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Small-convection limit for two-dimensional chemotaxis-Navier–Stokes system with logarithmic sensitivity and logistic-type source

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Abstract

In this paper, we consider the small-convection limit of chemotaxis-Navier–Stokes system with logarithmic sensitivity and logistic-type source

$$\begin{cases} n_t^\kappa + \boldsymbol{u}^\kappa \cdot \nabla n^\kappa = \Delta n^\kappa - \chi \nabla \cdot (n^\kappa \nabla \log c^\kappa) + f(n^\kappa), & x \in \Omega, t > 0, \\ c_t^\kappa + \boldsymbol{u}^\kappa \cdot \nabla c^\kappa = \Delta c^\kappa - c^\kappa + n^\kappa, & x \in \Omega, t > 0, \\ \boldsymbol{u}_t^\kappa + \kappa (\boldsymbol{u}^\kappa \cdot \nabla)\boldsymbol{u}^\kappa = \Delta \boldsymbol{u}^\kappa + \nabla P^\kappa + n^\kappa \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot \boldsymbol{u}^\kappa = 0, & x \in \Omega, t > 0, \end{cases}$$

in a bounded convex domain $\Omega \subseteq \mathbb{R}^2$ with smooth boundary, where $\kappa \in \mathbb{R}$, $f(s) = \mu_1 s - \mu_2 s^\lambda$, $\lambda > 1$, and $\phi : \Omega \to \mathbb{R}$ is a given smooth potential with second-order partial derivatives. When the chemotaxis sensitivity χ satisfies the appropriate conditions, it is proved that the unique global classical solutions $(n^\kappa, c^\kappa, \boldsymbol{u}^\kappa)$ will stabilize to $(n^0, c^0, \boldsymbol{u}^0)$ as $\kappa \to 0$.

MSC: 35K55; 35Q92; 35Q35; 92C17

Keywords: Chemotaxis-Navier–Stokes; Logarithmic sensitivity; Logistic-type source; Small-convection limit; Convergence

1 Introduction

One of the first mathematical models of chemotaxis was investigated by Keller and Segel [14] to describe the aggregation of certain types of bacteria. In mathematics, it is described as a fully parabolic system

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n \chi(n,c) \nabla c), & x \in \Omega, t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega, t > 0. \end{cases} \tag{1.1}$$

Here, the unknowns $n = n(t,x)$ and $c = c(t,x)$ denote the cell density and concentration of chemical, respectively. The physical domain $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. The chemotaxis function $\chi(\cdot)$ denotes the chemotactic sensitivity. In particular, model (1.1) in which chemotactic sensitivity function choices $\chi(n,c) = \frac{\chi}{c}$ with $\chi > 0$ is an important class of chemotaxis models, its form is suggested by the Weber–Fechner laws and supported by experimental [13] and theoretical evidence [52]. When the chemical reaction is much faster than cell diffusion, system (1.1) could be simplified to parabolic-elliptic equations

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\chi(n,c)\nabla c), & x \in \Omega, t > 0, \\ 0 = \Delta c - c + n, & x \in \Omega, t > 0, \end{cases}$$

this limit process was proved by Wang, Winkler, and Xiang in [36].

In order to study the dynamic behavior of cells under the action of fluid, Tuval et al. [31] took into account the experiment of the collective behavior of *Bacillus subtilis* in suspension. They observed the formation of plume-like structures and large-scale convection patterns. As an extension of the classical Keller–Segel model, it was used in the case of chemical diffusion and cell migration in a nontrivial interactive fluid environment and was coupled with chemotaxis-fluid equations of the form

$$\begin{cases} n_t + \boldsymbol{u} \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(n,c)\nabla c), & x \in \Omega, t > 0, \\ c_t + \boldsymbol{u} \cdot \nabla c = \Delta c + g(n,c), & x \in \Omega, t > 0, \\ \boldsymbol{u}_t + \kappa(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \Delta \boldsymbol{u} + \nabla P + n\nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot \boldsymbol{u} = 0, & x \in \Omega, t > 0. \end{cases} \tag{1.2}$$

Here, $\boldsymbol{u} = \boldsymbol{u}(t,x)$ and $P = P(t,x)$ denote the velocity field and the pressure of fluid, respectively, and Ω is a spatial domain where the cells and the fluid interact with each other and move. The given functions $\chi(n,c)$ and $g(n,c)$ are the chemotactic sensitivity and the signal function of consumption or production. The potential function ϕ is a scalar-valued function. It can be produced by the different physical mechanism such as gravity, centrifugal. The parameter $\kappa \in \{0,1\}$ denotes the case of Stokes and Navier–Stokes flows, respectively.

Mathematically, analyzing the above fluid model is very challenging. Tao and Winkler in [28] gave the globally bounded large initial value solution of the problem with Neumann boundary value in a two-dimensional condition for three conditions, whereas for large data up to now only global weak solutions could be established and become eventually smooth and classical. At the same time, there is a globally bounded solution with small initial data. In recent years, there have been many related research works in this regard. For more reference about the chemotaxis-Navier–Stokes system, the corresponding global solvability of classical solutions has been investigated by [6–8, 15, 17, 24, 25, 34, 39, 43, 44, 46, 47, 49, 53] in two- or three-dimensional situation. We also mention complicated variants, e.g., involving rotational flux [4, 5, 18, 19, 22] and logistic source terms [3, 16, 30, 37, 48] as well as nonlinear diffusion [6, 12, 19, 38, 40, 54]. For chemotaxis-Stokes system, the interested reader can refer to earlier works of the global solvability of classical solutions in [1, 20, 27] and nonlinear diffusion in [9, 19, 23, 29, 33, 45] as well as rotational flux in [32, 41, 42].

