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Optimal time-decay rates of the Keller–Segel system coupled to compressible Navier–Stokes equation in three dimensions



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Abstract

Recently, Hattori–Lagha established the global existence and asymptotic behavior of the solutions for a three-dimensional compressible chemotaxis system with chemoattractant and repellent (Hattori and Lagha in Discrete Contin. Dyn. Syst. 41(11):5141–5164, 2021). Motivated by Hattori–Lagha's work, we further investigated the optimal time-decay rates of strong solutions with small perturbation to the three-dimensional Keller–Segel system coupled to the compressible Navier–Stokes equations, which models for the motion of swimming bacteria in a compressible viscous fluid. First, we reformulate the system into a perturbation form. Then we establish *a prior* estimates of solutions and prove the existence of the global-in-time solutions based on the local existence of unique solutions. Finally, we will establish the optimal time-decay rates of the nonhomogeneous system by the decomposition technique of both low and high frequencies of solutions as in (Wang and Wen in Sci. China Math., 2020, https://doi.org/10.1007/s11425-020-1779-7). Moreover, the decay rate is optimal since it agrees with the solutions of the linearized system.

Keywords: Compressible chemotactic fluids; Global existence; Uniqueness; Fourier theory; Optimal time-decay rates

1 Introduction

As described in the pioneering literature Keller–Segel [26], chemotaxis as a biological process is responsible for some instances of such demeanor, which is the directed movement of living cells (e.g., bacteria) that move towards a chemically more favorable environment under the effects of chemical gradients. We shall note that when no chemicals are present, the movement of cells is completely random. When an attractant chemical is present, the motility changes, and the tumbles become less frequent so that the cells move towards the chemical attractant. It is important for microorganism to find food (e.g., glucose) by swimming toward the highest concentration of food molecules. A lot of relevant mathematical models have been developed; see [3, 14, 15] for examples. Furthermore, in [8], it can be observed experimentally that bacteria are suspended in the fluid, which is influenced by the gravitational forcing generated by the aggregation of cells. Moreover, oxygen plays an important role in the reproduction of aerobic bacteria. For instance, bacillus subtilis often

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live in the thin fluid layers near the solid-air-water contact line in which the swimming bacteria move towards a higher concentration of oxygen according to the mechanism of chemotaxis. Further, we also note that oxygen concentration, chemical attractant, and bacteria density are transported by the fluid and diffuse through the fluid [6, 8, 29, 38].

Concerning the chemotaxis models based on fluid dynamics, i.e., the chemotaxis-fluid system, there are two approaches: incompressible and compressible. For the incompressible case, Chae-Kang-Lee [4] and Duan-Lorz-Markowich [9] showed the global-in-time existence for the incompressible chemotaxis equations near the constant states if the initial data is sufficiently small. Rodriguez, Ferreira, and Villamizar-Roa [10] showed the global existence of an attraction-repulsion chemotaxis-fluid system with a logistic source. Tan–Zhou [35, 36] proved the global existence and time-decay estimate of solutions to the chemotaxis-fluid system in \mathbb{R}^3 with small initial data. Later Tan-Zhong-Wu further obtained the time-decay estimates of time-periodic strong solutions [34]. The interested readers are referred to [27, 28, 37, 39, 43-47, 49-52] for more mathematical results concerning the well-posedness and regularity of solutions of the various types of the chemotaxis-fluid system. For the compressible case, Ambrosi-Bussolino-Preziosi [1] discussed vasculogenesis using the compressible fluid dynamics for the cells and the diffusion equation for the attractant. Modeling aspects of vasculogenesis are studied in [2, 12, 33]. Recently Hattori-Lagha established the global existence and the temporal decay of the solutions for a three-dimensional compressible chemotaxis system with chemoattractant and repellent [13]. We mention that the temporal decay of solutions is hydrodynamic equations are hot topics; see [11, 16–24, 42] and the references cited therein.

Motivated by Hattori–Lagha's temporal decay results in [13], we further investigated the optimal time-decay rates of strong solutions with small perturbation to the threedimensional Keller–Segel system coupled to the compressible Navier–Stokes equations, which models for the motion of swimming bacteria in a compressible viscous fluid in \mathbb{R}^3 and reads as follows:

$$\begin{cases}
\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\
\rho \mathbf{u}_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \lambda_1 \Delta \mathbf{u} + \lambda_2 \nabla \operatorname{div} \mathbf{u} - n \nabla \phi, \\
n_t + \mathbf{u} \cdot \nabla n = \Delta n - \nabla \cdot (nS(n) \nabla c), \\
c_t + \mathbf{u} \cdot \nabla c = \Delta c - nf(c).
\end{cases}$$
(1.1)

Next we shall introduce the notations in the above system of equations, which is called a (single) compressible chemotaxis-fluid system or compressible Keller–Segel–Navier– Stokes system.

The unknown functions $\rho = \rho(t, x)$ and $\mathbf{u} = \mathbf{u}(t, x)$ denote the density and velocity of fluids, resp. The unknown functions n = n(t, x) and c = c(t, x) represent density of amoebae and oxygen concentration, resp. $\lambda_1 > 0$ is the coefficient of shear viscosity, and $\lambda_2 := \nu + \lambda_1/3$ with ν being the positive bulk viscosity. $\phi = \phi(t, x)$ is a given potential function. The smooth function $P(\cdot) > 0$ is the pressure of fluid (depending on ρ), S(n) a given sensitivity parameter function, and the consumption rate of oxygen f(c) a step function [40], please refer to [52] for the different mathematical expressions of S(n) and f(c) corresponding to different environments. However, for the sake of simplicity, we assume that in this paper,

$$\phi = 0$$
, $S(n) = 1$ and $f(c) = c$.

For the investigation of the Cauchy problem of the system (1.1), we should pose the initial condition:

$$(\rho, \mathbf{u}, n, c)|_{t=0} = (\rho_0, \mathbf{u}_0, n_0, c_0)$$
(1.2)

In this paper, we prove the global existence of small perturbation solutions around some rest state for the Cauchy problem (1.1)-(1.2) and provide time-decay rates for the strong solutions. Moreover, the decay rate is optimal since it agrees with the solutions of the linearized system. It should be noted that Hattori–Lagha only gave the time-decay rates for the zero-order derivative of solutions in [13]. However, the novelty of this paper is that we further provide time-decay rates for *all-order derivatives of solutions* using a decomposition technique of both low and high frequencies of solutions as in [41].

1.1 Notations

Before stating our main result, we shall introduce some notations, which are used frequently throughout the paper.

The notation $C_i > 0$ $(i \in \mathbb{Z}^+)$ represents a fixed constant. For simplicity, we use the expression $m \leq n$ to mean $m \leq Cn$, where *C* is a positive constant and varies from line to line. $\nabla = (\partial_1, \partial_2, \partial_3)^T$, where $\partial_1 = \partial_{x_1}$. $\partial_x^{\alpha} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ with a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. We set $\langle \cdot, \cdot \rangle$ to represent the inner product in $L^2(\mathbb{R}^3)$, i.e.

$$\langle f,g\rangle = \int_{\mathbb{R}^3} f(x)g(x) \,\mathrm{d}x \quad \text{for } f(x),g(x) \in L^2(\mathbb{R}^3).$$

For $m \ge 0$ and $p \ge 1$, the norms of Sobolev spaces $H^m(\mathbb{R}^3)$ and $W^{m,p}(\mathbb{R}^3)$ are denoted by $\|\cdot\|_{H^m}$ and $\|\cdot\|_{W^{m,p}}$, resp. In particular, we will switch to use $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^p}$ for m = 0, resp. In addition, $\widehat{f}(\xi)$ is the Fourier transform of f(x) with respect to the variables $x \in \mathbb{R}^3$, that is $\widehat{f}(\xi) = \mathcal{F}(f)(\xi)$. We further define

$$\Lambda^m f = \mathcal{F}^{-1}(|\xi|^m \widehat{f}) \quad \text{for } m \in \mathbb{R},$$

where Λ^m is a pseudo-differential operator.

Let $\chi_0(\xi)$ and $\chi_1(\xi)$ be two smooth cut-off functions satisfying $0 \le \chi_0(\xi)$, $\chi_1(\xi) \le 1$ ($\xi \in \mathbb{R}^3$) and

$$\chi_0(\xi) = \begin{cases} 1, & |\xi| < \frac{r_0}{2}; \\ 0, & |\xi| > r_0, \end{cases} \qquad \chi_1(\xi) = \begin{cases} 0, & |\xi| < R_0; \\ 1, & |\xi| > R_0 + 1 \end{cases}$$

for r_0 and R_0 satisfying

$$0 < r_0 \leq \min\left\{\frac{1}{2}\sqrt{\frac{\mu}{\nu}}, \frac{1}{2}\right\} \text{ and } R_0 > \max\left\{2\sqrt{\frac{\mu_2 + \mu}{\mu_1}}, 1\right\}.$$

Let $\chi_0(D_x)$ and $\chi_1(D_x)$ be quasi-differential operators of $\chi_0(\xi)$ and $\chi_1(\xi)$ resp., then for any given function $f(x) \in L^2(\mathbb{R}^3)$, we can define its frequency distribution $(f^l(x), f^m(x), f^h(x))$ as follows

$$f^{l}(x) = \chi_{0}(D_{x})f(x), \qquad f^{m}(x) = \left(I - \chi_{0}(D_{x}) - \chi_{1}(D_{x})\right)f(x), \qquad f^{h}(x) = \chi_{1}(D_{x})f(x),$$

where $D_x = \frac{1}{\sqrt{-1}} \nabla = \frac{1}{\sqrt{-1}} (\partial_1, \partial_2, \partial_3)$. Notice that f(x) can be expressed as follows

$$f(x) = f^{l}(x) + f^{m}(x) + f^{h}(x),$$

where we have defined that $f^{L}(x) = f^{l}(x) + f^{m}(x)$ and $f^{H}(x) = f^{m}(x) + f^{h}(x)$.

1.2 Main results

Now we state the main result of this paper.

Theorem 1.1 Suppose that $(\rho_0 - \rho_\infty, \mathbf{u}_0, n_0 - n_\infty, c_0) \in H^2(\mathbb{R}^3)$ for some constants $\rho_\infty > 0$ and $n_\infty > 0$. There exists a constant $\varepsilon > 0$ such that if

$$\left\| \left(\rho_0 - \rho_\infty, \mathbf{u}_0, n_0 - n_\infty, c_0 \right) \right\|_{H^2(\mathbb{R}^3)} \le \varepsilon, \tag{1.3}$$

and

$$0 \leq c_0 \leq 1$$
,

then the Cauchy problem of (1.1)-(1.2) with initial data admits a unique global-in-time solution (ρ, \mathbf{u}, n, c) , which satisfies

$$\rho - \rho_{\infty} \in C^{0}([0,\infty); H^{2}(\mathbb{R}^{3})) \cap C^{1}([0,\infty); H^{1}(\mathbb{R}^{3})),$$

$$\mathbf{u}, n - n_{\infty}, c \in C^{0}([0,\infty); H^{2}(\mathbb{R}^{3})) \cap C^{1}([0,\infty); L^{2}(\mathbb{R}^{3})),$$

$$0 \le c(t,x) \le 1.$$
(1.4)

Furthermore, if the initial data $(\rho_0 - \rho_\infty, \mathbf{u}_0, n_0 - n_\infty, c_0)$ is bounded in $L^1(\mathbb{R}^3)$, then there exists a constant C > 0, such that, for any $t \ge 0$,

$$\left\|\nabla^{k}(\rho - \rho_{\infty}, \mathbf{u}, n - n_{\infty}, c)\right\|_{L^{2}(\mathbb{R}^{3})} \le C(1+t)^{-\frac{3}{4} - \frac{k}{2}}, \quad k = 0, 1, 2,$$
(1.5)

$$\|c(t)\|_{H^2(\mathbb{R}^3)} \le Ce^{-Ct}.$$
(1.6)

Now we shall introduce our main idea for deriving the optimal time-decay rates in (1.5). The main difficulty focuses on obtaining the energy estimates, which include only the highest-order spatial derivative of the solution $\nabla^2(\rho - 1, \mathbf{u})$, which is essentially caused by the "degenerate" dissipative structure of the hyperbolic parabolic system. To get the dissipative estimate for $\nabla^2 \rho$, the usual energy method is to construct the interaction energy functional between u and $\nabla \rho$ using the pressure term in linearized momentum equations; see (3.27). It implies that both the first and second orders of the spatial derivatives of the velocity and the density should be involved in the Lyapunov functional

$$\mathcal{L}(t) = \left\|\nabla\rho\right\|_{H^{1}}^{2} + \left\|\nabla\mathbf{u}(t)\right\|_{H^{1}}^{2} + \int_{\mathbb{R}^{3}} \nabla\mathbf{u} \cdot \nabla\nabla\rho \, \mathrm{d}x \sim \left\|\nabla(\rho, \mathbf{u})(t)\right\|_{H^{1}}^{2}.$$

Consequently, the L^2 -norms of the highest order and the first-order derivative of solutions have the same time-decay rate.

One of the main goals of this paper is to develop a way to capture the optimal time-decay rates for the highest order derivative of the solution to the Cauchy problem (1.1)–(1.2) if the initial perturbation is bounded in $L^1(\mathbb{R}^3)$. Firstly, using the standard energy method, we establish estimate (3.24) of the energy functional $\mathfrak{D}_h(t)$ in (3.22). Secondly, motivated by the decomposition technique of both the low and high frequencies of solutions in [41], to get rid of the obstacle from the term $\int_{\mathbb{R}^3} \nabla u \cdot \nabla \rho \, dx$, we shall remove the low-medium-frequency part of the term from $\mathfrak{D}_h(t)$ in (4.12), which requires a new estimate for the low-medium-frequency term (see Lemma 4.1 for detailed derivation).

The rest of this paper is organized as follows. In Sect. 2, for the convenience of analysis, we write the original system (1.1) as a perturbation form (2.3). In Sect. 3, we establish *a prior* estimates of solutions, and provide the global unique solvability for the Cauchy problem of (1.1)-(1.2). Finally, in Sect. 4, we will derive the optimal time decay rate for the non-homogeneous system (2.3) by the decomposition technique of both low and high frequencies of solutions as in [41].

