El-Gaber et al. *Boundary Value Problems* (2022) 2022:41
https://doi.org/10.1186/s13661-022-01620-2

Boundary Value Problems
a SpringerOpen Journal

REVIEW

Open Access

Oscillation of super-linear fourth-order differential equations with several sub-linear neutral terms

A.A. El-Gaber[1*], M.M.A. El-Sheikh[1] and E.I. El-Saedy[1]

*Correspondence: amina.aboalnour@science.menofia.edu.eg
[1]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin El-Koom, Egypt

Abstract

In this paper, we discuss the oscillatory behavior of solutions of a class of Super-linear fourth-order differential equations with several sub-linear neutral terms using the Riccati and generalized Riccati transformations. Some Kamenev–Philos-type oscillation criteria are established. New oscillation criteria are deduced in both canonical and non-canonical cases. An illustrative example is given.

MSC: Primary 34C10; 34K11

Keywords: Oscillation; Fourth order; Neutral differential equations

1 Introduction

The aim of this paper is to discuss the oscillatory behavior of solutions of a class of super-linear fourth-order neutral differential equations of the type,

$$\bigl(r(t)\bigl(z'''(t)\bigr)^{\gamma}\bigr)' + \sum_{i=1}^{m} f_i\bigl(t, x\bigl(\tau_i(t)\bigr)\bigr) = 0, \quad t \geq t_0, \tag{1.1}$$

where $z(t) = x(t) + \sum_{j=1}^{n} a_j(t)x^{\alpha_j}(\sigma_j(t))$, m, n are positive integers, and α_j, γ are ratios of odd positive integers and $0 < \alpha_j \leq 1$, $\gamma \geq 1$, under the conditions

$$R(t_0) = \int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(t)}\, dt = \infty, \tag{1.2}$$

and

$$R(t_0) = \int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(t)}\, dt < \infty. \tag{1.3}$$

Throughout the paper, we assume the following assumptions

(A_1) $r(t) \in C^1([t_0, \infty), (0, \infty))$, $r'(t) \geq 0$;

(A_2) $a_j(t), \sigma_j(t), \tau_i(t) \in C[t_0, \infty))$, $\sigma_j(t) \leq t$, $\lim_{t\to\infty} \sigma_j(t) = \infty$;

(A_3) there exists a function $\tau \in C^1([t_0,\infty),R)$ such that $\tau(t) \leq \tau_i(t)$ for $i = 1,2,\ldots,m$, $\tau(t) \leq t$, $\tau'(t) > 0$ and $\lim_{t\to\infty} \tau(t) = \infty$;

(A_4) $0 \leq a_j(t) \leq a_{0j}(t)$, $\sum_{j=1}^{n} a_{0j}(t) < 1$, $f_i(t,x) \in C([t_0,\infty) \times R,R)$ satisfy $xf_i(t,x) > 0$ for all $x \neq 0$, and there exist positive continuous functions $q_i(t)$ defined on $[t_0,\infty)$ such that $|f_i(t,x)| \geq q_i(t)|x|^\gamma$.

By a solution of (1.1), we mean a nontrivial real function $x(t)$ such that $r(t)([x(t) + \sum_{j=1}^{n} a_j(t)x^{\alpha_j}(\sigma_j(t))]''')^\gamma$ is continuously differentiable satisfying (1.1) for any $t_1 \geq t_0$.

A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Oscillation phenomena take part in different models from real-world applications; see, e.g., paper [8] for more details. In the last three decades, there has been considerable interest in studying the oscillation of solutions of several kinds of differential equations [1–5, 7, 8, 10–20, 22–24, 26–39]. The half-linear equations have numerous applications in the study of p-Laplace equations, non-Newtonian fluid theory, porous medium, etc.; see, e.g., papers [6, 21, 25] for more details. In particular, papers [11, 24] were concerned with the oscillation of various classes of half-linear differential equations, whereas the papers [3–5, 7, 10, 20, 26, 38] were concerned with the oscillatory behavior of the fourth-order differential equation (1.1) and its special cases. In what follows, we briefly comment on a number of closely related results which motivated our work. The authors in [3, 4, 26] discussed in their recent papers, the special case of (1.1) of the form,

$$\left(r(t)\left(\left[x(t)+p(t)x\left(\tau(t)\right)\right]'''\right)^{\alpha}\right)'+q(t)x^{\beta}\left(\delta(t)\right)=0. \tag{1.4}$$

Under the condition (1.2), Dassios and Bazighifan in [10] discussed the oscillation of the same equation under condition (1.3). In [20], Li et al. studied the oscillatory behavior of a class of fourth-order differential equations with the p-Laplacian-like operator of the type,

$$\left(r(t)\left|z'''(t)\right|^{p-2}z'''(t)\right)'+\sum_{i=1}^{l}q_i(t)\left|x\left(\tau_i(t)\right)\right|^{p-2}x\left(\tau_i(t)\right)=0, \tag{1.5}$$

where $z(t) = x(t) + a(t)x(\sigma(t))$. Under the condition $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{p-2}}(t)}\,dt < \infty$, they used the Riccati transformation and integral averaging technique and presented a Kamenev-type oscillation criterion.

More recently, Bazighifan et al. [5] studied the asymptotic behavior of solutions of the fourth-order neutral differential equation with the continuously distributed delay of the form

$$\left(r(t)\left(\left[x(t)+p(t)x\left(\phi(t)\right)\right]'''\right)^{\alpha}\right)'+\int_a^b q(t,\theta)x^{\beta}\left(\delta(t,\theta)\right)d\theta=0, \tag{1.6}$$

where α, β are quotients of odd positive integers, and $\beta \geq \alpha$ under the condition (1.2).

2 Preliminaries

The following preliminary results will be needed for our proofs.

