\operatorname*{ess\,inf}_{x\in\Omega}\big(p^*(x)-s(x)\big) > 0. \tag{1.9}$$

This paper is arranged as follows: In Sect. 2, we state the required preliminaries. The existence results are discussed in Sect. 3 using the Faedo-Galerkin approximation method. Blow-up analysis of the solutions is done in Sects. 4 and 5. In this paper, C and $C(\epsilon)$ are generic constants, which may vary accordingly.

2 Preliminaries

To discuss the problem (1.6), we need the following facts about generalized Lebesgue and Sobolev spaces. For more details, one can refer to [11]. In this section, we take $p, s : \Omega \longrightarrow [1,\infty)$ as measurable functions, and $\Omega \subset \mathbb{R}^n$ is bounded.

Definition 2.1 ([5]) Let X be a Banach space. Then $L^p(0,T,X)$ is defined as the set of measurable functions $w : [0,T] \longrightarrow X$ such that
if $1 \le p < \infty$,

$$\|w\|_{L^p(0,T;X)} = \left(\int_0^T \|w(t)\|_X^p \, dt\right)^{\frac{1}{p}} < \infty,$$

and if $p = \infty$,

$$\|w\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0\le t\le T} \|w(t)\|_X < \infty.$$

Remark 2.1 For $1 \le p \le \infty$, $L^p(0,T;X)$ is a Banach space with the above norms.

Definition 2.2 ([11]) The variable exponent Lebesgue space with exponent $p(x)$ is defined by

$$L^{p(x)}(\Omega) := \left\{w : \Omega \longrightarrow \mathbb{R} | \rho_{p(x)}(\lambda w) < \infty, \text{ for some } \lambda > 0\right\},$$

where

$$\rho_{p(x)}(w) = \int_\Omega |w(x)|^{p(x)} \, dx.$$

Theorem 2.1 ([11]) *The space $L^{p(x)}(\Omega)$ endowed with the Luxembourg norm*

$$\|w\|_{p(x)} = \inf\left\{\lambda > 0 | \rho_{p(x)}\left(\frac{w}{\lambda}\right) \le 1\right\},$$

is a Banach space and

$$\min\left\{\|w\|_{p(x)}^{p_-}, \|w\|_{p(x)}^{p_+}\right\} \le \int_\Omega |w|^{p(x)} \, dx \le \max\left\{\|w\|_{p(x)}^{p_-}, \|w\|_{p(x)}^{p_+}\right\}$$

where $p_- = \min p(x)$ and $p_+ = \max p(x)$ on Ω.

Remark 2.2 ([11]) $L^{p'(x)}(\Omega)$ denotes the dual space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Definition 2.3 ([11]) The variable exponent Sobolev space is defined as

$$W^{k,p(x)}(\Omega) = \{w \in L^{p(x)}(\Omega) | D^\alpha w \in L^{p(x)}(\Omega), |\alpha| \le k\},$$

where $k \geq 1$, $D^\alpha w$ is the α^{th} weak partial derivative with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, a multi-index and $|\alpha| = \sum_{j=1}^{N} \alpha_j$.

Theorem 2.2 ([11]) *The variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ endowed with the norm $\|w\|_{k,p(x)} := \sum_{|\alpha| \le k} \|D^\alpha w\|_{p(x)}$ is a Banach space.*

Observe that $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Lemma 2.1 ([11]) *If $p(x)$, $s(x)$ are variable exponents satisfying $p(x) \le s(x)$ a.e. in Ω, then there is a continuous embedding from $L^{s(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.*

Lemma 2.2 *There exists a continuous and compact Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$, where the variable exponents $p(x) \in C(\overline{\Omega})$, $s : \Omega \longrightarrow [1, \infty)$ are measurable functions and satisfy*

$$\operatorname*{ess\,inf}_{x \in \overline{\Omega}} \bigl(p^*(x) - s(x)\bigr) > 0, \quad \text{where } p^* = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if } p(x) < n, \\ \infty, & \text{if } p(x) \ge n. \end{cases}$$

Lemma 2.3 ([9, 37] [lemma 2]) *For all $w \in [1, \infty)$*

$$|\log w| \le \frac{w^\eta}{e\eta},$$

where η is a positive number.

3 Weak solutions

Here we prove the existence of weak solutions to the equation (1.6). The main result Theorem 3.1 can be proved using the Faedo-Galerkin approximation method and Sobolev embeddings as in [22].

Definition 3.1 A function $w \in L^2(0, T; W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)) \cap C(0, T; H_0^1(\Omega))$ is said to be a weak solution to (1.6) if $w_0 \in W_0^{1,p(x)}(\Omega) \backslash \{0\}$, $w_t \in L^2(0, T; H_0^1(\Omega))$, and w satisfies

$$\begin{aligned} &\int_0^T \int_\Omega w_t \phi \, dx \, dt + \int_0^T \int_\Omega \nabla w_t \nabla \phi \, dx \, dt + \int_0^T \int_\Omega |\nabla w|^{p(x)-2} \nabla w \nabla \phi \, dx \, dt \\ &\quad = \int_0^T \int_\Omega |w|^{s(x)-2} w \phi \, dx \, dt + \int_0^T \int_\Omega |w|^{h-2} w \log |w| \phi \, dx \, dt, \end{aligned} \tag{3.1}$$

$\forall \phi \in C^\infty(0, T; C_0^\infty(\Omega))$.

