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Mountain pass solution for the weighted Dirichlet $(p(z), q(z))$-problem

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Abstract

We consider the Dirichlet boundary value problem for equations involving the $(p(z), q(z))$-Laplacian operator in the principal part on an open bounded domain $\Omega \subset \mathbb{R}^n$. Here, the $p(z)$-Laplacian is weighted by a function $a \in L^\infty(\Omega)_+$, and the nonlinearity in the reaction term is allowed to depend on the solution without imposing the Ambrosetti–Rabinowitz condition. The proof of the existence of solution to our problem is based on a mountain pass critical point approach with the Cerami condition at level c.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a smooth boundary. Here, we focus on the following Dirichlet problem:

$$(P_g) \quad \begin{cases} -\operatorname{div}(a(z)|\nabla u|^{p(z)-2}\nabla u) + b(z)|u(z)|^{p(z)} = \operatorname{div}(|\nabla u|^{q(z)-2}\nabla u) + g(z, u(z)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is the nonlinearity (namely reaction term), $a, b : \Omega \to [0, +\infty[$ are weight functions, both belonging to $L^\infty(\Omega)$ and with $a(z) \geq a_0 > 0$ for all $z \in \Omega$. The variable exponents $p, q \in C(\overline{\Omega})$ are related by the strict inequality $q(z) < p(z)$ for all $z \in \overline{\Omega}$, and separately they satisfy the conditions:

$$1 < q^- := \inf_{z \in \Omega} q(z) \leq q(z) \leq q^+ := \sup_{z \in \Omega} q(z) < +\infty,$$

$$1 < p^- := \inf_{z \in \Omega} p(z) \leq p(z) \leq p^+ := \sup_{z \in \Omega} p(z) < +\infty.$$

We assume the following regularities on the nonlinearity g:

(g_0) $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z, \xi) = 0$ for all $z \in \Omega$ and all $\xi \leq 0$;

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(g_1) there exist $a_1, a_2 \in [0, +\infty[$ and $\alpha \in C(\overline{\Omega})$ with $p(z) < \alpha(z) < p^*(z)$ for all $z \in \overline{\Omega}$ satisfying

$$\left| g(z, \xi) \right| \le a_1 + a_2 |\xi|^{\alpha(z)-1} \quad \text{for all } (z, \xi) \in \Omega \times \mathbb{R},$$

with $p^*(z) = \frac{np(z)}{n - p(z)}$ if $p(z) < n$ and $p^*(z) = +\infty$ if $p(z) \ge n$;

(g_2) if $G(z, t) = \int_0^t g(z, \xi) \, d\xi$, then we have

$$\lim_{t \to +\infty} \frac{G(z, t)}{t^{p^+}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(g_3) $\lim_{t \to 0^+} \frac{g(z,t)}{t^{p(z)-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(g_4) there exists $r_0 > 0$ such that $\mathcal{G}(z, t) = g(z, t)t - p^+ G(z, t) \ge 0$ for all $z \in \Omega$ and $t \ge r_0$;

(g_5) there exist $c_0 > 0$ and $\beta \in C(\overline{\Omega})$ with $1 < \beta(z)$ such that $p^- \beta'(z) < p^*(z)$ for all $z \in \overline{\Omega}$ such that

$$\left| \frac{G(z, t)}{t^{p^-}} \right|^{\beta(z)} \le c_0 \mathcal{G}(z, t) \quad \text{for all } z \in \Omega \text{ and } t \ge r_0,$$

where r_0 is given in (g_4).

The above set of hypotheses is derived from the consolidated literature on the use of energy functional methods to solve partial differential equations (for (g_5), see Zhou and Wang [28]). Mainly we impose polynomial growth conditions on both the nonlinearity g and its integral G. On the other side, we require $(p(z) - 1)$-sublinearity of g at zero, and p^+-superlinearity of G at infinity.

A large and interesting class of nonlinear partial differential equations presents as leading operator the $(p(z), q(z))$-Laplacian operator (namely often $(p(z), q(z))$-elliptic equations). So, here we also consider the sum of a $p(z)$-Laplacian and of a $q(z)$-Laplacian, but the first one is weighted using the function $a \in L^\infty(\Omega)$. This study applies in the general framework of Lebesgue and Sobolev spaces, with the structure of variable exponents (namely, $L^{p(z)}(\Omega)$ and $W^{1,p(z)}(\Omega)$, respectively; see [9, 20]). The practical applications of these spaces originate from the analysis of different physical phenomena. In particular, they model the behavior of non-Newtonian fluids that change viscosity (recall the variable exponent $p(z)$) in the presence of an electromagnetic field; see Rădulescu and co-workers [22, 25] and Ružička [21] (electrorheological fluids). See also the recent works of Gasiński and Papageorgiou [14] (resonant reaction), Barile and Figueiredo [4] (constant exponents case), Papageorgiou and Vetro [18], and Vetro and Vetro [24] (variable exponents case), and Vetro [23] (variable exponents depending on the unknown solution).

If $W_0^{1,p(z)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(z)}(\Omega)$, for a weak solution of the problem, (P_g) we mean a function $u \in W_0^{1,p(z)}(\Omega)$ such that

$$\int_\Omega a(z)|\nabla u|^{p(z)-2} \nabla u \nabla v \, dz + \int_\Omega |\nabla u|^{q(z)-2} \nabla u \nabla v \, dz + \int_\Omega b(z)|u|^{p(z)-2} uv \, dz$$

$$= \int_\Omega g(z, u)v \, dz \quad \text{for each } v \in W_0^{1,p(z)}(\Omega).$$

We recall here some facts on the development of this kind of (double phase) $(p(z), q(z))$-problems, focusing on the Italian school. So, we fix attention to the results of Marcellini

[15–17], Mingione and co-workers [1, 5–7], but we do not forget the pioneering papers of Zhikov [26, 27], where the interested reader can find a deep investigation over variational integrals related to the total energy associated with special forms of integrand functions. Also, we mention the very recent work of Alves and Molica Bisci [3] about compact embeddings results in variable exponent Sobolev spaces with applications. We refer to the above literature and references therein for precise information and details, but here we mention the fact that a crucial aspect of this research focuses on nonstandard growth conditions of (p, q)-type, according to the pioneering work of Marcellini. These are functionals where the energy density satisfies a condition of the form

$$|\xi|^p \le g(x, \xi) \le |\xi|^q + 1, \quad 1 \le p \le q, \ \xi \in \mathbb{R}.$$