As for the Navier–Stokes subsystem of (1.2), despite decades of efforts, the global well-posedness still stays in the Dirichlet problem of bounded domain in a two-dimensional case, but the research on the three-dimensional smooth global solution still lacks the viewpoint of mathematical theory. However, the corresponding Stokes case is much more complete [26]. Therefore, some works focus on the chemotaxis-Stokes variant model under a coupled system of the form (1.2). A natural problem is when the chemotactic Navier–Stokes system can approach the chemotactic-Stokes system. The corresponding experimental observations, such as $Re \approx 10^{-5}$, are in the case of Reynolds number very small. However, rigorous mathematical results are very few. Wang, Winkler, and Xiang [35] and Wu and Xiang [49] gave the mathematical theoretical proof when the solutions of chemotaxis-Navier–Stokes systems will approximate the solution of corresponding chemotaxis-Stokes system in the case of signal consumption and signal generation, respectively.

Under the interaction of chemical signals, a very important coupled system is the chemotaxis model with a logarithmic sensitivity function and a logistic source. Black et al. [2] investigated the model of signal production with logarithmic sensitivity and proved the global existence and uniqueness of classical solutions. Zhao and Zheng in [55] gave the global existence and boundedness of solutions to a chemotaxis system with singular sensitivity and logistic-type source without fluid. Further, Wu and Natal [51] researched the model in [55] coupled with fluid equations and gave the decay rate of the solutions.

Motivated by the above work, we study the small-convection limit of the following chemotaxis-Navier–Stokes system with logarithmic sensitivity and logistic-type source:

$$\begin{cases} n_t^\kappa + \boldsymbol{u}^\kappa \cdot \nabla n^\kappa = \Delta n^\kappa - \chi \nabla \cdot (n^\kappa \nabla \log c^\kappa) + f(n^\kappa), & x \in \Omega, t > 0, \\ c_t^\kappa + \boldsymbol{u}^\kappa \cdot \nabla c^\kappa = \Delta c^\kappa - c^\kappa + n^\kappa, & x \in \Omega, t > 0, \\ \boldsymbol{u}_t^\kappa + \kappa (\boldsymbol{u}^\kappa \cdot \nabla) \boldsymbol{u}^\kappa = \Delta \boldsymbol{u}^\kappa + \nabla P^\kappa + n^\kappa \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot \boldsymbol{u} = 0, & x \in \Omega, t > 0. \end{cases} \tag{1.3}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with smooth boundary, ν denotes an outer normal vector of $\partial\Omega$, $f(s) = \mu_1 s - \mu_2 s^\lambda$, $\lambda > 1$ is a logistic source, the parameters μ_1 and μ_2 are positive constants, the chemotactic sensitivity parameter $\chi > 0$ satisfies

$$\begin{cases} 0 < \chi < \min\{2\sqrt{\mu_1}, 1\}, & \text{if } 1 < \lambda \le 2, \\ 0 < \chi < \min\{\frac{\sqrt{4\mu_1(\mu_1+1)\lambda + \mu_1^2} - \mu_1}{\lambda}, \frac{2}{\sqrt{\lambda(\lambda-1)(\lambda-2)}}, 1\}, & \text{if } \lambda > 2, \end{cases} \tag{1.4}$$

and the initial data satisfy

$$n^\kappa(x,0) = n_0(x), \qquad c^\kappa(x,0) = c_0(x), \qquad \boldsymbol{u}^\kappa(x,0) = \boldsymbol{u}_0(x), \tag{1.5}$$

and the boundary conditions satisfy

$$\partial_\nu n^\kappa = \partial_\nu c^\kappa = 0, \qquad \boldsymbol{u}^\kappa = \boldsymbol{0}, \quad x \in \partial\Omega, t > 0. \tag{1.6}$$

For simplicity, we shall assume that n_0, c_0, $\boldsymbol{u}_0$, ϕ satisfy

$$\begin{cases} 0 \le n_0(x) \in C^0(\bar{\Omega}) \quad \text{and} \quad n_0(x) \not\equiv 0, \quad x \in \Omega, \\ c_0(x) \in W^{1,\vartheta}(\Omega), \qquad \inf_{x\in\Omega} c_0(x) > 0, \quad \vartheta > 8, \\ \boldsymbol{u}_0 \in D(A^\alpha), \quad \alpha \in (\frac{1}{2}, 1), \\ \phi \in W^{2,\infty}(\bar{\Omega}), \end{cases} \tag{1.7}$$

where $A := -\mathcal{P}\Delta$ denotes the realization of the Stokes operator in $L^2(\Omega;\mathbb{R}^2)$, $D(A) := W^{2,2}(\Omega;\mathbb{R}^2) \cap W_0^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega)$ denotes the domain and $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega;\mathbb{R}^2) \mid \nabla \cdot \boldsymbol{u} = 0\}$. The map $\mathcal{P} : L^2(\Omega;\mathbb{R}^2) \to L^2_\sigma(\Omega)$ is a Helmholtz projection operator.

Theorem 1.1 *For all $\kappa \in (0,1)$, $p \in (1,\vartheta)$, $\alpha \in (\frac{1}{2},1)$, suppose that the initial data $(n_0, c_0, \boldsymbol{u}_0)$ satisfy (1.7) and that $(n^\kappa, c^\kappa, \boldsymbol{u}^\kappa)$ is the unique global solution of system (1.3)–(1.6). Then, for all $p \in (1,\infty)$ and any time $T > 0$, there exists a constant $C(p,T) > 0$ such that*

$$\begin{aligned} &\left\|n^\kappa(\cdot,t) - n^0(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|c^\kappa(\cdot,t) - c^0(\cdot,t)\right\|_{W^{1,p}(\Omega)} + \left\|A^\alpha \boldsymbol{u}^\kappa(\cdot,t) - A^\alpha \boldsymbol{u}^0(\cdot,t)\right\|_{L^2(\Omega)} \\ &\quad \le C(p,T)\kappa. \end{aligned}$$

In particular, there is a constant $C(p,T) > 0$ such that

$$\begin{aligned} &\left\|n^\kappa(\cdot,t) - n^0(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|c^\kappa(\cdot,t) - c^0(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|\boldsymbol{u}^\kappa(\cdot,t) - \boldsymbol{u}^0(\cdot,t)\right\|_{L^\infty(\Omega)} \\ &\quad \le C(p,T)\kappa. \end{aligned}$$

2 Preliminaries

It is necessary for us to give Lemma 2.1, which ensures the global existence and uniqueness of the solutions of our problems.