Definition 3.2 For $w_0 \in W_0^{1,p(x)}(\Omega) \backslash \{0\}$, define an energy functional as

$$\mathfrak{N}(w) = -\int_\Omega \frac{|w|^{s(x)}}{s(x)} \, dx + \int_\Omega \frac{|\nabla w|^{p(x)}}{p(x)} \, dx + \frac{1}{h^2} \int_\Omega |w|^h \, dx - \frac{1}{h} \int_\Omega |w|^h \log |w| \, dx. \tag{3.2}$$

Theorem 3.1 *Suppose $w_0 \in W_0^{1,p(x)}(\Omega)\backslash\{0\}$ and $p(x)$, $s(x)$, h satisfy the conditions (1.7), (1.8), and (1.9). Then, the equation (1.6) has a weak solution.*

Proof Now we consider an orthonormal basis of $L^2(\Omega)$, which is orthogonal in $H_0^1(\Omega)$ given by $\{r_j\}_{j=1}^{\infty}$ and a collection of eigenfunctions of Δ corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$. We seek for finite dimensional approximation solutions to (1.6) as

$$w_m = \sum_{j=1}^{m} c_{mj}(t) r_j(x), \tag{3.3}$$

where c_{mj} are unknown and satisfy

$$\begin{aligned} &\int_\Omega w_m' r_i\,dx + \int_\Omega \nabla w_m' \nabla r_i\,dx + \int_\Omega |\nabla w_m|^{p(x)-2}\nabla w_m \nabla r_i\,dx \\ &\quad = \int_\Omega |w_m|^{s(x)-2} w_m r_i\,dx + \int_\Omega |w_m|^{h-2} w_m \log|w_m| r_i\,dx, \end{aligned} \tag{3.4}$$

and

$$w_{0m} = \sum_{j=1}^{m} c_{mj}(0) r_j(x) \longrightarrow w_0 \quad \text{in } W_0^{1,p(x)}(\Omega). \tag{3.5}$$

This generates an initial value problem for a system of ordinary differential equations in $\{c_{mi}(t)\}_{i=1}^{m}$, namely,

$$\begin{cases} (1+\lambda_i)\frac{d}{dt} c_{mi}(t) = F(c_{m1}, c_{m2}, \dots, c_{mm}), \\ c_{mi}(0) = (w_0, r_i)_{L^2}, \end{cases}$$

where

$$\begin{aligned} &F(c_{m1}, c_{m2}, \dots, c_{mm}) \\ &\quad = \int_\Omega \bigl(-|\nabla w_m|^{p(x)-2}\nabla w_m \nabla r_i + |w_m|^{s(x)-2} w_m r_i + |w_m|^{h-2} w_m \log|w_m| r_i\bigr)\,dx. \end{aligned}$$

Since, $F(c_{m1}, c_{m2}, \dots, c_{mm})$ depends on $(c_{m1}, c_{m2}, \dots, c_{mm})$ continuously, Peano's theorem gives the existence of a local solution to this problem.

Now multiply (3.4) by $c_{mi}(t)$ and sum over i to get

$$\begin{aligned} &\int_\Omega w_m' w_m\,dx + \int_\Omega \nabla w_m' \nabla w_m\,dx + \int_\Omega |\nabla w_m|^{p(x)-2}\nabla w_m \nabla w_m\,dx \\ &\quad = \int_\Omega |w_m|^{s(x)-2} w_m w_m\,dx + \int_\Omega |w_m|^{h-2} w_m \log|w_m| w_m\,dx. \end{aligned} \tag{3.6}$$

This gives

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |w_m|^2 + |\nabla w_m|^2\,dx + \int_0^t \int_\Omega \bigl|\nabla w_m(x,\tau)\bigr|^{p(x)}\,dx\,d\tau\right) \\ &\quad = \int_\Omega |w_m|^{s(x)}\,dx + \int_\Omega |w_m|^h \log|w_m|\,dx. \end{aligned} \tag{3.7}$$

Integrating (3.7) over $(0,t)$, we get

$$\mathcal{H}_m(t) = \mathcal{H}_m(0) + \int_0^t \int_\Omega |w_m|^{s(x)}\,dx\,d\tau + \int_0^t \int_\Omega |w_m|^h \log|w_m|\,dx\,d\tau, \tag{3.8}$$

where $\mathcal{H}_m(t) = \frac{1}{2}\int_\Omega |w_m|^2 + |\nabla w_m|^2\,dx + \int_0^t \int_\Omega |\nabla w_m|^{p(x)}\,dx\,d\tau$.

Now, we look for estimates to (3.8). By Lemma 2.3 and following the calculations similar to [22], we get

$$\begin{aligned}
&\int_\Omega |w_m|^h \log|w_m|\,dx \\
&\quad\le \int_{\{x\in\Omega:|w_m|<1\}} |w_m|^h \log|w_m|\,dx + \int_{\{x\in\Omega:|w_m|\ge 1\}} |w_m|^h \log|w_m|\,dx \\
&\quad\le \int_{\{x\in\Omega:|w_m|\ge 1\}} |w_m|^h \log|w_m|\,dx, \quad \bigl[\text{since } \log|w_m| < 0 \text{ for } |w_m| < 1\bigr] \\
&\quad\le \frac{1}{e\eta}\int_{\{x\in\Omega:|w_m|\ge 1\}} |w_m|^{h+\eta}\,dx \\
&\quad\le \frac{1}{e\eta}\|w_m\|_{h+\eta}^{h+\eta}.
\end{aligned} \tag{3.9}$$

Choose η such that $p_- < \eta < \frac{np_-}{n-p_-} - h$. Then, by the interpolation inequality, we obtain

$$\int_\Omega |w_m|^h \log|w_m|\,dx \le C\|w_m\|_2^{(1-\theta)(h+\eta)}\|w_m\|_{\frac{np_-}{n-p_-}}^{\theta(h+\eta)},$$

where $\theta \in (0,1)$ is given by $\frac{1}{h+\eta} = \frac{\theta(n-p_-)}{np_-} + \frac{1-\theta}{2}$. We have the following continuous embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ and $L^{p^*(x)}(\Omega) \hookrightarrow L^{\frac{np_-}{n-p_-}}(\Omega)$ that together give

$$\int_\Omega |w_m|^h \log|w_m|\,dx \le C\|w_m\|_2^{(1-\theta)(h+\eta)}\|w_m\|_{W_0^{1,p(x)}(\Omega)}^{\theta(h+\eta)}.$$

