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Necessary and sufficient conditions on the existence of solutions for the exterior Dirichlet problem of Hessian equations

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Abstract

In this paper, we consider the exterior Dirichlet problem of Hessian equations $\sigma_k(\lambda(D^2u)) = g(x)$ with g being a perturbation of a general positive function at infinity. By estimating the eigenvalues of the solution, we obtain the necessary and sufficient conditions of existence of radial symmetric solutions with asymptotic behavior at infinity.

Keywords: Hessian equations; Exterior Dirichlet problem; Necessary and sufficient conditions

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $n \geq 3$. In this paper, we consider the exterior Dirichlet problem of Hessian equations

$$\sigma_k\big(\lambda\big(D^2u\big)\big) = \omega(x), \quad x \in \mathbb{R}^n\backslash\overline{\Omega}, \tag{1.1}$$

$$u = \phi(x), \quad x \in \partial\Omega, \tag{1.2}$$

where $\lambda(D^2u)$ are the eigenvalues $\lambda_1, \dots, \lambda_n$ of the Hessian matrix D^2u,

$$\sigma_k\big(\lambda\big(D^2u\big)\big) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}$$

is the kth elementary symmetric function for $k = 1, \dots, n$, $\omega \in C^0(\mathbb{R}^n\backslash\Omega)$ is positive and $\phi \in C^2(\partial\Omega)$. Note that, for $k = 1$, (1.1) is the Poisson equation $\Delta u = \omega(x)$ which is a linear elliptic equation; for $k = n$, (1.1) is the notable Monge–Ampère equation $\det D^2u = \omega(x)$ which is a fully nonlinear elliptic equation.

The exterior Dirichlet problem of Monge–Ampère equations is closely related to the classical theorem of Jörgens [18] ($n = 2$), Calabi [8] ($n \leq 5$), and Pogorelov [26] ($n \geq 2$) which states that any classical convex solution of $\det D^2u = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. Cheng and Yau [10], Caffarelli [6], Jost and Xin [19], and Trudinger and Wang [27] also gave related results with the Jörgens–Calabi–Pogorelov theorem. The cases of

$\det D^2u = f$ in $\mathbb{R}^n$ with f being a periodic function can be referred to Li and Lu [25] and the references therein.

In 2003, Caffarelli and Li [7] extended the Jörgens–Calabi–Pogorelov theorem to exterior domains and also investigated the existence of solutions to the exterior Dirichlet problem

$$\begin{cases} \det D^2u = 1, & x \in \mathbb{R}^n \backslash \overline{\Omega}, \\ u = \phi, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

They got that if Ω is a smooth, bounded, strictly convex open subset and $\phi \in C^2(\partial\Omega)$, then for any given $b \in \mathbb{R}^n$ and any given $n \times n$ real symmetric positive definite matrix A with $\det A = 1$, there exists some constant c^* depending only on n, Ω, ϕ, b, and A, such that for every $c > c^*$ there exists a unique function $u \in C^\infty(\mathbb{R}^n \backslash \overline{\Omega}) \cap C^0(\overline{\mathbb{R}^n \backslash \Omega})$ which satisfies (1.3) and

$$\limsup_{|x| \to \infty} \left(|x|^{n-2} \left| u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c \right) \right| \right) < \infty.$$

Since then, many results of the exterior problem for the fully nonlinear elliptic equations have been obtained. For instance, in 2011, the first author and Bao [13], the first author [11] studied the Dirichlet problem of Hessian equation

$$\sigma_k\big(\lambda\big(D^2u\big)\big) = 1 \tag{1.4}$$

and got the existence and uniqueness of viscosity solutions with the asymptotic behavior

$$\limsup_{x \to \infty} \left(|x|^{\alpha-2} \left| u(x) - \left(\frac{c_*}{2} |x|^2 + c \right) \right| \right) < \infty, \tag{1.5}$$

where $\alpha = n$ or $k, c \in \mathbb{R}$ and

$$c_* = \big(1/C_n^k\big)^{\frac{1}{k}}.$$

In 2013, Wang and Bao [28] studied the necessary and sufficient conditions on the existence of radially symmetric solutions for the Dirichlet problem outside a unit ball $B_1 = B_1(0)$,

$$\begin{cases} \sigma_k(\lambda(D^2u)) = 1, & x \in \mathbb{R}^n \setminus \overline{B_1}, \\ u = \text{constant}, & x \in \partial B_1, \end{cases}$$