Lemma 2.1 (Theorem 1.1 in [51]) *Let χ satisfy (1.4) and $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. If $(n_0, c_0, \boldsymbol{u}_0, \phi)$ fulfils (1.7), then for each $\kappa \in \mathbb{R}$, system (1.3)–(1.6) admits a unique solution $(n^\kappa, c^\kappa, \boldsymbol{u}^\kappa)$ such that*

$$\begin{cases} n^\kappa \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ c^\kappa \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty([0,\infty); W^{1,\vartheta}(\Omega)), \\ \boldsymbol{u}^\kappa \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ P^\kappa \in C^{1,0}(\bar{\Omega} \times [0,\infty)) \end{cases} \tag{2.1}$$

and that n^κ and c^κ are positive in $\Omega \times (0,\infty)$.

Moreover, this solution is uniformly bounded in the sense that

$$\left\|n^\kappa(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|c^\kappa(\cdot,t)\right\|_{W^{1,\vartheta}(\Omega)} + \left\|A^\alpha \boldsymbol{u}^\kappa(\cdot,t)\right\|_{L^2(\Omega)} \le M$$

with some positive constant M.

Next, we will use the L^1-mass conservation technique to deal with the logarithmic sensitive function to obtain the uniform lower bound estimation of c, which aims to eliminate the singularity of the logarithmic function.

Lemma 2.2 (Lemma 2.2 and Lemma 2.4 in [51]) *For each* $\kappa \in \mathbb{R}$, $\lambda > 1$, *it holds that*

$$\|n^\kappa(\cdot,t)\|_{L^1(\Omega)} \le m_* := \max\left\{\|n_0(\cdot)\|_{L^1(\Omega)}, |\Omega|\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{\lambda-1}}\right\} \quad \text{for all } t > 0 \tag{2.2}$$

and

$$c^\kappa(\cdot,t) \ge \theta_0 \quad \text{for all } t > 0, \tag{2.3}$$

as well as

$$\|c^\kappa(\cdot,t)\|_{L^1(\Omega)} \le \max\{\|c_0(\cdot)\|_{L^1(\Omega)}, m_*\} \quad \text{for all } t > 0, \tag{2.4}$$

where θ_0 *is a positive constant.*

In order to use semigroup estimates, we need the following auxiliary lemma.

Lemma 2.3 (Lemma 3.3 in [10]) *Let* $p \in (1,\infty]$ *and let* $\lambda_1 > 0$ *denote the first nonzero eigenvalue of* $-\Delta$ *in* $\Omega \subset \mathbb{R}^N$. *Then there exists* $C > 0$ *such that, for all* $\varphi \in C^1(\bar{\Omega},\mathbb{R}^N)$ *fulfilling* $\varphi \cdot \nu = 0$ *on* $\partial\Omega$, *we have*

$$\|e^{t\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le C\left(1 + t^{-\frac{1}{2}-\frac{N}{2p}}\right)e^{-\lambda_1 t}\|\varphi\|_{L^p(\Omega)} \quad \text{for all } t > 0.$$

3 Convergence as $\kappa \to 0$

In order to obtain the convergence of $(n^\kappa, c^\kappa, \boldsymbol{u}^\kappa) \to (n^0, c^0, \boldsymbol{u}^0)$, we need the following transformation.

Let

$$\hat{n} := n^\kappa - n^0, \qquad \hat{c} := c^\kappa - c^0, \qquad \hat{\boldsymbol{u}} := \boldsymbol{u}^\kappa - \boldsymbol{u}^0, \quad \text{and} \quad \hat{P} := P^\kappa - P^0$$

for each $\kappa \in (0,1)$, where P^0 denotes the pressure of $\kappa = 0$. System (1.3) is transformed into

$$\begin{cases} \partial_t\hat{n} + \boldsymbol{u}^\kappa\cdot\nabla\hat{n} + \hat{\boldsymbol{u}}\cdot\nabla n^0 & \\ \quad = \Delta\hat{n} - \chi\nabla\cdot[\frac{\hat{n}}{c^\kappa}\nabla c^\kappa - \frac{n^0\hat{c}}{c^0c^\kappa}\nabla c^\kappa + \frac{n^0\nabla\hat{c}}{c^0}] + \mu_1\hat{n} + \mu_2[(n^\kappa)^\lambda - (n^0)^\lambda], & x \in \Omega, t > 0, \\ \partial_t\hat{c} + \boldsymbol{u}^\kappa\cdot\nabla\hat{c} + \hat{\boldsymbol{u}}\cdot\nabla c^0 = \Delta\hat{c} - \hat{c} + \hat{n}, & x \in \Omega, t > 0, \\ \partial_t\hat{\boldsymbol{u}} + \kappa(\boldsymbol{u}^\kappa\cdot\nabla)\boldsymbol{u}^\kappa = \Delta\hat{\boldsymbol{u}} + \nabla\hat{P} + \hat{n}\nabla\phi, & x \in \Omega, t > 0, \\ \nabla\cdot\hat{\boldsymbol{u}} = 0, & x \in \Omega, t > 0 \end{cases} \tag{3.1}$$

under the initial conditions

$$\hat{n}(x,0) = \hat{c}(x,0) = 0, \qquad \hat{\boldsymbol{u}}(x,0) = \boldsymbol{0}, \quad x \in \Omega, \tag{3.2}$$

and the boundary conditions

$$\partial_\nu \hat{n} = \partial_\nu \hat{c} = 0, \qquad \hat{\boldsymbol{u}} = \boldsymbol{0}, \quad x \in \partial\Omega, t > 0. \tag{3.3}$$

Our key step toward Theorem 1.1 is to derive the corresponding estimate for $(\hat{n}, \hat{c}, \hat{\boldsymbol{u}})$ with respect to the norm in $(L^2(\Omega))^4$. So, we give the following three lemmas.