Here we assume $\|w_m\|_{W_0^{1,p(x)}(\Omega)} \ge 1$. Then by Theorem 1.3 of [14] and following the calculations similar to [15], we get

$$\int_\Omega |w_m|^h \log|w_m|\,dx \le C\|w_m\|_2^{(1-\theta)(h+\eta)}\left(\int_\Omega |\nabla w_m|^{p(x)}\,dx\right)^{\frac{\theta(h+\eta)}{p_-}}.$$

Since $p_- \ge 2$, $\frac{\theta(h+\eta)}{p_-} < 1$. Similarly, if $\|w_m\|_{W_0^{1,p(x)}(\Omega)} < 1$, we get

$$\int_\Omega |w_m|^h \log|w_m|\,dx \le C\|w_m\|_2^{(1-\theta)(h+\eta)}\left(\int_\Omega |\nabla w_m|^{p(x)}\,dx\right)^{\frac{\theta(h+\eta)}{p_+}}.$$

However, in this paper, we proceed with the calculations under the assumption $\|w_m\|_{W_0^{1,p(x)}(\Omega)} \ge 1$, since we can do the other case in the same way. Now employing Young's inequality with $\epsilon > 0$, we obtain

$$\int_\Omega |w_m|^h \log|w_m|\,dx \le C(\epsilon)\bigl(\|w_m\|_2^2\bigr)^\nu + \epsilon\int_\Omega |\nabla w_m|^{p(x)}\,dx, \tag{3.10}$$

where $\nu = \frac{(1-\theta)(h+\eta)p_-}{2p_- - 2\theta(h+\eta)} > 1$.

To proceed further choose the sets $\Omega_1^+ = \{x \in \Omega : |w_m| \geq 1\}$ and $\Omega_1^- = \{x \in \Omega : |w_m| < 1\}$. Hence, we can write

$$\int_\Omega |w_m|^{s(x)}\,dx \leq \int_{\Omega_1^-} |w_m|^{s_-}\,dx + \int_{\Omega_1^+} |w_m|^{s_+}\,dx \leq \|w_m\|_{s_-}^{s_-} + \|w_m\|_{s_+}^{s_+}. \tag{3.11}$$

This gives

$$\int_\Omega |w_m|^{s(x)}\,dx \leq 2\|w_m\|_{s_+}^{s_+}. \tag{3.12}$$

Applying the Gagliardo-Nirenberg interpolation inequality to (3.12), we get

$$\|w_m\|_{s_+}^{s_+} \leq C\|\nabla w_m\|_{p_-}^{\vartheta s_+}\|w_m\|_2^{(1-\vartheta)s_+}, \tag{3.13}$$

where $\vartheta = \frac{(2-s_+)np_-}{s_+(2n-2p_--np_-)}$. Now Young's inequality gives us

$$\|w_m\|_{s_+}^{s_+} \leq \epsilon\|\nabla w_m\|_{p_-}^{p_-} + C(\epsilon)\|w_m\|_2^{\frac{p_-(1-\vartheta)s_+}{p_--\vartheta s_+}}, \quad \epsilon > 0. \tag{3.14}$$

Application of the inequalities (3.10) and (3.11) together with (3.14) in (3.8) gives

$$\begin{aligned}\mathcal{H}_m(t) \leq \mathcal{H}_m(0) + \epsilon\int_0^t \|\nabla w_m\|_{p_-}^{p_-}\,d\tau + C(\epsilon)\int_0^t \|w_m\|_2^{\frac{p_-(1-\vartheta)s_+}{p_--\vartheta s_+}}\,d\tau \\ + C(\epsilon)\int_0^t \left(\|w_m\|_2^2\right)^\nu d\tau + \epsilon\int_0^t\int_\Omega |\nabla w_m|^{p(x)}\,dx\,d\tau.\end{aligned}$$

Let $\delta = \max\{2\nu, \frac{p_-(1-\vartheta)s_+}{p_--\vartheta s_+}\}$. Now, by Lemma 2.1 and Theorem 2.1, we get

$$\mathcal{H}_m(t) \leq \mathcal{H}_m(0) + 2\epsilon\int_0^t\int_\Omega |\nabla w_m|^{p(x)}\,dx\,d\tau + C(\epsilon)\int_0^t \|w_m\|_2^\delta\,d\tau. \tag{3.15}$$

Further, by putting $\epsilon = \frac{1}{4}$ and using the definition of $\mathcal{H}_m(t)$, we arrive at the following inequality

$$\mathcal{H}_m(t) \leq 2\mathcal{H}_m(0) + C\int_0^t \mathcal{H}_m^\delta(s)\,d\tau.$$

To carry forward, we apply the Gronwall-Bellman-Bihari-type integral inequality and obtain

$$\mathcal{H}_m(t) \leq C_T, \tag{3.16}$$

where the constant C_T depends on T. Hence

$$\frac{1}{2}\int_\Omega \left(|w_m|^2 + |\nabla w_m|^2\right)dx + \int_0^T\int_\Omega |\nabla w_m|^{p(x)}\,dx\,d\tau \leq C_T. \tag{3.17}$$

Assuming $\min\{\|\nabla w_m\|_{p(x)}^{p_-}, \|\nabla w_m\|_{p(x)}^{p_+}\} = \|\nabla w_m\|_{p(x)}^{p_-}$, by (3.17), (1.7) and Theorem 2.1, we get

$$\int_0^T \|\nabla w_m\|_{p(x)}^2 \, d\tau \le \int_0^T \int_\Omega |\nabla w_m|^{p(x)} \, dx \, d\tau \le C_T. \tag{3.18}$$

Now consider the functional $\mathfrak{N}(w)$ defined in definition (3.2). Since it is continuous and we have $w_{0m} \longrightarrow w_0$ in $W_0^{1,p(x)}(\Omega)$, we get a constant C with

$$\mathfrak{N}(w_{0m}) \le C, \tag{3.19}$$

for any integer $m > 0$ large enough.