2 Reformation of motion equations

To facilitate the proof of Theorem 1.1, we shall first reformulate the Cauchy problem (1.1)-(1.2). Obviously, we have

$$\begin{cases}
\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\
(\rho \mathbf{u})_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} + P'(\rho) \nabla \rho = \lambda_1 \Delta \mathbf{u} + \lambda_2 \nabla \operatorname{div} \mathbf{u}, \\
n_t + \mathbf{u} \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c), \\
c_t + \mathbf{u} \cdot \nabla c = \Delta c - nc.
\end{cases}$$
(2.1)

Let

(

$$\sigma = \rho - \rho_{\infty}, \qquad \mathbf{u} = \mathbf{u}, \qquad N = n - n_{\infty}, \qquad c = c, \tag{2.2}$$

then the following inhomogeneous system of equations is equivalent to (1.1):

$$\begin{cases} \sigma_t + \rho_{\infty} \operatorname{div} \mathbf{u} = \tilde{M}_1, \\ \mathbf{u}_t + \frac{P'(\rho_{\infty})}{\rho_{\infty}} \nabla \sigma - \frac{\lambda_1}{\rho_{\infty}} \Delta \mathbf{u} - \frac{\lambda_2}{\rho_{\infty}} \nabla \operatorname{div} \mathbf{u} = \tilde{M}_2, \\ N_t - \Delta N + n_{\infty} \nabla^2 c = \tilde{M}_3, \\ c_t - \Delta c + n_{\infty} c = \tilde{M}_4, \end{cases}$$

$$(2.3)$$

where we have defined that

$$\begin{split} \tilde{M}_1 &= -\operatorname{div}(\sigma \mathbf{u}), \\ \tilde{M}_2 &= -\mathbf{u} \cdot \nabla \mathbf{u} - h_1 \nabla \sigma + \lambda_1 g_1 \Delta \mathbf{u} + \lambda_2 g_1 \nabla \operatorname{div} \mathbf{u}, \\ \tilde{M}_3 &= -\mathbf{u} \cdot \nabla N - \nabla N \nabla c - N \nabla^2 c, \\ \tilde{M}_4 &= -\mathbf{u} \cdot \nabla c - N c \end{split}$$
(2.4)

with

$$\begin{cases} h_1 = \frac{P'(\sigma + \rho_\infty)}{\rho_\infty + \sigma} - \frac{P'(\rho_\infty)}{\rho_\infty}, \\ g_1 = \frac{1}{\rho_\infty + \sigma} - \frac{1}{\rho_\infty}. \end{cases}$$
(2.5)

From now on, we renew to define $\beta \mathbf{u}$ by u, then the system (2.3) is reformulated as

$$\begin{cases} \sigma_t + \mu \operatorname{div} \mathbf{u} = M_1, \\ \mathbf{u}_t + \mu \nabla \sigma - \mu_1 \Delta \mathbf{u} - \mu_2 \nabla \operatorname{div} \mathbf{u} = M_2, \\ N_t - \Delta N + n_\infty \nabla^2 c = M_3, \\ c_t - \Delta c + n_\infty c = M_4, \end{cases}$$
(2.6)

with the initial data

$$\begin{aligned} (\sigma, \mathbf{u}, N, c)|_{t=0} &= (\sigma_0, \mathbf{u}_0, N_0, c_0)(x) \\ &:= (\rho_0 - \rho_\infty, \mathbf{u}_0, n_0 - n_\infty, c_0)(x) \to 0 \quad \text{as } |x| \to +\infty, \end{aligned}$$
(2.7)

where

$$\mu_{1} = \frac{\lambda_{1}}{\rho_{\infty}}, \qquad \mu_{2} = \frac{\lambda_{2}}{\rho_{\infty}},$$

$$\beta = \frac{\rho_{\infty}}{\sqrt{P'(\rho_{\infty}) + n_{\infty}\phi'(\rho_{\infty})}}, \qquad \mu = \sqrt{\left[P'(\rho_{\infty}) + n_{\infty}\phi'(\rho_{\infty})\right]}, \qquad (2.8)$$

$$(M_1, M_2, M_3, M_4) := (\tilde{M}_1, \beta \tilde{M}_2, \tilde{M}_3, \tilde{M}_4) \bigg(\sigma, \frac{1}{\beta} \mathbf{u}, N, c \bigg).$$
(2.9)

3 Global existence and uniqueness for the nonlinear system

In this section, we will prove the global well-posedness result in Theorem 1.1, that is, the global existence and uniqueness for the solutions of the chemotaxis–fluid system.

3.1 Unique solvability

First of all, we define a work space for the Cauchy problem of (2.6) and (2.7) as follows

$$\begin{aligned} X(0,T) &= \left\{ (\sigma,\mathbf{u},N,c) \mid \sigma \in C^0((0,T); H^2(\mathbb{R}^3)) \cap C^1((0,T); H^1(\mathbb{R}^3)), \\ \mathbf{u},N,c \in C^0((0,T); H^2(\mathbb{R}^3)) \cap C^1((0,T); L^2(\mathbb{R}^3)), \\ \nabla \sigma \in L^2((0,T); H^2(\mathbb{R}^3), \nabla \mathbf{u}, \nabla N \in L^2((0,T); H^1(\mathbb{R}^3), \\ c \in L^2((0,T); H^3(\mathbb{R}^3)) \right\} \end{aligned}$$

for any $0 \le T \le +\infty$.

Then, we further introduce the results of local existence and *a priori* estimates of solutions in sequence.

Proposition 3.1 (Local existence) Let $(\sigma_0, \mathbf{u}_0, N_0, c_0) \in H^2(\mathbb{R}^3)$ and

$$\inf_{x\in\mathbb{R}^3}\{\sigma_0+\rho_\infty,N+n_\infty\}>0\quad and\quad 0\leq c_0\leq 1.$$

Then, there exists a constant $T_0 > 0$ depending on $\|\sigma_0, \mathbf{u}_0, N_0, c_0\|_{H^2(\mathbb{R}^3)}$ such that the Cauchy problem (2.6) and (2.7) has a unique solution (σ, \mathbf{u}, N, c) $\in (0, T_0)$, which satisfies

$$\inf_{x\in\mathbb{R}^3,0\leq t\leq T_0}\{\sigma+\rho_\infty,N+n_\infty\}>0\quad and\quad 0\leq c(x,t)\leq 1.$$

Proof We can easily prove the above conclusion using an iterative method, the fixed point theorem, and the maxima principle. Interested readers can refer to [5, 31] for the proof.

Proposition 3.2 (A priori estimate) Suppose that the Cauchy problem of (2.6) and (2.7) has a solution $(\sigma, \mathbf{u}, N, c) \in (0, T)$, where T > 0, then there exists a sufficiently small constant $\delta > 0$ and a positive constant C_1 independent of T, such that if the solution satisfies

$$\sup_{0 \le t \le T} \left\| (\sigma, \mathbf{u}, N, c)(t) \right\|_{H^2} \le \delta,$$
(3.1)

then we have

$$\| (\sigma, \mathbf{u}, N, c)(t) \|_{H^{2}}^{2} + \int_{0}^{t} \left(\| \nabla \sigma(\tau) \|_{H^{1}}^{2} + \| \nabla (\mathbf{u}, N)(\tau) \|_{H^{2}}^{2} + \| c(\tau) \|_{H^{3}}^{2} \right) d\tau$$

$$\leq C_{1} \| (\sigma_{0}, \mathbf{u}_{0}, N_{0}, c_{0})(t) \|_{H^{2}}^{2}$$
 (3.2)

hold for any $t \in [0, T]$.

Proof The proof of Proposition 3.2 will be given in Sect. 3.2.

Remark 3.1 Here C_1 is independent of ε and δ , and $\delta = \max \{2\varepsilon, \frac{3\sqrt{C_1\varepsilon}}{2}\}$ such that

$$\|(\sigma, \mathbf{u}, N, c)(t)\|_{H^2}^2 \le C_1 \|(\sigma_0, \mathbf{u}_0, N_0, c_0)(t)\|_{H^2}^2 \le \left(\frac{2\delta}{3}\right)^2.$$

In addition, by (3.1), we have

$$\left| \left(h_1(\rho), g_1(\rho) \right) \right| \le C |\rho|, \qquad \left| \nabla^k \left(h_1(\rho), g_1(\rho) \right) \right| \le C, \text{ for any given } k \ge 1.$$

Thanks to Propositions 3.1 and 3.2, we immediately get the global existence of unique solutions of the Cauchy problem (2.6)-(2.7) using a standard continuity argument.

3.2 Proof of Proposition 3.2

In this section, we aim to complete the proof of Proposition 3.2. The key step is to derive the energy estimates of the lower and higher derivatives of the solution (σ , **u**, N, c) of the Cauchy problem (2.6)–(2.7).

Lemma 3.1 Let the functional $\mathfrak{D}_l(t)$ be defined as follows

$$\mathfrak{D}_{l}(t) := \frac{1}{2} \bigg\{ \|\sigma\|_{H^{1}}^{2} + 2\alpha_{1} \int_{\mathbb{R}^{3}} \nabla \sigma \cdot \mathbf{u} \, \mathrm{d}x + \|\mathbf{u}\|_{H^{1}}^{2} + \|N\|_{H^{1}}^{2} + 2\gamma \|c\|_{H^{1}}^{2} \bigg\},$$
(3.3)

then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{D}_{l}(t) + \frac{\alpha_{1}\mu}{4} \|\nabla\sigma\|_{L^{2}}^{2} + \frac{\mu_{1}}{4} \|\nabla\mathbf{u}\|_{H^{1}}^{2} + \frac{\mu_{2}}{4} \|\operatorname{div}\mathbf{u}\|_{H^{1}}^{2}
+ \frac{1}{4} \|\nabla N\|_{H^{1}}^{2} + \frac{\gamma}{2} \|\nabla c\|_{H^{1}}^{2} + \frac{n_{\infty}^{2}\gamma}{4} \|c\|_{H^{1}}^{2} \leq 0,$$
(3.4)

where α_1 and γ are two given constants.

Proof Multiplying $\nabla^k(2.6)_1 - \nabla^k(2.6)_3$ by $\nabla^k \sigma$, $\nabla^k \mathbf{u}$ and $\nabla^k N$ in $L^2(\mathbb{R}^3)$ resp., then summing the resulting identities up, and finally, using Young inequality and the integral by parts, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\left\| \nabla^{k} \sigma \right\|_{L^{2}}^{2} + \left\| \nabla^{k} \mathbf{u} \right\|_{L^{2}}^{2} + \left\| \nabla^{k} N \right\|_{L^{2}}^{2} \right)
+ \mu_{1} \left\| \nabla^{k} \nabla \mathbf{u} \right\|_{L^{2}}^{2} + \mu_{2} \left\| \nabla^{k} \operatorname{div} \mathbf{u} \right\|_{L^{2}}^{2} + \left\| \nabla^{k} \nabla N \right\|_{L^{2}}^{2}
= -n_{\infty} \int_{\mathbb{R}^{3}} \nabla^{k} \nabla^{2} c \nabla^{k} N \, \mathrm{d}x + \int_{\mathbb{R}^{3}} \nabla^{k} \sigma \nabla^{k} M_{1} \, \mathrm{d}x
+ \int_{\mathbb{R}^{3}} \nabla^{k} \mathbf{u} \nabla^{k} M_{2} \, \mathrm{d}x + \int_{\mathbb{R}^{3}} \nabla^{k} N \nabla^{k} M_{3} \, \mathrm{d}x
\leq \frac{n_{\infty}^{2}}{2} \left\| \nabla^{k} \nabla c \right\|_{L^{2}}^{2} + \frac{1}{2} \left\| \nabla^{k} \nabla N \right\|_{L^{2}}^{2} + \int_{\mathbb{R}^{3}} \nabla^{k} \sigma \nabla^{k} M_{1} \, \mathrm{d}x
+ \int_{\mathbb{R}^{3}} \nabla^{k} \mathbf{u} \nabla^{k} M_{2} \, \mathrm{d}x + \int_{\mathbb{R}^{3}} \nabla^{k} N \nabla^{k} M_{3} \, \mathrm{d}x.$$
(3.5)

Multiplying ∇ (2.6)₁, (2.6)₂ by **u** and $\nabla \sigma$, resp., and then integrating by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{3}} \nabla \sigma \cdot \mathbf{u} \, \mathrm{d}x + \mu \int_{\mathbb{R}^{3}} |\nabla \sigma|^{2} \, \mathrm{d}x$$

$$= \mu \| \operatorname{div} \mathbf{u} \|_{L^{2}}^{2} + \mu_{1} \int_{\mathbb{R}^{3}} \nabla \sigma \cdot \Delta \mathbf{u} \, \mathrm{d}x$$

$$+ \mu_{2} \int_{\mathbb{R}^{3}} \nabla \sigma \cdot \nabla \operatorname{div} \mathbf{u} \, \mathrm{d}x + \int_{\mathbb{R}^{3}} \mathbf{u} \cdot \nabla M_{1} \, \mathrm{d}x + \int_{\mathbb{R}^{3}} \nabla \sigma \cdot M_{2} \, \mathrm{d}x.$$
(3.6)

For any given constant $\alpha_1 > 0$, we use Young's inequality to get

$$\begin{aligned} &\alpha_1 \mu_1 \int_{\mathbb{R}^3} \nabla \sigma \cdot \Delta \mathbf{u} \, \mathrm{d}x \leq \frac{\alpha_1 \mu}{4} \| \nabla \sigma \|_{L^2}^2 + \frac{\alpha_1 \mu_1^2}{\mu} \| \Delta \mathbf{u} \|_{L^2}^2, \\ &\alpha_1 \mu_2 \int_{\mathbb{R}^3} \nabla \sigma \cdot \nabla \operatorname{div} \mathbf{u} \, \mathrm{d}x \leq \frac{\alpha_1 \mu}{4} \| \nabla \sigma \|_{L^2}^2 + \frac{\alpha_1 \mu_2^2}{\mu} \| \nabla \operatorname{div} \mathbf{u} \|_{L^2}^2. \end{aligned}$$

Adding $\sum_{0 \le k \le 1} (3.5)$ to (3.6) and then using the above two inequalities, we get

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \bigg\{ \|\sigma\|_{H^{1}}^{2} + 2\alpha_{1} \int_{\mathbb{R}^{3}} \nabla\sigma \cdot \mathbf{u} \,\mathrm{d}x + \|\mathbf{u}\|_{H^{1}}^{2} + \|N\|_{H^{1}}^{2} \bigg\} \\ &+ \frac{\alpha_{1}\mu}{2} \|\nabla\sigma\|_{L^{2}}^{2} + \mu_{1} \|\nabla\mathbf{u}\|_{H^{1}}^{2} + \mu_{2} \|\operatorname{div}\mathbf{u}\|_{H^{1}}^{2} + \frac{1}{2} \|\nabla N\|_{H^{1}}^{2} \\ &\leq \frac{\alpha_{1}\mu_{1}^{2}}{\mu} \|\Delta\mathbf{u}\|_{L^{2}}^{2} + \frac{\alpha_{1}\mu_{2}^{2}}{\mu} \|\nabla\operatorname{div}\mathbf{u}\|_{L^{2}}^{2} + \alpha_{1}\mu \|\operatorname{div}\mathbf{u}\|_{L^{2}}^{2} + \frac{n_{\infty}^{2}}{2} \|\nabla c\|_{H^{1}}^{2} \end{aligned}$$

$$+ \int_{\mathbb{R}^{3}} \sigma M_{1} dx + \int_{\mathbb{R}^{3}} \nabla \sigma \nabla M_{2} dx + \int_{\mathbb{R}^{3}} \mathbf{u} \cdot M_{2} dx$$
$$+ \int_{\mathbb{R}^{3}} \nabla \mathbf{u} \cdot \nabla M_{2} dx + \int_{\mathbb{R}^{3}} NM_{3} dx + \int_{\mathbb{R}^{3}} \nabla N \nabla M_{3} dx$$
$$+ \alpha_{1} \int_{\mathbb{R}^{3}} \mathbf{u} \cdot \nabla M_{1} dx + \alpha_{1} \int_{\mathbb{R}^{3}} \nabla \sigma \cdot M_{2} dx.$$
(3.7)