Lemma 3.1 *There is a positive constant C independent of time such that, for any* $\kappa \in (0,1)$, *we have*

$$\|\hat{n}\|_{L^2(\Omega)} + \|\hat{c}\|_{H^1(\Omega)} + \|\hat{\boldsymbol{u}}\|_{L^2(\Omega)} \le \kappa e^{Ct} \quad \text{for all } t > 0.$$

Proof This proof is based on the standard energy methods. We divide it into five steps.

Step 1. Multiplying equation $(3.1)_1$ by $\hat{n}$ and integrating by parts, one has

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\hat{n}\|_{L^2(\Omega)}^2 + \|\nabla \hat{n}\|_{L^2(\Omega)}^2 \\ &\quad = \int_\Omega n^0 \hat{\boldsymbol{u}} \cdot \nabla \hat{n} + \chi \int_\Omega \frac{\hat{n}}{c^\kappa} \nabla c^\kappa \cdot \nabla \hat{n} - \chi \int_\Omega \frac{n^0 \hat{c}}{c^0 c^\kappa} \nabla c^\kappa \cdot \nabla \hat{n} + \chi \int_\Omega \frac{n^0}{c^0} \nabla \hat{c} \cdot \nabla \hat{n} \\ &\qquad + \mu_1 \|\hat{n}\|_{L^2(\Omega)}^2 + \mu_2 \int_\Omega \hat{n}\big[\big(n^\kappa\big)^\lambda - \big(n^0\big)^\lambda\big] \\ &\quad := I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{3.4}$$

For I_1, we can use Hölder's inequality and Young's inequality to deduce that

$$\begin{aligned} I_1 &\le \big\|n^0\big\|_{L^\infty(\Omega)} \|\hat{\boldsymbol{u}}\|_{L^2(\Omega)} \|\nabla \hat{n}\|_{L^2(\Omega)} \le \frac{1}{8}\|\nabla \hat{n}\|_{L^2(\Omega)}^2 + 2\big\|n^0\big\|_{L^\infty(\Omega)}^2 \|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2 \\ &\le \frac{1}{8}\|\nabla \hat{n}\|_{L^2(\Omega)}^2 + C_0 \|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2, \end{aligned}$$

where $C_0 > 0$ is constant.

For I_2, thanks to $\int_\Omega \hat{n} = 0$ and using Hölder's inequality, Young's inequality, the Gagliardo–Nirenberg inequality, and Poincaré's inequality, we can infer

$$\begin{aligned} I_2 &\le \frac{\chi}{\theta_0} \|\hat{n}\|_{L^4(\Omega)} \big\|\nabla c^\kappa\big\|_{L^4(\Omega)} \|\nabla \hat{n}\|_{L^2(\Omega)} \\ &\le C_1\big(\|\hat{n}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \hat{n}\|_{L^2(\Omega)}^{\frac{1}{2}} + \|\hat{n}\|_{L^2(\Omega)}\big) \big\|\nabla c^\kappa\big\|_{L^4(\Omega)} \|\nabla \hat{n}\|_{L^2(\Omega)} \\ &= C_1\big(\|\hat{n}\|_{L^2(\Omega)}^{\frac{1}{2}} \big\|\nabla c^\kappa\big\|_{L^4(\Omega)} \|\nabla \hat{n}\|_{L^2(\Omega)}^{\frac{3}{2}} + \|\hat{n}\|_{L^2(\Omega)}^{\frac{1}{2}} \big\|\nabla c^\kappa\big\|_{L^4(\Omega)} \|\hat{n}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \hat{n}\|_{L^2(\Omega)}\big) \\ &\le C_2 \|\hat{n}\|_{L^2(\Omega)}^{\frac{1}{2}} \big\|\nabla c^\kappa\big\|_{L^4(\Omega)} \|\nabla \hat{n}\|_{L^2(\Omega)}^{\frac{3}{2}} \\ &\le \frac{1}{8}\|\nabla \hat{n}\|_{L^2(\Omega)}^2 + \frac{C_3}{2}\|\hat{n}\|_{L^2(\Omega)}^2, \end{aligned}$$

where C_1, C_2, and $C_3 := \max\{2(\mu_1 + \lambda\mu_2 M^{\lambda-1}), 512 C_2^4 \|\nabla c^\kappa\|_{L^4(\Omega)}^4\}$ are positive constants.