Multiplying (3.4) by $c'_{mi}(t)$ and summing over i, then integrating with respect to t gives

$$\int_0^t \left\| w'_m(s) \right\|_{H_0^1(\Omega)}^2 \, d\tau + \mathfrak{N}\big(w_m(t)\big) = \mathfrak{N}\big(w_m(0)\big).$$

The inequality (3.19) gives

$$\int_0^t \left\| w'_m(s) \right\|_{H_0^1(\Omega)}^2 \, d\tau + \mathfrak{N}\big(w_m(t)\big) \le C. \tag{3.20}$$

From the estimates (3.17), (3.18), and (3.20), together with the standard compactness arguments, we get

$$w_m \longrightarrow w \quad \text{weakly}^* \text{ in } L^\infty\big(0, T; H_0^1(\Omega)\big), \tag{3.21}$$

$$w_m \longrightarrow w \quad \text{weakly in } L^2\big(0, T; W_0^{1,p(x)}(\Omega)\big), \tag{3.22}$$

$$w'_m \longrightarrow w' \quad \text{weakly in } L^2\big(0, T; H_0^1(\Omega)\big), \tag{3.23}$$

$$|\nabla w_m|^{p(x)-2} \nabla w_m \longrightarrow \xi \quad \text{weakly in } L^2\big(0, T; L^{p'(x)}(\Omega)\big). \tag{3.24}$$

Since $w_m \in W_0^{1,p(x)}(\Omega)$, the Sobolev embedding gives

$$\int_0^T \|w_m\|_{s(x)}^2 \, d\tau \le C \int_0^T \|\nabla w_m\|_{p(x)}^2 \, d\tau \le C_T \quad \text{by (3.22)}.$$

This implies

$$w_m \rightharpoonup w \quad \text{in } L^2\big(0, T; L^{s(x)}(\Omega)\big).$$

Since we have the convergences (3.21) and (3.23), by employing the Aubin-Lions lemma [40], we get

$$w_m \longrightarrow w \quad \text{in } C\big(0, T; L^2(\Omega)\big),$$

which implies

$$w_m \longrightarrow w \quad \text{a.e. on } \Omega \times (0, T).$$

Thus, we get

$$
\begin{aligned}
&|w_m|^{h-2}w_m\log|w_m| \longrightarrow |w|^{h-2}w\log|w| \quad \text{a.e. on } \Omega\times(0,T),\\
&|w_m|^{s(x)-2}w_m \longrightarrow |w|^{s(x)-2}w \quad \text{a.e. on } \Omega\times(0,T).
\end{aligned}
\tag{3.25}
$$

Since we have $p_- < \eta < \frac{np_-}{n-p_-} - h$, we can choose $\gamma > 0$ such that $p_- < (h-1+\gamma)h' < p^*$. Now, following the trick in [22], we get

$$
\begin{aligned}
\int_\Omega \left|\psi_m(x,t)\right|^{h'}dx &= \int_{\Omega_1^-}\left|\psi_m(x,t)\right|^{h'}dx + \int_{\Omega_1^+}\left|\psi_m(x,t)\right|^{h'}dx\\
&\le \left(e(h-1)\right)^{-h'}|\Omega| + (\gamma)^{-h'}\int_{\Omega_1^+}\left(|w_m|^{h-1+\gamma}\right)^{h'}dx\\
&= \left(e(h-1)\right)^{-h'}|\Omega| + (\gamma)^{-h'}\|w_m\|_{(h-1+\gamma)h'}^{(h-1+\gamma)h'},
\end{aligned}
\tag{3.26}
$$

where $\psi_m(x,t) = |w_m(x,t)|^{h-1}\log|w_m(x,t)|$. Choosing $\eta = \frac{\gamma h}{h-1}$ in (3.9) and following the calculations up to (3.10), we get

$$
\int_\Omega \left|\psi_m(x,t)\right|^{h'}dx \le C(\epsilon)\left(\|w_m\|_2^2\right)^{\nu} + \epsilon\int_\Omega |\nabla w_m|^{p(x)}\,dx. \tag{3.27}
$$

Integrating this inequality over $(0,T)$ and applying (3.17), we get

$$
\int_0^T\int_\Omega \left|\psi_m(x,t)\right|^{h'}dx\,dt \le C_T. \tag{3.28}
$$

Also,

$$
\int_0^T\int_\Omega \left(|w_m|^{s(x)-1}\right)^{s'(x)}dx \le \int_0^T\int_\Omega |w_m|^{s(x)}\,dx \le C_T. \tag{3.29}
$$

Hence, from (3.25), (3.29), and Lion's lemma (see [40], Lemma 1.3, p.12), we have

$$
|w_m|^{h-2}w_m\log|w_m| \longrightarrow |w|^{h-2}w\log|w| \quad \text{weakly star in } L^\infty\left(0,T;L^{h'}(\Omega)\right)
$$

Now, since we have the monotonicity of $|\zeta|^{p(x)-2}\zeta$, making use of the Minty-Browder condition, we get $\xi = |\nabla w|^{p(x)-2}\nabla w$. Hence the proof. □

4 Upper bound for blow-up time

Here our objective is to seek an upper bound for the time at which the solutions to the problem (1.6) become unbounded.