Lemma 1 ([9]) *Let $h > 0$. Then*

$$h^{\alpha} \leq \alpha h + (1 - \alpha), \quad 0 < \alpha \leq 1.$$

Lemma 2 ([28]) *Let $z(t)$ be a positive and n-times differentiable function on an interval $[T, \infty)$ with non-positive nth derivative $z^{(n)}(t)$ on $[T, \infty)$, which is not identically zero on any interval of the form $[T', \infty)$, $T' \geq T$ and such that $z^{(n-1)}(t)z^{(n)}(t) \leq 0$. Then, there exist constants $0 < \theta < 1$ and $N > 0$ such that $z'(\theta t) \geq Nt^{n-2}z^{(n-1)}(t)$ for all sufficient large t.*

Lemma 3 ([26]) *Let $z^{(n)}(t)$ be of fixed sign and $z^{(n-1)}(t)z^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} z(t) \neq 0$, then for every $\lambda \in (0,1)$, there exists $t_{\lambda} \geq t$ such that $z(t) \geq \frac{\lambda}{(n-1)!}t^{n-1} \times |z^{(n-1)}(t)|$ for $t \geq t_{\lambda}$.*

Lemma 4 ([2]) *Let α is a ratio of two odd numbers. Suppose that U, V are constants with $V > 0$. Then, $UY - VY^{\frac{(\gamma+1)}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{U^{\gamma+1}}{V^{\gamma}}$.*

Lemma 5 *Assume that $x(t)$ is an eventually positive solution of (1.1), $z'(t) > 0$, and there exists a positive decreasing function $\delta(t) \in C([t_0, \infty))$ tending to zero such that $\theta(\tau_i(t)) > 0$ for $t \geq t_0$ where $\theta(t) = 1 - \sum_{j=1}^{n} \alpha_j a_j(t) - \frac{1}{\delta(t)} \sum_{j=1}^{n}(1 - \alpha_j)a_j(t)$. Then,*

$$\bigl(r(t)\bigl(z'''(t)\bigr)^{\gamma}\bigr)' \leq -\sum_{i=1}^{m} q_i(t)\theta^{\gamma}\bigl(\tau_i(t)\bigr)z^{\gamma}\bigl(\tau(t)\bigr). \tag{2.1}$$

Proof Let x be an eventually positive solution of Eq. (1.1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\sigma_j(t)) > 0$ and $x(\tau_i(t)) > 0$ for $t \geq t_1$. Now from the definition of z, we have

$$x(t) = z(t) - \sum_{j=1}^{n} a_j(t)x^{\alpha_j}\bigl(\sigma_j(t)\bigr) \geq z(t) - \sum_{j=1}^{n} a_j(t)z^{\alpha_j}\bigl(\sigma_j(t)\bigr) \geq z(t) - \sum_{j=1}^{n} a_j(t)z^{\alpha_j}(t).$$

Then, by Lemma 1, we have

$$x(t) \geq \left(1 - \sum_{j=1}^{n} \alpha_j a_j(t)\right)z(t) - \sum_{j=1}^{n}(1 - \alpha_j)a_j(t).$$

Now since $z(t)$ is positive and increasing, and $\delta(t)$ is a positive decreasing function tending to zero, then there exists a $t_2 \geq t_1$ such that $z(t) \geq \delta(t)$, and

$$x(t) \geq \left[1 - \sum_{j=1}^{n} \alpha_j a_j(t) - \frac{1}{\delta(t)} \sum_{j=1}^{n}(1 - \alpha_j)a_j(t)\right]z(t), \quad \text{for } t \geq t_2.$$

That is $x(t) \geq \theta(t)z(t)$. Therefore, from (1.1), it follows that

$$\bigl(r(t)\bigl(z'''(t)\bigr)^{\gamma}\bigr)' \leq -\sum_{i=1}^{m} q_i(t)\theta^{\gamma}\bigl(\tau_i(t)\bigr)z^{\gamma}\bigl(\tau_i(t)\bigr) \leq -\sum_{i=1}^{m} q_i(t)\theta^{\gamma}\bigl(\tau_i(t)\bigr)z^{\gamma}\bigl(\tau(t)\bigr).$$

Thus, the proof is completed. □

The following two auxiliary results are very similar to those reported in [3] and [10].

Lemma 6 *Let $x(t)$ be a positive solution of (1.1). If (1.2) is satisfied, then there exists $t \geq t_1$ such that*

$$z(t) > 0, \qquad z'(t) > 0, \qquad z'''(t) > 0, \qquad z^{(4)}(t) < 0, \qquad \bigl(r(t)\bigl(z'''(t)\bigr)^{\gamma}\bigr)' \leq 0.$$

Lemma 7 *Let $x(t)$ be a positive solution of (1.1). If (1.3) is satisfied, then there exist three possible cases for sufficiently large $t \geq t_1$*

(S_1) $z(t) > 0, z'(t) > 0, z'''(t) > 0, z^{(4)}(t) \leq 0$;

(S_2) $z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) < 0$;

(S_3) $z(t) > 0, z'(t) < 0, z''(t) > 0, z'''(t) < 0$.

3 Main results

We first consider the case $R(t_0) = \infty$.

Theorem 8 *If there exist $\eta(t) \in C^1([t_0,\infty),(0,\infty))$, $b(t) \in C^1([t_0,\infty),[0,\infty))$, $\zeta \in (0,1)$ and $\epsilon > 0$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[Q(s) - \frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}} \frac{[\frac{\eta'(s)}{\eta(s)} + (\gamma+1)\zeta\epsilon\tau'(s)\tau^2(s)b^{\frac{1}{\gamma}}(s)]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)]^{\gamma}} \right] ds = \infty, \tag{3.1}$$

then (1.1) is oscillatory, where $Q(t) = \eta(t)\sum_{i=1}^{m} q_i(t)\theta^{\gamma}(\tau_i(t)) - \eta(t)[r(t)b(t)]' + \zeta\epsilon\tau'(t)\tau^2(t) \times r(t)\eta(t)b^{1+\frac{1}{\gamma}}(t)$.