Interesting models with (p, q)-growth for geometrically constrained problems were the focus of a recent paper by De Filippis [8]. Our approach here uses geometrical conditions to depict a mountain pass geometry and obtain critical points of the energy functional associated with (P_g). We know that the Ambrosetti–Rabinowitz condition ensures the boundedness of a convergent sequence (namely the Palais–Smale sequence) of such a kind of functional. This is a crucial aspect in dealing with the critical point theory. The Ambrosetti–Rabinowitz condition says that there exist $\eta > p$ and $M > 0$ such that

$$0 < \eta G(z, t) \le g(z, t)t \quad \text{for a.a. } z \in \Omega, \text{all } |t| \ge M,$$
$$0 < \operatorname*{essinf}_{\Omega} G(\cdot, \pm M).$$

Integrating the first inequality and using the second one, we obtain the following weaker condition

$$c_1 |t|^\eta \le G(z, t) \quad \text{for a.a. } z \in \Omega, \text{all } |t| \ge M, \text{some } c_1 > 0,$$
$$\Rightarrow \quad c_1 |t|^\eta \le g(z, t)t \quad \text{for a.a. } z \in \Omega, \text{all } |t| \ge M.$$

We remark that we do not impose the Ambrosetti–Rabinowitz condition, but we employ alternative conditions involving the integral function G and the function $\mathcal{G}$ (see (g_4), (g_5)), which incorporates in our setting also nonlinearities with slower growth.

2 Mathematical background

We collect some classical notions and notation from the variational calculus. By (X, X^*), we mean the couple of a Banach space X and its topological dual X^*. Since we work in a variable exponent framework space, we recall the basic definition of a variable exponent Lebesgue space:

$$L^{p(z)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \rho_p(u) := \int_\Omega |u(z)|^{p(z)} \, dz < +\infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p(z)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(z)}{\lambda} \right|^{p(z)} \, dz \le 1 \right\}.$$

Then we provide the notion of variable exponent Sobolev space as follows:

$$W^{1,p(z)}(\Omega) := \left\{ u \in L^{p(z)}(\Omega) : |\nabla u| \in L^{p(z)}(\Omega) \right\}.$$

In $W^{1,p(z)}(\Omega)$, we use the norm

$$\|u\|_{1,p} = \|u\|_{W^{1,p(z)}(\Omega)} = \|u\|_{L^{p(z)}(\Omega)} + \big\| |\nabla u| \big\|_{L^{p(z)}(\Omega)}.$$

In $L^{p(z)}(\Omega)$, the norm of $u \in W_0^{1,p(z)}(\Omega)$ and the norm of $|\nabla u|$ satisfy the inequality:

$$\|u\|_{L^{p(z)}(\Omega)} \leq m \big\| |\nabla u| \big\|_{L^{p(z)}(\Omega)} \quad \text{for all } u \in W_0^{1,p(z)}(\Omega), \text{some } m > 0$$

(see Theorem 8.2.18, p. 263, Diening et al. [9]). It means that the norms $\|u\|_{W^{1,p(z)}(\Omega)}$ and $\big\| |\nabla u| \big\|_{L^{p(z)}(\Omega)}$ are equivalent norms on $W_0^{1,p(z)}(\Omega)$. This remark gives us the key to use the last one to replace $\|u\|_{W^{1,p(z)}(\Omega)}$. So, we put

$$\|u\| = \big\| |\nabla u| \big\|_{L^{p(z)}(\Omega)} \quad \text{in } W_0^{1,p(z)}(\Omega).$$

A crucial aspect of the methods of the variational calculus leads to the embedding results. Adopting the Fan and Zhang arguments in [10], we know that the above norms make both the variable Lebesgue and Sobolev spaces separable, reflexive and uniformly convex Banach spaces. Also, in Fan and Zhao [11], we find the following version of the classical Sobolev embedding:

Proposition 1 ([11], Theorem 2.3) *Let $p \in C(\overline{\Omega})$ with $p(z) > 1$ for all $z \in \overline{\Omega}$. If $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(z) < p^*(z)$ for all $z \in \Omega$, then there exists a continuous and compact embedding $W^{1,p(z)}(\Omega) \hookrightarrow L^{\alpha(z)}(\Omega)$.*

Moreover, [11, Theorem 1.11] gives us the continuity of the embedding $L^{p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega)$, provided that $p, q \in C(\overline{\Omega})$ with $1 < q(z) \leq p(z)$ for all $z \in \Omega$. Finally, the following linking theorem is given in [11] (see Theorem 1.3).

Theorem 1 *Let $u \in L^{p(z)}(\Omega)$, then we have*:
  (i) $\|u\|_{L^{p(z)}(\Omega)} < 1 (= 1, > 1) \Leftrightarrow \rho_P(u) < 1 (= 1, > 1)$;
  (ii) *if $\|u\|_{L^{p(z)}(\Omega)} > 1$, then $\|u\|_{L^{p(z)}(\Omega)}^{p^-} \leq \rho_P(u) \leq \|u\|_{L^{p(z)}(\Omega)}^{p^+}$*;
  (iii) *if $\|u\|_{L^{p(z)}(\Omega)} < 1$, then $\|u\|_{L^{p(z)}(\Omega)}^{p^+} \leq \rho_P(u) \leq \|u\|_{L^{p(z)}(\Omega)}^{p^-}$*.

The last ingredient we mention here is the following lemma by Fu [12] (see Lemma 2.14).

Lemma 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $p(z) \in L^\infty(\Omega)$ and $u \in W_0^{1,p}(\Omega)$, then*

$$\int_\Omega |u|^{p(z)}\, dz \leq C \int_\Omega |\nabla u|^{p(z)}\, dz$$

for some Ω-dependent constant C.