Theorem 4.1 *Let w be a weak solution of* (1.6) *and assume that w_0 satisfies*

$$
\int_\Omega\left(\frac{|w_0|^{s(x)}}{s(x)} - \frac{|\nabla w_0|^{p(x)}}{p(x)}\right)dx - \frac{1}{h^2}\int_\Omega |w_0|^h\,dx + \frac{1}{h}\int_\Omega |w_0|^h\log|w_0|\,dx \ge 0. \tag{4.1}
$$

Then the solution w blows up at a finite time $T^\star > 0$. In addition, there exists an upper bound for the time as given below

$$T^\star \leq \frac{2[N(0)]^{(\frac{2-p_-}{2})}}{(p_- - 2)\theta}, \tag{4.2}$$

where $\theta > 0$ is some constant.

Proof We have the energy functional related to the problem (1.6) given by

$$\mathfrak{N}(t) = -\int_\Omega \frac{|w|^{s(x)}}{s(x)}\,dx + \int_\Omega \frac{|\nabla w|^{p(x)}}{p(x)}\,dx + \frac{1}{h^2}\int_\Omega |w|^h\,dx - \frac{1}{h}\int_\Omega |w|^h \log|w|\,dx, \tag{4.3}$$

which gives

$$\mathfrak{N}'(t) = -\int_\Omega |w_t|^2 + |\nabla w_t|^2\,dx \leq 0. \tag{4.4}$$

Now we set an auxiliary functional

$$N(t) = \int_\Omega |w|^2 + |\nabla w|^2\,dx. \tag{4.5}$$

Multiply (1.6) by w and integrate over Ω to get

$$\int_\Omega w w_t\,dx + \int_\Omega \nabla w \nabla w_t\,dx = \int_\Omega |w|^{s(x)}\,dx - \int_\Omega |\nabla w|^{p(x)}\,dx + \int_\Omega |w|^h \log|w|\,dx. \tag{4.6}$$

Now differentiate $N(t)$ with respect to t to obtain

$$\begin{aligned}
N'(t) &= 2\int_\Omega (w w_t + \nabla w \nabla w_t)\,dx \\
&= 2\left[\int_\Omega |w|^{s(x)}\,dx - \int_\Omega |\nabla w|^{p(x)}\,dx + \int_\Omega |w|^h \log|w|\,dx\right] \\
&= 2\left[\int_\Omega s(x)\left[\frac{|w|^{s(x)}}{s(x)} - \frac{|\nabla w|^{p(x)}}{p(x)}\right]dx - \int_\Omega \frac{s(x)}{h^2}|w|^h\,dx + \int_\Omega \frac{s(x)}{h}|w|^h \log|w|\,dx\right. \\
&\quad + \int_\Omega s(x)\left[\frac{1}{p(x)} - \frac{1}{s(x)}\right]|\nabla w|^{p(x)}\,dx + \int_\Omega \frac{s(x)}{h^2}|w|^h\,dx \\
&\quad \left. - \int_\Omega \frac{s(x)}{h}|w|^h \log|w|\,dx + \int_\Omega |w|^h \log|w|\,dx\right].
\end{aligned} \tag{4.7}$$

Since we have $\mathfrak{N}'(t) \leq 0$, we get

$$\begin{aligned}
&\int_\Omega s(x)\left[\frac{|w|^{s(x)}}{s(x)} - \frac{|\nabla w|^{p(x)}}{p(x)} - \frac{1}{h^2}|w|^h + \frac{1}{h}|w|^h \log|w|\right]dx \\
&\quad \geq s_- \int_\Omega \left[\frac{|w_0|^{s(x)}}{s(x)} - \frac{|\nabla w_0|^{p(x)}}{p(x)} - \frac{1}{h^2}|w_0|^h + \frac{1}{h}|w_0|^h \log|w_0|\right]dx \geq 0.
\end{aligned}$$

Hence

$$N'(t) \geq 2\left[\int_\Omega s(x)\left[\frac{1}{p(x)} - \frac{1}{s(x)}\right]|\nabla w|^{p(x)}\,dx + \int_\Omega \frac{s(x)}{h^2}|w|^h\,dx + \int_\Omega \left(1 - \frac{s(x)}{h}\right)|w|^h \log|w|\,dx\right],$$

by (1.7), we know $(1 - \frac{s(x)}{h}) > 0$. So,

$$\begin{aligned} N'(t) &\geq 2\int_\Omega s(x)\left[\frac{1}{p(x)} - \frac{1}{s(x)}\right]|\nabla w|^{p(x)}\,dx \\ &\geq 2\int_\Omega s_-\left[\frac{1}{p_+} - \frac{1}{s_-}\right]|\nabla w|^{p(x)}\,dx = \beta_1 \int_\Omega |\nabla w|^{p(x)}\,dx \geq 0, \end{aligned} \tag{4.8}$$

where $\beta_1 = 2s_-[\frac{1}{p_+} - \frac{1}{s_-}]$. Now define the sets $\Omega_2^+ = \{x \in \Omega : |\nabla w| \geq 1\}$ and $\Omega_2^- = \{x \in \Omega : |\nabla w| \leq 1\}$. Since we have $\|\nabla w\|_2 \leq C\|\nabla w\|_\gamma$ for all $\gamma \geq 2$, we get

$$\begin{aligned} N'(t) &\geq \beta_1\left[\int_{\Omega_2^-} |\nabla w|^{p_+}\,dx + \int_{\Omega_2^+} |\nabla w|^{p_-}\,dx\right] \\ &\geq \beta_2\left[\left(\int_{\Omega_2^-} |\nabla w|^2\,dx\right)^{\frac{p_+}{2}} + \left(\int_{\Omega_2^+} |\nabla w|^2\,dx\right)^{\frac{p_-}{2}}\right]. \end{aligned}$$