with the asymptotic behavior

$$u(x) = \frac{c_*}{2}|x|^2 + c + O\big(|x|^{2-n}\big), \quad |x| \to \infty, n \geq 3,$$

and

$$u(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\ln|x| + c + O\big(|x|^{2-n}\big), \quad |x| \to \infty, n = 2,$$

where $c, d \in \mathbb{R}$. Recently, Li and Lu [24] characterized the existence and nonexistence of solutions for exterior problem of Monge–Ampère equations

$$\begin{cases} \det D^2 u = 1, & x \in \mathbb{R}^n \backslash \overline{\Omega}, \\ u = \phi(x), & x \in \partial\Omega, \\ \lim_{|x| \to \infty} |u(x) - (\frac{1}{2}x' A x + \tilde{b} \cdot x + \tilde{c}))| = 0, \end{cases}$$

with $\det A = 1$, $\tilde{b} \in \mathbb{R}^n$, $\tilde{c} \in \mathbb{R}$. Bao, Li, and Li [2] and Cao and Bao [9] studied the solutions with the generalized asymptotic behavior for exterior Dirichlet problem of Hessian equation (1.1). The results of the exterior Dirichlet problem for Monge–Ampère equations can also be referred to [1, 3–5, 17, 20] and the references therein. However, for the Hessian quotient equations

$$\frac{\sigma_k(\lambda(D^2 u))}{\sigma_l(\lambda(D^2 u))} = 1,$$

where $0 \leq l < k \leq n$, $n \geq 3$, and $\sigma_0(\lambda) = 1$, one can refer to [12, 21–23]. Note that if $l = 0$, the Hessian quotient equation is the Hessian equation. Moreover, for $n = 2$, the exterior Dirichlet problem of Monge–Ampère equations can be referred to the earlier works by Ferrer, Martínez, and Milán [15, 16] using the complex variable methods. One can also refer to Delanoë [14].

To work in the realm of elliptic equations, we restrict the class of functions. Let

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, j = 1, \dots, k\}.$$

Suppose that $u \in C^2(\mathbb{R}^n \backslash \overline{\Omega})$. If $\lambda(D^2 u) \in \overline{\Gamma}_k$ in $\mathbb{R}^n \backslash \overline{\Omega}$, we say that u is k-convex.

We shall discuss the necessary and sufficient conditions of existence for radially symmetric solutions to the exterior Dirichlet problem of Hessian equation.

Let $\omega_0 \in C^0(\mathbb{R}^n)$ be positive and radially symmetric in x,

$$0 < \inf_{\mathbb{R}^n} \omega_0 \leq \sup_{\mathbb{R}^n} \omega_0 < +\infty,$$

and $\omega \in C^0(\mathbb{R}^n \backslash B_1)$ be a radially symmetric function satisfying for $\beta > 2$

$$\omega(x) = \omega(|x|) = \omega_0(|x|) + O(|x|^{-\beta}), \quad |x| \to \infty, \tag{1.6}$$

and

$$0 < \inf_{\mathbb{R}^n \backslash B_1} \omega \leq \sup_{\mathbb{R}^n \backslash B_1} \omega < +\infty. \tag{1.7}$$

Suppose that, for $k \leq m \leq n$,

$$b_1 := \inf_{\mathbb{R}^n \backslash B_1} \left(\frac{m|x|^n \omega(x)}{m - k} - \int_1^{|x|} n t^{n-1} \omega(t)\, dt \right) > 0. \tag{1.8}$$

For $l = 1, 2, \dots, n$, let

$$\Phi_l := \{u \in C^1(\mathbb{R}^n \setminus B_1) \cap C^2(\mathbb{R}^n \setminus \overline{B_1}) | u \text{ is an } l\text{-convex radially symmetric function}\},$$

and the radially symmetric function

$$f_0(|x|) = \int_0^{|x|} c_* s^{1-\frac{n}{k}} \left[\int_0^s n t^{n-1} \omega_0(t)\, dt \right]^{\frac{1}{k}} ds, \quad x \in \mathbb{R}^n.$$