For I_3, thanks to Lemma 2.2 and using Hölder's inequality, Young's inequality, and the Gagliardo–Nirenberg inequality, we find that

$$I_3 \le \frac{\chi \|n^0\|_{L^\infty(\Omega)}}{\theta_0^2} \int_\Omega \hat{c}\big|\nabla c^\kappa\big| |\nabla \hat{n}|$$

$$
\begin{aligned}
&\le \frac{\chi\|n^0\|_{L^\infty(\Omega)}}{\theta_0^2}\big\|\nabla c^\kappa\big\|_{L^4(\Omega)}\|\hat{c}\|_{L^4(\Omega)}\|\nabla\hat{n}\|_{L^2(\Omega)}\\
&\le \frac{1}{8}\|\nabla\hat{n}\|^2_{L^2(\Omega)} + \frac{2\chi^2\|n^0\|^2_{L^\infty(\Omega)}\|\nabla c^\kappa\|^2_{L^4(\Omega)}}{\theta_0^4}\|\hat{c}\|^2_{L^4(\Omega)}\\
&\le \frac{1}{8}\|\nabla\hat{n}\|^2_{L^2(\Omega)} + \frac{2\chi^2 C_{GN}\|n^0\|^2_{L^\infty(\Omega)}\|\nabla c^\kappa\|^2_{L^4(\Omega)}}{\theta_0^4}\big(\|\hat{c}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla\hat{c}\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\hat{c}\|_{L^2(\Omega)}\big)^2\\
&\le \frac{1}{8}\|\nabla\hat{n}\|^2_{L^2(\Omega)} + \|\nabla\hat{c}\|^2_{L^2(\Omega)}\\
&\quad + \frac{2\chi^2 C_{GN}\|n^0\|^2_{L^\infty(\Omega)}\|\nabla c^\kappa\|^2_{L^4(\Omega)}}{\theta_0^4}\left(\frac{2\chi^2 C_{GN}\|n^0\|^2_{L^\infty(\Omega)}\|\nabla c^\kappa\|^2_{L^4(\Omega)}}{\theta_0^4} + 2\right)\|\hat{c}\|^2_{L^2(\Omega)}\\
&\le \frac{1}{8}\|\nabla\hat{n}\|^2_{L^2(\Omega)} + \|\nabla\hat{c}\|^2_{L^2(\Omega)} + C_4\|\hat{c}\|^2_{L^2(\Omega)}
\end{aligned}
$$

with some $C_4 > 0$ and $C_{GN} > 0$.

For I_4, using Hölder's inequality and Young's inequality, we see that

$$
I_4 \le \frac{\chi\|n^0\|_{L^\infty(\Omega)}}{\theta_0}\|\nabla\hat{c}\|_{L^2(\Omega)}\|\nabla\hat{n}\|_{L^2(\Omega)} \le \frac{1}{8}\|\nabla\hat{n}\|^2_{L^2(\Omega)} + C_5\|\nabla\hat{c}\|^2_{L^2(\Omega)}
$$

with some $C_5 > 0$.

For I_5 and I_6, using Lagrange's mean value theorem and Hölder's inequality, we deduce that

$$
\begin{aligned}
I_5 + I_6 &= \mu_1\|\hat{n}\|^2_{L^2(\Omega)} + \mu_2\int_\Omega \hat{n}\big[(n^\kappa)^\lambda - (n^0)^\lambda\big]\\
&\le \big(\mu_1 + \lambda\mu_2\max\big\{\big\|n^\kappa\big\|^{\lambda-1}_{L^\infty(\Omega)}, \big\|n^0\big\|^{\lambda-1}_{L^\infty(\Omega)}\big\}\big)\|\hat{n}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Substituting $I_1, I_2, I_3, I_4, I_5, I_6$ into (3.4) and using Lemma 2.1, we have

$$
\begin{aligned}
&\frac{d}{dt}\|\hat{n}\|^2_{L^2(\Omega)} + \|\nabla\hat{n}\|^2_{L^2(\Omega)}\\
&\quad\le 2C_3\|\hat{n}\|^2_{L^2(\Omega)} + 2C_4\|\hat{c}\|^2_{L^2(\Omega)} + 2(1 + C_5)\|\nabla\hat{c}\|^2_{L^2(\Omega)} + 2C_0\|\hat{\boldsymbol{u}}\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.5}
$$

Step 2. We multiply equation $(3.1)_2$ with $\hat{c}$, integrate the resulting equation in Ω, and use the integration by parts to obtain

$$
\frac{1}{2}\frac{d}{dt}\|\hat{c}\|^2_{L^2(\Omega)} + \|\hat{c}\|^2_{L^2(\Omega)} + \|\nabla\hat{c}\|^2_{L^2(\Omega)} = \int_\Omega c^0\hat{\boldsymbol{u}}\cdot\nabla\hat{c} + \int_\Omega \hat{n}\hat{c} := I_7.
\tag{3.6}
$$

For I_7, we use Hölder's inequality and Young's inequality to get

$$
\begin{aligned}
I_7 &\le \big\|c^0\big\|_{L^\infty(\Omega)}\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}\|\nabla\hat{c}\|_{L^2(\Omega)} + \|\hat{n}\|_{L^2(\Omega)}\|\hat{c}\|_{L^2(\Omega)}\\
&\le \frac{1}{2}\|\nabla\hat{c}\|^2_{L^2(\Omega)} + \frac{\|c^0\|^2_{L^\infty(\Omega)}}{2}\|\hat{\boldsymbol{u}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\hat{n}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\hat{c}\|_{L^2(\Omega)}.
\end{aligned}
\tag{3.7}
$$

Let $C_7 := \|c^0\|^2_{L^\infty(\Omega)}$. Thus substituting (3.7) into (3.6), we have

$$
\frac{d}{dt}\|\hat{c}\|^2_{L^2(\Omega)} + \|\hat{c}\|^2_{L^2(\Omega)} + \|\nabla\hat{c}\|^2_{L^2(\Omega)} \le C_7\|\hat{\boldsymbol{u}}\|^2_{L^2(\Omega)} + \|\hat{n}\|^2_{L^2(\Omega)}.
\tag{3.8}
$$

Step 3. Testing the second equation of (3.1) with $-\Delta\hat{c}$ and using the integration by parts, we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\hat{c}\|_{L^2(\Omega)}^2+\|\nabla\hat{c}\|_{L^2(\Omega)}^2+\|\Delta\hat{c}\|_{L^2(\Omega)}^2\\&\quad=\int_\Omega\Delta\hat{c}\boldsymbol{u}^\kappa\cdot\nabla\hat{c}+\int_\Omega\Delta\hat{c}\hat{\boldsymbol{u}}\cdot\nabla c^0+\int_\Omega\nabla\hat{n}\cdot\nabla\hat{c}:=I_8.\end{aligned}\tag{3.9}$$