Proof Suppose for the contrary that x is an eventually positive solution of (1.1). Then there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\sigma_j(t)) > 0$ and $x(\tau_i(t)) > 0$ for $t \geq t_1$. Using Lemma 5, we obtain (2.1). Define

$$\psi(t) = \eta(t)\left[\frac{r(t)(z'''(t))^{\gamma}}{z^{\gamma}(\zeta\tau(t))} + r(t)b(t)\right], \quad \mathbf{t} \geq \mathbf{t}_1. \tag{3.2}$$

It is clear that $\psi(t) > 0$ for $t \geq t_1$, and

$$\begin{aligned}\psi'(t) = {}& \frac{\eta'(t)}{\eta(t)}\psi(t) + \eta(t)\bigl[r(t)b(t)\bigr]' + \eta(t)\frac{(r(t)(z'''(t))^{\gamma})'}{z^{\gamma}(\zeta\tau(t))} \\ & - \eta(t)\frac{\gamma\zeta r(t)\tau'(t)(z'''(t))^{\gamma}z'(\zeta\tau(t))}{z^{\gamma+1}(\zeta\tau(t))}.\end{aligned}$$

Thus, by (2.1), it follows that

$$\begin{aligned}\psi'(t) \leq {}& \frac{\eta'(t)}{\eta(t)}\psi(t) + \eta(t)\bigl[r(t)b(t)\bigr]' - \eta(t)\frac{\sum_{i=1}^{m} q_i(t)\theta^{\gamma}(\tau_i(t))z^{\gamma}(\tau(t))}{z^{\gamma}(\zeta\tau(t))} \\ & - \eta(t)\frac{\gamma\zeta r(t)\tau'(t)(z'''(t))^{\gamma}z'(\zeta\tau(t))}{z^{\gamma+1}(\zeta\tau(t))}.\end{aligned}$$

By Lemma 2, we have

$$z'\bigl(\zeta\tau(t)\bigr) \geq \epsilon\tau^2(t)z'''\bigl(\tau(t)\bigr) \geq \epsilon\tau^2(t)z'''(t).$$

However, since $z(t)$ is increasing, then $z(\tau(t)) \geq z(\zeta\tau(t))$. Therefore,

$$\psi'(t) \leq \frac{\eta'(t)}{\eta(t)}\psi(t) + \eta(t)\big[r(t)b(t)\big]' - \eta(t)\sum_{i=1}^{m} q_i(t)\theta^{\gamma}\big(\tau_i(t)\big) - \eta(t)\frac{\gamma\zeta\epsilon r(t)\tau'(t)\tau^2(t)(z'''(t))^{\gamma+1}}{z^{\alpha+1}(\zeta\tau(t))}.$$

Moreover, since from (3.2), we have

$$\frac{z'''(t)}{z(\zeta\tau(t))} = \frac{1}{r^{\frac{1}{\gamma}}(t)}\left[\frac{\psi(t)}{\eta(t)} - \big[r(t)b(t)\big]\right]^{\frac{1}{\gamma}},$$

then

$$\psi'(t) \leq \frac{\eta'(t)}{\eta(t)}\psi(t) + \eta(t)\big[r(t)b(t)\big]' - \eta(t)\sum_{i=1}^{m} q_i(t)\theta^{\gamma}\big(\tau_i(t)\big) - \gamma\zeta\epsilon\tau'(t)\tau^2(t)\frac{\eta(t)}{r^{\frac{1}{\gamma}}(t)}\left(\frac{\psi(t)}{\eta(t)} - \big[r(t)b(t)\big]\right)^{\frac{\gamma+1}{\gamma}}. \tag{3.3}$$

As in [35], we use the inequality

$$M^{1+\frac{1}{\gamma}} - (M-N)^{1+\frac{1}{\gamma}} \leq N^{\frac{1}{\gamma}}\left[\left(1+\frac{1}{\gamma}\right)M - \frac{1}{\gamma}N\right], \quad MN \geq 0, \gamma \geq 1,$$

with

$$M = \frac{\psi(t)}{\eta(t)} \quad \text{and} \quad N = r(t)b(t),$$

to get

$$\left(\frac{\psi(t)}{\eta(t)} - \big[r(t)b(t)\big]\right)^{\frac{\gamma+1}{\gamma}} \geq \left[\frac{\psi(t)}{\eta(t)}\right]^{1+\frac{1}{\gamma}} + \frac{1}{\gamma}\big[r(t)b(t)\big]^{1+\frac{1}{\gamma}} - \left(1+\frac{1}{\gamma}\right)\frac{[r(t)b(t)]^{\frac{1}{\gamma}}}{\eta(t)}\psi(t). \tag{3.4}$$

Using inequalities (3.3) and (3.4), for $t \geq T$, we have

$$\begin{aligned}\psi'(t) \leq{}& \frac{\eta'(t)}{\eta(t)}\psi(t) + \eta(t)\big[r(t)b(t)\big]' - \eta(t)\sum_{i=1}^{m} q_i(t)\theta^{\gamma}\big(\tau_i(t)\big) \\ &+ \gamma\zeta\epsilon\tau'(t)\tau^2(t)\frac{\eta(t)}{r^{\frac{1}{\gamma}}(t)}\left[\left(1+\frac{1}{\gamma}\right)\frac{[r(t)b(t)]^{\frac{1}{\gamma}}}{\eta(t)}\psi(t) \right.\\ &\left. - \frac{1}{\gamma}\big[r(t)b(t)\big]^{1+\frac{1}{\gamma}} - \frac{\psi^{1+\frac{1}{\gamma}}(t)}{\eta^{1+\frac{1}{\gamma}}(t)}\right].\end{aligned}$$

Then,

$$\psi'(t) \le \eta(t)\left(\left[r(t)b(t)\right]' - \sum_{i=1}^{m} q_i(t)\theta^{\gamma}\left(\tau_i(t)\right)\right) + \left[\frac{\eta'(t)}{\eta(t)} + (\gamma+1)\zeta\epsilon\tau'(t)\tau^2(t)b^{\frac{1}{\gamma}}(t)\right]\psi(t) - \frac{\gamma\zeta\epsilon\tau'(t)\tau^2(t)}{r^{\frac{1}{\gamma}}(t)\eta^{\frac{1}{\gamma}}(t)}\psi^{1+\frac{1}{\gamma}}(t) - \zeta\epsilon\tau'(t)\tau^2(t)r(t)\eta(t)b^{1+\frac{1}{\gamma}}(t),$$

i.e.

$$\psi'(t) \le -Q(t) + \left[\frac{\eta'(t)}{\eta(t)} + (\gamma+1)\zeta\epsilon\tau'(t)\tau^2(t)b^{\frac{1}{\gamma}}(t)\right]\psi(t) - \frac{\gamma\zeta\epsilon\tau'(t)\tau^2(t)}{r^{\frac{1}{\gamma}}(t)\eta^{\frac{1}{\gamma}}(t)}\psi^{1+\frac{1}{\gamma}}(t). \tag{3.5}$$