Theorem 1.1 *Let $n \geq 3$, $2 \leq k \leq m \leq n$, ω satisfy (1.6)–(1.8), and $\hat{c}$ be a constant. Then, for $m = k$, there exists a unique radially symmetric function $u \in \Phi_m$ satisfying*

$$\sigma_k\big(\lambda\big(D^2 u\big)\big) = \omega(x), \quad x \in \mathbb{R}^n \setminus \overline{B_1}, \tag{1.9}$$

$$u = \hat{c}, \quad x \in \partial B_1, \tag{1.10}$$

and as $|x| \to \infty$,

$$u(x) = \begin{cases} c + f_0(|x|) + O(|x|^{2-\min\{\beta,n\}}), & \text{if } \beta \neq n, \\ c + f_0(|x|) + O(|x|^{2-n} \ln |x|), & \text{if } \beta = n, \end{cases} \tag{1.11}$$

if and only if $c \in [\mu(0), +\infty)$; for $m > k$, there exists a unique radially symmetric function $u \in \Phi_m$ satisfying (1.9)–(1.11) if and only if $c \in [\mu(0), \mu(b_1)]$, where

$$\begin{aligned} \mu(\tau) = \hat{c} &+ \int_1^\infty c_* s^{1-\frac{n}{k}} \left[\left(\int_1^s n t^{n-1} \omega(t)\, dt + \tau \right)^{\frac{1}{k}} - \left(\int_0^s n t^{n-1} \omega_0(t)\, dt \right)^{\frac{1}{k}} \right] ds \\ &- \int_0^1 c_* s^{1-\frac{n}{k}} \left(\int_0^s n t^{n-1} \omega_0(t)\, dt \right)^{\frac{1}{k}} ds. \end{aligned} \tag{1.12}$$

Remark 1.2 In fact, $f_0(|x|)$ satisfies $\sigma_k(\lambda(D^2 f_0)) = \omega_0(|x|)$, $x \in \mathbb{R}^n \backslash \{0\}$.

Remark 1.3 From Theorem 1.1, we know that if $c < \mu(0)$, then (1.9)–(1.11) has no solution.

2 Proof of Theorem 1.1

We first give several lemmas in order to prove Theorem 1.1.

Lemma 2.1 ([28]) *Assume that $\lambda = (\hat{\beta}, \hat{\delta}, \ldots, \hat{\delta}) \in \Gamma_m$, $n \geq m \geq 2$, then $\hat{\delta} > 0$.*

Lemma 2.2 *Assume that $\lambda = (\hat{\beta}, \hat{\delta}, \ldots, \hat{\delta})$, $\sigma_k(\lambda) = \omega(\hat{r})$, $\hat{r} = |x| > 1$, $2 \leq k \leq n$, then $\lambda \in \Gamma_m$, $k \leq m \leq n$ if and only if $0 < \hat{\delta} < \hat{\delta}_m(\hat{r})$, where*

$$\hat{\delta}_m(\hat{r}) = \begin{cases} c_* \big(\frac{\omega(\hat{r})}{1-\frac{k}{m}}\big)^{\frac{1}{k}}, & m > k, \\ +\infty, & m = k, \end{cases} \tag{2.1}$$

and $c_* = (C_n^k)^{-\frac{1}{k}}$.

Proof Since $\sigma_k(\lambda) = \omega(\hat{r})$, $\hat{r} > 1$, then

$$C_{n-1}^{k-1} \hat{\beta} \hat{\delta}^{k-1} + C_{n-1}^{k} \hat{\delta}^k = \omega(\hat{r}).$$

So,

$$\hat{\beta} = \frac{\hat{\delta}}{k}\left[nc_*^k\omega(\hat{r})\hat{\delta}^{-k} - n + k\right]. \tag{2.2}$$

Because $\lambda \in \Gamma_m$, then for $j = 1, 2, \ldots, m$,

$$\sigma_j(\lambda) = C_{n-1}^{j-1}\hat{\beta}\hat{\delta}^{j-1} + C_{n-1}^{j}\hat{\delta}^j > 0,$$

and so

$$\hat{\delta}^{j-1}\big(j\hat{\beta} + (n-j)\hat{\delta}\big) > 0.$$

From Lemma 2.1, we know that $\hat{\delta} > 0$, so

$$j\hat{\beta} + (n-j)\hat{\delta} > 0.$$

Then from (2.2) we have that

$$j\frac{\hat{\delta}}{k}\big(nc_*^k\hat{\delta}^{-k}\omega(\hat{r}) - n + k\big) + (n-j)\hat{\delta} > 0.$$