We work to construct the energy functional associated to (P_g) in some steps. Indeed, starting from the integral function $G : \Omega \times \mathbb{R} \to \mathbb{R}$ given as

$$G(z,t) = \int_0^t g(z,\xi)\,d\xi \quad \text{for all } t \in \mathbb{R}, \text{ all } z \in \Omega,$$

we obtain the functional $B : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$B(u) = \int_\Omega G\big(z, u(z)\big)\,dz, \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

The assumption (g_1) implies that $B \in C^1(W_0^{1,p(z)}(\Omega), \mathbb{R})$. Also, Proposition 1 leads to the following compact derivative of B:

$$\big\langle B'(u), v \big\rangle = \int_\Omega g\big(z, u(z)\big) v(z)\,dz, \quad \text{for all } u, v \in W_0^{1,p(z)}(\Omega).$$

Next, using the weight functions $a, b \in L^\infty(\Omega)$, we introduce the functionals $A_1, A_2, A_3 :$ $W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$A_1(u) = \int_\Omega \frac{a(z)}{p(z)} \big|\nabla u(z)\big|^{p(z)}\,dz \quad \text{and} \quad A_2(u) = \int_\Omega \frac{1}{q(z)} \big|\nabla u(z)\big|^{q(z)}\,dz$$

and

$$A_3 = \int_\Omega \frac{b(z)}{p(z)} \big|u(z)\big|^{p(z)}\,dz, \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

We stress that $A_1, A_2, A_3 \in C^1(W_0^{1,p(z)}(\Omega), \mathbb{R})$, and the following derivatives hold:

$$\big\langle A_1'(u), v \big\rangle = \int_\Omega a(z) |\nabla u|^{p(z)-2} \nabla u \nabla v\,dz, \qquad \big\langle A_2'(u), v \big\rangle = \int_\Omega |\nabla u|^{q(z)-2} \nabla u \nabla v\,dz$$

and

$$\big\langle A_3'(u), v \big\rangle = \int_\Omega b(z) |u|^{p(z)-2} u v\,dz, \quad \text{for all } u, v \in W_0^{1,p(z)}(\Omega).$$

Remark 1 $A_1' : W_0^{1,p(z)}(\Omega) \to W_0^{1,p(z)}(\Omega)^*$ is a mapping of type (S_+), that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p(z)}(\Omega)$ and $\limsup_{n \to +\infty} \langle A_1'(u_n), u_n - u \rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p(z)}(\Omega)$ (see Gasiński and Papageorgiou [13], p. 279). The same holds for A_2'. Consequently, $A_1' + A_2'$ is a mapping of type (S_+) too.

We combine the above functionals to obtain the functional $I : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = A_1(u) + A_2(u) + A_3(u) - B(u) \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

Trivially, we have that $I(0) = 0$.

3 Main results

In this section, we apply the mountain pass approach to the functional I under the Cerami condition at level c (for short (C_c)-condition).

Here, we recall the general definition of (C_c)-condition in a Banach space X.

Definition 1 Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. We say that I satisfies the (C_c)-condition if any sequence $\{u_n\} \subset X$ such that $I(u_n) \to c \in \mathbb{R}$ and $(1 + \|u_n\|)I'(u_n) \to 0$ in X^* as $n \to +\infty$ has a convergent subsequence.

We will consider the following version of the mountain pass theorem as can be found in Afrouzi et al. [2] (see Lemma 3.3).

Theorem 2 *Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$ satisfies the (C_c)-condition for any $c \in \mathbb{R}$, $I(0) = 0$ and*

(i) *there exist $\rho > 0$ and $\delta > 0$ such that $I|_{\partial B_\rho} \geq \delta$, where B_ρ is a ball of radius ρ;*

(ii) *there exists $v \in X \setminus B_\rho$ such that $I(v) \leq 0$.*

Then,

$$c_0 = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \geq \delta$$

is a critical value of I where

$$\Gamma = \left\{ \gamma \in C^0([0,1], X) : \gamma(0) = 0, \gamma(1) = v \right\}.$$

The first step to cover is "creating" the convergent subsequence in $W_0^{1,p(z)}(\Omega)$.

Lemma 2 *Let $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ be a bounded sequence such that $(1 + \|u_n\|)I'(u_n) \to 0$ in $W_0^{1,p(z)}(\Omega)^*$ as $n \to +\infty$. If the assumption (g_1) is satisfied, then the sequence $\{u_n\}$ has a subsequence convergent in $W_0^{1,p(z)}(\Omega)$.*

Proof Let $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ be a bounded sequence such that $(1 + \|u_n\|)I'(u_n) \to 0$ in $W_0^{1,p(z)}(\Omega)^*$ as $n \to +\infty$. Note that $W_0^{1,p(z)}(\Omega)$ is a reflexive Banach space and so, passing to a subsequence if necessary, there exists $u \in W_0^{1,p(z)}(\Omega)$ such that $u_n \overset{w}{\to} u$ in $W_0^{1,p(z)}(\Omega)$. Then Proposition 1 (embedding result) leads to $u_n \to u$ in $L^{\alpha(z)}(\Omega)$. An Hölder inequality can be applied, so that we have

$$\left| \int_\Omega \left[g(z, u_n(z)) - g(z, u(z)) \right] (u_n(z) - u(z)) \, dz \right|$$

$$\leq \int_\Omega \left(\left| g(z, u_n(z)) \right| + \left| g(z, u(z)) \right| \right) \left| u_n(z) - u(z) \right| \, dz$$

$$\leq \int_\Omega \left(2a_1 + a_2 \left| u_n(z) \right|^{\alpha(z)-1} + a_2 \left| u(z) \right|^{\alpha(z)-1} \right) \left| u_n(z) - u(z) \right| \, dz$$

$$\leq 2 \left\| 2a_1 + a_2 |u_n|^{\alpha(z)-1} + a_2 |u|^{\alpha(z)-1} \right\|_{L^{\alpha'(z)}(\Omega)} \|u_n - u\|_{L^{\alpha(z)}(\Omega)}.$$

Passing to the limit as $n \to +\infty$, we deduce that

$$\lim_{n \to +\infty} \int_\Omega \left[g(z, u_n(z)) - g(z, u(z)) \right] (u_n(z) - u(z)) \, dz = 0.$$

Since $(1 + \|u_n\|)I'(u_n) \to 0$ in $W_0^{1,p(z)}(\Omega)^*$, we obtain

$$\lim_{n \to +\infty} \langle I'(u_n) - I'(u), u_n - u \rangle = 0.$$

Recalling the definition of the functional I in Sect. 2, we have

$$\langle A_1'(u_n) - A_1'(u), u_n - u \rangle + \langle A_2'(u_n) - A_2'(u), u_n - u \rangle + \langle A_3'(u_n) - A_3'(u), u_n - u \rangle$$

$$= \int_\Omega \big(g(z, u_n(z)) - g(z, u(z))\big)(u_n - u)\, dz + \langle I'(u_n) - I'(u), u_n - u \rangle$$

$$\to 0 \quad \text{as } n \to +\infty.$$

Since

$$\langle A_2'(u_n) - A_2'(u), u_n - u \rangle + \langle A_3'(u_n) - A_3'(u), u_n - u \rangle \geq 0$$

for all $n \in \mathbb{N}$, we deduce that

$$\limsup_{n \to +\infty} \langle A_1'(u_n), u_n - u \rangle = \limsup_{n \to +\infty} \langle A_1'(u_n) - A_1'(u), u_n - u \rangle \leq 0.$$