This will give

$$\bigl(N'(t)\bigr)^{\frac{2}{p_+}} \geq \beta_3 \int_{\Omega_2^-} |\nabla w|^2\,dx \geq 0, \tag{4.9}$$

$$\bigl(N'(t)\bigr)^{\frac{2}{p_-}} \geq \beta_4 \int_{\Omega_2^+} |\nabla w|^2\,dx \geq 0. \tag{4.10}$$

From the Poincare inequality, we can deduce that $\|\nabla w\|_2^2 \geq \kappa\|w\|_2^2$, where κ is the first eigenvalue of $-\Delta$. Therefore, we get

$$\begin{aligned} \|\nabla w\|_2^2 &= \frac{1}{1+\kappa}\|\nabla w\|_2^2 + \frac{\kappa}{1+\kappa}\|\nabla w\|_2^2 \\ &\geq \frac{\kappa}{1+\kappa}\|w\|_2^2 + \frac{\kappa}{1+\kappa}\|\nabla w\|_2^2 = \frac{\kappa}{1+\kappa}\|\nabla w\|_{H_0^1(\Omega)}^2. \end{aligned} \tag{4.11}$$

Now set $\beta_5 = \min\{\beta_3, \beta_4\}$. Combining (4.9) and (4.10) and using (4.11), we obtain

$$\bigl(N'(t)\bigr)^{\frac{2}{p_+}} + \bigl(N'(t)\bigr)^{\frac{2}{p_-}} \geq \beta_5\|\nabla w\|_2^2 \geq \frac{\kappa\beta_5}{1+\kappa}\|\nabla w\|_{H_0^1\Omega}^2 = \beta_6 N(t), \tag{4.12}$$

where $\beta_6 = \frac{\kappa\beta_5}{1+\kappa}$. Since we have the fact that $N(t) > N(0) > 0$, from (4.12), we get

$$\bigl(N'(t)\bigr)^{\frac{2}{p_+}} \geq \frac{\beta_6}{2}N(0) \quad \text{or} \quad \bigl(N'(t)\bigr)^{\frac{2}{p_-}} \geq \frac{\beta_6}{2}N(0).$$

Consequently,

$$\bigl(N'(t)\bigr) \geq \frac{\beta_7}{2}N(0)^{\frac{p_+}{2}} \quad \text{or} \quad \bigl(N'(t)\bigr) \geq \frac{\beta_8}{2}N(0)^{\frac{p_-}{2}},$$

where $\beta_7 = (\frac{\beta_6}{2})^{\frac{p_+}{2}}$ and $\beta_8 = (\frac{\beta_6}{2})^{\frac{p_-}{2}}$. Now put $\beta_9 = \min\{\frac{\beta_7}{2}N(0)^{\frac{p_+}{2}}, \frac{\beta_8}{2}N(0)^{\frac{p_-}{2}}\}$, then we get

$$\big(N'(t)\big) \geq \beta_9. \tag{4.13}$$

(4.12) implies that

$$\big(N'(t)\big)^{\frac{2}{p_-}}\big(1 + \big(N'(t)\big)^{(\frac{2}{p_+}-\frac{2}{p_-})}\big) \geq \beta_6 N(t). \tag{4.14}$$

From (1.7), we observe that $\frac{2}{p_+} - \frac{2}{p_-} \leq 0$. Making use of (4.13) consequently, we get

$$\big(N'(t)\big) \geq \theta\big(N(t)\big)^{\frac{p_-}{2}}, \tag{4.15}$$

where the constant $\theta = (\frac{\beta_6}{1+\beta_9^{(\frac{2}{p_+}-\frac{2}{p_-})}})^{\frac{p_-}{2}}$. Integrating from 0 to t, (4.15) gives

$$N(t) \geq \frac{1}{[(N(0))^{(1-\frac{p_-}{2})} + (\frac{2-p_-}{2})\theta t]^{\frac{2}{p_--2}}}. \tag{4.16}$$

This gives the finite time blow-up of the solution w at $T^\star$ with

$$T^\star \leq \frac{2[N(0)]^{(\frac{2-p_-}{2})}}{(p_- - 2)\theta}. \tag{4.17}$$

Hence the proof. □

5 Lower bound for blow-up time

Here we obtain a lower bound for the blow-up time of the solutions of (1.6).

Theorem 5.1 *If the weak solution w of the problem (1.6) blows up at finite time $T^\star$, then $T^\star$ has a lower bound given by*

$$T^\star \geq \int_{N(0)}^{\infty} \frac{d\sigma}{2\alpha_2^{s_-}(\sigma)^{\frac{s_-}{2}} + 2\alpha_3^{s_+}(\sigma)^{\frac{s_+}{2}} + 2C\alpha_1(\sigma)^{\frac{h+\eta}{2}}}, \tag{5.1}$$

where C, α_1, α_2 and α_3 are constants.

Proof Consider $N(t)$ as in (4.5). From the previous section, we have

$$N'(t) = 2\bigg[\int_\Omega |w|^{s(x)}\,dx - \int_\Omega |\nabla w|^{p(x)}\,dx + \int_\Omega |w|^h \log|w|\,dx\bigg] \tag{5.2}$$

$$\leq 2\bigg[\int_\Omega |w|^{s(x)}\,dx + \int_\Omega |w|^h \log|w|\,dx\bigg]. \tag{5.3}$$

Since we have $w^{-\eta}\log w \leq (e\eta)^{-1}$ for all $\eta > 0$ and $w \geq 1$, we can deduce

$$\int_\Omega |w|^h \log|w|\,dx \leq \int_{\{x\in\Omega:|w|\geq 1\}} |w|^h \log|w|\,dx$$

$$\leq (e\eta)^{-1}\int_{\{x\in\Omega:|w|\geq 1\}} |w|^{h+\eta}\,dx \leq C\|w\|_{h+\eta}^{h+\eta} \leq C\alpha_1\|\nabla w\|_2^{h+\eta} \tag{5.4}$$

using Sobolev embedding theorem, where α_1 is the embedding constant.