Next, we estimate the nonlinear part on the right-hand side of (3.7). By exploiting the Hölder inequality, Young inequality, Lemmas A.4–A.5, assumption (3.1), and integral by parts, we can get

$$\begin{split} \int_{\mathbb{R}^{3}} \sigma M_{1} \, \mathrm{d}x &\leq C \|\sigma\|_{L^{6}} \Big(\|\nabla\sigma\|_{L^{2}} \|\mathbf{u}\|_{L^{3}} + \|\sigma\|_{L^{3}} \|\nabla\mathbf{u}\|_{L^{2}} \Big) \\ &\leq C \|\nabla\sigma\|_{L^{2}} (\|\nabla\sigma\|_{L^{2}} \|\mathbf{u}\|_{H^{1}} + \left(\|\sigma\|_{H^{1}} \|\nabla\mathbf{u}\|_{L^{2}} \right) \\ &\leq C \delta \|\nabla(\sigma, \mathbf{u})\|_{L^{2}}^{2} \end{split}$$
(3.8)

and

$$\begin{split} \int_{\mathbb{R}^{3}} \nabla \sigma \nabla M_{1} \, \mathrm{d}x &\leq C \left\| \nabla^{2} \sigma \right\|_{L^{2}} \left(\| \nabla \sigma \|_{L^{2}} \| \mathbf{u} \|_{L^{\infty}} + \| \sigma \|_{L^{\infty}} \| \nabla \mathbf{u} \|_{L^{2}} \right) \\ &\leq C \left\| \nabla^{2} \sigma \right\|_{L^{2}} \left(\| \nabla \sigma \|_{L^{2}} \| \mathbf{u} \|_{H^{2}} + \| \sigma \|_{H^{2}} \| \nabla \mathbf{u} \|_{L^{2}} \right) \\ &\leq C \delta \left(\left\| \nabla (\sigma, \mathbf{u}) \right\|_{L^{2}}^{2} + \left\| \nabla^{2} \sigma \right\|_{L^{2}}^{2} \right). \end{split}$$
(3.9)

Recalling the definition of h_i (i = 1, 2, 3) and then using the Hölder inequality, Young inequality, assumption (3.1), and integral by parts, we know that

$$\int_{\mathbb{R}^{3}} \mathbf{u} \cdot M_{2} \, dx \leq C \|\mathbf{u}\|_{L^{6}} \left(\|\nabla \mathbf{u}\|_{L^{2}} \|\mathbf{u}\|_{L^{3}} + \|\nabla \sigma\|_{L^{2}} \|h_{1}(\sigma)\|_{L^{3}} + \|\nabla \sigma\|_{L^{2}} \|h_{2}(\sigma)\|_{L^{3}} + \|\nabla \sigma\|_{L^{2}} \|h_{3}(\sigma)\|_{L^{3}} \right)
+ C \|\mathbf{u}\|_{L^{6}} \|g_{1}(\sigma)\|_{L^{3}} \|\nabla^{2}\mathbf{u}\|_{L^{3}} \leq C \|\nabla \mathbf{u}\|_{L^{2}} (\|\nabla \mathbf{u}\|_{L^{2}} \|\mathbf{u}\|_{H^{1}} + \|\nabla \sigma\|_{L^{2}} \|h_{1}(\sigma)\|_{L^{3}}
+ \|\nabla \sigma\|_{L^{2}} \|h_{2}(\sigma)\|_{L^{3}} + \|\nabla \sigma\|_{L^{2}} \|N\|_{L^{\infty}} \|h_{3}(\sigma)\|_{L^{3}}
+ \|g_{1}(\sigma)\|_{L^{3}} \|\nabla^{2}\mathbf{u}\|_{L^{3}})
\leq C \delta \left(\|\nabla (\sigma, \mathbf{u})\|_{L^{2}}^{2} + \|\nabla^{2}\mathbf{u}\|_{L^{2}}^{2} \right)$$
(3.10)

and

$$\begin{split} \int_{\mathbb{R}^{3}} \nabla \mathbf{u} \cdot \nabla M_{2} \, \mathrm{d}x &\leq C \| \nabla^{2} \mathbf{u} \|_{L^{2}} (\| \nabla \mathbf{u} \|_{L^{2}} \| \mathbf{u} \|_{L^{\infty}} + \| \nabla \sigma \|_{L^{2}} \| h_{1}(\sigma) \|_{L^{\infty}} \\ &+ \| \nabla \sigma \|_{L^{2}} \| h_{2}(\sigma) \|_{L^{\infty}} + \| N \nabla \sigma \|_{L^{2}} \| h_{3}(\sigma) \|_{L^{\infty}} \\ &+ \| g_{1}(\sigma) \|_{L^{\infty}} \| \nabla^{2} \mathbf{u} \|_{L^{2}}) \\ &\leq C \| \nabla \mathbf{u} \|_{L^{2}} (\| \nabla \mathbf{u} \|_{L^{2}} \| \mathbf{u} \|_{H^{2}} + \| \nabla \sigma \|_{L^{2}} \| h_{1}(\sigma) \|_{H^{2}} \end{split}$$

+
$$\|\nabla \sigma\|_{L^{2}} \|h_{2}(\sigma)\|_{H^{2}} + \|\nabla \sigma\|_{L^{2}} \|N\|_{L^{\infty}} \|h_{3}(\sigma)\|_{L^{\infty}}$$

 $\leq C\delta(\|\nabla(\sigma, \mathbf{u})\|_{L^{2}}^{2} + \|\nabla^{2}\mathbf{u}\|_{L^{2}}^{2}).$ (3.11)

Similarly, we have

$$\int_{\mathbb{R}^{3}} NM_{3} \, \mathrm{d}x \leq C \|N\|_{L^{6}} \left(\|\nabla N\|_{L^{2}} \|\mathbf{u}\|_{L^{3}} + \|\nabla N\|_{L^{3}} \|\nabla c\|_{L^{2}} \right)$$

$$\leq C \|\nabla N\|_{L^{2}} \left(\|\nabla N\|_{L^{2}} \|\mathbf{u}\|_{H^{1}} + \|N\|_{H^{2}} \|\nabla c\|_{L^{2}} \right)$$

$$\leq C \delta \|\nabla (N, c)\|_{L^{2}}^{2}$$
(3.12)

and

$$\begin{split} \int_{\mathbb{R}^{3}} \nabla N \nabla M_{3} \, \mathrm{d}x &\leq C \| \nabla^{2} N \|_{L^{2}} \big(\| \nabla N \|_{L^{2}} \| \mathbf{u} \|_{L^{\infty}} + \| N \|_{L^{\infty}} \| \nabla^{2} c \|_{L^{2}} \big) \\ &\leq C \| \nabla^{2} N \|_{L^{2}} \big(\| \nabla N \|_{L^{2}} \| \mathbf{u} \|_{H^{2}} + \| N \|_{H^{2}} \| \nabla^{2} c \|_{L^{2}} \big) \\ &\leq C \delta \big(\| \nabla^{2} (N, c) \|_{L^{2}}^{2} \big). \end{split}$$
(3.13)

For the last two nonlinear terms in (3.7), we can use the integral by parts, Young inequality, Hölder inequality, assumption (3.1), and Lemmas A.4–A.5 to estimate that

$$\alpha_{1} \int_{\mathbb{R}^{3}} \mathbf{u} \cdot \nabla M_{1} \, \mathrm{d}x = -\alpha_{1} \int_{\mathbb{R}^{3}} \operatorname{div} \mathbf{u} M_{1} \, \mathrm{d}x$$

$$\leq C \alpha_{1} \| \operatorname{div} \mathbf{u} \|_{L^{2}} \| M_{1} \|_{L^{2}}$$

$$\leq C \alpha_{1} \| \operatorname{div} \mathbf{u} \|_{L^{2}} \left(\| \sigma \|_{L^{\infty}} \| \nabla \mathbf{u} \|_{L^{2}} + \| \mathbf{u} \|_{L^{\infty}} \| \nabla \sigma \|_{L^{2}} \right)$$

$$\leq C \alpha_{1} \delta \| \nabla (\sigma, \mathbf{u}) \|_{L^{2}}^{2}$$
(3.14)

and

$$\begin{aligned} \alpha_{1} \int_{\mathbb{R}^{3}} \nabla \sigma \cdot M_{1} \, \mathrm{d}x &\leq C \alpha_{1} \| \nabla \sigma \|_{L^{2}} \| M_{2} \|_{L^{2}} \\ &\leq C \alpha_{1} \| \nabla \sigma \|_{L^{2}} \left(\| \mathbf{u} \|_{L^{\infty}} \| \nabla \mathbf{u} \|_{L^{2}} + \left\| h_{1}(\sigma) \right\|_{L^{\infty}} \| \nabla \sigma \|_{L^{2}} \\ &+ \left\| h_{2}(\sigma) \right\|_{L^{\infty}} \| \nabla \sigma \|_{L^{2}} + \left\| h_{3}(\sigma) \right\|_{L^{\infty}} \| N \| \|_{L^{\infty}} \nabla \sigma \|_{L^{2}} \right) \\ &\leq C \alpha_{1} \delta \left\| \nabla (\sigma, \mathbf{u}) \right\|_{L^{2}}^{2}. \end{aligned}$$

$$(3.15)$$

Putting (3.8)-(3.15) into (3.7) yields

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \|\sigma\|_{H^{1}}^{2} + 2\alpha_{1} \int_{\mathbb{R}^{3}} \nabla\sigma \cdot \mathbf{u} \,\mathrm{d}x + \|\mathbf{u}\|_{H^{1}}^{2} + \|N\|_{H^{1}}^{2} \right\}
+ \frac{\alpha_{1}\mu}{2} \|\nabla\sigma\|_{L^{2}}^{2} + \mu_{1} \|\nabla\mathbf{u}\|_{H^{1}}^{2} + \mu_{2} \|\operatorname{div}\mathbf{u}\|_{H^{1}}^{2} + \frac{1}{2} \|\nabla N\|_{H^{1}}^{2}
\leq \frac{\alpha_{1}\mu_{1}^{2}}{\mu} \|\Delta\mathbf{u}\|_{L^{2}}^{2} + \frac{\alpha_{1}\mu_{2}^{2}}{\mu} \|\nabla\operatorname{div}\mathbf{u}\|_{L^{2}}^{2} + \alpha_{1}\mu \|\operatorname{div}\mathbf{u}\|_{L^{2}}^{2} + \frac{n_{\infty}^{2}}{2} \|\nabla c\|_{H^{1}}^{2}
+ C(1 + \alpha_{1})\delta(\|\nabla(\sigma, \mathbf{u}, N, c)\|_{L^{2}}^{2} + \|\nabla^{2}(\sigma, \mathbf{u}, N, c)\|_{L^{2}}^{2}).$$
(3.16)

Now we proceed to estimate for *c*. Multiplying $\nabla^k(2.6)_4$ by $\nabla^k c$ in \mathbb{R}^3 , and then we integrate by parts to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^{k}c\|_{L^{2}}^{2} + \|\nabla^{k}\nabla c\|_{L^{2}}^{2} + n_{\infty}\|\nabla^{k}c\|_{L^{2}}^{2} = \int_{\mathbb{R}^{3}} \nabla^{k}c\nabla^{k}M_{4}\,\mathrm{d}x.$$
(3.17)

It is easy to estimate that

$$\begin{split} \int_{\mathbb{R}^3} cM_4 \, \mathrm{d}x &\leq C \int_{\mathbb{R}^3} c\mathbf{u} \cdot \nabla c \, \mathrm{d}x + C \int_{\mathbb{R}^3} N|c|^2 \, \mathrm{d}x \\ &\leq C \big(\|\mathbf{u}\|_{L^\infty} \|\nabla c\|_{L^\infty} \|c\|_{L^2} + \|N\|_{L^\infty} \|c\|_{L^2} + \|c\|_{L^2} \big) \\ &\leq C\delta \big(\|c\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 \big) \end{split}$$
(3.18)

and

$$\begin{split} \int_{\mathbb{R}^{3}} \nabla c \nabla M_{4} \, \mathrm{d}x &\leq C \left\| \nabla^{2} c \right\|_{L^{2}} \|M_{4}\|_{L^{2}} \\ &\leq C \left\| \nabla^{2} c \right\|_{L^{2}} \|\mathbf{u}\|_{L^{\infty}} \|\nabla c\|_{L^{2}} + C \left\| \nabla^{2} c \right\|_{L^{2}} \|N\|_{L^{\infty}} \|c\|_{L^{2}} \\ &\leq C \delta \left(\|c\|_{L^{2}}^{2} + \|\nabla c\|_{L^{2}}^{2} + \left\| \nabla^{2} c \right\|_{L^{2}}^{2} \right). \end{split}$$
(3.19)

Putting (3.18) and (3.19) into $\sum_{0 \le k \le 1}$ (3.17) and using the smallness of δ , we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|c\|_{H^1}^2 + \|\nabla c\|_{H^1}^2 + n_\infty \|c\|_{H^1}^2 \le 0.$$
(3.20)

By (3.16) and (3.20), we get

$$\frac{1}{2} \frac{d}{dt} \left\{ \|\sigma\|_{H^{1}}^{2} + 2\alpha_{1} \int_{\mathbb{R}^{3}} \nabla\sigma \cdot \mathbf{u} \, dx + \|\mathbf{u}\|_{H^{1}}^{2} + \|N\|_{H^{1}}^{2} + 2\gamma \|c\|_{H^{1}}^{2} \right\}
+ \frac{\alpha_{1}\mu}{2} \|\nabla\sigma\|_{L^{2}}^{2} + \mu_{1} \|\nabla\mathbf{u}\|_{H^{1}}^{2} + \mu_{2} \|\operatorname{div}\mathbf{u}\|_{H^{1}}^{2}
+ \frac{1}{2} \|\nabla N\|_{H^{1}}^{2} + \gamma \|\nabla c\|_{H^{1}}^{2} + n_{\infty}\gamma \|c\|_{H^{1}}^{2}
\leq \frac{\alpha_{1}\mu_{1}^{2}}{\mu} \|\Delta\mathbf{u}\|_{L^{2}}^{2} + \frac{\alpha_{1}\mu_{2}^{2}}{\mu} \|\nabla\operatorname{div}\mathbf{u}\|_{L^{2}}^{2} + \alpha_{1}\mu \|\operatorname{div}\mathbf{u}\|_{L^{2}}^{2} + \frac{n_{\infty}^{2}}{2} \|\nabla c\|_{H^{1}}^{2}
+ C(1 + \alpha_{1})\delta (\|c\|_{L^{2}}^{2} + \|\nabla(\sigma,\mathbf{u},N,c)\|_{L^{2}}^{2} + \|\nabla^{2}(\sigma,\mathbf{u},N,c)\|_{L^{2}}^{2}),$$
(3.21)

where α_1 is a fixed parameter that satisfies the following definition

$$0 \le \alpha_1 \le \min\left\{\frac{\mu}{4\mu_1}, \frac{\mu}{4\mu_2}, \frac{\mu_2}{4\mu}, \frac{1}{2}\right\},\$$

and $\gamma := n_{\infty}^2$. This completes the proof of Lemma 3.1.

Next, we focus on the energy estimate of the highest derivatives of solutions.

Lemma 3.2 Let the functional $\mathfrak{D}_h(t)$ be defined as follows

$$\mathfrak{D}_{h}(t) := \frac{1}{2} \left\{ \left\| \nabla^{2} \sigma \right\|_{L^{2}}^{2} + 2\alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma \cdot \nabla \mathbf{u} \, \mathrm{d}x + \left\| \nabla^{2} \mathbf{u} \right\|_{L^{2}}^{2} + \left\| \nabla^{2} N \right\|_{L^{2}}^{2} + 2\gamma \left\| \nabla^{2} c \right\|_{L^{2}}^{2} \right\}.$$
(3.22)

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{D}_{h}(t) + \frac{\alpha_{2}\mu}{4} \|\nabla^{2}\sigma\|_{L^{2}}^{2} + \frac{\mu_{1}}{2} \|\nabla^{3}\mathbf{u}\|_{L^{2}}^{2} + \frac{\mu_{2}}{4} \|\nabla^{2}\operatorname{div}\mathbf{u}\|_{L^{2}}^{2}
+ \frac{1}{4} \|\nabla^{3}N\|_{L^{2}}^{2} + \frac{\gamma}{2} \|\nabla^{3}c\|_{L^{2}}^{2} + \frac{n_{\infty}\gamma}{2} \|\nabla^{2}c\|_{L^{2}}^{2}
\leq \frac{\mu_{2}}{8} \|\nabla\operatorname{div}\mathbf{u}\|_{L^{2}}^{2} + C\delta \|c\|_{H^{1}}^{2} + C\delta \|\nabla^{2}(\mathbf{u},N)\|_{L^{2}}^{2},$$
(3.23)

where α_1 and γ are two given constant.