As A_1' is a mapping of type (S_+) (see Remark 1), we conclude that the sequence $\{u_n\}$ converges to u in $W_0^{1,p(z)}(\Omega)$. $\qquad \square$

We point out some facts about our set of assumptions. In particular, (g_1) ensures that for each $s > 0$, there exists a constant $C_s > 0$ such that

$$|G(z,t)|, |\mathcal{G}(z,t)| \leq C_s \quad \text{for all } (z,t) \in \Omega \times [0,s], \tag{1}$$

and (g_0) says that

$$G(z,t) = \mathcal{G}(z,t) = 0 \quad \text{for all } (z,t) \in \Omega \times\,]{-\infty}, 0]. \tag{2}$$

Again assumption (g_2) ensures that there exists $s_0 > 0$ such that

$$G(z,t) \geq 0 \quad \text{for a.a. } z \in \Omega \text{ and } t \in [s_0, +\infty[. \tag{3}$$

Remark 2 Let $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ be a sequence such that $(1 + \|u_n\|)I'(u_n) \to 0$ in $W_0^{1,p(z)}(\Omega)^*$ as $n \to +\infty$. If (g_0) holds, then the sequence $\{u_n^-\}$ converges to zero in $W_0^{1,p(z)}(\Omega)$. Indeed, from

$$a_0 \int_\Omega |\nabla u_n^-|^{p(z)}\, dz \leq |\langle I'(u_n), u_n^- \rangle|$$

$$\leq \big(1 + \|u_n\|\big) \big\| I'(u_n) \big\| \to 0 \quad \text{as } n \to +\infty,$$

we deduce that $\|\nabla u_n^-\|_{L^{p(z)}(\Omega)} \to 0$, and hence $\|u_n^-\| \to 0$ in $W_0^{1,p(z)}(\Omega)$.

Remark 3 Let $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ be a (C_c)-sequence, then the sequence $\{u_n^+\}$ satisfies the following:

$$I(u_n^+) \to c \quad \text{and} \quad \langle I'(u_n^+), u_n^+ \rangle \to 0 \quad \text{as } n \to +\infty.$$

Indeed, from $\nabla u = \nabla(u^+ - u^-) = \nabla u^+ - \nabla u^-$ and $\nabla u^+ = \nabla u$ a.e. in $\{u \geq 0\}$, $\nabla u^- = -\nabla u$ a.e. in $\{u < 0\}$, we get

$$
\begin{aligned}
I(u_n) &= \int_\Omega \left[\frac{a(z)}{p(z)} |\nabla u_n|^{p(z)} + \frac{1}{q(z)} |\nabla u_n|^{q(z)} + \frac{b(z)}{p(z)} |u_n|^{p(z)} \right] dz - \int_\Omega G(z, u_n(z)) \, dz \\
&= \int_{\{u<0\}} \left[\frac{a(z)}{p(z)} |\nabla u_n^-|^{p(z)} + \frac{1}{q(z)} |\nabla u_n^-|^{q(z)} + \frac{b(z)}{p(z)} |u_n^-|^{p(z)} \right] dz \\
&\quad + \int_{\{u \geq 0\}} \left[\frac{a(z)}{p(z)} |\nabla u_n^+|^{p(z)} + \frac{1}{q(z)} |\nabla u_n^+|^{q(z)} + \frac{b(z)}{p(z)} |u_n^+|^{p(z)} \right] dz \\
&\quad - \int_\Omega G(z, u_n^+(z)) \, dz \\
&= I(u_n^-) + I(u_n^+) \\
&\Rightarrow I(u_n^+) = I(u_n) - I(u_n^-) \to c \quad \text{as } n \to +\infty \text{ (by Remark 2).}
\end{aligned}
$$

We also have

$$\langle I'(-u_n^-), u_n^- \rangle = \int_{\{u<0\}} \left[a(z) |\nabla u_n^-|^{p(z)} + |\nabla u_n^-|^{q(z)} + b(z) |u_n^-|^{p(z)} \right] dz$$

and

$$\langle I'(u_n^+), u_n^+ \rangle = \int_{\{u \geq 0\}} \left[a(z) |\nabla u_n^+|^{p(z)} + |\nabla u_n^+|^{q(z)} + b(z) |u_n^+|^{p(z)} - g(z, u_n^+) u_n^+ \right] dz.$$

From

$$\langle I'(u_n), u_n \rangle = \langle I'(u_n), u_n^+ \rangle + \langle I'(u_n), -u_n^- \rangle = \langle I'(u_n^+), u_n^+ \rangle - \langle I'(-u_n^-), u_n^- \rangle,$$

we deduce that

$$\langle I'(u_n^+), u_n^+ \rangle = \langle I'(u_n), u_n \rangle + \langle I'(-u_n^-), u_n^- \rangle \to 0 \quad \text{as } n \to +\infty.$$

The second step of our finding gives us a boundedness result for (C_c)-sequences in $W_0^{1,p(z)}(\Omega)$.

Lemma 3 *If the assumptions (g_0)–(g_2), (g_4), (g_5) hold, then any (C_c)-sequence $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ is bounded in $W_0^{1,p(z)}(\Omega)$.*

Proof By Remark 3, the hypothesis that $\{u_n\}$ is a (C_c)-sequence gives us that $I(u_n^+) \to c$ and $\langle I'(u_n^+), u_n^+ \rangle \to 0$ as $n \to +\infty$. Consequently, we can find a constant $M > 0$ such that

$$
\begin{aligned}
M &\geq I(u_n^+) - \frac{1}{p^+} \langle I'(u_n^+), u_n^+ \rangle \\
&= A_1(u_n^+) + A_2(u_n^+) + A_3(u_n^+) - B(u_n^+)
\end{aligned}
$$

$$- \frac{1}{p^+} \left[\left\langle A_1'(u_n^+), u_n^+ \right\rangle + \left\langle A_2'(u_n^+), u_n^+ \right\rangle + \left\langle A_3'(u_n^+), u_n^+ \right\rangle - \int_\Omega g(z, u_n^+) u_n^+ \, dz \right]$$

$$= \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p^+} \right] |\nabla u_n^+|^{p(z)} \, dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p^+} \right] |\nabla u_n^+|^{q(z)} \, dz$$

$$+ \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p^+} \right] b(z) |u_n^+|^{p(z)} \, dz + \frac{1}{p^+} \int_\Omega \left[g(z, u_n^+) u_n^+ - p^+ G(z, u_n^+(z)) \right] \, dz$$

$$\geq \frac{1}{p^+} \int_\Omega \mathcal{G}(z, u_n^+(z)) \, dz \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

Inequality (4) and the information in (1) and (2) give us

$$\int_{\{u_n \geq s\}} \mathcal{G}(z, u_n^+(z)) \, dz \leq p^+ M - \int_{\{u_n < s\}} \mathcal{G}(z, u_n^+(z)) \, dz$$

$$\leq p^+ M + C_s |\Omega| \quad \text{for all } s > 0, \tag{5}$$

where $|\Omega|$ means the Lebesgue measure of Ω.