Using Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}I_8&\le\|\boldsymbol{u}^\kappa\|_{L^\infty(\Omega)}\|\Delta\hat{c}\|_{L^2(\Omega)}\|\nabla\hat{c}\|_{L^2(\Omega)}+\|\Delta\hat{c}\|_{L^2(\Omega)}\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}\|\nabla c^0\|_{L^4(\Omega)}\\&\quad+\|\nabla\hat{n}\|_{L^2(\Omega)}\|\nabla\hat{c}\|_{L^2(\Omega)}\\&\le\frac{1}{2}\|\Delta\hat{c}\|_{L^2(\Omega)}^2+\left(\|\boldsymbol{u}^\kappa\|_{L^\infty(\Omega)}^2+\frac{1}{2}\right)\|\nabla\hat{c}\|_{L^2(\Omega)}^2\\&\quad+\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}^2\|\nabla c^0\|_{L^4(\Omega)}^2+\frac{1}{2}\|\nabla\hat{n}\|_{L^2(\Omega)}^2.\end{aligned}\tag{3.10}$$

Using the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}^2&\le C_{GN}\big(\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^{\frac{1}{2}}+\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}\big)^2\\&\le 2C_{GN}\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}+2C_{GN}\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2\\&\le\frac{1}{2\|\nabla c^0\|_{L^4(\Omega)}^2}\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2+2C_{GN}\big(C_{GN}\|\nabla c^0\|_{L^4(\Omega)}^2+1\big)\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2.\end{aligned}\tag{3.11}$$

Substituting (3.11) into (3.10), one has

$$\begin{aligned}I_8&\le\frac{1}{2}\|\nabla\hat{n}\|_{L^2(\Omega)}^2+\frac{1}{2}\|\Delta\hat{c}\|_{L^2(\Omega)}^2+\left(\|\boldsymbol{u}^\kappa\|_{L^\infty(\Omega)}^2+\frac{1}{2}\right)\|\nabla\hat{c}\|_{L^2(\Omega)}^2\\&\quad+\frac{1}{2}\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2+2C_{GN}\|\nabla c^0\|_{L^4(\Omega)}^2\big(C_{GN}\|\nabla c^0\|_{L^4(\Omega)}^2+1\big)\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2.\end{aligned}\tag{3.12}$$

Thus, we can substitute (3.12) into (3.9) to deduce that

$$\begin{aligned}&\frac{d}{dt}\|\nabla\hat{c}\|_{L^2(\Omega)}^2+2\|\nabla\hat{c}\|_{L^2(\Omega)}^2+\|\Delta\hat{c}\|_{L^2(\Omega)}^2\\&\quad\le\|\nabla\hat{n}\|_{L^2(\Omega)}^2+\big(2\|\boldsymbol{u}^\kappa\|_{L^\infty(\Omega)}^2+1\big)\|\nabla\hat{c}\|_{L^2(\Omega)}^2\\&\qquad+4C_{GN}\|\nabla c^0\|_{L^4(\Omega)}^2\big(C_{GN}\|\nabla c^0\|_{L^4(\Omega)}^2+1\big)\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2+\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2\\&\quad\le\|\nabla\hat{n}\|_{L^2(\Omega)}^2+C_8\big(\|\nabla\hat{c}\|_{L^2(\Omega)}^2+\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2\big)+\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2,\end{aligned}\tag{3.13}$$

where $C_8:=\max\{2\|\boldsymbol{u}^\kappa\|_{L^\infty(\Omega)}^2+1,4C_{GN}\|\nabla c^0\|_{L^4(\Omega)}^2(C_{GN}\|\nabla c^0\|_{L^4(\Omega)}^2+1)\}$.

Step 4. Taking the inner product $(1.3)_3$ by $\hat{\boldsymbol{u}}$, integrating by parts, and using Hölder's inequality, Poincaré's inequality, and Young's inequality, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2+\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2&=\kappa\int_\Omega\left(\boldsymbol{u}^\kappa\otimes\boldsymbol{u}^\kappa\right):\nabla\hat{\boldsymbol{u}}-\int_\Omega\phi\nabla\hat{n}\cdot\hat{\boldsymbol{u}}\\&\le|\kappa|\left\|\boldsymbol{u}^\kappa\otimes\boldsymbol{u}^\kappa\right\|_{L^2(\Omega)}\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}+\|\phi\|_{L^\infty(\Omega)}\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}\|\hat{n}\|_{L^2(\Omega)}\\&\le|\kappa|\left\|\boldsymbol{u}^\kappa\right\|_{L^4(\Omega)}^2\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}+c_1\|\phi\|_{L^\infty(\Omega)}\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}\|\hat{n}\|_{L^2(\Omega)}\\&\le\frac{1}{2}\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2+c_1^2\|\phi\|_{L^\infty(\Omega)}^2\|\hat{n}\|_{L^2(\Omega)}^2+|\kappa|^2\left\|\boldsymbol{u}^\kappa\right\|_{L^4(\Omega)}^4.\end{aligned}$$

That is,

$$\begin{aligned}\frac{d}{dt}\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2+\|\nabla\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2&\le2c_1^2\|\phi\|_{L^\infty(\Omega)}^2\|\hat{n}\|_{L^2(\Omega)}^2+2|\kappa|^2\left\|\boldsymbol{u}^\kappa\right\|_{L^4(\Omega)}^4\\&\le C_9\left(\|\hat{n}\|_{L^2(\Omega)}^2+|\kappa|^2\right),\end{aligned}\tag{3.14}$$

where $C_9:=\max\{2c_1^2\|\phi\|_{L^\infty(\Omega)}^2,\|\boldsymbol{u}^\kappa\|_{L^4(\Omega)}^4\}$.