Proof Multiplying $\nabla^2(2.6)_1 - \nabla^2(2.6)_3$ by $\nabla^2 \sigma$, $\nabla^2 \mathbf{u}$ and $\nabla^2 N$ in $L^2(\mathbb{R}^3)$, resp., then summing the resulting identities up, and finally, using the Young inequality and integral by parts, we can get

$$\frac{1}{2} \frac{d}{dt} \left(\left\| \nabla^2 \sigma \right\|_{L^2}^2 + \left\| \nabla^2 \mathbf{u} \right\|_{L^2}^2 + \left\| \nabla^2 N \right\|_{L^2}^2 \right) \\
+ \mu_1 \left\| \nabla^2 \nabla \mathbf{u} \right\|_{L^2}^2 + \mu_2 \left\| \nabla^2 \operatorname{div} \mathbf{u} \right\|_{L^2}^2 + \left\| \nabla^2 \nabla N \right\|_{L^2}^2 \\
= n_\infty \int_{\mathbb{R}^3} \nabla^2 \nabla c \nabla^2 \nabla N \, dx + \int_{\mathbb{R}^3} \nabla^2 \sigma \nabla^2 M_1 \, dx \\
+ \int_{\mathbb{R}^3} \nabla^2 \mathbf{u} \nabla^2 M_2 \, dx + \int_{\mathbb{R}^3} \nabla^2 N \nabla^2 M_3 \, dx \\
\leq \frac{n_\infty^2}{2} \left\| \nabla^2 \nabla c \right\|_{L^2}^2 + \frac{1}{2} \left\| \nabla^2 \nabla N \right\|_{L^2}^2 + \int_{\mathbb{R}^3} \nabla^2 \sigma \nabla^2 M_1 \, dx \\
+ \int_{\mathbb{R}^3} \nabla^2 \mathbf{u} \nabla^2 M_2 \, dx + \int_{\mathbb{R}^3} \nabla^2 N \nabla^2 M_3 \, dx.$$
(3.24)

Multiplying $\nabla^2(2.6)_1$, $\nabla(2.6)_2$ by $\nabla \mathbf{u}$ and $\nabla^2 \sigma$, resp., and integrating by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{3}} \nabla \sigma \cdot \mathbf{u} \, \mathrm{d}x + \mu \int_{\mathbb{R}^{3}} |\nabla \sigma|^{2} \, \mathrm{d}x$$

$$= \mu \|\nabla \operatorname{div} \mathbf{u}\|_{L^{2}}^{2} + \mu_{1} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma \cdot \nabla \Delta \mathbf{u} \, \mathrm{d}x + \mu_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma \cdot \nabla \nabla \operatorname{div} \mathbf{u} \, \mathrm{d}x$$

$$+ \int_{\mathbb{R}^{3}} \nabla \mathbf{u} \cdot \nabla \nabla M_{1} \, \mathrm{d}x + \int_{\mathbb{R}^{3}} \nabla \nabla \sigma \cdot \nabla M_{2} \, \mathrm{d}x$$

$$\leq \frac{\mu}{2} \|\nabla \nabla \sigma\|_{L^{2}}^{2} + \mu \|\nabla \operatorname{div} \mathbf{u}\|_{L^{2}}^{2} + \frac{\mu_{1}^{2}}{\mu} \|\nabla \Delta \mathbf{u}\|_{L^{2}}^{2}$$

$$+ \frac{\mu_{2}^{2}}{\mu} \|\nabla \nabla \operatorname{div} \mathbf{u}\|_{L^{2}}^{2} + \int_{\mathbb{R}^{3}} \nabla \mathbf{u} \cdot \nabla \nabla M_{1} \, \mathrm{d}x + \int_{\mathbb{R}^{3}} \nabla \nabla \sigma \cdot \nabla M_{2} \, \mathrm{d}x.$$
(3.25)

Let α_2 be a fixed constant. Addition of $\alpha_2 \times (3.25)$ and (3.24) yields

$$\frac{1}{2} \frac{d}{dt} \left\{ \left\| \nabla^{2} \sigma \right\|_{L^{2}}^{2} + 2\alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma \cdot \nabla \mathbf{u} \, dx + \left\| \nabla^{2} \mathbf{u} \right\|_{L^{2}}^{2} + \left\| \nabla^{2} N \right\|_{L^{2}}^{2} \right\}
+ \frac{\alpha_{2} \mu}{2} \left\| \nabla \nabla \sigma \right\|_{L^{2}}^{2} + \mu_{1} \left\| \nabla^{2} \nabla \mathbf{u} \right\|_{L^{2}}^{2} + \mu_{2} \left\| \nabla^{2} \operatorname{div} \mathbf{u} \right\|_{L^{2}}^{2} + \left\| \nabla^{2} \nabla N \right\|_{L^{2}}^{2}
\leq \frac{\alpha_{2} \mu_{1}^{2}}{\mu} \left\| \nabla \Delta \mathbf{u} \right\|_{L^{2}}^{2} + \frac{\alpha_{2} \mu_{2}^{2}}{\mu} \left\| \nabla \operatorname{div} \mathbf{u} \right\|_{L^{2}}^{2} + \alpha_{1} \mu \left\| \nabla \operatorname{div} \mathbf{u} \right\|_{L^{2}}^{2} + \frac{n_{\infty}^{2}}{2} \left\| \nabla^{2} \nabla c \right\|_{L^{2}}^{2}
+ \frac{1}{2} \left\| \nabla^{2} \nabla N \right\|_{L^{2}}^{2} + \int_{\mathbb{R}^{3}} \nabla^{2} \sigma \nabla^{2} M_{1} \, dx + \int_{\mathbb{R}^{3}} \nabla^{2} \mathbf{u} \nabla^{2} M_{2} \, dx
+ \int_{\mathbb{R}^{3}} \nabla^{2} N \nabla^{2} M_{3} \, dx + \alpha_{2} \int_{\mathbb{R}^{3}} \mathbf{u} \cdot \nabla M_{1} \, dx + \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \sigma \cdot M_{2} \, dx.$$
(3.26)

Next, we estimate the nonlinear term on the right-hand side of formula (3.26). By the Hölder inequality, Young inequality, assumption (3.1), Lemmas A.3–A.4, and the integral by parts, we have

$$\begin{split} \int_{\mathbb{R}^{3}} \nabla^{2} \sigma \nabla^{2} M_{1} \, \mathrm{d}x &\leq C \left| \left\langle \nabla^{2} \sigma, \nabla^{2} (\sigma \operatorname{div} \mathbf{u}) \right\rangle \right| + C \left| \left\langle \nabla^{2} \sigma, \nabla^{2} (\nabla \sigma \cdot \mathbf{u}) \right\rangle \right| \\ &\leq C \left\| \nabla^{2} \sigma \right\|_{L^{2}} \left(\left\| \nabla^{2} \sigma \right\|_{L^{2}} \right\| \operatorname{div} \mathbf{u} \|_{L^{\infty}} + \|\sigma\|_{L^{\infty}} \left\| \nabla^{2} \operatorname{div} \mathbf{u} \right\|_{L^{2}} \right) \\ &+ C \| \operatorname{div} \mathbf{u} \|_{L^{\infty}} \left\| \nabla^{2} \sigma \right\|_{L^{2}}^{2} + C \left\| \nabla^{2} \sigma \right\|_{L^{2}} \left\| \nabla^{2} (\nabla \sigma \cdot \mathbf{u}) - \nabla^{2} \nabla \sigma \cdot \mathbf{u} \right\|_{L^{2}} \\ &\leq C \left\| \nabla^{2} \sigma \right\|_{L^{2}}^{2} \| \operatorname{div} \mathbf{u} \|_{L^{\infty}} + C \left\| \nabla^{2} \sigma \right\|_{L^{2}} \|\sigma\|_{L^{\infty}} \left\| \nabla^{2} \operatorname{div} \mathbf{u} \right\|_{L^{2}} \\ &+ C \left\| \nabla^{2} \sigma \right\|_{L^{2}} \left(\left\| \nabla^{2} \sigma \right\|_{L^{2}} \| \nabla \mathbf{u} \|_{L^{\infty}} + \| \nabla \sigma \|_{L^{3}} + \| \nabla^{2} \mathbf{u} \|_{L^{6}} \right) \\ &\leq C \left\| \nabla^{2} \sigma \right\|_{L^{2}}^{2} \| \nabla \mathbf{u} \|_{H^{2}} + C \left\| \nabla^{2} \sigma \right\|_{L^{2}} \|\sigma\|_{H^{2}} \left\| \nabla^{3} \mathbf{u} \right\|_{L^{2}} \\ &+ C \left\| \nabla^{2} \sigma \right\|_{L^{2}} \left(\left\| \nabla^{2} \sigma \right\|_{L^{2}} \| \nabla \mathbf{u} \|_{H^{2}} + \| \nabla \sigma \|_{H^{1}} \left\| \nabla^{3} \mathbf{u} \right\|_{L^{2}} \right) \\ &\leq C \delta \left(\left\| \nabla^{2} (\sigma, \mathbf{u}) \right\|_{L^{2}}^{2} + \left\| \nabla^{3} \mathbf{u} \right\|_{L^{2}}^{2} \right). \end{split}$$
(3.27)

Exploiting the integral by parts, we have

$$\int_{\mathbb{R}^{3}} \nabla^{2} \mathbf{u} \nabla^{2} M_{2} \, \mathrm{d}x \leq C \left| \left\langle \nabla^{3} \mathbf{u}, \nabla(\mathbf{u} \cdot \operatorname{div} \mathbf{u}) \right\rangle \right| + C \left| \left\langle \nabla^{3} \mathbf{u}, \nabla[h_{1}(\sigma) \nabla\sigma] \right\rangle \right| \\ + C \left| \left\langle \nabla^{3} \mathbf{u}, \nabla[h_{2}(\sigma) N \nabla\sigma] \right\rangle \right| + C \left| \left\langle \nabla^{3} \mathbf{u}, \nabla[g_{1}(\sigma) \Delta \mathbf{u}] \right\rangle \right| \\ + C \left| \left\langle \nabla^{3} \mathbf{u}, \nabla[g_{1}(\sigma) \nabla \operatorname{div} \mathbf{u}] \right\rangle \right| + C \left| \left\langle \nabla^{3} \mathbf{u}, \nabla[h_{3}(\sigma) N \nabla\sigma] \right\rangle \right|.$$
(3.28)

Making use of Lemmas A.3–A.5, assumption (3.1), and the definition of h_1 , we can derive from the above inequality that

$$\begin{split} \int_{\mathbb{R}^{3}} \nabla^{2} \mathbf{u} \nabla^{2} M_{2} \, dx &\leq C \| \nabla^{3} \mathbf{u} \|_{L^{2}} \big(\| \nabla \mathbf{u} \|_{L^{6}} \| \nabla \mathbf{u} \|_{L^{3}} + \| \mathbf{u} \|_{L^{\infty}} \| \nabla^{2} \mathbf{u} \|_{L^{2}} \big) \\ &+ C \| \nabla^{3} \mathbf{u} \|_{L^{2}} \big(\| h_{1}(\sigma) \|_{L^{\infty}} \| \nabla^{2} \sigma \|_{L^{2}} + \| \nabla h_{1}(\sigma) \|_{L^{6}} \| \nabla \sigma \|_{L^{3}} \big) \\ &+ C \| \nabla^{3} \mathbf{u} \|_{L^{2}} \big(\| h_{2}(\sigma) \|_{L^{\infty}} \| \nabla^{2} \sigma \|_{L^{2}} + \| \nabla h_{2}(\sigma) \|_{L^{6}} \| \nabla \sigma \|_{L^{3}} \big) \\ &+ C \| \nabla^{3} \mathbf{u} \|_{L^{2}} \big(\| g_{1}(\sigma) \|_{L^{\infty}} \| \nabla^{3} \mathbf{u} \|_{L^{2}} + \| \nabla g_{1}(\sigma) \|_{L^{6}} \| \nabla \mathbf{u} \|_{L^{3}} \big) \end{split}$$

$$+ C \|\nabla^{3}\mathbf{u}\|_{L^{2}} \|\nabla^{2}\sigma\|_{L^{2}} \|h_{3}(\sigma)\|_{L^{\infty}} \|N\|_{L^{\infty}} \\
+ C \|\nabla^{3}\mathbf{u}\|_{L^{2}} \|\nabla\sigma\|_{L^{\infty}} (\|h_{3}(\sigma)\|_{L^{\infty}} \|\nabla N\|_{L^{2}} + \|\nabla h_{3}(\sigma)\|_{L^{6}} \|N\|_{L^{3}}) \\
\leq C \|\nabla^{3}\mathbf{u}\|_{L^{2}} (\|\nabla^{2}\mathbf{u}\|_{L^{2}} \|\mathbf{u}\|_{H^{2}} + \|\mathbf{u}\|_{H^{2}} \|\nabla^{2}\mathbf{u}\|_{L^{2}}) \\
+ C \|\nabla^{3}\mathbf{u}\|_{L^{2}} (\|h_{1}(\sigma)\|_{H^{2}} \|\nabla^{2}\sigma\|_{L^{2}} + \|\nabla^{2}\sigma\|_{L^{2}} \|\sigma\|_{H^{2}}) \\
+ C \|\nabla^{3}\mathbf{u}\|_{L^{2}} (\|h_{2}(\sigma)\|_{H^{2}} \|\nabla^{2}\sigma\|_{L^{2}} + \|\nabla^{2}\sigma\|_{L^{2}} \|\sigma\|_{H^{2}}) \\
+ C \|\nabla^{3}\mathbf{u}\|_{L^{2}} \|\nabla^{2}\sigma\|_{L^{2}} \|h_{3}(\sigma)\|_{L^{\infty}} \|N\|_{L^{\infty}} \\
+ C \|\nabla^{3}\mathbf{u}\|_{L^{2}} \|\nabla\sigma\|_{L^{6}} (\|h_{3}(\sigma)\|_{L^{6}} \|\nabla N\|_{L^{6}} + \|\nabla h_{3}(\sigma)\|_{L^{6}} \|N\|_{L^{6}}) \\
\leq C \delta (\|\nabla^{2}(\sigma,\mathbf{u})\|_{L^{2}}^{2} + \|\nabla^{3}(\mathbf{u})\|_{L^{2}}^{2}), \qquad (3.29)$$

where we have used the fact that

$$\left\|\nabla h_i(\sigma)\right\|_{L^6} \le C \left\|\nabla \sigma\right\|_{L^6} \le C \left\|\nabla^2 \sigma\right\|_{L^2} \quad \text{for } i = 1, 2.$$