Now let

$$U = \frac{\eta'(t)}{\eta(t)} + (\gamma+1)\zeta\epsilon\tau'(t)\tau^2(t)b^{\frac{1}{\gamma}}(t),$$

$$V = \frac{\gamma\zeta\epsilon\tau'(t)\tau^2(t)}{r^{\frac{1}{\gamma}}(t)\eta^{\frac{1}{\gamma}}(t)} \quad \text{and} \quad Y = \psi(t).$$

Then, by Lemma 4, we obtain

$$\left[\frac{\eta'(t)}{\eta(t)} + (\gamma+1)\zeta\epsilon\tau'(t)\tau^2(t)b^{\frac{1}{\gamma}}(t)\right]\psi(t) - \frac{\gamma\zeta\epsilon\tau'(t)\tau^2(t)}{r^{\frac{1}{\gamma}}(t)\eta^{\frac{1}{\gamma}}(t)}\psi^{1+\frac{1}{\gamma}}(t) \le \frac{\gamma^{\gamma}r(t)\eta(t)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(t)}{\eta(t)} + (\gamma+1)\zeta\epsilon\tau'(t)\tau^2(t)b^{\frac{1}{\gamma}}(t)]^{\gamma+1}}{\gamma^{\gamma}[\zeta\epsilon\tau'(t)\tau^2(t)]^{\gamma}}.$$

Thus, we have

$$\psi'(t) \le -Q(t) + \frac{r(t)\eta(t)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(t)}{\eta(t)} + (\gamma+1)\zeta\epsilon\tau'(t)\tau^2(t)b^{\frac{1}{\gamma}}(t)]^{\gamma+1}}{[\zeta\epsilon\tau'(t)\tau^2(t)]^{\gamma}}. \tag{3.6}$$

Integrating (3.6) from T to t, we get

$$\int_T^t \left[Q(s) - \frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(s)}{\eta(s)} + (\gamma+1)\zeta\epsilon\tau'(s)\tau^2(s)b^{\frac{1}{\gamma}}(s)]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)]^{\gamma}}\right]ds \le \psi(T),$$

which contradicts (3.1), and this completes the proof. □

The following result deals with the Kamenev-type oscillation for Eq. (1.1) under the condition (1.2).

Theorem 9 *If*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left[Q(s)-\frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(s)}{\eta(s)}+(\gamma+1)\zeta\epsilon\tau'(s)\tau^2(s)b^{\frac{1}{\gamma}}(s)]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)]^{\gamma}}\right]ds$$
$$=\infty, \tag{3.7}$$

then (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1) on $[t_0,\infty)$. Without loss of generality, we may assume that x is an eventually positive solution. Define $\psi(t)$ as in (3.2). Then, following the same steps as in the proof of Theorem 8, we arrive at (3.6). Multiplying (3.6) by $(t-s)^n$ and integrating the resulting inequality from t_0 to t, we have

$$-\int_{t_0}^{t}(t-s)^n\psi'(s)\,ds$$
$$\geq\int_{t_0}^{t}(t-s)^n\left[Q(s)-\frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(s)}{\eta(s)}+(\gamma+1)\zeta\epsilon\tau'(s)\tau^2(s)b^{\frac{1}{\gamma}}(s)]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)]^{\gamma}}\right]ds. \tag{3.8}$$

However, since

$$\int_{t_0}^{t}(t-s)^n\psi'(s)\,ds=n\int_{t_0}^{t}(t-s)^{n-1}\psi(s)\,ds-(t-t_0)^n\psi(t_0),$$

then from (3.8), we get

$$(t-t_0)^n\psi(t_0)-n\int_{t_0}^{t}(t-s)^{n-1}\psi(s)\,ds$$
$$\geq\int_{t_0}^{t}(t-s)^n\left[Q(s)-\frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(s)}{\eta(s)}+(\gamma+1)\zeta\epsilon\tau'(s)\tau^2(s)b^{\frac{1}{\gamma}}(s)]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)]^{\gamma}}\right]ds.$$

Hence,

$$\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left[Q(s)-\frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(s)}{\eta(s)}+(\gamma+1)\zeta\epsilon\tau'(s)\tau^2(s)b^{\frac{1}{\gamma}}(s)]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)]^{\gamma}}\right]ds$$
$$\leq\left(\frac{t-t_0}{t}\right)^n\psi(t_0),$$

and so

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left[Q(s)-\frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(s)}{\eta(s)}+(\gamma+1)\zeta\epsilon\tau'(s)\tau^2(s)b^{\frac{1}{\gamma}}(s)]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)]^{\gamma}}\right]ds$$
$$\to\psi(t_0),$$

which contradicts (3.7), and this completes the proof. □

Now we are going to discuss the so called Philos-type oscillation criteria for Eq. (1.1) under condition (1.2), but we first outline the following definition.

Definition 10 Let $D = \{(t,s) \in R^2 : t \geq s \geq t_0\}$ and $D_0 = \{(t,s) \in R^2 : t > s \geq t_0\}$. The functions $K_i(t,s) \in C(D,R)$, $i = 1,2$ are said to belong to the class X (written $K_i \in X$) if they satisfy

(I) $K_i(t,t) = 0$ for $t \geq t_0$, $K_i(t,s) > 0$, $(t,s) \in D_0$

(II) $\frac{\partial K_i(t,s)}{\partial s} \leq 0$, and there exist $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ and $L_i(t,s) \in C(D,R)$ such that

$$-\frac{\partial K_1(t,s)}{\partial s} = K_1(t,s)\left[\frac{\eta'(t)}{\eta(t)} + (\gamma+1)\zeta\epsilon\tau'(t)\tau^2(t)b^{\frac{1}{\gamma}}(t)\right] + L_1(t,s),$$

and

$$\frac{\partial K_2(t,s)}{\partial s} + \frac{\rho'(t)}{\rho(t)}K_2(t,s) = \frac{L_2(t,s)}{\rho(t)}\left[K_2(t,s)\right]^{\frac{\gamma}{\gamma+1}}.$$

Theorem 11 *Assume that there exists a function $K_1 \in X$ such that*

$$\limsup_{t\to\infty}\frac{1}{K_1(t,t_0)}\int_{t_0}^{t}\left[K_1(t,s)Q(s) - \frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}}\frac{[|L_1(t,s)|]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)K_1(t,s)]^{\gamma}}\right]ds = \infty. \quad (3.9)$$