Thus

$$c_*^k\hat{\delta}^{-k}\omega(\hat{r}) > 1 - \frac{k}{j}, \quad j = 1, 2, \ldots, m,$$

which is equivalent to

$$c_*^k\hat{\delta}^{-k}\omega(\hat{r}) > 1 - \frac{k}{m}.$$

That is, for any $\hat{r} > 1$,

$$0 < \hat{\delta} < \hat{\delta}_m(\hat{r}),$$

where $\hat{\delta}_m$ is defined by (2.1). □

Lemma 2.3 *Assume that $u \in C^1(\mathbb{R}^n \setminus B_1) \cap C^2(\mathbb{R}^n \setminus \overline{B_1})$ is a radially symmetric solution to (1.9) and (1.10). Let*

$$\tau := C_n^k\big(u'(1)^k\big).$$

Then u is k-convex if and only if $\tau \in [0, +\infty)$, and u is m-convex if and only if $\tau \in [0, b_1]$ for $m = k+1, \ldots, n$, where b_1 is defined by (1.8).

Proof Let

$$u(x) = u(\hat{r}) = u\big(|x|\big) \in C^1\big(\mathbb{R}^n \setminus B_1\big) \cap C^2\big(\mathbb{R}^n \setminus \overline{B_1}\big)$$

be a radially symmetric solution to (1.9) and (1.10). By a direct computation, we have

$$D_{ij}u = (\hat{r}u'' - u')\frac{x_i x_j}{\hat{r}^3} + u'\frac{\delta_{ij}}{\hat{r}}, \quad i,j = 1,\dots,n, \hat{r} > 1,$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Then the eigenvalues of the Hessian matrix $D^2 u$ are

$$\lambda_1 = u'', \qquad \lambda_2 = \cdots = \lambda_n = \frac{u'}{\hat{r}}.$$

By Lemma 2.1, we know that

$$\hat{\delta} = \frac{u'}{\hat{r}} > 0 \quad \text{for } \hat{r} > 1.$$

So $\tau \geq 0$. From (1.9), we have that

$$C_{n-1}^{k-1} u'' \left(\frac{u'}{\hat{r}}\right)^{k-1} + C_{n-1}^{k}\left(\frac{u'}{\hat{r}}\right)^{k} = \omega(\hat{r}),$$

i.e.,

$$\left(\hat{r}^{n-k}\left(u'\right)^k\right)' = \frac{n\hat{r}^{n-1}\omega(\hat{r})}{C_n^k}.$$

Then

$$\left(u'\right)^k = \frac{\hat{r}^{k-n}}{C_n^k}\left[\int_1^{\hat{r}} nt^{n-1}\omega(t)\,dt + C_n^k\left(u'(1)\right)^k\right], \quad \hat{r} > 1. \tag{2.3}$$

According to Lemma 2.2 and (2.3), we can get that u is m-convex for $k \leq m \leq n$ if and only if

$$0 < \hat{\delta}^k = \left(\frac{u'}{\hat{r}}\right)^k = \frac{\hat{r}^{-n}[\int_1^{\hat{r}} nt^{n-1}\omega(t)\,dt + \tau]}{C_n^k} < \hat{\delta}_m^k(r),$$

which is equivalent to

$$0 \leq \tau < +\infty, \quad \text{if } m = k,$$

and

$$0 \leq \tau < \frac{m\hat{r}^n\omega(\hat{r})}{m-k} - \int_1^{\hat{r}} nt^{n-1}\omega(t)\,dt, \quad \hat{r} > 1, \text{if } m > k. \tag{2.4}$$

(2.4) is equivalent to

$$0 \le \tau \le b_1 \quad \text{if } m > k.$$

Then the lemma is proved. □

Lemma 2.4 *Let $n \ge 3$, and $\mu(\tau)$ be defined by (1.12). Then $\mu(\tau)$ is strictly increasing in $[0,+\infty)$ and $\mu(+\infty) = +\infty$.*

Proof It is clear that $\mu(\tau)$ is strictly increasing in $[0,+\infty)$ and $\mu(+\infty) = +\infty$. □

Proof of Theorem 1.1 In virtue of (2.3), we can get that

$$u(x) = \hat{c} + \int_1^{|x|} c_* s^{1-\frac{n}{k}} \left[\int_1^s nt^{n-1}\omega(t)\,dt + \tau\right]^{\frac{1}{k}} ds.$$