Now, if the sequence $\{u_n\}$ is unbounded, by Remark 2, we assume that $\|u_n^+\| \to +\infty$ as $n \to +\infty$ (going to a subsequence if necessary). So, we also suppose that $\|u_n^+\| > 1$ for all $n \in \mathbb{N}$. From

$$\frac{I(u_n^+)}{\|u_n^+\|^{p^-}} = \frac{1}{\|u_n^+\|^{p^-}} \left[A_1(u_n^+) + A_2(u_n^+) + A_3(u_n^+) - B(u_n^+) \right]$$

$$\geq \frac{1}{p^+} \int_\Omega a(z) \frac{|\nabla u_n^+|^{p(z)}}{\|u_n^+\|^{p^-}} \, dz - \int_\Omega \frac{G(z, u_n^+(z))}{\|u_n^+\|^{p^-}} \, dz$$

$$\geq \frac{a_0}{p^+} - \int_\Omega \frac{|G(z, u_n^+(z))|}{\|u_n^+\|^{p^-}} \, dz \quad \left(\text{by Theorem 1 and } \|u_n^+\| > 1 \right),$$

and since

$$\frac{I(u_n^+)}{\|u_n^+\|^{p^-}} \to 0 \quad \text{as } n \to +\infty,$$

we get

$$\limsup_{n \to +\infty} \int_\Omega \frac{|G(z, u_n^+(z))|}{\|u_n\|^{p^-}} \, dz \geq \frac{a_0}{p^+}. \tag{6}$$

Using (1) and (2), we obtain that for each $s > 0$

$$\int_{\{u_n^+ < s\}} \frac{|G(z, u_n^+(z))|}{\|u_n^+\|^{p^-}} \, dz = \int_{\{0 < u_n^+ < s\}} \frac{|G(z, u_n^+(z))|}{\|u_n^+\|^{p^-}} \, dz$$

$$\leq \frac{C_s |\Omega|}{\|u_n^+\|^{p^-}} \to 0 \quad \text{as } n \to +\infty.$$

We also put

$$v_n = \frac{u_n^+}{\|u_n^+\|} \quad \text{for all } n \in \mathbb{N}.$$

Clearly, $\|v_n\| = 1$ for all $n \in \mathbb{N}$. Thus, considering a subsequence if necessary, we suppose that there exists $v \in W_0^{1,p(z)}(\Omega)$ such that

$$v_n \xrightarrow{w} v \quad \text{in } W_0^{1,p(z)}(\Omega),$$

$$v_n \to v \quad \text{in } L^{s(z)}(\Omega), 1 < s(z) < p^*(z) \text{ (compactly)},$$

$$v_n(z) \to v(z) \quad \text{a.e. in } \Omega.$$

Denote $\Omega_0 := \{z \in \Omega : v(z) > 0\}$. We claim that $|\Omega_0| = 0$ ($|\Omega_0|$ means the Lebesgue measure of Ω_0). We argue by contradiction again. So, suppose that $|\Omega_0| > 0$. We note that $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega_0$. Now, let s_0 as in (3), and we consider

$$\frac{I(u_n^+)}{\|u_n^+\|^{p^+}} = \frac{1}{\|u_n^+\|^{p^+}} \left[A_1\left(u_n^+\right) + A_2\left(u_n^+\right) + A_3\left(u_n^+\right) - B\left(u_n^+\right) \right]$$

$$\leq C - \int_\Omega \frac{G(z, u_n^+(z)) + C_{s_0}}{\|u_n^+\|^{p^+}} \, dz + \int_\Omega \frac{C_{s_0}}{\|u_n^+\|^{p^+}} \, dz.$$

This implies

$$0 = \lim_{n \to +\infty} \frac{I(u_n^+)}{\|u_n^+\|^{p^+}}$$

$$\leq \limsup_{n \to +\infty} \left[C - \int_\Omega \frac{G(z, u_n^+(z)) + C_{s_0}}{\|u_n^+\|^{p^+}} \, dz \right] + \limsup_{n \to +\infty} \int_\Omega \frac{C_{s_0}}{\|u_n^+\|^{p^+}} \, dz$$

$$= C - \liminf_{n \to +\infty} \int_\Omega \frac{G(z, u_n^+(z)) + C_{s_0}}{\|u_n^+\|^{p^+}} \, dz$$

$$\leq C - \liminf_{n \to +\infty} \int_{\Omega_0} \frac{G(z, u_n^+(z)) + C_{s_0}}{\|u_n^+\|^{p^+}} \, dz$$

$$\leq C - \int_{\Omega_0} \liminf_{n \to +\infty} \frac{G(z, u_n^+(z)) + C_{s_0}}{|u_n^+(z)|^{p^+}} v_n(z)^{p^+} \, dz = -\infty,$$

a contradiction and hence $|\Omega_0| = 0$. Then we have $v(z) = 0$ for a.a. $z \in \Omega$.

Now, we get

$$\int_{\{u_n^+ \geq s\}} \frac{|G(z, u_n^+(z))|}{\|u_n^+\|^{p^-}} \, dz$$

$$= \int_{\{u_n^+ \geq s\}} \frac{|G(z, u_n^+(z))|}{|u_n^+|^{p^-}} |v_n|^{p^-} \, dz \leq 2 \left\| \frac{|G(z, u_n^+(z))|}{|u_n^+|^{p^-}} \right\|_{L^{\beta(z)}(\{u_n^+ \geq s\})} \left\| |v_n|^{p^-} \right\|_{L^{\beta'(z)}(\{u_n^+ \geq s\})}$$

$$\leq C \max \left\{ \left(\int_{\{u_n^+ \geq s\}} |v_n|^{p^- \beta'(z)} \, dz \right)^{1/(\beta')^-}, \left(\int_{\{u_n^+ \geq s\}} |v_n|^{p^- \beta'(z)} \, dz \right)^{1/(\beta')^+} \right\}$$

$$\to 0 \quad \text{as } n \to +\infty \text{ for all } s \geq r_0,$$

where

$$C = 2 \max \left\{ \left(c_0 p^+ M + c_0 C_s |\Omega| \right)^{\frac{1}{\beta^-}}, \left(p^+ M + C_s |\Omega| \right)^{\frac{1}{\beta^+}} \right\} \quad \text{(by (5))}$$