Step 5. Summarily, from Step 1–Step 4, we can close the evolution estimates for $\hat{n}$, $\nabla\hat{c}$, and $\boldsymbol{u}$.

Combining inequalities (3.5), (3.8), (3.13), and (3.14), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\hat{n}\|_{L^2(\Omega)}^2+\|\hat{c}\|_{L^2(\Omega)}^2+\|\nabla\hat{c}\|_{L^2(\Omega)}^2+\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2\right)+\|\Delta\hat{c}\|_{L^2(\Omega)}^2\\&\quad\le(2C_3+C_9+1)\|\hat{n}\|_{L^2(\Omega)}^2+2C_4\|\hat{c}\|_{L^2(\Omega)}^2+(C_8-2)\|\nabla\hat{c}\|_{L^2(\Omega)}^2\\&\qquad+(2C_0+C_7+C_8)\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2+C_9|\kappa|^2\\&\quad\le C_{10}\left(\|\hat{n}\|_{L^2(\Omega)}^2+\|\hat{c}\|_{L^2(\Omega)}^2+\|\nabla\hat{c}\|_{L^2(\Omega)}^2+\|\hat{\boldsymbol{u}}\|_{L^2(\Omega)}^2\right)+C_{10}|\kappa|^2,\end{aligned}\tag{3.15}$$

where $C_{10}:=\max\{2C_3+C_9+1,2,2C_0+C_7+C_8\}$. Then, applying Gronwall's inequality to (3.15) and $\hat{n}_0=\hat{c}_0=\nabla\hat{c}_0=0$, $\hat{\boldsymbol{u}}_0=\mathbf{0}$, we conclude that Lemma 3.1 holds. □

Lemma 3.2 *There exists $\widetilde{C}>0$ dependent of time such that, for any $\kappa\in(0,1)$, we find that*

$$\left\|A^\alpha\hat{\boldsymbol{u}}(\cdot,t)\right\|_{L^2(\Omega)}\le\widetilde{C}|\kappa|\quad\text{for all } t>0\tag{3.16}$$

and

$$\left\|\hat{\boldsymbol{u}}(\cdot,t)\right\|_{L^\infty(\Omega)}\le\widetilde{C}|\kappa|\quad\text{for all } t>0.\tag{3.17}$$

Proof Using that $\alpha<1$ and relying on known regularization properties of the Stokes semigroup in Ω, we obtain $C_{11}>0$ and $C_{12}>0$ such that (3.18), where $f_1:=\mathcal{P}[\hat{n}\nabla\phi]-\kappa\mathcal{P}[\boldsymbol{u}^\kappa\cdot\nabla]\boldsymbol{u}^\kappa$, $\alpha<1$. Lemma 2.1 and Lemma 3.1 provide

$$\begin{aligned}\left\|A^\alpha\hat{\boldsymbol{u}}(\cdot,t)\right\|_{L^2(\Omega)}&\le\int_0^t\left\|A^\alpha e^{-(t-s)A}f_1(\cdot,s)\right\|_{L^2(\Omega)}ds\\&\le C_{11}\int_0^t(t-s)^{-\alpha}\left\|f_1(\cdot,s)\right\|_{L^2(\Omega)}ds\end{aligned}$$

$$\begin{aligned}&\le C_{11}\int_0^t (t-s)^{-\alpha}\big(\|\nabla\phi\|_{L^\infty(\Omega)}\|\hat{n}\|_{L^2(\Omega)} + \kappa\left\|\boldsymbol{u}^\kappa\right\|_{L^\infty(\Omega)}\left\|\nabla\boldsymbol{u}^\kappa\right\|_{L^2(\Omega)}\big)\,ds\\ &\le C_{12}\kappa \quad \text{for all } t>0. \end{aligned}\tag{3.18}$$

Meanwhile, $\alpha > \frac{1}{2}$ warrants that $D(A^\alpha) \hookrightarrow L^\infty(\Omega;\mathbb{R}^2)$ [11] holds. □

Lemma 3.3 *There is $\widetilde{C} > 0$ such that, for any $\kappa \in (0,1)$, we see that*

$$\left\|\hat{n}(\cdot,t)\right\|_{L^p(\Omega)} + \left\|\hat{c}(\cdot,t)\right\|_{W^{1,p}(\Omega)} \le \widetilde{C}\kappa \quad \textit{for all } p>3, t>0.$$

Proof Multiplying equation $(3.1)_1$ by $\hat{n}^{p-1}$ and integrating by parts, we see that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\hat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \frac{4(p-1)}{p^2}\|\nabla\hat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)}\\ &= (p-1)\int_\Omega n^0\hat{n}^{p-2}\hat{\boldsymbol{u}}\cdot\nabla\hat{n} + \chi(p-1)\int_\Omega \frac{\hat{n}^{p-1}}{c^\kappa}\nabla c^\kappa\cdot\nabla\hat{n}\\ &\quad - \chi(p-1)\int_\Omega \frac{n^0\hat{n}^{p-2}\hat{c}}{c^0c^\kappa}\nabla c^\kappa\cdot\nabla\hat{n} + \chi(p-1)\int_\Omega\frac{n^0\hat{n}^{p-2}}{c^0}\nabla\hat{c}\cdot\nabla\hat{n}\\ &\quad + \mu_1\|\hat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \mu_2\int_\Omega \hat{n}^{p-1}\big[\big(n^\kappa\big)^\lambda - \big(n^0\big)^\lambda\big]\\ &= \frac{2(p-1)}{p}\int_\Omega n^0\hat{n}^{\frac{p}{2}-1}\hat{\boldsymbol{u}}\cdot\nabla\hat{n}^{\frac{p}{2}} + \frac{2\chi(p-1)}{p}\int_\Omega\frac{\hat{n}^{\frac{p}{2}}}{c^\kappa}\nabla c^\kappa\cdot\nabla\hat{n}^{\frac{p}{2}}\\ &\quad - \frac{2\chi(p-1)}{p}\int_\Omega\frac{n^0\hat{n}^{\frac{p}{2}-1}\hat{c}}{c^0c^\kappa}\nabla c^\kappa\cdot\nabla\hat{n}^{\frac{p}{2}} + \frac{2\chi(p-1)}{p}\int_\Omega\frac{n^0\hat{n}^{\frac{p}{2}-1}}{c^0}\nabla\hat{c}\cdot\nabla\hat{n}^{\frac{p}{2}}\\ &\quad + \mu_1\|\hat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \mu_2\int_\Omega\hat{n}^{p-1}\big[\big(n^\kappa\big)^\lambda - \big(n^0\big)^\lambda\big].\end{aligned}$$