Then, Eq. (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1). Now define $\psi(t)$ as in (3.2). Following the same steps as in the proof of Theorem 8, we arrive at (3.5). Multiplying (3.5) by $K_1(t,s)$ and integrating the resulting inequality from T to t, we have

$$\int_T^t K_1(t,s)Q(s)\,ds \leq \int_T^t K_1(t,s)\left[-\psi'(s) + A(s)\psi(s) - B(s)\psi^{1+\frac{1}{\gamma}}(s)\right]ds,$$

where

$$A(t) = \frac{\eta'(t)}{\eta(t)} + (\gamma+1)\zeta\epsilon\tau'(t)\tau^2(t)b^{\frac{1}{\gamma}}(t), \qquad B(t) = \frac{\zeta\gamma\epsilon\tau'(t)\tau^2(t)}{r^{\frac{1}{\gamma}}(t)\eta^{\frac{1}{\gamma}}(t)}.$$

Then, we have

$$\begin{aligned}\int_T^t K_1(t,s)Q(s)\,ds &\leq K_1(t,T)\psi(T) + \int_T^t\left[\frac{\partial K_1(t,s)}{\partial s} + K_1(t,s)A(s)\right]\psi(s)\,ds \\ &\quad - \int_T^t K_1(t,s)B(s)\psi^{1+\frac{1}{\gamma}}(s)\,ds \\ &= K_1(t,T)\psi(T) - \int_T^t L_1(t,s)\psi(s)\,ds - \int_T^t K_1(t,s)B(s)\psi^{1+\frac{1}{\gamma}}(s)\,ds \\ &\leq K_1(t,T)\psi(T) + \int_T^t\left[\left|L_1(t,s)\right|\psi(s) - K_1(t,s)B(s)\psi^{1+\frac{1}{\gamma}}(s)\right]ds.\end{aligned}$$

Putting $U = |L_1(t,s)|$, $V = K_1(t,s)B(s)$ and then using Lemma 4, we obtain

$$\left|L_1(t,s)\right|\psi(s) - K_1(t,s)B(s)\psi^{1+\frac{1}{\gamma}}(s) \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{|L_1(t,s)|^{\gamma+1}}{[K_1(t,s)B(s)]^{\gamma}}.$$

Then,

$$\int_T^t K_1(t,s)Q(s)\,ds \le K_1(t,T)\psi(T) + \int_T^t \frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}} \frac{[|L_1(t,s)|]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)K_1(t,s)]^{\gamma}}\,ds.$$

Hence,

$$\frac{1}{K_1(t,T)}\int_T^t \left[K_1(t,s)Q(s) - \frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}} \frac{[|L_1(t,s)|]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)K_1(t,s)]^{\gamma}}\right] ds \le \psi(T),$$

for all sufficiently large t, which contradicts (3.9). □

Theorem 12 *Assume that*

$$\liminf_{t\to\infty} \frac{1}{\phi_1^*(t)} \int_t^\infty \phi_2(s)\big[\phi_1^*(s)\big]^{\frac{\gamma+1}{\gamma}}\,ds > \frac{\gamma}{(\gamma+1)^{\frac{\gamma+1}{\gamma}}} \tag{3.10}$$

where

$$\phi_1(t) = \sum_{i=1}^m q_i(t)\theta^{\gamma}\big(\tau_i(t)\big), \qquad \phi_2(t) = \frac{\gamma\zeta\epsilon\tau'(t)\tau^2(t)}{r^{\frac{1}{\gamma}}(t)}, \quad \text{and}$$

$$\phi_1^*(t) = \int_t^\infty \phi_1(s)\,ds.$$

Then, (1.1) *is oscillatory.*

Proof Assume that $x(t)$ is an eventually positive solution of (1.1). Then, there exists a $t_1 \ge t_0$ such that $x(t) > 0$, $x(\sigma_j(t)) > 0$ and $x(\tau_i(t)) > 0$ for $t \ge t_1$. Using Lemma 5, we arrive at (2.1). Define

$$\omega(t) = \frac{r(t)(z'''(t))^{\gamma}}{z^{\gamma}(\zeta\tau(t))}.$$

Then, it is clear by (2.1) that

$$\omega'(t) \le -\frac{\sum_{i=1}^m q_i(t)\theta^{\gamma}(\tau_i(t))z^{\gamma}(\tau(t))}{z^{\gamma}(\zeta\tau(t))} - \frac{\gamma\zeta\tau'(t)r(t)(z'''(t))^{\gamma}z'(\zeta\tau(t))}{z^{\gamma+1}(\zeta\tau(t))}$$

Since, by Lemma 2, we have

$$z'\big(\zeta\tau(t)\big) \ge \epsilon\tau^2(t)z''\big(\tau(t)\big) \ge \epsilon\tau^2(t)z'''(t),$$

then

$$\omega'(t) \le -\sum_{i=1}^m q_i(t)\theta^{\gamma}\big(\tau_i(t)\big) - \frac{\gamma\zeta\epsilon\tau'(t)\tau^2(t)r(t)(z'''(t))^{\gamma+1}}{z^{\gamma+1}(\zeta\tau(t))}$$

i.e.

$$\omega'(t) + \phi_1(t) + \phi_2(t)\omega^{\frac{\gamma+1}{\gamma}}(t) \le 0.$$

Integrating the above inequality from t to l, we get

$$\omega(l) - \omega(t) + \int_t^l \phi_1(s)\,ds + \int_t^l \phi_2(s)\omega^{\frac{\gamma+1}{\gamma}}(s)\,ds \leq 0.$$

Letting $l \to \infty$ and using the fact that $\omega(t)$ is positive and decreasing, we get

$$\frac{\omega(t)}{\phi_1^*(t)} \geq 1 + \frac{1}{\phi_1^*(t)} \int_t^\infty \phi_2(s)\big[\phi_1^*(s)\big]^{\frac{\gamma+1}{\gamma}} \left[\frac{\omega(s)}{\phi_1^*(s)}\right]^{\frac{\gamma+1}{\gamma}} ds. \tag{3.11}$$

Let $\delta = \inf_{t\geq T} \frac{\omega(t)}{\phi_1^*(t)}$. Then obviously $\delta \geq 1$, and by (3.10) and (3.11), it follows that

$$\delta \geq 1 + \gamma\left(\frac{\delta}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}},$$

which contradicts the admissible values of $\delta \geq 1$ and $\gamma \geq 1$. Therefore, the proof is completed. □

4 The case $R(t_0) < \infty$

Now we are going to discuss the oscillatory behavior of Eq. (1.1) under the condition (1.3). First we need the following lemma.