$$\geq 2 \max \left\{ \left(\int_{\{u_n^+ \geq s\}} c_0 \mathcal{G}\left(z, u_n^+(z)\right) dz \right)^{\frac{1}{\beta^-}}, \left(\int_{\{u_n^+ \geq s\}} c_0 \mathcal{G}\left(z, u_n^+(z)\right) dz \right)^{\frac{1}{\beta^+}} \right\}$$

$$\geq 2\max\left\{\left(\int_{\{u_n^+\geq s\}}\frac{|G(z,u_n^+(z))|^{\beta(z)}}{|u_n^+|^{p^-\beta(z)}}\,dz\right)^{\frac{1}{\beta^-}},\left(\int_{\{u_n^+\geq s\}}\frac{|G(z,u_n^+(z))|^{\beta(z)}}{|u_n^+|^{p^-\beta(z)}}\,dz\right)^{\frac{1}{\beta^+}}\right\}$$

$$\geq 2\left\|\frac{|G(z,u_n^+(z))|}{|u_n^+|^{p^-}}\right\|_{L^{\beta(z)}(\{u_n^+\geq s\})}.$$

It follows that

$$\int_\Omega\frac{|G(z,u_n(z))|}{\|u_n\|^{p^-}}\,dz=\int_{\{u_n<s\}}\frac{|G(z,u_n(z))|}{\|u_n\|^{p^-}}\,dz+\int_{\{u_n\geq s\}}\frac{|G(z,u_n(z))|}{\|u_n\|^{p^-}}\,dz$$

$$\to 0\quad\text{as }n\to+\infty,$$

which leads to contradiction with (6), and in this case, the sequence $\{u_n^+\}$ is bounded. □

From Lemma 2 and Lemma 3, it follows the lemma.

Lemma 4 *If the assumptions* (g_0)–(g_2), (g_4), (g_5) *hold, then the functional I satisfies the* (C_c) *condition.*

The third and last step of our finding gives us the mountain pass geometry and hence the existence of a non-trivial critical point of the energy functional I.

Lemma 5 *If the assumptions* (g_0), (g_1), *and* (g_3) *hold, then we conclude that*:
  (i) *there exist* $\rho > 0$ *and* $\delta > 0$ *such that* $I(u) \geq \delta$ *for each* $u \in W_0^{1,p(z)}(\Omega)$ *with* $\|u\|_{W^{1,p(z)}(\Omega)} = \rho$;
  (ii) *there exists* $v \in W_0^{1,p(z)}(\Omega)$ *such that* $I(v) < 0$ *and* $\|v\|_{W^{1,p(z)}(\Omega)} > \rho$.

Proof (i) By the limit in (g_3), we deduce that for any $\varepsilon > 0$, there exists $t_0 > 0$ such that $G(z,t) \leq \varepsilon t^{p(z)}$, whenever $z \in \Omega$ and $0 \leq t < t_0$. The growth condition in (g_1) gives us a constant $C_0 = C(t_0) > 0$ such that $G(z,t) \leq C_0 t^{\alpha(z)}$, whenever $z \in \Omega$ and $t \geq t_0$. When combining these two inequalities, we find the following limitation from above:

$$G(z,t) \leq \varepsilon t^{p(z)} + C_0 t^{\alpha(z)}\quad\text{for all }z\in\Omega\text{ and }t\geq 0.$$

Consequently, we obtain the following limitation from below of the total energy functional:

$$I(u) = A_1(u) + A_2(u) + A_3(u) - B(u)$$

$$\geq \int_\Omega\frac{a_0}{p^+}|\nabla u|^{p(z)}\,dz - \int_\Omega G(z,t)\,dz$$

$$\geq \frac{a_0}{p^+}\int_\Omega|\nabla u|^{p(z)}\,dz - \varepsilon\int_\Omega|u|^{p(z)}\,dz - C_0\int_\Omega|u|^{\alpha(z)}\,dz.$$

Taking $\varepsilon \leq \frac{a_0}{3Cp^+}$, the general Poincaré inequality in Lemma 1 gives us

$$\varepsilon\int_\Omega|u|^{p(z)}\,dz \leq \varepsilon C\int_\Omega|\nabla u|^{p(z)}\,dz \leq \frac{a_0}{3p^+}\int_\Omega|\nabla u|^{p(z)}\,dz,$$

which refines the above limitation for the total energy functional as follows:

$$I(u) \geq K \int_{\Omega} |\nabla u|^{p(z)} \, dz + K \int_{\Omega} |u|^{p(z)} \, dz - C_0 \int_{\Omega} |u|^{\alpha(z)} \, dz, \tag{7}$$

where $K = \min\{\frac{a_0}{3p^+}, \frac{a_0}{3Cp^+}\}$.

We enter in the setting of Proposition 1, so we denote by $C_\alpha > 0$ the constant of the continuous and compact embedding $W_0^{1,p(z)}(\Omega) \hookrightarrow L^{\alpha(z)}(\Omega)$ there. Moreover, let $u \in W_0^{1,p(z)}(\Omega)$ satisfy $\|u\|_{W^{1,p(z)}(\Omega)} < \min\{1, C_\alpha^{-1}\}$, and so $\|u\|_{L^{\alpha(z)}(\Omega)} < 1$.

Clearly, for any $z = (z_1, \ldots, z_n) \in \overline{\Omega}$, as $p, \alpha \in C(\overline{\Omega})$, we can get

$$Q_r(z) = \left\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i - z_i| < r, i = 1, 2, \ldots, n \right\}$$

such that $|p(y) - p(z)| < \varepsilon$ and $|\alpha(y) - \alpha(z)| < \varepsilon$, whenever $y \in Q_r(z) \cap \overline{\Omega}$. Putting $\varepsilon := 4^{-1}(\alpha(z) - p(z))$, we have

$$p_z^+ = \sup_{y \in Q_r(z) \cap \overline{\Omega}} p(y) < \alpha_z^- = \inf_{y \in Q_r(z) \cap \overline{\Omega}} \alpha(y) \quad \text{for all } z \in \overline{\Omega}. \tag{8}$$