Similarly, it is easy to estimate that

$$\int_{\mathbb{R}^{3}} \nabla^{2} N \nabla^{2} M_{3} \, \mathrm{d}x \leq C \left| \left\langle \nabla^{3} N, \nabla (\mathbf{u} \cdot \nabla N) \right\rangle \right| + C \left| \left\langle \nabla^{2} N, \nabla^{2} (\nabla N \nabla c) \right\rangle \right|$$
$$+ C \left| \left\langle \nabla^{2} N, \nabla^{2} (N \nabla^{2} c) \right\rangle \right|, \tag{3.30}$$

which gives

$$\int_{\mathbb{R}^{3}} \nabla^{2} N \nabla^{2} M_{3} \, \mathrm{d}x \leq C \| \nabla^{3} N \|_{L^{2}} (\| \nabla \mathbf{u} \|_{L^{6}} \| \nabla N \|_{L^{3}} + \| \mathbf{u} \|_{L^{\infty}} \| \nabla^{2} N \|_{L^{2}})$$

$$+ C \| \nabla^{3} N \|_{L^{2}} \| \nabla^{2} N \|_{L^{6}} \| \nabla c \|_{L^{3}} + C \| \nabla^{3} c \|_{L^{2}} \| \nabla^{2} N \|_{L^{6}} \| \nabla N \|_{L^{3}}
+ C \| \nabla N \|_{L^{3}} \| \nabla^{2} \nabla N \|_{L^{2}} \| \nabla^{2} c \|_{L^{6}} + C \| \nabla N \|_{L^{3}} \| \nabla^{2} N \|_{L^{6}} \| \nabla^{3} c \|_{L^{2}}
+ C \| \nabla^{3} N \|_{L^{2}} \| N \|_{L^{\infty}} \| \nabla^{3} c \|_{L^{2}} + \| \nabla N \|_{L^{3}} \| \nabla^{3} c \|_{L^{2}} \| \nabla^{2} N \|_{L^{6}}
\leq C \delta (\| \nabla^{2} (\mathbf{u}, N, c) \|_{L^{2}}^{2} + \| \nabla^{3} (N, c) \|_{L^{2}}^{2}).$$
(3.31)

Using Lemma A.5, Young inequality, assumption (3.1), and Hölder inequality, we obtain

$$\begin{aligned} \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \mathbf{u} \cdot \nabla \nabla M_{1} \, \mathrm{d}x &= -\alpha_{2} \int_{\mathbb{R}^{3}} \nabla \operatorname{div} \mathbf{u} \nabla M_{1} \, \mathrm{d}x \\ &\leq C \alpha_{2} \| \nabla \operatorname{div} \mathbf{u} \|_{L^{2}} \| \nabla M_{1} \|_{L^{2}} \\ &\leq C \alpha_{2} \| \nabla^{2} \mathbf{u} \|_{L^{2}} (\| \nabla^{2} \mathbf{u} \|_{L^{2}} \| \sigma \|_{L^{\infty}} + \| \mathbf{u} \|_{L^{\infty}} \| \nabla^{2} \sigma \|_{L^{2}}) \\ &\leq C \alpha_{2} \delta \| \nabla^{2} (\sigma, \mathbf{u}) \|_{L^{2}}^{2} \end{aligned}$$
(3.32)

and

$$\alpha_2 \int_{\mathbb{R}^3} \nabla \nabla \sigma \cdot \nabla M_2 \, \mathrm{d}x$$
$$\leq C \alpha_2 \| \nabla \nabla \sigma \|_{L^2} \| \nabla M_2 \|_{L^2}$$

$$\leq C\alpha_{2} \|\nabla\nabla\sigma\|_{L^{2}} (\|\nabla\mathbf{u}\|_{L^{6}} \|\nabla\mathbf{u}\|_{L^{3}} + \|\mathbf{u}\|_{L^{\infty}} + \|\nabla^{2}\mathbf{u}\|_{L^{2}}) + C\alpha_{2} \|\nabla\nabla\sigma\|_{L^{2}} (\|\nabla h_{1}(\sigma)\|_{L^{6}} \|\nabla\sigma\|_{L^{3}} + \|h_{1}(\sigma)\|_{L^{\infty}} \|\nabla^{2}\sigma\|_{L^{2}}) + C\alpha_{2} \|\nabla\nabla\sigma\|_{L^{2}} (\|\nabla h_{2}(\sigma)\|_{L^{6}} \|\nabla\sigma\|_{L^{3}} + \|h_{2}(\sigma)\|_{L^{\infty}} \|\nabla^{2}\sigma\|_{L^{2}}) + C\alpha_{2} \|\nabla^{2}\sigma\|_{L^{2}} \|\nabla^{2}\sigma\|_{L^{2}} \|h_{3}(\sigma)\|_{L^{\infty}} \|N\|_{L^{\infty}} + C\alpha_{2} \|\nabla^{2}\sigma\|_{L^{2}} \|\nabla\sigma\|_{L^{6}} (\|h_{3}(\sigma)\|_{L^{6}} \|\nabla N\|_{L^{6}} + \|\nabla h_{3}(\sigma)\|_{L^{6}} \|N\|_{L^{6}}) \leq C\alpha_{2}\delta \|\nabla^{2}(\sigma,\mathbf{u})\|_{L^{2}}^{2}.$$
(3.33)

Substituting the above result into (3.26), we have

$$\frac{1}{2} \frac{d}{dt} \left\{ \left\| \nabla^{2} \sigma \right\|_{L^{2}}^{2} + 2\alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma \cdot \nabla \mathbf{u} \, dx + \left\| \nabla^{2} \mathbf{u} \right\|_{L^{2}}^{2} + \left\| \nabla^{2} N \right\|_{L^{2}}^{2} \right\}
+ \frac{\alpha_{2} \mu}{2} \left\| \nabla \nabla \sigma \right\|_{L^{2}}^{2} + \mu_{1} \left\| \nabla^{2} \nabla \mathbf{u} \right\|_{L^{2}}^{2} + \mu_{2} \left\| \nabla^{2} \operatorname{div} \mathbf{u} \right\|_{L^{2}}^{2} + \frac{1}{2} \left\| \nabla^{2} \nabla N \right\|_{L^{2}}^{2}
\leq \frac{\alpha_{2} \mu_{1}^{2}}{\mu} \left\| \nabla \Delta \mathbf{u} \right\|_{L^{2}}^{2} + \frac{\alpha_{2} \mu_{2}^{2}}{\mu} \left\| \nabla \nabla \operatorname{div} \mathbf{u} \right\|_{L^{2}}^{2} + \alpha_{2} \mu \left\| \nabla \operatorname{div} \mathbf{u} \right\|_{L^{2}}^{2} + \frac{n_{\infty}^{2}}{2} \left\| \nabla^{2} \nabla c \right\|_{L^{2}}^{2}
+ C(1 + \alpha_{2}) \delta \left(\left\| \nabla^{2} (\sigma, \mathbf{u}, N, c) \right\|_{L^{2}}^{2} + \left\| \nabla^{3} (\mathbf{u}, N, c) \right\|_{L^{2}}^{2} \right).$$
(3.34)

Multiplying ∇^2 (2.6)₄ by $\nabla^2 c$ in $L^2(\mathbb{R}^3)$, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^2 c\|_{L^2}^2 + \|\nabla^2 \nabla c\|_{L^2}^2 + n_\infty \|\nabla^2 c\|_{L^2}^2 = \int_{\mathbb{R}^3} \nabla^2 c \nabla^2 M_4 \,\mathrm{d}x.$$
(3.35)

It is also easy to estimate that

$$\begin{split} \int_{\mathbb{R}^{3}} \nabla^{2} c \nabla^{2} M_{4} \, \mathrm{d}x &\leq C \| \nabla^{3} c \|_{L^{2}} \| \nabla M_{4} \|_{L^{2}} \\ &\leq C \| \nabla^{3} c \|_{L^{2}} \big(\| \nabla \mathbf{u} \|_{L^{3}} \| \nabla c \|_{L^{6}} + \| \mathbf{u} \|_{L^{\infty}} \| \nabla^{2} c \|_{L^{2}} \big) \\ &+ C \| \nabla^{3} c \|_{L^{2}} \big(\| \nabla N \|_{L^{6}} \| c \|_{L^{3}} + \| N \|_{L^{3}} \| \nabla c \|_{L^{6}} \big) \\ &\leq C \delta \big(\| c \|_{L^{3}}^{2} + \| \nabla^{2} c \|_{L^{2}}^{2} + \| \nabla^{3} c \|_{L^{2}}^{2} \big) \\ &\leq C \delta \big(\| c \|_{H^{1}}^{2} + \| \nabla^{2} c \|_{L^{2}}^{2} + \| \nabla^{3} c \|_{L^{2}}^{2} \big). \end{split}$$
(3.36)

Combining (3.35) with (3.36) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^2 c\|_{L^2}^2 + \|\nabla^2 \nabla c\|_{L^2}^2 + n_\infty \|\nabla^2 c\|_{L^2}^2 \le C\delta \|c\|_{H^1}^2.$$
(3.37)

Let α_2 be a constant satisfying

$$0 < \alpha_2 \le \min\left\{\frac{\mu}{8\mu_1}, \frac{\mu}{8\mu_2}, \frac{\mu_2}{8\mu}, \frac{1}{4}\right\}.$$

By the smallness of δ , we immediately get Lemma 3.2.

With Lemmas 3.1–3.2 in hand, we easily further obtain Proposition 3.2. In fact, keeping in mind the Young inequality and the definitions of \mathfrak{D}_l , \mathfrak{D}_h , we have

$$\frac{1}{C_2} \left\| (\sigma, \mathbf{u}, N, c)(t) \right\|_{H^2}^2 \le \mathfrak{D}_l(t) + \mathfrak{D}_h(t) \le C_2 \left\| (\sigma, \mathbf{u}, N, c)(t) \right\|_{H^2}^2,$$
(3.38)

which yields

$$\mathfrak{D}_{l}(t) + \mathfrak{D}_{h}(t) \approx \left\| (\sigma, \mathbf{u}, N, c)(t) \right\|_{H^{2}}^{2},$$
(3.39)

where $C_2 > 0$ is a constant. Integrating the two inequalities in the above two lemmas over [0, t], thus (3.2) holds for the small enough δ . This completes the proof of Proposition 3.2.

4 Decay rates

In this section, we shall derive the decay-in-time rates for the Cauchy problem (2.6)–(2.7). The proof will be broken up into three subsections. First, in Sect. 4.1, we obtain the $L_t^{\infty}L_x^2$ -norm estimate of the second derivatives of solutions of the Cauchy problem. Secondly, we establish the decay estimate of the low-medium-frequency parts based on the idea of the decomposition technique of both low and high frequencies of solutions in Sect. 4.2. Finally, in Sect. 4.3, we estimate the nonlinear part and derive the time decay rates for solutions of the Cauchy problem.

4.1 Cancellation of a low-frequency part

Inspired by the observation of canceling the low-frequency part of solutions, we have the following conclusion.

Lemma 4.1 It holds that

$$\|\nabla^{2}(\sigma, \mathbf{u}, N, c)(t)\|_{L^{2}}^{2} \leq Ce^{-C_{3}t} \|\nabla^{2}(\sigma_{0}, \mathbf{u}_{0}, N_{0}, c_{0})(t)\|_{L^{2}}^{2} + C\delta \int_{0}^{t} e^{-C_{3}(t-\tau)} \|c(\tau)\|_{H^{1}}^{2} d\tau + C \int_{0}^{t} e^{-C_{3}(t-\tau)} \|\nabla^{2}(\sigma^{L}, \mathbf{u}^{L}, N^{L}, c^{L})(\tau)\|_{L^{2}}^{2} d\tau,$$

$$(4.1)$$

where the positive constants *C* are independent of δ .

Proof Multiplying $\nabla(2.6)_2$ by $\nabla \nabla \sigma^L$ in L^2 , we integrate by parts and use $(2.6)_1$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \nabla \nabla \sigma^L \cdot \nabla \mathbf{u} \, \mathrm{d}x = \mu \int_{\mathbb{R}^3} \left(\nabla \operatorname{div} \mathbf{u} \nabla \operatorname{div} \mathbf{u}^L - \nabla \nabla \sigma \cdot \nabla \nabla \sigma^L \right) \mathrm{d}x \\ + \mu_1 \int_{\mathbb{R}^3} \nabla \nabla \sigma^L \cdot \nabla \Delta \mathbf{u} \, \mathrm{d}x + \mu_2 \int_{\mathbb{R}^3} \nabla \nabla \sigma^L \cdot \nabla \nabla \operatorname{div} \mathbf{u} \, \mathrm{d}x \\ + \int_{\mathbb{R}^3} \nabla \nabla \sigma^L \cdot \nabla M_2 \, \mathrm{d}x - \int_{\mathbb{R}^3} \nabla \operatorname{div} \mathbf{u} \nabla M_1^L \, \mathrm{d}x.$$
(4.2)

Then, thanks to the Young inequality, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, \mathrm{d}x \leq \frac{\mu+1}{2} \|\nabla \operatorname{div} \mathbf{u}\|_{L^{2}}^{2} + \frac{\mu}{8} \|\nabla \nabla \sigma\|_{L^{2}}^{2} + \frac{\mu_{1}}{2} \|\nabla \Delta \mathbf{u}\|_{L^{2}}^{2} + \frac{\mu_{2}}{2} \|\nabla \nabla \operatorname{div} \mathbf{u}\|_{L^{2}}^{2} + \left(2\mu + \frac{1+\mu_{1}+\mu_{2}}{2}\right) \|\nabla \nabla \sigma^{L}\|_{L^{2}}^{2} + \frac{1}{2} \|\nabla \operatorname{div} \mathbf{u}^{L}\|_{L^{2}}^{2} + \frac{1}{2} \|\nabla M_{2}\|_{L^{2}}^{2} + \frac{1}{2} \|\nabla M_{1}^{L}\|_{L^{2}}^{2}.$$
(4.3)

By virtue of the Plancherel theorem and Lemma A.3, we have

$$\|\nabla M_2\|_{L^2}^2 + \|\nabla M_1^L\|_{L^2}^2 \le C\delta \|\nabla^2(\sigma, \mathbf{u})\|_{L^2}^2.$$
(4.4)

Adding $\alpha_2 \times (4.3)$ and (3.23) together and using (4.4) and (A.1), we estimate that

$$\frac{d}{dt} \left(\mathfrak{D}_{h}(t) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, dx \right) + \frac{\alpha_{2}\mu}{4} \| \nabla^{2} \sigma \|_{L^{2}}^{2}
+ \frac{\mu_{1}}{4} \| \nabla^{3} \mathbf{u} \|_{L^{2}}^{2} + \frac{\mu_{1}}{4} R_{0}^{2} \| \nabla^{2} \mathbf{u}^{h} \|_{L^{2}}^{2} + \frac{\mu_{2}}{4} \| \nabla^{2} \operatorname{div} \mathbf{u} \|_{L^{2}}^{2}
+ \frac{1}{8} \| \nabla^{3} N \|_{L^{2}}^{2} + \frac{1}{8} R_{0}^{2} \| \nabla^{2} N^{h} \|_{L^{2}}^{2} + \frac{\gamma}{2} \| \nabla^{3} c \|_{L^{2}}^{2} + \frac{n_{\infty} \gamma}{2} \| \nabla^{2} c \|_{L^{2}}^{2}
\leq \left[\frac{\mu_{2}}{8} + \frac{\alpha_{2}(\mu+1)}{2} \right] \| \nabla \operatorname{div} \mathbf{u} \|_{L^{2}}^{2} + \frac{\alpha_{2}\mu}{8} \| \nabla \nabla \sigma \|_{L^{2}}^{2} + \frac{\alpha_{2}\mu_{1}}{2} \| \nabla \Delta \mathbf{u} \|_{L^{2}}^{2}
+ \frac{\alpha_{2}\mu_{2}}{2} \| \nabla \nabla \operatorname{div} \mathbf{u} \|_{L^{2}}^{2} + C\alpha_{2} \| \nabla^{2} \sigma^{L} \|_{L^{2}}^{2} + C\alpha_{2} \| \nabla \operatorname{div} \mathbf{u}^{L} \|_{L^{2}}^{2}
+ C\delta(1+\alpha_{2}) \| \nabla^{2}(\sigma, \mathbf{u}, N, c) \|_{L^{2}}^{2} + C\delta \| c \|_{H^{1}}^{2}.$$
(4.5)