By (1.6), we can assume that $\omega(|x|) = \omega_0(|x|) + C_0|x|^{-\beta}$, $|x| > s_0$, where C_0, s_0 are positive constants and s_0 is sufficiently large. Again by (1.12), we have that

$$\begin{aligned}
u(x) &= \hat{c} + \int_1^{\infty} c_* s^{1-\frac{n}{k}} \left[\int_1^s nt^{n-1}\omega(t)\,dt + \tau\right]^{\frac{1}{k}} ds \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left[\int_1^s nt^{n-1}\omega(t)\,dt + \tau\right]^{\frac{1}{k}} ds \\
&= \hat{c} + \int_1^{\infty} c_* s^{1-\frac{n}{k}} \left[\int_1^s nt^{n-1}\omega(t)\,dt + \tau\right]^{\frac{1}{k}} ds \\
&\quad - \int_1^{\infty} c_* s^{1-\frac{n}{k}} \left[\int_0^s nt^{n-1}\omega_0(t)\,dt\right]^{\frac{1}{k}} ds \\
&\quad + \int_1^{\infty} c_* s^{1-\frac{n}{k}} \left[\int_0^s nt^{n-1}\omega_0(t)\,dt\right]^{\frac{1}{k}} ds \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left[\int_1^s nt^{n-1}\omega(t)\,dt + \tau\right]^{\frac{1}{k}} ds \\
&= \hat{c} + \int_1^{\infty} c_* s^{1-\frac{n}{k}} \left[\left(\int_1^s nt^{n-1}\omega(t)\,dt + \tau\right)^{\frac{1}{k}} - \left(\int_0^s nt^{n-1}\omega_0(t)\,dt\right)^{\frac{1}{k}}\right] ds \\
&\quad - \int_0^1 c_* s^{1-\frac{n}{k}} \left[\int_0^s nt^{n-1}\omega_0(t)\,dt\right]^{\frac{1}{k}} ds \\
&\quad + \int_0^{|x|} c_* s^{1-\frac{n}{k}} \left[\int_0^s nt^{n-1}\omega_0(t)\,dt\right]^{\frac{1}{k}} ds \\
&\quad + \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left[\int_0^s nt^{n-1}\omega_0(t)\,dt\right]^{\frac{1}{k}} ds \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left[\int_1^s nt^{n-1}\omega(t)\,dt + \tau\right]^{\frac{1}{k}} ds \\
&= \mu(\tau) + f_0\big(|x|\big)
\end{aligned}$$

$$\begin{aligned}
&- \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left[\left(\int_1^s nt^{n-1}\omega(t)\,dt + \tau \right)^{\frac{1}{k}} - \left(\int_0^s nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}} \right] ds \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left[\left(\int_{s_0}^s nt^{n-1}\omega(t)\,dt + d_1 \right)^{\frac{1}{k}} - \left(\int_0^s nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}} \right] ds \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left[\left(\int_{s_0}^s nt^{n-1}\bigl(\omega_0(t) + C_0 t^{-\beta}\bigr)\,dt + d_1 \right)^{\frac{1}{k}} \right. \\
&\quad \left. - \left(\int_0^s nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}} \right] ds,
\end{aligned} \tag{2.5}$$

where $d_1 = \tau + \int_1^{s_0} nt^{n-1}\omega(t)\,dt$.

If $\beta \neq n$, then (2.5) becomes

$$\begin{aligned}
&\mu(\tau) + f_0\bigl(|x|\bigr) \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left[\left(\int_0^s nt^{n-1}\omega_0(t)\,dt + d_4 s^{n-\beta} + d_5 - \int_0^{s_0} nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}} \right. \\
&\quad \left. - \left(\int_0^s nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}} \right] ds \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left(\int_0^s nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}} \left[\left(1 + \frac{d_4 s^{n-\beta} + d_6}{\int_0^s nt^{n-1}\omega_0(t)\,dt} \right)^{\frac{1}{k}} - 1 \right] ds,
\end{aligned} \tag{2.6}$$

where $d_4 = \frac{nC_0}{n-\beta}$, $d_5 = d_1 - d_4 s_0^{n-\beta}$, and $d_6 = d_5 - \int_0^{s_0} nt^{n-1}\omega_0(t)\,dt$. Since $\omega_0(t)$ is bounded, then as $s \to +\infty$,