Lemma 13 *Assume that x is an eventually positive solution of Eq.* (1.1) *and* (S_2) *holds. If*

$$\vartheta(t) = \rho(t)\frac{r(t)[z'''(t)]^\gamma}{[z''(t)]^\gamma}, \tag{4.1}$$

then

$$\vartheta'(t) \leq \frac{\rho'(t)}{\rho(t)}\vartheta(t) - \rho(t)\left[\frac{\lambda}{2}\tau^2(t)\right]^\gamma \sum_{i=1}^m q_i(t)\theta^\gamma\big(\tau_i(t)\big) - \frac{\gamma\vartheta^{\gamma+1}(t)}{r^{\frac{1}{\gamma}}(t)\rho^{\frac{1}{\gamma}}(t)}, \quad \lambda \in (0,1). \tag{4.2}$$

Proof Since x is an eventually positive solution of Eq. (1.1) and (S_2) holds, then using Lemma 5, we obtain (2.1). Now from Eq. (4.1), we see that $\vartheta(t) < 0$ for $t \geq t_1$, and

$$\vartheta'(t) = \frac{\rho'(t)}{\rho(t)}\vartheta(t) + \rho(t)\frac{[r(t)[z'''(t)]^\gamma]'}{[z''(t)]^\gamma} - \frac{\gamma\rho(t)r(t)[z'''(t)]^{\gamma+1}}{[z''(t)]^{\gamma+1}}.$$

This with (2.1) and (4.1) leads to

$$\vartheta'(t) \leq \frac{\rho'(t)}{\rho(t)}\vartheta(t) - \rho(t)\frac{\sum_{i=1}^m q_i(t)\theta^\gamma(\tau_i(t))z^\gamma(\tau(t))}{[z''(t)]^\gamma} - \frac{\gamma[\vartheta(t)]^{\gamma+1}}{r^{\frac{1}{\gamma}}(t)\rho^{\frac{1}{\gamma}}(t)},$$

i.e.

$$\vartheta'(t) \leq \frac{\rho'(t)}{\rho(t)}\vartheta(t) - \rho(t)\frac{\sum_{i=1}^m q_i(t)\theta^\gamma(\tau_i(t))z^\gamma(\tau(t))[z''(\tau(t))]^\gamma}{[z''(\tau(t))]^\gamma[z''(t)]^\gamma} - \frac{\gamma[\vartheta(t)]^{\gamma+1}}{r^{\frac{1}{\gamma}}(t)\rho^{\frac{1}{\gamma}}(t)}.$$

Now since $z''(t)$ is decreasing, then it follows that $-\frac{z''(\tau(t))}{z''(t)} \leq -1$. Consequently, by Lemma 3, we have $z(\tau(t)) \geq \frac{\lambda}{2}\tau^2(t)z''(\tau(t))$. Then

$$\vartheta'(t) \leq \frac{\rho'(t)}{\rho(t)}\vartheta(t) - \rho(t)\left[\frac{\lambda}{2}\tau^2(t)\right]^{\gamma}\sum_{i=1}^{m} q_i(t)\theta^{\gamma}\big(\tau_i(t)\big) - \frac{\gamma[\vartheta(t)]^{\gamma+1}}{r^{\frac{1}{\gamma}}(t)\rho^{\frac{1}{\gamma}}(t)}.$$

The proof is completed. □

Theorem 14 *Suppose that* (3.9) *holds, and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[K_2(t,s)\rho(s)\left[\frac{\lambda}{2}\tau^2(s)\right]^{\gamma}\sum_{i=1}^{m} q_i(s)\theta^{\gamma}\big(\tau_i(s)\big) - \frac{r(s)}{(\gamma+1)^{\gamma+1}\rho^{\gamma}(s)}\big[L_2(t,s)\big]^{\gamma+1}\right]ds > 0. \tag{4.3}$$

If

$$\int_{t_0}^{\infty} R(s)\,ds = \infty, \tag{4.4}$$

or

$$\int_{t_0}^{\infty}\int_{u}^{\infty} R(s)\,ds\,du = \infty, \tag{4.5}$$

then Eq. (1.1) *is oscillatory.*

Proof Suppose for the contrary that there exists a nonoscillatory solution $x(t) > 0$ of (1.1). Then, we have one of the three possible cases of Lemma 7. We first assume that (S_1) holds. Then by Theorem 11, if (3.9) holds, Eq. (1.1) is oscillatory. Secondly, if (S_2) holds, then by Lemma 13, we get (4.2). Multiplying (4.2) by $K_2(t,s)$ and integrating from t_1 to t, we obtain

$$\begin{aligned}
&\int_{t_1}^{t} K_2(t,s)\rho(s)\left[\frac{\lambda}{2}\tau^2(s)\right]^{\gamma}\sum_{i=1}^{m} q_i(s)\theta^{\gamma}\big(\tau_i(s)\big)\,ds \\
&\quad\leq K_2(t,t_1)\omega(t_1) + \int_{t_1}^{t}\left[\frac{\partial K_2(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)}K_2(t,s)\right]\omega(s)\,ds - \gamma\int_{t_1}^{t} K_2(t,s)\frac{\omega^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)\rho^{\frac{1}{\gamma}}(s)}\,ds \\
&\quad= K_2(t,t_1)\omega(t_1) + \int_{t_1}^{t}\frac{L_2(t,s)}{\rho(s)}\big[K_2(t,s)\big]^{\frac{\gamma}{\gamma+1}}\omega(s)\,ds - \gamma\int_{t_1}^{t} K_2(t,s)\frac{\omega^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)\rho^{\frac{1}{\gamma}}(s)}\,ds.
\end{aligned}$$

Setting

$$V = \frac{\gamma K_2(t,s)}{r^{\frac{1}{\gamma}}(s)\rho^{\frac{1}{\gamma}}(s)},\qquad U = \frac{L_2(t,s)}{\rho(s)}\big[K_2(t,s)\big]^{\frac{\gamma}{\gamma+1}}\quad\text{and}\quad Y = \omega(s).$$