In addition, using frequency decomposition and adding $\frac{\mu_1}{4}R_0^2 \|\nabla^2 \mathbf{u}^L\|_{L^2}^2 + \frac{1}{4}R_0^2 \|\nabla^2 N^L\|_{L^2}^2$ to both sides of (4.5), we can get

$$\frac{d}{dt} \left(\mathfrak{D}_{h}(t) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, dx \right) + \frac{\alpha_{2}\mu}{8} \| \nabla^{2} \sigma \|_{L^{2}}^{2}
+ \frac{\mu_{1}}{4} \| \nabla^{3} \mathbf{u} \|_{L^{2}}^{2} + \frac{\mu_{1}}{8} R_{0}^{2} \| \nabla^{2} \mathbf{u} \|_{L^{2}}^{2} + \frac{\mu_{2}}{4} \| \nabla^{2} \operatorname{div} \mathbf{u} \|_{L^{2}}^{2}
+ \frac{1}{8} \| \nabla^{3} N \|_{L^{2}}^{2} + \frac{1}{16} R_{0}^{2} \| \nabla^{2} N \|_{L^{2}}^{2} + \frac{\gamma}{2} \| \nabla^{3} c \|_{L^{2}}^{2} + \frac{n_{\infty} \gamma}{2} \| \nabla^{2} c \|_{L^{2}}^{2}
\leq C \alpha_{2} \| \nabla^{2} \sigma^{L} \|_{L^{2}}^{2} + \left(C \alpha_{2} + \frac{\mu_{1}}{4} R_{0}^{2} \right) \| \nabla^{2} \mathbf{u}^{L} \|_{L^{2}}^{2}
+ \left[\frac{\mu_{2}}{8} + \frac{\alpha_{2}(\mu + 1)}{2} \right] \| \nabla \operatorname{div} \mathbf{u} \|_{L^{2}}^{2} + \frac{\alpha_{2} \mu_{1}}{2} \| \nabla \Delta \mathbf{u} \|_{L^{2}}^{2}
+ \frac{\alpha_{2} \mu_{2}}{2} \| \nabla^{2} \operatorname{div} \mathbf{u} \|_{L^{2}}^{2} + \frac{1}{4} R_{0}^{2} \| \nabla^{2} N^{L} \|_{L^{2}}^{2}
+ C \delta (1 + \alpha_{2}) \| \nabla^{2}(\sigma, \mathbf{u}, N, c) \|_{L^{2}}^{2} + C \delta \| c \|_{H^{1}}^{2}.$$
(4.6)

Choosing $\alpha_2 < \frac{1}{4}$ and $R_0^2 > \max{\{\frac{4(\mu_2 + \mu + 1)}{\mu_1}, 1\}}$, and then using the smallness of δ , we get

$$\frac{d}{dt} \left(\mathfrak{D}_{h}(t) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, dx \right) + \frac{\alpha_{2} \mu}{16} \| \nabla^{2} \sigma \|_{L^{2}}^{2}
+ \frac{\mu_{1}}{8} \| \nabla^{3} \mathbf{u} \|_{L^{2}}^{2} + \frac{\mu_{1}}{16} R_{0}^{2} \| \nabla^{2} \mathbf{u} \|_{L^{2}}^{2} + \frac{\mu_{2}}{8} \| \nabla^{2} \operatorname{div} \mathbf{u} \|_{L^{2}}^{2}
+ \frac{1}{8} \| \nabla^{3} N \|_{L^{2}}^{2} + \frac{1}{32} R_{0}^{2} \| \nabla^{2} N \|_{L^{2}}^{2} + \frac{\gamma}{2} \| \nabla^{3} c \|_{L^{2}}^{2} + \frac{n_{\infty} \gamma}{8} \| \nabla^{2} c \|_{L^{2}}^{2}
\leq C \| \nabla^{2} (\sigma^{L}, \mathbf{u}^{L}, N^{L}, C^{L}) \|_{L^{2}}^{2} + C \delta \| c \|_{H^{1}}^{2}.$$
(4.7)

In view of frequency decomposition, one gets

$$\mathfrak{D}_{h}(t) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, \mathrm{d}x$$

$$= \frac{1}{2} \left\| \nabla^{2} \sigma \right\|_{L^{2}}^{2} + \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{h} \cdot \nabla \mathbf{u} \, \mathrm{d}x + \frac{1}{2} \left\| \nabla^{2} \mathbf{u} \right\|_{L^{2}}^{2} + \frac{1}{2} \left\| \nabla^{2} N \right\|_{L^{2}}^{2} + \gamma \left\| \nabla^{2} c \right\|_{L^{2}}^{2}.$$
(4.8)

It follows from the Young inequality and integral by parts that

$$\alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{h} \cdot \nabla \mathbf{u} \, \mathrm{d}x = -\alpha_{2} \int_{\mathbb{R}^{3}} \nabla \sigma^{h} \nabla \operatorname{div} \mathbf{u} \, \mathrm{d}x$$
$$\leq \frac{\alpha_{2}}{2} \| \nabla \sigma^{h} \|_{L^{2}}^{2} + \frac{\alpha_{2}}{2} \| \nabla \operatorname{div} \mathbf{u} \|_{L^{2}}^{2}$$
$$\leq \frac{\alpha_{2}}{2} \| \nabla^{2} \sigma \|_{L^{2}}^{2} + \frac{\alpha_{2}}{2} \| \nabla^{2} \mathbf{u} \|_{L^{2}}^{2}, \tag{4.9}$$

which implies

$$\mathfrak{D}_{h}(t) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, \mathrm{d}x \sim \left\| \nabla^{2}(\sigma, \mathbf{u}, N, c) \right\|_{L^{2}}^{2}, \tag{4.10}$$

where we have used the fact $0 < \alpha_2 < \frac{1}{4}$.

Thanks to (4.7) and (4.10), we can deduce that for a suitable constant C_3 ,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathfrak{D}_{h}(t) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, \mathrm{d}x \right) + C_{3} \left(\mathfrak{D}_{h}(t) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, \mathrm{d}x \right) \\
\leq C \left\| \nabla^{2} \left(\sigma^{L}, \mathbf{u}^{L}, N^{L}, c^{L} \right) \right\|_{L^{2}}^{2} + C \delta \|c\|_{H^{1}}^{2}.$$
(4.11)

Consequently, by the Gronwall inequality, we conclude that

$$\mathfrak{D}_{h}(t) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma^{L} \cdot \nabla \mathbf{u} \, \mathrm{d}x \leq e^{-C_{3}t} \left(\mathfrak{D}_{h}(0) - \alpha_{2} \int_{\mathbb{R}^{3}} \nabla \nabla \sigma_{0}^{L} \cdot \nabla \mathbf{u}_{0} \, \mathrm{d}x \right) + C \int_{0}^{t} e^{-C_{3}(t-\tau)} \left\| \nabla^{2} \left(\sigma^{L}, \mathbf{u}^{L}, N^{L}, c^{L} \right)(\tau) \right\|_{L^{2}}^{2} \mathrm{d}\tau + C \delta \int_{0}^{t} e^{-C_{3}(t-\tau)} \left\| c \right\|_{H^{1}}^{2} \mathrm{d}\tau.$$

$$(4.12)$$

This completes the proof of Lemma 4.1.

4.2 Decay estimates of the low-medium-frequency parts

Based on the temporal decay estimates from Fourier analysis of linearized systems, we can derive the estimates of the low-medium frequency part of solutions of the Cauchy problem. Next, we divide the derivation into three steps.

Step 1: we decouple the velocity **u**.

First, we define that

$$\Lambda =: (-\Delta)^{\frac{1}{2}}, \qquad b := \Lambda^{-1} \operatorname{div} \mathbf{u} \quad \text{and} \quad \mathfrak{p} \mathbf{u} = \Lambda^{-1} \operatorname{curl} \mathbf{u}.$$

Then we have div $\mathbf{u} = \Lambda b$ and $(\operatorname{curl} \mathbf{u})_{ij} := \partial_j \mathbf{u}^i - \partial_i \mathbf{u}^j$. The system (2.6) can be decoupled into the following systems:

$$\begin{cases} \sigma_t + \mu \Lambda b = M_1, \\ b_t - \mu \Lambda \sigma - \nu \Delta b - n_\infty \phi'(\rho_\infty) \Lambda \sigma = \mathfrak{M}_2, \\ N_t - \Delta N + n_\infty \Lambda^2 c = M_3, \\ c_t - \Delta c + n_\infty c = M_4, \\ (\sigma, b, N, c))(x, t)|_{t=0} = (\sigma_0, b_0, N_0, c_0)(x) \end{cases}$$

$$(4.13)$$

and

$$\begin{aligned} (\mathfrak{p}\mathbf{u})_t &- \Delta \mathfrak{p}\mathbf{u} = \mathfrak{p}M_2, \\ \mathfrak{p}\mathbf{u}(x,t)|_{t=0} &= \mathfrak{p}\mathbf{u}_0(x), \end{aligned} \tag{4.14}$$

where

$$v := \mu_1 + \mu_2$$
, $\mathfrak{M}_2 := \Lambda^{-1} \operatorname{div} M_2$, $b_0 := \Lambda^{-1} \operatorname{div} \mathbf{u}_0$.

In fact, the estimate of **u** translates into the estimate of b and the estimate of **pu**.

Applying the Fourier transform to (4.13), we get that

$$\begin{cases} \widehat{\sigma_t} + \mu |\xi| b = \widehat{M_1}, \\ \widehat{b_t} - \mu |\xi| \widehat{\sigma} + \nu |\xi|^2 \widehat{b} - n_\infty \phi'(\rho_\infty) |\xi| \widehat{\sigma} = \widehat{\mathfrak{M}_2}, \\ \widehat{N_t} + |\xi|^2 \widehat{N} + n_\infty |\xi|^2 \widehat{c} = \widehat{M_3}, \\ \widehat{c_t} + |\xi|^2 \widehat{c} + n_\infty \widehat{c} = \widehat{M_4}. \end{cases}$$

$$(4.15)$$

We rewrite the above equations in a vector form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \widehat{\sigma} \\ \widehat{b} \\ \widehat{N} \\ \widehat{c} \end{pmatrix} + H(|\xi|) \begin{pmatrix} \widehat{\sigma} \\ \widehat{b} \\ \widehat{N} \\ \widehat{c} \end{pmatrix} = \begin{pmatrix} \widehat{M_1} \\ \widehat{\mathfrak{M}_2} \\ \widehat{M_3} \\ \widehat{M_4} \end{pmatrix}, \qquad (4.16)$$

where

$$H(|\xi|) = \begin{pmatrix} 0 & \mu|\xi| & 0 & 0\\ -\mu|\xi| - n_{\infty}\phi'(\rho_{\infty})|\xi| & \nu|\xi|^2 & 0 & 0\\ 0 & 0 & |\xi|^2 & n_{\infty}|\xi|^2\\ 0 & 0 & 0 & |\xi|^2 + n_{\infty} \end{pmatrix}.$$
 (4.17)

Define $\mathfrak{g} = |\xi|$, then the characteristic polynomial of matrix *H* is given as follows:

$$P(\lambda) = |H(\mathfrak{g}) - \lambda I|$$

$$:= a_0 \lambda^4 - a_1 \lambda^3 + a_2 \lambda^2 - a_3 \lambda + a_4, \qquad (4.18)$$

where

$$a_{0} = 1, \qquad a_{1} = (2 + \nu)\mathfrak{g}^{2} + n_{\infty},$$

$$a_{2} = (1 + 2\nu)\mathfrak{g}^{4} + (\mu^{2} + n_{\infty} + n_{\infty}\nu + n_{\infty}\phi'(\rho_{\infty})\mu)\mathfrak{g}^{2},$$

$$a_{3} = \nu\mathfrak{g}^{6} + (2\mu^{2} + n_{\infty}\nu + 2n_{\infty}\phi'(\rho_{\infty})\mu)\mathfrak{g}^{4} + (n_{\infty}\mu^{2} + n_{\infty}^{2}\phi'(\rho_{\infty})\mu)\mathfrak{g}^{2},$$

$$a_{4} = (\mu^{2} + n_{\infty}\phi'(\rho_{\infty})\mu)\mathfrak{g}^{6} + (n_{\infty}\mu^{2} + n_{\infty}^{2}\phi'(\rho_{\infty})\mu)\mathfrak{g}^{4}.$$
(4.19)

The four solutions of the equation $P(\lambda) = 0$ are

$$\Lambda_1 = \mathfrak{g}^2 + O(\mathfrak{g}^3), \qquad \Lambda_2, \Lambda_3 = \pm i\mathfrak{g}\frac{\sqrt{|\nu^2|\mathfrak{g}^2 - 4(\mu^2 + n_\infty\phi'(\rho_\infty)\mu)}}{2} + \frac{\nu}{2}\mathfrak{g}^2 + O(\mathfrak{g}^3)$$

and

$$\Lambda_4 = \mathfrak{g}^2 + n_\infty + O(\mathfrak{g}^3).$$

Step 2: We shall analyze the asymptotic of the low-intermediate frequency.

Proposition 4.1 For a solution to $(\hat{\sigma}, \hat{b}, \hat{N}, \hat{c})$, there exists a constant C_4 , and the following inequality holds

$$\left| (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(t, \xi) \right|^2 \le \left| C e^{-c_4 |\xi|^2 t} \right| \left| (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(0, \xi) \right|^2.$$

$$(4.20)$$

Proof We can derive from the homogeneous linear equation

$$\begin{cases} \widehat{\sigma_t} + \mu |\xi| \widehat{b} = 0, \\ \widehat{b_t} - \mu |\xi| \widehat{\sigma} + \nu |\xi|^2 \widehat{b} - n_\infty \phi'(\rho_\infty) |\xi| \widehat{\sigma} = 0, \\ \widehat{N_t} + |\xi|^2 \widehat{N} + n_\infty |\xi|^2 \widehat{c} = 0, \\ \widehat{c_t} + |\xi|^2 \widehat{c} + n_\infty \widehat{c} = 0 \end{cases}$$

$$(4.21)$$

that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{|\widehat{\sigma}|^{2}+|\widehat{b}|^{2}+|\widehat{N}|^{2}\right\}+\nu|\xi|^{2}|\widehat{b}|^{2}+|\xi|^{2}|\widehat{N}|^{2}$$

$$x=n_{\infty}\phi'(\rho_{\infty})|\xi|\operatorname{Re}(\widehat{\sigma}\widehat{b})-n_{\infty}|\xi|^{2}\operatorname{Re}(\widehat{c}\widehat{N}),$$
(4.22)

and

$$\frac{1}{2}\frac{d}{dt}|\hat{c}|^2 + |\xi|^2|\hat{c}|^2 + n_{\infty}|\hat{c}|^2 = 0.$$
(4.23)

Multiplying $(4.21)_1$ and $\overline{(4.21)_2}$ by $\overline{\hat{b}}$ and $\widehat{\sigma}$, resp., and then adding the resulting identities up, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}(\widehat{\sigma}\overline{\widehat{b}}) - \left[\mu + n_{\infty}\phi'(\rho_{\infty})\right]|\xi|\widehat{\sigma}^{2} + \mu|\xi|\widehat{b}^{2} = -\nu|\xi|^{2}\operatorname{Re}(\overline{\widehat{b}}\widehat{\sigma}).$$
(4.24)