Thus,

$$\int_{\Omega} |w|^h \log|w|\,dx \le C\alpha_1 \|\nabla w\|_2^{h+\eta}. \tag{5.5}$$

Now by Sobolev embedding theorem,

$$\begin{aligned}\int_{\Omega} |w|^{s(x)}\,dx &\le \int_{\Omega} |w|^{s_-}\,dx + \int_{\Omega} |w|^{s_+}\,dx \\ &\le \alpha_2^{s_-}\left(\int_{\Omega} |\nabla w|^2\,dx\right)^{\frac{s_-}{2}} + \alpha_3^{s_+}\left(\int_{\Omega} |\nabla w|^2\,dx\right)^{\frac{s_+}{2}},\end{aligned} \tag{5.6}$$

where α_2 and α_3 are the corresponding embedding constants. The inequalities (5.5) and (5.6) together imply

$$\begin{aligned}N'(t) &\le 2\alpha_2^{s_-}\left(\int_{\Omega} |\nabla w|^2\,dx\right)^{\frac{s_-}{2}} + 2\alpha_3^{s_+}\left(\int_{\Omega} |\nabla w|^2\,dx\right)^{\frac{s_+}{2}} + 2C\alpha_1\left(\int_{\Omega} |\nabla w|^2\,dx\right)^{\frac{h+\eta}{2}} \\ &\le 2\alpha_2^{s_-}\big(N(t)\big)^{\frac{s_-}{2}} + 2\alpha_3^{s_+}\big(N(t)\big)^{\frac{s_+}{2}} + 2C\alpha_1\big(N(t)\big)^{\frac{h+\eta}{2}}.\end{aligned} \tag{5.7}$$

Integrating (5.7) from 0 to t, we get

$$\int_{N(0)}^{N(t)} \frac{d\sigma}{2\alpha_2^{s_-}(\sigma)^{\frac{s_-}{2}} + 2\alpha_3^{s_+}(\sigma)^{\frac{s_+}{2}} + 2C\alpha_1(\sigma)^{\frac{h+\eta}{2}}} \le t. \tag{5.8}$$

Theorem 4.1 ensures the existence of finite time blow-up. Thus, from (5.8), we get a lower bound as below

$$T^\star \ge \int_{N(0)}^{\infty} \frac{d\sigma}{2\alpha_2^{s_-}(\sigma)^{\frac{s_-}{2}} + 2\alpha_3^{s_+}(\sigma)^{\frac{s_+}{2}} + 2C\alpha_1(\sigma)^{\frac{h+\eta}{2}}}, \tag{5.9}$$

which completes the proof. □

6 Conclusion

In history, there are many studies devoted to logarithmic nonlinearity or polynomial nonlinearity. The work in this paper is about what happens to the solutions when we combine these two nonlinearities together. Here we established the existence and finite time blow-up of solutions for the case when $s(x) < h$. Also, we obtained upper and lower bounds for the blow-up time under suitable conditions. The case $s(x) > h$ is still under study.

Acknowledgements

The authors are thankful to the referee for the improvement of the paper.

Funding

The first author would like to thank the Department of Science and Technology, New Delhi, for their financial support under the INSPIRE fellowship program. This research was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. 2021R1F1A1048937) to the last author.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors have contributed equally to this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Central University of Kerala, Kasaragod 671320, India. [2]Department of Mathematics, Bharathiar University, Coimbatore 641046, India. [3]Department of Applied Mathematics, Kongju National University, Chungcheongnam-do 32588, Republic of Korea.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 November 2021 Accepted: 14 April 2022 Published online: 03 May 2022