$$\frac{d_4 s^{n-\beta} + d_6}{\int_0^s nt^{n-1}\omega_0(t)\,dt} \to 0.$$

Therefore, (2.6) approximately equals

$$\begin{aligned}
&\mu(\tau) + f_0\bigl(|x|\bigr) - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \left(\int_0^s nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}} \frac{1}{k} \frac{d_4 s^{n-\beta} + d_6}{\int_0^s nt^{n-1}\omega_0(t)\,dt}\,ds \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) - \int_{|x|}^{\infty} \frac{1}{k} c_* d_4 \left(\int_0^s nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}-1} s^{1-\frac{n}{k}+n-\beta}\,ds \\
&\quad - \int_{|x|}^{\infty} \frac{1}{k} c_* d_6 \left(\int_0^s nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}-1} s^{1-\frac{n}{k}}\,ds \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) + O\left(\left(\int_0^{|x|} nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}-1} |x|^{2-\frac{n}{k}+n-\beta} \right) \\
&\quad + O\left(\left(\int_0^{|x|} nt^{n-1}\omega_0(t)\,dt \right)^{\frac{1}{k}-1} |x|^{2-\frac{n}{k}} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \mu(\tau) + f_0\bigl(|x|\bigr) + O\bigl(|x|^{2-\beta}\bigr) + O\bigl(|x|^{2-n}\bigr) \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) + O\bigl(|x|^{2-\min\{\beta,n\}}\bigr) \quad \text{as } |x| \to \infty.
\end{aligned}
$$

If $\beta = n$, then (2.5) becomes

$$
\begin{aligned}
&\mu(\tau) + f_0\bigl(|x|\bigr) \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \biggl[\biggl(\int_0^s nt^{n-1}\omega_0(t)\,dt + C_0 n \ln s + d_2\biggr)^{\frac{1}{k}} \\
&\quad - \biggl(\int_0^s nt^{n-1}\omega_0(t)\,dt\biggr)^{\frac{1}{k}}\biggr]\,ds \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) \\
&\quad - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \biggl(\int_0^s nt^{n-1}\omega_0(t)\,dt\biggr)^{\frac{1}{k}} \biggl[\biggl(1 + \frac{C_0 n \ln s + d_2}{\int_0^s nt^{n-1}\omega_0(t)\,dt}\biggr)^{\frac{1}{k}} - 1\biggr]\,ds,
\end{aligned}
\tag{2.7}
$$

where $d_2 = d_1 - C_0 n \ln s_0 - \int_0^{s_0} nt^{n-1}\omega_0(t)\,dt$. Since

$$
\frac{C_0 n \ln s + d_2}{\int_0^s nt^{n-1}\omega_0(t)\,dt} \to 0, \quad s \to +\infty,
$$

therefore (2.7) approximately equals

$$
\begin{aligned}
&\mu(\tau) + f_0\bigl(|x|\bigr) - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \biggl(\int_0^s nt^{n-1}\omega_0(t)\,dt\biggr)^{\frac{1}{k}} \frac{1}{k} \frac{C_0 n \ln s + d_2}{\int_0^s nt^{n-1}\omega_0(t)\,dt}\,ds \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) - \int_{|x|}^{\infty} c_* s^{1-\frac{n}{k}} \frac{1}{k} \frac{C_0 n \ln s + d_2}{(\int_0^s nt^{n-1}\omega_0(t)\,dt)^{1-\frac{1}{k}}}\,ds \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) + O\biggl(\biggl(\int_0^{|x|} nt^{n-1}\omega_0(t)\,dt\biggr)^{\frac{1}{k}-1} |x|^{2-\frac{n}{k}} \ln|x|\biggr) \\
&= \mu(\tau) + f_0\bigl(|x|\bigr) + O\bigl(|x|^{2-n}\ln|x|\bigr) \quad \text{as } |x| \to \infty.
\end{aligned}
$$

Consequently, we have that as $|x| \to \infty$,

$$
u(x) = \begin{cases} \mu(\tau) + f_0(|x|) + O(|x|^{2-\min\{\beta,n\}}), & \text{if } \beta \neq n, \\ \mu(\tau) + f_0(|x|) + O(|x|^{2-n}\ln|x|), & \text{if } \beta = n. \end{cases}
\tag{2.8}
$$

Comparing (2.8) with (1.11), by Lemmas 2.3 and 2.4, we know that, for $m = k$, u is m-convex if and only if $c \in [\mu(0), +\infty)$; for $m > k$, u is m-convex if and only if $c \in [\mu(0), \mu(b_1)]$. Theorem 1.1 is proved. □

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The first author proposed the idea of this paper and performed all the steps of the proofs. The second author wrote the whole paper. All authors read and approved the final manuscript.

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