Combining $-\alpha_3|\xi| \times (4.24)$ and (4.22) yields

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ |\widehat{\sigma}|^2 - 2\alpha_3 |\xi| \operatorname{Re}(\widehat{\sigma}\overline{\hat{b}}) + |\widehat{b}|^2 + |\widehat{N}|^2 \right\} \\ + \left[\mu + n_\infty \phi'(\rho_\infty) \right] \alpha_3 |\xi|^2 \widehat{\sigma}^2 + (\nu - \mu \alpha_3) |\xi|^2 \widehat{b}^2 + |\xi|^2 |\widehat{N}|^2 \\ = n_\infty \phi'(\rho_\infty) |\xi| \operatorname{Re}(\widehat{\sigma}\widehat{b}) - n_\infty |\xi|^2 \operatorname{Re}(\widehat{c}\widehat{N}) + \nu \alpha_3 |\xi|^3 \operatorname{Re}(\widehat{b}\widehat{\sigma}).$$

$$(4.25)$$

For any fixed constant $\alpha_3 > 0$, by the Young inequality and a simple calculation, one gets

$$\frac{1}{2} \frac{d}{dt} \left\{ |\widehat{\sigma}|^{2} - 2\alpha_{3}|\xi| \operatorname{Re}(\widehat{\sigma}\widehat{b}) + |\widehat{b}|^{2} + |\widehat{N}|^{2} \right\} \\
+ \left[\mu + n_{\infty} \phi'(\rho_{\infty}) \right] \alpha_{3} |\xi|^{2} \widehat{\sigma}^{2} + (\nu - \mu \alpha_{3}) |\xi|^{2} \widehat{b}^{2} + |\xi|^{2} |\widehat{N}|^{2} \\
\leq \frac{n_{\infty} \phi'(\rho_{\infty})}{2} |\xi|^{2} |\widehat{\sigma}|^{2} + \frac{1}{2} |\widehat{b}|^{2} + \frac{1}{2} |\xi|^{2} |\widehat{N}|^{2} \\
+ \frac{n_{\infty}}{2} |\xi|^{2} |\widehat{c}|^{2} + \frac{\alpha_{3} \mu}{2} |\xi|^{2} |\widehat{\sigma}|^{2} + \frac{\nu^{2} \alpha_{3}}{\mu} |\xi|^{4} |\widehat{b}|^{2}.$$
(4.26)

Now we choose the constant α_3 satisfying

$$0 < \alpha_3 \le \min\left\{\frac{1}{2}, \frac{\nu}{4\mu}\right\}.$$

Then, we can get from (4.25) and (4.26) that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ |\widehat{\sigma}|^{2} - 2\alpha_{3}|\xi| \operatorname{Re}(\widehat{\sigma}\widehat{b}) + |\widehat{b}|^{2} + |\widehat{N}|^{2} \right\} \\
+ \frac{\mu\alpha_{3}}{2} |\xi|^{2} \widehat{\sigma}^{2} + \frac{\nu}{4} |\xi|^{2} \widehat{b}^{2} + \frac{1}{2} |\xi|^{2} |\widehat{N}|^{2} \\
\leq + \frac{1}{2} |\widehat{b}|^{2} + \frac{n_{\infty}}{2} |\xi|^{2} |\widehat{c}|^{2} + \frac{\nu^{2}\alpha_{3}}{\mu} |\xi|^{4} |\widehat{b}|^{2}.$$
(4.27)

Let the small constant r_0 satisfy $|\xi| \le r_0 \le \min\{\frac{1}{2}\sqrt{\frac{\mu}{\nu}}, \frac{1}{2}\}$. We derive from (4.23) and (4.27) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{L}_{l}(t,\xi) + \frac{\alpha_{3}\mu}{2}|\xi|^{2}|\widehat{\sigma}|^{2} + \frac{\nu}{8}|\xi|^{2}|\widehat{b}|^{2} + \frac{1}{2}|\xi|^{2}|\widehat{N}|^{2} + \frac{\bar{r}_{0}}{2}|\widehat{b}|^{2} + \frac{n_{\infty}\bar{r}_{0}}{2}|\xi|^{2}|\widehat{c}|^{2} \le 0, \quad (4.28)$$

where $\bar{\gamma_0} = 2$ and

$$\mathfrak{L}_{l}(t,\xi) := \frac{1}{2} |\widehat{\sigma}|^{2} - 2\alpha_{3}|\xi| \operatorname{Re}(\widehat{\sigma}\overline{\widehat{b}}) + \frac{1}{2}|\widehat{b}|^{2} + \frac{1}{2}|\widehat{N}|^{2} + \bar{\gamma}_{0}|\widehat{c}|^{2}.$$
(4.29)

Since $2\alpha_3 r_0 \leq \frac{1}{2}$, we have

$$\mathfrak{L}_{l}(t,\xi) \sim |\widehat{\sigma}|^{2} + |\widehat{b}|^{2} + |\widehat{N}|^{2} + |\widehat{c}|^{2}, \qquad (4.30)$$

which implies that there is a positive constant C_4 , such that for any $|\xi| \le r_0$,

$$C_{4}|\xi|^{2}\mathcal{L}_{l}(t,\xi) \leq \frac{\alpha_{3}\mu}{2}|\xi|^{2}|\widehat{\sigma}|^{2} + \frac{\nu}{8}|\xi|^{2}|\widehat{b}|^{2} + \frac{1}{2}|\xi|^{2}|\widehat{N}|^{2} + \frac{\bar{\gamma}_{0}}{2}|\widehat{b}|^{2} + \frac{n_{\infty}\bar{\gamma}_{0}}{2}|\xi|^{2}|\widehat{c}|^{2}.$$
(4.31)

Consequently, we immediately get (4.20).

Lemma 4.2 For any given constants r and R with 0 < r < R, there exists a positive constant j such that

$$\left|e^{-tG(|\xi|)}\right| \le Ce^{(jt)} \quad \text{for all } r \le |\xi| \le R \text{ and } t \in \mathbb{R}^+.$$

$$(4.32)$$

For the system (4.21), the inequality (4.32) yields

$$\begin{aligned} \left| (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(t, \xi) \right| &= \left| e^{-tG(|\xi|)} (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(0, \xi) \right| \\ &\leq C e^{-Jt} \left| (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(0, \xi) \right| \quad for \ all \ |\xi| \in [r, R], \end{aligned}$$

$$(4.33)$$

where r and R are any given positive constants.

Proof It is easy to check that the eigenvalues of $H(\xi)$ have positive real parts for sufficiently small g. Next, we further extend this fact to the case that no condition is required for large g.

By the Routh–Hurwitz theorem, the roots of the function $P(\lambda)$ have a positive real part if and only if the following determinants are positive:

$$H_{1} := a_{1}, \qquad H_{2} := \begin{vmatrix} a_{1} & a_{0} \\ a_{3} & a_{2} \end{vmatrix},$$

$$H_{3} := \begin{vmatrix} a_{1} & a_{0} & 0 \\ a_{3} & a_{2} & a_{1} \\ 0 & a_{4} & a_{3} \end{vmatrix}, \qquad H_{4} := \begin{vmatrix} a_{1} & a_{0} & 0 & 0 \\ a_{3} & a_{2} & a_{1} & 0 \\ 0 & a_{4} & a_{3} & 0 \\ 0 & 0 & 0 & a_{4} \end{vmatrix}.$$

$$(4.34)$$

It is clear that $H_1 > 0$ and $sgnH_3 = sgnH_4$. Then, we can check that

$$H_2 = a_1 a_2 - a_0 a_3$$

:= $H_{21} \mathfrak{g}^6 + H_{22} \mathfrak{g}^4 + H_{23} \mathfrak{g}^2 > 0,$ (4.35)

where the coefficients H_{21} , H_{22} , and H_{23} are defined by

$$H_{21} := 2(1 + 2\nu + \nu^{2}) > 0,$$

$$H_{22} := \mu^{2}\nu + 3n_{\infty} + 4n_{\infty}\nu + n_{\infty}\nu^{2} + n_{\infty}\phi'(\rho_{\infty})\mu\nu,$$

$$H_{23} := n_{\infty}^{2} + n_{\infty}^{2}\nu.$$
(4.36)

By calculation, we obtain $H_3 = a_3(a_1a_2 - a_0a_3) - a_1^2a_4 > 0$. We mediately see the conclusions in Lemma 4.2 hold; please refer to Sect. 3.3 in [7] for details.

Step 3: Next, we estimate for $\widehat{pu}(t,\xi)$.

The linearized equations of (4.14) under Fourier transform take the following form:

$$\frac{d}{dt}\widehat{\mathbf{pu}} + \mu|\xi|^2\widehat{\mathbf{pu}} = 0.$$
(4.37)

By a direct calculation, it follows from (4.37) that for all $|\xi| \ge 0$,

$$\left|\widehat{\mathfrak{pu}}(t,\xi)\right|^{2} \le C e^{-\mu|\xi|^{2}t} \left|\widehat{\mathfrak{pu}}(0,\xi)\right|^{2}.$$
(4.38)

Finally, exploiting the Fourier analysis of linear systems, we can show the temporal decay estimates for the low-intermediate part of the Cauchy problem solution in $L_t^2 L_x^2$ -norm. Let \mathbb{H} be a matrix of the differential operators, which enjoys the following form

$$\mathbb{H} = \begin{pmatrix} 0 & \mu \operatorname{div} & 0 & 0 \\ \mu \nabla + n_{\infty} \phi'(\rho_{\infty}) \nabla & -\mu_{1} \Delta - \mu_{2} \nabla \operatorname{div} & 0 & 0 \\ 0 & 0 & -\Delta & n_{\infty} \nabla^{2} \\ 0 & 0 & 0 & |\xi|^{2} + n_{\infty} \end{pmatrix}$$
(4.39)

and

$$\mathbb{U}(t) := (\sigma(t), \mathbf{u}(t), N(t), c(t))^{\mathrm{T}}, \qquad \mathbb{U}(0) := (\sigma_0, \mathbf{u}_0, N_0, c_0)^{\mathrm{T}}.$$
(4.40)

Then, we can get the corresponding linear equation problem

$$\begin{cases} \frac{d}{dt}\mathbb{U}(t) + \mathbb{H}\mathbb{U} = 0 \quad \text{for } t > 0, \\ \mathbb{U}|_{t=0} = \mathbb{U}(0). \end{cases}$$

$$(4.41)$$

Applying the Fourier transform to (4.41) with respect to the variables *x* and solving the ordinary equation with respect to *t*, we obtain

$$\mathbb{U}(t) = \mathfrak{H}(t)\mathbb{U}(0), \tag{4.42}$$

where $\mathfrak{H}(t) = e^{-t\mathbb{H}}$ $(t \ge 0)$ is the semigroup that generated by the linear operator \mathbb{H} and $\mathfrak{H}(t)f := F^{-1}(e^{-t\mathbb{H}_{\xi}}\hat{f}(\xi))$ with

$$\mathbb{H}_{\xi} = \begin{pmatrix} 0 & i\xi^{\mathrm{T}} & 0 & 0\\ i\xi & \mu_{1}|\xi|^{2}\delta_{ij} + \mu_{2}\xi_{i}\xi_{j} & 0 & 0\\ 0 & 0 & |\xi|^{2} & n_{\infty}|\xi|^{2}\\ 0 & 0 & 0 & |\xi|^{2} + n_{\infty} \end{pmatrix}.$$
(4.43)

Then, we have the following decay estimate.

Lemma 4.3 Let $1 \le p \le 2$. Then, for any integer $k \ge 0$,

$$\left\|\nabla^{k}\left(\mathfrak{h}(t)\mathbb{U}^{L}(0)\right)\right\|_{L^{2}} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\left\|\mathbb{U}(0)\right\|_{L^{p}}.$$
(4.44)

Proof Exploiting the Plancherel theorem and (4.20) and then taking $r = r_0$ and $R = R_0$ in (4.33), we obtain

$$\begin{split} \left\| \partial_{x}^{k} \left(\sigma^{L}, b^{L}, N^{L}, c^{L} \right)(t) \right\|_{L^{2}} &= \left\| (i\xi)^{k} \left(\widehat{\sigma^{L}}, \widehat{b^{L}}, \widehat{N^{L}}, \widehat{c^{L}} \right) \right\|_{L^{2}_{\xi}} \\ &= \left(\int_{\mathbb{R}^{3}} \left| (i\xi)^{k} \left(\widehat{\sigma^{L}}, \widehat{b^{L}}, \widehat{N^{L}}, \widehat{c^{L}} \right)(\xi, t) \right|^{2} \mathrm{d}\xi \right)^{\frac{1}{2}} \\ &\leq C \bigg(\int_{|\xi| \leq R_{0}} \left| (\xi) \right|^{2k} \left| (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(\xi, t) \right|^{2} \mathrm{d}\xi \bigg)^{\frac{1}{2}} \\ &\leq C \bigg(\int_{|\xi| \leq R_{0}} \left| (\xi) \right|^{2k} e^{-c_{6}|\xi|^{2}t} \left| (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(\xi, 0) \right|^{2} \mathrm{d}\xi \bigg)^{\frac{1}{2}} \\ &+ C \bigg(\int_{r_{0} \leq |\xi| \leq R_{0}} \left| (\xi) \right|^{2k} e^{-jt} \left| (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(\xi, 0) \right|^{2} \mathrm{d}\xi \bigg)^{\frac{1}{2}}. \end{split}$$
(4.45)

Using the Hausdorff–Young inequality and Hölder inequality, we get from (4.45) that

$$\begin{aligned} \left\| \partial_{x}^{k} \big(\sigma^{L}, b^{L}, N^{L}, c^{L} \big)(t) \right\|_{L^{2}} &\leq C \left\| (\widehat{\sigma}, \widehat{b}, \widehat{N}, \widehat{c})(0) \right\|_{L^{q}_{\xi}} (1+t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{q}) - \frac{k}{2}} \\ &\leq C \left\| (\sigma, \mathbf{u}, N, c)(0) \right\|_{L^{p}} (1+t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}}. \end{aligned}$$

$$(4.46)$$

Here $1 \le p \le 2 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Similarly to the estimate (4.46), using (4.38), we get

$$\left\|\partial_{x}^{k}(\mathbf{p}\mathbf{u})^{L}(t)\right\|_{L^{2}} \leq C \left\|\mathbf{u}(0)\right\|_{L^{p}} (1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}.$$
(4.47)

Thanks to (4.46) and (4.47), we immediately get the desired estimate (4.44).

4.3 Decay rates for the nonlinear system

Next, we establish the time decay estimates of solutions to the nonlinear problem (2.6) and (2.7). Let us consider the nonhomogeneous problem:

$$\begin{cases} \frac{d}{dt}\mathbb{U} + \mathbb{H}\mathbb{U} = S(\mathbb{U}) & \text{for } t > 0, \\ \mathbb{U}|_{t=0} = \mathbb{U}(0), \end{cases}$$

$$(4.48)$$

where

$$S(\mathbb{U}) = (M_1, M_2, M_3, M_4)^{\mathrm{T}}.$$
(4.49)

Based on Duhamel's principle, the solution of (4.48) can be written as follows

$$\mathbb{U}(t) = \mathfrak{h}(t)\mathbb{U}(0) + \int_0^t \mathfrak{h}(t-\tau)S(\mathbb{U})(\tau)\,\mathrm{d}\tau.$$
(4.50)

Thus, we have the following conclusion.

Lemma 4.4 Suppose that $1 \le p \le 2$, then for any integer $k \ge 0$, there is a positive constant C_5 such that

$$\begin{aligned} \left\| \nabla^{k} \mathbb{U}^{L}(t) \right\|_{L^{2}} &\leq C_{5}(1+t)^{-\frac{3}{4}-\frac{k}{2}} \left\| \mathbb{U}(0) \right\|_{L^{1}} \\ &+ C_{5} \int_{0}^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} \left\| S(\mathbb{U})(\tau) \right\|_{L^{1}} \mathrm{d}\tau \\ &+ C_{5} \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{k}{2}} \left\| S(\mathbb{U})(\tau) \right\|_{L^{2}} \mathrm{d}\tau. \end{aligned}$$

$$(4.51)$$

By combining Lemma 3.2 with Lemma 4.4, we get the time decay rates of solutions to the nonlinear problem.