Since $\{Q_r(z)\}_{z \in \overline{\Omega}}$ is an open covering of $\overline{\Omega}$, the Lebesgue number lemma (see, for example, Proposition 1.5.34, p. 51, Papageorgiou and Winkert [19]) provides us with a number $\lambda > 0$ such that every subset of $\overline{\Omega}$, having diameter less than λ, is contained in some member of the cover. If we define $u = 0$ on $\mathbb{R}^n \setminus \Omega$, then $u \in W_0^{1,p(z)}(U)$ for any open hypercube U such that $\Omega \subset U$. Using the number of Lebesgue, we find a finite pairwise family of open hypercubes $\{Q_j\}_{j=1}^J$ having a diameter smaller than λ such that $Q_j \subset U$ is contained in some member of the cover $\{Q_r(z)\}_{z \in \overline{\Omega}}$, $Q_j \cap \overline{\Omega} \neq \emptyset$ for $j = 1, \ldots, J$ and $\overline{\Omega} \subseteq \bigcup_{j=1}^J \overline{Q}_j$. Put

$$\alpha_j^- = \inf_{z \in Q_j \cap \Omega} \alpha(z) \quad \text{and} \quad p_j^+ = \sup_{z \in Q_j \cap \Omega} p(z) \quad \text{for } j = 1, \ldots, J.$$

From (8), we get that $p_j^+ < \alpha_j^-$ for $j = 1, \ldots, J$. Also, we have $u \in W^{1,p(z)}(Q_j)$ ($j = 1, \ldots, J$). By Theorem 1 and Proposition 1, we deduce

$$\int_{Q_j \cap \Omega} |u|^{\alpha(z)} \, dz \leq \|u\|_{L^{\alpha(z)}(Q_j \cap \Omega)}^{\alpha_j^-} \leq \left(C_{\alpha,j} \|u\|_{W^{1,p(z)}(Q_j \cap \Omega)} \right)^{\alpha_j^-}, \tag{9}$$

where $C_{\alpha,j} > 0$ is the constant of the continuous and compact embedding $W^{1,p(z)}(Q_j \cap \Omega) \hookrightarrow L^{\alpha(z)}(Q_j \cap \Omega)$. As $\|\nabla u\|_{L^p(Q_j \cap \Omega)} < 1$ and $\|u\|_{L^p(Q_j \cap \Omega)} < 1$, by Theorem 1, we have

$$\int_{Q_j \cap \Omega} |\nabla u|^{p(z)} \, dz \geq \|\nabla u\|_{L^{p(z)}(Q_j \cap \Omega)}^{p_j^+} \quad \text{and} \quad \int_{Q_j \cap \Omega} |u|^{p(z)} \, dz \geq \|u\|_{L^{p(z)}(Q_j \cap \Omega)}^{p_j^+}. \tag{10}$$

As $|\Omega \setminus \bigcup_{j=1}^J (Q_j \cap \Omega)| = 0$, from (7), we deduce

$$I(u) \geq K \left(\int_{\bigcup_{j=1}^J (Q_j \cap \Omega)} |\nabla u|^{p(z)} \, dz + \int_{\bigcup_{j=1}^J (Q_j \cap \Omega)} |u|^{p(z)} \, dz \right) - C_0 \int_{\bigcup_{j=1}^J (Q_j \cap \Omega)} |u|^{\alpha(z)} \, dz$$

$$= \sum_{j=1}^J \left(K \int_{Q_j \cap \Omega} |\nabla u|^{p(z)} \, dz + K \int_{Q_j \cap \Omega} |u|^{p(z)} \, dz - C_0 \int_{Q_j \cap \Omega} |u|^{\alpha(z)} \, dz \right).$$

Again from (9)–(10), we have

$$K \int_{Q_j \cap \Omega} |\nabla u|^{p(z)} \, dz + K \int_{Q_j \cap \Omega} |u|^{p(z)} \, dz - C_0 \int_{Q_j \cap \Omega} |u|^{\alpha(z)} \, dz$$

$$\geq K_j \big(\|u\|_{W^{1,p(z)}(Q_j \cap \Omega)} \big)^{p_j^+} - C_j \big(\|u\|_{W^{1,p(z)}(Q_j \cap \Omega)} \big)^{\alpha_j^-}$$

for some $K_j, C_j > 0$, with $j = 1, \ldots, J$. As $\alpha_j^- > p_j^+$, there exists $\rho_j > 0$ such that

$$K_j - C_j \big(\|u\|_{W^{1,p(z)}(Q_j \cap \Omega)} \big)^{\alpha_j^- - p_j^+} \geq \delta_j > 0$$

if $\|u\|_{W^{1,p(z)}(Q_j \cap \Omega)} \leq \rho_j$.

Let $u \in W_0^{1,p(z)}(\Omega)$ be such that $\|u\|_{W^{1,p(z)}(\Omega)} = \rho = \min\{\rho_j : 1 \leq j \leq J\}$. From

$$\rho = \|u\|_{W^{1,p(z)}(\Omega)}$$

$$= \left\| u \sum_{j=1}^{J} \chi_{Q_j \cap \Omega} \right\|_{W^{1,p(z)}(\Omega)}$$

$$\leq \sum_{j=1}^{J} \|u \chi_{Q_j \cap \Omega}\|_{W^{1,p(z)}(\Omega)}$$

$$= \sum_{j=1}^{J} \|u\|_{W^{1,p(z)}(Q_j \cap \Omega)},$$

we obtain that there exists at least one $\|u\|_{W^{1,p(z)}(Q_j \cap \Omega)}$ satisfying

$$\frac{\rho}{J} \leq \|u\|_{W^{1,p(z)}(Q_j \cap \Omega)} \leq \rho.$$

Then we have

$$I(u) \geq \sum_{j=1}^{J} \big(K_j \big(\|u\|_{W^{1,p(z)}(Q_j \cap \Omega)} \big)^{p_j^+} - C_j \big(\|u\|_{W^{1,p(z)}(Q_j \cap \Omega)} \big)^{\alpha_j^-} \big)$$

$$\geq \left(\frac{\rho}{J} \right)^{p^+} \big(K_j - C_j(\rho)^{\alpha_j^- - p_j^+} \big)$$

$$\geq \left(\frac{\rho}{J} \right)^{p^+} \delta_j = \delta > 0 \quad \text{if } \|u\|_{W^{1,p(z)}(\Omega)} = \rho > 0.$$

(ii) Using (g_1) and (g_2), we deduce that for all $M > 0$, there exists $C_M > 0$ such that

$$G(z,t) \geq M|t|^{p^+} - C_M \quad \text{for a.a. } z \in \Omega, \text{ all } t \in \mathbb{R}. \tag{11}$$