Then, by Lemma 4, we have

$$\frac{L_2(t,s)}{\rho(s)}\big[K_2(t,s)\big]^{\frac{\gamma}{\gamma+1}}\omega(s) - \frac{\gamma K_2(t,s)\omega^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)\rho^{\frac{1}{\gamma}}(s)}$$

$$\le \frac{1}{(\gamma+1)^{\gamma+1}}\big[L_2(t,s)\big]^{(\gamma+1)}\frac{r(s)}{\rho^\gamma(s)}.$$

Hence,

$$\int_{t_1}^t \left[K_2(t,s)\rho(s)\left[\frac{\lambda}{2}\tau^2(s)\right]^\gamma \sum_{i=1}^m q_i(s)\theta^\gamma\big(\tau_i(s)\big) - \frac{r(s)}{(\gamma+1)^{\gamma+1}\rho^\gamma(s)}\big[L_2(t,s)\big]^{\gamma+1}\right] ds$$
$$\le K_2(t,t_1)\omega(t_1) < 0.$$

This contradicts (4.3). Finally, assume the case (S_3). Hence, since $r(t)(z'''(t))^\gamma$ is nonincreasing, then for $s \ge t \ge t_1$, we have

$$r^{\frac{1}{\gamma}}(s)\big(z'''(s)\big) \le r^{\frac{1}{\gamma}}(t)\big(z'''(t)\big).$$

Going through as in the proof of Theorem 2.3 case 1 in [20], we get a contradiction with (4.4) and (4.5), and so the proof is completed. □

Remark 15 Theorem 14 remains true if we used (3.1), or (3.7), or (3.10) instead of (3.9).

5 Example

Example 16 Consider the fourth-order differential equation

$$\left(t\left[x(t) + \frac{1}{t^3}x^{\frac{1}{3}}(t-2) + \frac{1}{t^4}x^{\frac{1}{5}}(t-3)\right]'''\right)' + \frac{3}{t}x(t) + \frac{1}{t^3}x(2t) = 0, \quad t \ge 2. \tag{5.1}$$

Here $\gamma = 1$, $r(t) = t$, $a_1 = \frac{1}{t^3}$, $a_2 = \frac{1}{t^4}$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{5}$, $q_1 = \frac{3}{t}$, $q_2 = \frac{1}{t^3}$, $\tau_1(t) = t$, $\tau_2(t) = 2t$. Let $\tau(t) = \frac{t}{2} \to \tau(t) \le \tau_i(t)$, $\lim_{t\to\infty} \tau(t) = \infty$, $\tau'(t) = \frac{1}{2} > 0$. Therefore, the conditions $(A_1) - (A_5)$ and (1.2) are satisfied. Choosing $\delta(t) = \frac{1}{t}$. Then $\delta(t) \to 0$ for $t \to \infty$. Moreover, $\theta(\tau_1(t)) = \theta(t) = [1 - \frac{2}{3t^2} - \frac{17}{15t^3} - \frac{1}{5t^4}] > 0$ for $t \ge 2$, and $\theta(\tau_2(t)) = \theta(2t) = [1 - \frac{1}{6t^2} - \frac{17}{120t^3} - \frac{1}{80t^4}] > 0$ for $t \ge 2$. Choosing $\eta(t) = 1$, $b(t) = \frac{1}{t^2}$, we have

$$Q(t) = \eta(t)\sum_{i=1}^m q_i(t)\theta^\gamma\big(\tau_i(t)\big) - \eta(t)\big[r(t)b(t)\big]' + \zeta\epsilon\tau'(t)\tau^2(t)r(t)\eta(t)b^{1+\frac{1}{\gamma}}(t)$$
$$= \frac{1}{t}\left[\left(3 + \frac{\zeta\epsilon}{8}\right) + \frac{1}{t} - \frac{1}{t^2} - \frac{17}{5t^3} - \frac{23}{30t^4} - \frac{17}{120t^5} - \frac{1}{80t^6}\right],$$

$$\limsup_{t\to\infty}\int_{t_0}^t \left[Q(s) - \frac{r(s)\eta(s)}{(\gamma+1)^{\gamma+1}}\frac{[\frac{\eta'(s)}{\eta(s)} + (\gamma+1)\zeta\epsilon\tau'(s)\tau^2(s)b^{\frac{1}{\gamma}}(s)]^{\gamma+1}}{[\zeta\epsilon\tau'(s)\tau^2(s)]^\gamma}\right] ds$$
$$= \limsup_{t\to\infty}\int_2^t \frac{1}{s}\left[3 + \frac{1}{s} - \frac{1}{s^2} - \frac{17}{5s^3} - \frac{23}{30s^4} - \frac{17}{120s^5} - \frac{1}{80s^6}\right] ds = \infty.$$

Therefore, by Theorem 8, every solution of (5.1) is oscillatory.

6 Conclusions

In this paper, we consider a general class of super-linear fourth-order differential equations with several sub-linear neutral terms of the type (1.1). Using the Riccati and generalized Riccati transformations, we establish new oscillation criteria in both cases of canonical case $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt = \infty$ and non-canonical case $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt < \infty$. With the help of the methods given in this paper, we derive some the Kamenev–Philos-type oscillation criteria for (1.1). An illustrative example is given. For interested researchers, there is a good deal of finding new results for (1.1) when $z(t) = x(t) - \sum_{j=1}^{n} a_j(t)x^{\alpha_j}(\sigma_j(t))$.

Acknowledgements
The authors of the paper are grateful to the editorial board and reviewers for the careful reading and helpful suggestions, which led to an improvement of our original manuscript.

Funding
This research was not supported by any project.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Authors read and approved the final manuscript.