Lemma 4.5 *By the assumption of Theorem* 1.1, *we have*

$$\left\|\nabla^{k}(\sigma, \mathbf{u}, N)(t)\right\|_{L^{2}} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \quad for \ k = 0, 1, 2,$$
(4.52)

$$\|c(t)\|_{H^2} \le Ce^{-Ct}.$$
(4.53)

Proof Adding (3.20) and (3.37) together and then using the smallness of δ , we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| c(t) \right\|_{H^2}^2 + \frac{1}{2} \left\| \nabla c(t) \right\|_{H^2}^2 + \frac{n_\infty}{2} \left\| \nabla c(t) \right\|_{H^2}^2 \le 0.$$
(4.54)

Multiplying (4.54) by $e^{\frac{n_{\infty}}{2}t}$ and integrating the resulting identity over [0, *t*], we have

$$\left\|\nabla c(t)\right\|_{H^2}^2 \le e^{-\frac{n_\infty}{2}t} \|c_0\|_{H^2}^2.$$
(4.55)

Thus, we obtain (4.53).

Denote that

$$G(t) := \sup_{0 \le \tau \le t} \sum_{m=0}^{2} (1+\tau)^{\frac{3}{4}+\frac{m}{2}} \left\| \nabla^{m}(\sigma, \mathbf{u}, N)(\tau) \right\|_{L^{2}}.$$
(4.56)

It is easy to see that G(t) is non-decreasing, and we have for $0 \le m \le 2$

$$\left\|\nabla^{m}(\sigma, \mathbf{u}, N)(\tau)\right\|_{L^{2}} \le C_{6}(1+\tau)^{-\frac{3}{4}-\frac{m}{2}}G(t)$$
(4.57)

for some positive constant C_6 independent of δ , where $0 \le \tau \le t$.

Thanks to the Hölder inequality (4.57) and assumption (3.1), we get

$$\begin{split} \left\| S\big(\mathbb{U}(\tau) \big) \right\|_{L^{1}} &\lesssim \left\| (\sigma, \mathbf{u}, N) \right\|_{L^{2}} \left\| \nabla(\sigma, \mathbf{u}, N, c) \right\|_{L^{2}} \\ &+ \left\| N \right\|_{L^{2}} \| c \|_{L^{2}} + \left\| \nabla N \right\|_{L^{2}} \| \nabla c \|_{L^{2}} \\ &\lesssim \delta G(t) (1+\tau)^{-\frac{5}{4}} + \delta \| c_{0} \|_{H^{2}} e^{-\frac{n_{\infty}}{4}\tau} \end{split}$$
(4.58)

and

$$\begin{split} \left\| S\big(\mathbb{U}(\tau) \big) \right\|_{L^{2}} &\lesssim \left\| (\sigma, \mathbf{u}, N) \right\|_{L^{3}} \left\| \nabla(\sigma, \mathbf{u}, N, c) \right\|_{L^{6}} \\ &+ \left\| N \right\|_{L^{\infty}} \| c \|_{L^{2}} + \| \nabla N \|_{L^{3}} \| \nabla c \|_{L^{6}} \\ &\lesssim \left\| (\sigma, \mathbf{u}, N) \right\|_{H^{1}} \left\| \nabla^{2}(\sigma, \mathbf{u}, N, c) \right\|_{L^{2}} \\ &+ \left\| \nabla N \right\|_{H^{1}} \| c \|_{L^{2}} + \left\| \nabla N \right\|_{H^{1}} \left\| \nabla^{2} c \right\|_{L^{2}} \\ &\lesssim \delta^{1 - \varepsilon_{1}} G^{1 + \varepsilon_{1}}(t) (1 + \tau)^{-(\frac{7}{4} + \frac{3}{4} \varepsilon_{1})} + \delta \| c_{0} \|_{H^{2}} e^{-\frac{n_{\infty}}{4} \tau}, \end{split}$$
(4.59)

where $\varepsilon_1 \in (0, \frac{1}{2})$ is a small fixed position constant.

By Lemma 4.4, (4.58), and (4.59), we have for $0 \le k \le 2$

$$\begin{split} \left\| \nabla^{k} \mathbb{U}^{L}(t) \right\|_{L^{2}} &\leq C(1+t)^{-\frac{3}{4} - \frac{k}{2}} \left\| \mathbb{U}(0) \right\|_{L^{1}} + C\delta \|c_{0}\|_{H^{2}} \int_{0}^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4} - \frac{k}{2}} e^{-\frac{n_{\infty}}{4}\tau} \, \mathrm{d}\tau \\ &+ C\delta \|c_{0}\|_{H^{2}} \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{k}{2}} e^{-\frac{n_{\infty}}{4}\tau} \, \mathrm{d}\tau \\ &+ C\delta G(t) \int_{0}^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4} - \frac{k}{2}} (1+\tau)^{-\frac{5}{4}} \, \mathrm{d}\tau \\ &+ C\delta^{1-\varepsilon_{1}} G(t)^{1+\varepsilon_{1}} \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{k}{2}} (1+\tau)^{-\frac{7}{4} - \frac{3}{4}\varepsilon_{1}} \, \mathrm{d}\tau \\ &\leq C(\left\| \mathbb{U}(0) \right\|_{L^{1}} + \|c_{0}\|_{H^{2}} + \delta G(t) + \delta^{1-\varepsilon_{1}} G(t)^{1+\varepsilon_{1}})(1+t)^{-\frac{3}{4} - \frac{k}{2}}. \end{split}$$
(4.60)

From (4.1) and (4.60), we obtain

$$\left\| \nabla^{2} \mathbb{U}(t) \right\|_{L^{2}}^{2} \leq C e^{-C_{2}t} \left\| \nabla^{2} \mathbb{U}(0) \right\|_{L^{2}}^{2} + C \delta \int_{0}^{t} e^{-C_{2}(t-\tau)} \left\| c(\tau) \right\|_{H^{1}}^{2} \mathrm{d}\tau + C \left(\left\| \mathbb{U}(0) \right\|_{L^{1}}^{2} + \left\| c_{0} \right\|_{H^{2}}^{2} + \delta^{2} G^{2}(t) + \delta^{2-2\varepsilon_{1}} G(t)^{2+2\varepsilon_{1}} \right) \int_{0}^{t} e^{-C_{2}(t-\tau)} (1+\tau)^{-\frac{7}{2}} \mathrm{d}\tau.$$

$$(4.61)$$

Putting (4.55) into (4.61) yields

$$\left\|\nabla^{2}\mathbb{U}(t)\right\|_{L^{2}}^{2} \leq C\left(\left\|\mathbb{U}(0)\right\|_{H^{2}\cap L^{1}}^{2} + \delta^{2}G^{2}(t) + \delta^{2-2\varepsilon_{1}}G(t)^{2+2\varepsilon_{1}}\right)(1+\tau)^{-\frac{7}{2}}.$$
(4.62)

Moreover, by the Frequency decomposition and (A.1), we get for $0 \le k \le 2$

$$\begin{aligned} \left\| \nabla^{k} \mathbb{U}(t) \right\|_{L^{2}}^{2} &\leq C \left\| \nabla^{k} \mathbb{U}^{L}(t) \right\|_{L^{2}}^{2} + \left\| \nabla^{k} \mathbb{U}^{h}(t) \right\|_{L^{2}}^{2} \\ &\leq C \left\| \nabla^{k} \mathbb{U}^{L} \right\|_{L^{2}}^{2} + \left\| \nabla^{2} \mathbb{U} \right\|_{L^{2}}^{2}. \end{aligned}$$
(4.63)

Exploiting (4.60), (4.62), and (4.63) for $0 \le k \le 2$, we have

$$\left\|\nabla^{k}\mathbb{U}(t)\right\|_{L^{2}}^{2} \leq C\left(\left\|\mathbb{U}(0)\right\|_{H^{2}\cap L^{1}}^{2} + \delta^{2}G^{2}(t) + \delta^{2-2\varepsilon_{1}}G^{2+2\varepsilon_{1}}(t)\right)(1+t)^{-\frac{3}{2}-k}.$$
(4.64)

Recalling the definition of G(t) and the smallness of δ , we derive from (4.64) that there is a positive constant C_7 independent of δ such that

$$G^{2}(t) \leq \frac{C_{7}}{2} \left\{ \left\| (\sigma, \mathbf{u}, N, c)(0) \right\|_{H^{2} \cap L^{1}}^{2} + \delta^{2} G^{2}(t) + \delta^{2 - 2\varepsilon_{1}} G^{2 + 2\varepsilon_{1}}(t) \right\}.$$
(4.65)

Thanks to the Young inequality, we obtain

$$C_{7}\delta^{2-2\varepsilon_{1}}G^{2+2\varepsilon_{1}}(t) \leq \frac{1-\varepsilon_{1}}{2}C_{7}^{\frac{2}{1-\varepsilon_{1}}} + \frac{1+\varepsilon_{1}}{2}\delta^{\frac{2(2-\varepsilon_{1})}{1+\varepsilon_{1}}}G^{4}(t).$$
(4.66)

Now we denote

$$K_0 = C_7 \left\| (\sigma, \mathbf{u}, N, c)(0) \right\|_{H^2 \cap L^1}^2 + \frac{1 - \varepsilon_1}{2} C_7^{\frac{2}{1 - \varepsilon_1}}$$
(4.67)

and

$$C_{\delta} = \frac{1 + \varepsilon_1}{2} \delta^{\frac{4(1 - \varepsilon_1)}{1 + \varepsilon_1}}.$$
(4.68)

In view of (4.65) and the smallness of δ , we have

$$G^{2}(t) \le K_{0} + C_{\delta}G^{4}(t). \tag{4.69}$$

Now we claim $G(t) \leq C$. Assume that $G^2(t) > 2K_0$ for any $t \in [\bar{t}, +\infty)$ with a constant $\bar{t} > 0$. Since $G^2(0) = \|(\sigma_0, \mathbf{u}_0, N_0, c_0)\|_{H^2}$ is small and $G(t) \in C^0[0, +\infty)$, there is $t_0 \in (0, \bar{t})$ such that $G^2(t_0) = 2K_0$.

We obtain from (4.69) that

$$G^2(t_0) \le K_0 + C_\delta G^4(t_0).$$

By direct calculation, we obtain

$$G^{2}(t_{0}) \leq \frac{K_{0}}{1 - C_{\delta}G^{2}(t_{0})}.$$
(4.70)

Suppose that δ is a small constant such that $C_{\delta} < \frac{1}{4K_0}$, i.e. $C_{\delta}G^2(t_0) < \frac{1}{2}$. Then we get $G^2(t_0) < 2K_0$ by (4.70). That is a contradiction with the assumption $G^2(t_0) < 2K_0$. So, $G^2(t_0) \le 2K_0$ for any $t \in [\bar{t}, +\infty)$. Noticing that G(t) is non-decreasing, we further get $G(t) \le C$ for any $t \in [0, +\infty)$. By the definition of G(t) in (4.56), we arrive at (4.52).

Appendix: Analytic tools

In this section, we will introduce some well-known Sobolev inequalities and a decay estimate, which have been used in the previous sections. **Lemma A.1** ([41]) For any given integers q, q_0 , q_1 with $q_0 \le q \le q_1 \le m$, it holds that

$$\left\|\nabla^{q} f^{l}\right\|_{L^{2}} \leq r_{0}^{q-q_{0}} \left\|\nabla^{q_{0}} f^{l}\right\|_{L^{2}}, \qquad \left\|\nabla^{q} f^{l}\right\|_{L^{2}} \leq \left\|\nabla^{q_{1}} f\right\|_{L^{2}}, \tag{A.1}$$

$$\left\|\nabla^{q}f^{h}\right\|_{L^{2}} \leq \frac{1}{R_{0}^{q_{1}-q}} \left\|\nabla^{q_{1}}f^{h}\right\|_{L^{2}}, \qquad \left\|\nabla^{q}f^{h}\right\|_{L^{2}} \leq \left\|\nabla^{q_{1}}f\right\|_{L^{2}}$$
(A.2)

and

$$r_0^q \| f^m \|_{L^2} \le \| \nabla^q f^m \|_{L^2} \le R_0^q \| f^m \|_{L^2}.$$
(A.3)

Proof The above inequalities can be easily verified by the definition of the frequency distribution and using the Plancherel theorem. \Box

Lemma A.2 ([25]) Let $m \ge 1$ be an integer, then we have

$$\|\nabla^{m}(fg)\|_{L^{p}(\mathbb{R}^{n})} \leq C\|f\|_{L^{p_{1}}(\mathbb{R}^{n})} \|\nabla^{m}g\|_{L^{p_{2}}(\mathbb{R}^{n})} + C\|\nabla^{m}f\|_{L^{p_{3}}(\mathbb{R}^{n})} \|g\|_{L^{p_{4}}(\mathbb{R}^{n})},$$
(A.4)

where $1 \le p_i \le +\infty$ $(1 \le i \le 4)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$
(A.5)

Lemma A.3 ([30]) *Let* $f \in H^2(\mathbb{R}^3)$ *. Then*

$$\begin{cases} \|f\|_{L^{\infty}} \leq C \|\nabla f\|^{\frac{1}{2}} \|\nabla\|^{\frac{1}{2}}_{H^{1}} \leq C \|\nabla f\|_{H^{1}}; \\ \|f\|_{L^{6}} \leq C \|\nabla f\|; \\ \|f\|_{L^{q}} \leq C \|\nabla f\|_{H^{1}} \quad for \ 2 \leq q \leq 6. \end{cases}$$
(A.6)

Lemma A.4 ([48]) Assume that $\|\psi\|_{L^{\infty}(\mathbb{R}^n)} \leq 1$. Let $f(\psi)$ be a smooth function of ψ with bounded derivatives of any order, then for any integer $m \geq 1$ and $1 \leq p \leq +\infty$, we have

$$\left\|\nabla^{m} f(\psi)\right\|_{L^{p}(\mathbb{R}^{n})} \leq \left\|\nabla^{m} \psi\right\|_{L^{p}(\mathbb{R}^{n})}.$$
(A.7)

Lemma A.5 ([32]) *If* $0 \le i, j \le k$, we get

$$\left\|\nabla^{i}f\right\|_{L^{q}} \lesssim \left\|\nabla^{k}f\right\|_{L^{q_{1}}}^{1-\delta} \left\|\nabla^{k}f\right\|_{L^{q_{2}}}^{\delta}.$$
(A.8)

In particular, when $q = \infty$, we require that δ must satisfy $0 < \delta < 1$.

Lemma A.6 ([53]) Let $a_1, a_2, a_3 \in \mathbb{R}$ and $a_2 > 1, 0 \le a_1 \le a_2, a_3 > 0$, so we have

$$\int_0^t (1+t-\tau)^{-a_1} (1+\tau)^{-a_2} \, \mathrm{d}\tau \le C(a_1,a_2)(1+t)^{-a_1},\tag{A.9}$$

$$\int_0^t (1+t-\tau)^{-a_1} e^{-a_3(t-\tau)} \,\mathrm{d}\tau \le C(a_1,a_3)(1+t)^{-a_1},\tag{A.10}$$

where $t \in \mathbb{R}_+$ *,* $C(a_1, a_2) > 0$ *and* $C(a_1, a_3) > 0$ *.*

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Declarations

Ethics

We certify that this manuscript is original and has not been published and will not be submitted elsewhere for publication while being considered by *Boundary Value Problems*. And the study is not split up into several parts to increase the number of submissions and submitted to various journals or to one journal over time.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

This work was carried out in collaboration between the three authors. WW designed the study and guided the research. YG and RS performed the analysis and wrote the first draft of the manuscript. YG, RS, and WW managed the analysis of the study. The three authors read and approved the final manuscript.

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