References
1. Barrow, J., Parsons, P.: In inflationary models with logarithmic potentials. Phys. Rev. D **52**, 5576–5587 (1995)
2. Bartkowski, K., Gorka, P.: One-dimensional Klein-Gordon equation with logarithmic non-linearities. J. Phys. A **41**, 355201 (2008)
3. Bernini, F., Mugnai, D.: On a logarithmic Hartree equation. Adv. Nonlinear Anal. **9**, 850–865 (2020)
4. Bialynicki-Birula, I., Mycielski, J.: Wave equations with logarithmic nonlinearities. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **23**, 461–466 (1975)
5. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011)
6. Cao, Y., Liu, C.: Initial boundary value problem for a mixed pseudo parabolic p-Laplacian type equation with logarithmic nonlinearity. Electron. J. Differ. Equ. **116**, 1 (2018)
7. Chen, H., Luo, P., Liu, G.: Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. J. Math. Anal. Appl. **422**, 84–98 (2015)
8. Chen, H., Tian, S.: Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity. J. Differ. Equ. **258**, 4424–4442 (2015)
9. Dai, P., Mu, C., Xu, G.: Blow-up phenomena for a pseudo-parabolic equation with p-Laplacian and logarithmic nonlinearity terms. J. Math. Anal. Appl. **481**, 123439 (2020)
10. Di, H., Shang, Y., Peng, X.: Blow-up phenomena for a pseudo-parabolic equation with variable exponents. Appl. Math. Lett. **64**, 67–73 (2017)
11. Diening, L., Hästö, P., Harjulehto, P., Ružička, M.M.: Lebesgue and Sobolev Spaces with Variable Exponents. Springer, Berlin (2011)
12. Ding, H., Zhou, J.: Global existence and blow-up for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity. J. Math. Anal. Appl. **478**, 393–420 (2019)
13. Enqvist, K., McDonald, J.: Q-balls and baryogenesis in the MSSM. Phys. Lett. B **425**, 309–321 (1998)
14. Fan, X., Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. **263**, 424–446 (2001)
15. Gao, Y., Guo, B., Gao, W.: Weak solutions for a high-order pseudo-parabolic equation with variable exponents. Appl. Anal. **93**, 322–338 (2014)
16. Gorka, P.: Logarithmic Klein-Gordon equation. Acta Phys. Pol. B **40**, 59–66 (2009)
17. Gurusamy, A., Balachandran, K.: Blow-up of solutions to reaction-diffusion system with nonstandard growth conditions. J. Appl. Nonlinear Dyn. **6**, 407–425 (2017)
18. Gurusamy, A., Balachandran, K.: Global existence and blow-up of solutions to a coupled parabolic system. Indian J. Ind. Appl. Math. **11**, 138–156 (2020)
19. Gurusamy, A., Shangerganesh, L., Balachandran, K.: Blow-up solutions of nonlinear parabolic system with variable exponents. Nonlinear Funct. Anal. Appl. **21**, 449–461 (2016)
20. Han, Y.: Finite time blowup for a semilinear pseudo-parabolic equation with general nonlinearity. Appl. Math. Lett. **99**, 105986 (2020)
21. Han, Y., Cao, C., Sun, P.: A p p-Laplace equation with logarithmic nonlinearity at high initial energy level. Acta Appl. Math. **164**, 155–164 (2019)
22. He, Y., Gao, H., Wang, H.: Blow-up and decay for a class of pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity. Comput. Math. Appl. **75**, 459–469 (2018)
23. Huan, Y.: Blow-up at infinity of solutions to a semilinear heat equation with logarithmic nonlinearity. J. Math. Anal. Appl. **474**, 513–517 (2019)
24. Korpusov, M.O., Sveshnikov, A.G.: Three dimensional non-linear evolutionary pseudo - parabolic equations in mathematical physics. Ž. Vyčisl. Mat. Mat. Fiz. **43**, 1835–1869 (2003)
25. Lian, W., Ahmed, M.S., Xu, R.: Global existence and blow up of solution for semi-linear hyperbolic equation with the product of logarithmic and power-type nonlinearity. Opusc. Math. **40**, 111–130 (2020)
26. Lian, W., Wang, J., Xu, R.: Global existence and blow up of solutions for pseudo-parabolic equation with singular potential. J. Differ. Equ. **269**, 4914–4959 (2020)
27. Lian, W., Xu, R.: Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term. Adv. Nonlinear Anal. **9**, 613–632 (2020)
28. Liao, M.: Non-global existence of solutions to pseudo-parabolic equations with variable exponents and positive initial energy. C. R. Mecanique **347**, 710–715 (2019)
29. Liao, M., Guo, B., Li, Q.: Global existence and energy decay estimates for weak solutions to the pseudo-parabolic equation with variable exponents. Math. Methods Appl. Sci. **43**, 2516–2527 (2020)

30. Luo, P.: Blow-up phenomena for a pseudo-parabolic equation. Math. Methods Appl. Sci. **38**, 2636–2641 (2015)
31. Marras, M., Vernier-Piro, S., Viglialoro, G.: Blow-up phenomena for nonlinear pseudo parabolic equations with gradient term. Discrete Contin. Dyn. Syst., Ser. B **22**, 2291–2300 (2017)
32. Meyvaci, M.: Blow up of solutions of pseudo parabolic equations. J. Math. Anal. Appl. **352**, 629–633 (2009)
33. Meyvaci, M.: Bounds for blow-up time in nonlinear pseudo-parabolic equations. Mediterr. J. Math. **15**, Article ID 8 (2018)
34. Nhan, L.C., Truong, L.X.: Global solution and blow-up for a class of pseudo p-Laplacian evolution equations with logarithmic nonlinearity. Comput. Math. Appl. **73**, 2076–2091 (2017)
35. Nhan, L.C., Truong, L.X.: Global solution and blow-up for a class of p-Laplacian evolution equations with logarithmic nonlinearity. Acta Appl. Math. **151**(1), 149–169 (2017)
36. Peng, X., Shang, Y., Zheng, X.: Blow-up phenomena for some nonlinear pseudo-parabolic equations. Appl. Math. Lett. **56**, 17–22 (2016)
37. Sarra, T., Zara, A., Boulaaras, S.: Decay estimate and non-extinction of solutions of p-Laplacian nonlocal heat equations. AIMS Math. **5**, 1663–1679 (2020)
38. Shangerganesh, L., Gurusamy, A., Balachandran, K.: Weak solutions for nonlinear parabolic equations with variable exponents. Commun. Math. **25**, 55–70 (2017)
39. Showalter, R.E., Ting, T.W.: Pseudoparabolic partial differential equations. SIAM J. Math. Anal. **1**, 1–26 (1970)
40. Simon, J.: Compact sets in the space $L^p(0,T;B)$. Annali di Matematica Pura et Applicanta **146**, 65–96 (1987)
41. Sun, F., Liu, L., Wu, Y.: Finite time blow-up for a class of parabolic or pseudo-parabolic equations. Comput. Math. Appl. **75**, 3685–3701 (2018)
42. Wang, X., Chen, Y., Yang, Y., Li, J., Xu, R.: Kirchhoff-type system with linear weak damping and logarithmic nonlinearities. Nonlinear Anal. **188**, 475–499 (2019)
43. Wang, X., Xu, R.: Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation. Adv. Nonlinear Anal. **10**, 261–288 (2021)
44. Xu, R., Su, J.: Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations. J. Funct. Anal. **264**, 2732–2763 (2013)
45. Xu, R., Wang, X., Yang, Y.: Blowup and blowup time for a class of semilinear pseudo-parabolic equations with high initial energy. Appl. Math. Lett. **83**, 176–181 (2018)
46. Zhu, X., Li, F., Li, Y.: Some sharp results about the global existence and blowup of solutions to a class of pseudo-parabolic equations. Proc. R. Soc. Edinb. **147A**, 1311–1331 (2017)