Publisher's Note

Received: 6 December 2021 Accepted: 4 May 2022 Published online: 10 June 2022

References

1. Agarwal, R.P., Bohner, M., Li, T., Zhang, C.: Oscillation of third-order nonlinear delay differential equations. Taiwan. J. Math. **17**(2), 545–558 (2013)
2. Agarwal, R.P., Zhang, C., Li, T.: Some remarks on oscillation of second order neutral differential equations. Appl. Math. Comput. **274**, 178–181 (2016)
3. Bazighifan, O.: Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations. Adv. Differ. Equ. **201**, 1–12 (2020)
4. Bazighifan, O., Cesarano, C.: A Philos-type oscillation criteria for fourth-order neutral differential equations. Symmetry **379**(12), 1–10 (2020)
5. Bazighifan, O., Minhos, F., Moaaz, O.: Sufficient conditions for oscillation of fourth-order neutral differential equations with distributed deviating arguments. Axioms **39**(9), 1–11 (2020)
6. Bohner, M., Li, T.: Kamenev-type criteria for nonlinear damped dynamic equations. Sci. China Math. **58**(7), 1445–1452 (2015)
7. Chatzarakis, G.E., Elabbasy, E.M., Bazighifan, O.: An oscillation criterion in 4th-order neutral differential equations with a continuously distributed delay. Adv. Differ. Equ. **3366**, 1 (2019)
8. Chiu, K.S., Li, T.: Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments. Math. Nachr. **292**(10), 2153–2164 (2019)
9. Cloud, M.J., Drachman, B.C.: Inequalities with Applications to Enginering. Springer, New York (1998)
10. Dassios, I., Bazighifan, O.: Oscillation conditions for certain fourth-order non-linear neutral differential equation. Symmetry **1096**(12), 1–9 (2020)
11. Džuurina, J., Grace, S.R., Jadlovská, I., Li, T.: Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. Math. Nachr. **293**(5), 910–922 (2020)
12. El-Sheikh, M.M.A.: Oscillation and nonoscillation criteria for second order nonlinear differential equations. J. Math. Anal. Appl. **179**(1), 14–27 (1993)
13. El-Sheikh, M.M.A., Sallam, R.A., Elimy, D.: Oscillation criteria for second order nonlinear equations with damping. Adv. Differ. Equ. Control Process. **8**(2), 127–142 (2011)
14. El-Sheikh, M.M.A., Sallam, R.A., Salem, S.: Oscillation of nonlinear third-order differential equations with several sublinear neutral terms. Math. Slovaca **71**(6), 1411–1426 (2021)
15. Fu, Y., Tian, Y., Jiang, C., Li, T.: On the asymptotic properties of nonlinear third-order neutral delay differential equations with distributed deviating arguments. J. Funct. Spaces **2016**, 1–5 (2016)
16. Jiang, C., Jiang, Y., Li, T.: Asymptotic behavior of third-order differential equations with nonpositive neutral coefficients and distributed deviating arguments. Adv. Differ. Equ. **105**, 1–14 (2016)
17. Jiang, C., Li, T.: Oscillation criteria for third-order nonlinear neutral differential equations with distributed deviating arguments. J. Nonlinear Sci. Appl. **9**, 6170–6182 (2016)
18. Jiang, C., Tian, Y., Jiang, Y., Li, T.: Some oscillation results for nonlinear second-order differential equations with damping. Adv. Differ. Equ. **2015**, 354 (2015)
19. Jiang, Y., Jianng, C., Li, T.: Oscillatory behavior of third-order nonlinear neutral delay differential equations. Adv. Differ. Equ. **2016**, 171 (2016)

20. Li, T., Baculikova, B., Dzurina, J., Zhang, C.: Oscillation of fourth-order neutral differential equations with *p*-Laplacian like operators. Bound. Value Probl. **56**, 1–9 (2014)
21. Li, T., Pintus, N., Viglialoro, G.: Properties of solutions to porous medium problems with different sources and boundary conditions. Z. Angew. Math. Phys. **70**(3), 86 (2019)
22. Li, T., Rogovchenko, Y.V.: Asymptotic behavior of an odd-order delay differential equation. Bound. Value Probl. **2014**, 1 (2014)
23. Li, T., Rogovchenko, Y.V.: On asymptotic behavior of solutions to higher order sublinear Emden-Fowler delay differential equations. Appl. Math. Lett. **667**, 53–59 (2017)
24. Li, T., Rogovchenko, Y.V.: On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations. Appl. Math. Lett. **105**, 106293 (2020)
25. Li, T., Viglialoro, G.: Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime. Differ. Integral Equ. **34**(5–6), 315–336 (2021)
26. Moaaz, O., Cesarano, C., Muhib, A.: Some new oscillation results for fourth-order neutral differential equations. Eur. J. Appl. Math. **13**(2), 185–199 (2020)
27. Moaaz, O., Metwally, E., Elabbasy, M., Shaaban, E.: Oscillation criteria for a class of third order damped differential. Arab J. Math. Sci. **24**(1), 16–30 (2018)
28. Philos, C.: On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations will positive delays. Arch. Math. **36**, 168–178 (1981)
29. Qin, G., Huang, C., Xie, Y., Wen, F.: Asymptotic behavior for third-order quasi-linear differential equations. Adv. Differ. Equ. **305**, 1–8 (2013)
30. Qiu, Y.-C., Jadlovska, I., Chiu, K.-S., Li, T.: Existence of nonoscillatory solutions tending to zero of third-order neutral dynamic equations on time scales. Adv. Differ. Equ. **231**, 1–9 (2020)
31. Qiu, Y.C., Zada, A., Qin, H., Li, T.: Oscillation criteria for nonlinear third-order neutral dynamic equations with damping on time scales. J. Funct. Spaces **2017**, 1–18 (2017)
32. Sallam, R.A., El-Sheikh, M.M.A., El-Saedy, E.I.: On the oscillation of second order nonlinear neutral delay differential equations. Math. Slovaca **71**(4), 859–870 (2021)
33. Sallam, R.A., Salem, S., El-Sheikh, M.M.A.: Oscillation of solutions of third order nonlinear neutral differential equations. Adv. Differ. Equ. **314**, 1–25 (2020)
34. Thandapani, E., El-Sheikh, M.M.A., Sallam, R., Salem, S.: On the oscillatory behavior of third order differential equations with a sublinear neutral term. Math. Slovaca **70**(1), 95–106 (2020)
35. Tian, Y., Cail, Y., Fu, Y., Li, T.: Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. Adv. Differ. Equ. **267**, 1–14 (2015)
36. Tiryaki, A., Aktas, M.F.: Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping. J. Math. Anal. Appl. **325**, 54–68 (2007)
37. Wang, H., Chen, G., Jiang, Y., Jiang, C., Li, T.: Asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. J. Math. Comput. Sci. **17**, 194–199 (2017)
38. Zhang, C., Agarwal, R.P., Bohner, M., Li, T.: Oscillation of fourth-order delay dynamic equations. Sci. China Math. **58**(1), 143–160 (2015)
39. Zhang, Q., Gao, L., Yu, Y.: Oscillation criteria for third-order neutral differential equations with continuously distributed delay. Appl. Math. Lett. **25**, 1514–1519 (2012)