Let $\zeta \in C_0^\infty(\Omega) \setminus \{0\}$ be such that $\zeta(z) \geq 0$ for all $z \in \Omega$. From (11), for all $t > 1$, we get

$$I(t\zeta) = \int_\Omega a(z) \frac{t^{p(z)}}{p(z)} |\nabla \zeta|^{p(z)} \, dz + \int_\Omega \frac{t^{q(z)}}{q(z)} |\nabla \zeta|^{q(z)} \, dz + \int_\Omega b(z) \frac{t^{p(z)}}{p(z)} |\zeta|^{p(z)} \, dz$$

$$- \int_\Omega G(z, t\zeta) \, dz$$

$$\leq t^{p^+}\left[\int_\Omega \frac{a(z)}{p(z)}|\nabla\zeta|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|\nabla\zeta|^{q(z)}\,dz + \int_\Omega \frac{b(z)}{p(z)}|\zeta|^{p(z)}\,dz - M\int_\Omega \zeta^{p^+}\,dz\right]$$
$$+ C_M|\Omega|.$$

If we choose $M > 0$ such that

$$\int_\Omega \frac{a(z)}{p(z)}|\nabla\zeta|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|\nabla\zeta|^{q(z)}\,dz + \int_\Omega \frac{b(z)}{p(z)}|\zeta|^{p(z)}\,dz - M\int_\Omega \zeta^{p^+}\,dz < 0,$$

we obtain that $\lim_{n\to+\infty} I(t\zeta) = -\infty$. It follows that there exists $v = t_0\zeta \in W_0^{1,p(z)}(\Omega)$ such that $I(v) < 0$ and $\|v\|_{W^{1,p(z)}(\Omega)} > \rho$. $\square$

Summarizing, Lemma 5 and Theorem 2 say that the functional I admits a non-zero critical point, which is exactly a nontrivial solution to (P_g), under suitable assumptions. Precisely, we establish the following main result.

Theorem 3 *If the assumptions (g_0)–(g_5) hold, then problem (P_g) admits at least a nontrivial solution $u \in W_0^{1,p(z)}(\Omega)$.*

Declarations

Ethics approval and consent to participate
This item is not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the conceptualization and writing of original draft. All authors read and approved the final manuscript.

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References
1. Acerbi, E., Mingione, G.: Regularity results for a class of functionals with nonstandard growth. Arch. Ration. Mech. Anal. **156**, 121–140 (2001)
2. Afrouzi, G.A., Mirzapour, M., Rădulescu, V.: Variational analysis of anisotropic Schrödinger equations without Ambrosetti–Rabinowitz-type condition. Z. Angew. Math. Phys. **69**, 9 (2018), 17 pp.
3. Alves, C., Molica Bisci, G.: A compact embedding result for anisotropic Sobolev spaces associated to a strip-like domain and some applications. J. Math. Anal. Appl. **501**, 123490 (2019) 24 pp.
4. Barile, S., Figueiredo, G.M.: Existence of least energy positive, negative and nodal solutions for a class of p&q-problems with potentials vanishing at infinity. J. Math. Anal. Appl. **427**, 1205–1233 (2015)

5. Baroni, P., Colombo, M., Mingione, G.: Harnack inequalities for double phase functionals. Nonlinear Anal. **121**, 206–222 (2015)
6. Baroni, P., Colombo, M., Mingione, G.: Nonautonomous functionals, borderline cases and related function classes. St. Petersburg Math. J. **27**(3), 347–379 (2016)
7. Baroni, P., Colombo, M., Mingione, G.: Regularity for general functionals with double phase. Calc. Var. Partial Differ. Equ. **57**(2), 62 (2018), 48 pp.
8. De Filippis, C.: Higher integrability for constrained minimizers of integral functionals with (p, q)-growth in low dimension. Nonlinear Anal. **170**, 1–20 (2018)
9. Diening, L., Harjulehto, P., Hästö, P., Růžička, M.: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Math., vol. 2017. Springer, Heidelberg (2011)
10. Fan, X.L., Zhang, Q.H.: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal. **52**, 1843–1852 (2003)
11. Fan, X.L., Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. **263**, 424–446 (2001)
12. Fu, Y.: The existence of solutions for elliptic systems with nonuniform growth. Stud. Math. **151**, 227–246 (2002)
13. Gasiński, L., Papageorgiou, N.S.: Exercises in Analysis. Part 2. Nonlinear Analysis. Problem Books in Mathematics. Springer, Cham (2016)
14. Gasiński, L., Papageorgiou, N.S.: Resonant anisotropic (p, q)-equations. Mathematics **8**(8), 1332 (2020), 21 p.
15. Marcellini, P.: On the definition and the lower semicontinuity of certain quasiconvex integrals. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **3**, 391–409 (1986)
16. Marcellini, P.: Regularity and existence of solutions of elliptic equations with p, q-growth conditions. J. Differ. Equ. **90**, 1–30 (1991)
17. Marcellini, P.: Everywhere regularity for a class of elliptic systems without growth conditions. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) **23**, 1–25 (1996)
18. Papageorgiou, N.S., Vetro, C.: Superlinear $(p(z), q(z))$-equations. Complex Var. Elliptic Equ. **64**, 8–25 (2019)
19. Papageorgiou, N.S., Winkert, P.: Applied Nonlinear Functional Analysis: An Introduction. de Gruyter, Berlin (2018)
20. Rădulescu, V., Repovš, D.: Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis. CRC Press, Boca Raton (2015)
21. Ružička, M.: Electrorheological Fluids Modeling and Mathematical Theory. Springer, Berlin (2002)
22. Shi, X., Rădulescu, V., Repovš, D., Zhang, Q.: Multiple solutions of double phase variational problems with variable exponent. Adv. Calc. Var. **13**, 385–401 (2020)
23. Vetro, C.: The existence of solutions for local Dirichlet $(r(u), s(u))$-problems. Mathematics **10**, 237 (2022), 17 pp.
24. Vetro, C., Vetro, F.: On problems driven by the $(p(\cdot), q(\cdot))$-Laplace operator. Mediterr. J. Math. **17**, 24 (2020), 11 pp.
25. Zhang, Q., Rădulescu, V.: Double phase anisotropic variational problems and combined effects of reaction and absorption terms. J. Math. Pures Appl. **118**, 159–203 (2018)
26. Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR, Izv. **29**, 33–66 (1987)
27. Zhikov, V.V.: On Lavrentiev's phenomenon. Russ. J. Math. Phys. **3**(2), 249–269 (1995)
28. Zhou, Q.-M., Wang, K.-Q.: Infinitely many weak solutions for $p(x)$-Laplacian-like problems with sign-changing potential. Electron. J. Qual. Theory Differ. Equ. **2020**, 10 (2020), 14 pp.