That is,

$$\begin{aligned}&\frac{d}{dt}\|\hat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \frac{4(p-1)}{p}\|\nabla\hat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)}\\ &= 2(p-1)\int_\Omega n^0\hat{n}^{\frac{p}{2}-1}\hat{\boldsymbol{u}}\cdot\nabla\hat{n}^{\frac{p}{2}} + 2\chi(p-1)\int_\Omega\frac{\hat{n}^{\frac{p}{2}}}{c^\kappa}\nabla c^\kappa\cdot\nabla\hat{n}^{\frac{p}{2}}\\ &\quad - 2\chi(p-1)\int_\Omega\frac{n^0\hat{n}^{\frac{p}{2}-1}\hat{c}}{c^0c^\kappa}\nabla c^\kappa\cdot\nabla\hat{n}^{\frac{p}{2}} + 2\chi(p-1)\int_\Omega\frac{n^0\hat{n}^{\frac{p}{2}-1}}{c^0}\nabla\hat{c}\cdot\nabla\hat{n}^{\frac{p}{2}}\\ &\quad + \mu_1 p\|\hat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + \mu_2 p\int_\Omega\hat{n}^{p-1}\big[\big(n^\kappa\big)^\lambda - \big(n^0\big)^\lambda\big]\\ &:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6.\end{aligned}\tag{3.19}$$

For J_1, aided by Lemma 2.1 in [50], Hölder's inequality, Young's inequality, the Gagliardo–Nirenberg inequality, and Lemma 3.1, we have

$$\begin{aligned}J_1 &= 2(p-1)\left\|n^0\right\|_{L^\infty(\Omega)}\left\|\hat{n}^{\frac{p}{2}-1}\right\|_{L^4(\Omega)}\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}\\ &= 2(p-1)\left\|n^0\right\|_{L^\infty(\Omega)}\left\|\hat{n}^{\frac{p}{2}}\right\|^{\frac{p-2}{p}}_{L^{\frac{4(p-2)}{p}}(\Omega)}\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{p-1}{4p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + 4p(p-1)\big\|n^0\big\|_{L^\infty(\Omega)}^2\big\|\hat{n}^{\frac{p}{2}}\big\|_{L^{\frac{4(p-2)}{p}}(\Omega)}^{\frac{2(p-2)}{p}}\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}^2 \\
&\le \frac{p-1}{4p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + 4p(p-1)C_{GN}\big\|n^0\big\|_{L^\infty(\Omega)}^2\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}^2 \\
&\quad\times\Big(\big\|\hat{n}^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{1}{p-2}}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\frac{p-3}{p-2}} + \big\|\hat{n}^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}\Big)^{\frac{2(p-2)}{p}} \\
&\le \frac{p-1}{4p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + 2^{\frac{3p-4}{p}}p(p-1)C_{GN}\big\|n^0\big\|_{L^\infty(\Omega)}^2\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}^2 \\
&\quad\times\Big(\|\hat{n}\|_{L^2(\Omega)}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\frac{2(p-3)}{p}} + \|\hat{n}\|_{L^2(\Omega)}^{p-2}\Big) \\
&\le \frac{p-1}{2p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + c_1\|\hat{n}\|_{L^2(\Omega)}^{\frac{p}{3}}\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}^{\frac{2p}{3}} + c_1\|\hat{\boldsymbol{u}}\|_{L^4(\Omega)}^2\|\hat{n}\|_{L^2(\Omega)}^{p-2} \\
&\le \frac{p-1}{2p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + c_2\kappa^p \quad \text{for all } p>3.
\end{aligned}
\tag{3.20}
$$

For J_2, J_3, and J_4, using Hölder's inequality, Young's inequality, the Gagliardo–Nirenberg inequality, and Lemma 3.1 provides some $c_i>0$, $(i=3,\dots,10)$ such that

$$
\begin{aligned}
J_2 &= \frac{2\chi(p-1)}{\theta_0}\big\|\hat{n}^{\frac{p}{2}}\big\|_{L^4(\Omega)}\big\|\nabla c^\kappa\big\|_{L^4(\Omega)}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)} \\
&\le \frac{p-1}{4p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + \frac{4\chi^2p(p-1)}{\theta_0^2}\big\|\hat{n}^{\frac{p}{2}}\big\|_{L^4(\Omega)}^2\big\|\nabla c^\kappa\big\|_{L^4(\Omega)}^2 \\
&\le \frac{p-1}{4p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 \\
&\quad+\frac{4C_{GN}\chi^2p(p-1)\|\nabla c^\kappa\|_{L^4(\Omega)}^2}{\theta_0^2}\Big(\big\|\hat{n}^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{1}{p}}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\frac{p-1}{p}} + \big\|\hat{n}^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}\Big)^2 \\
&\le \frac{p-1}{2p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + c_3\|\hat{n}\|_{L^2(\Omega)}^p \\
&\le \frac{p-1}{2p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + c_4\kappa^p
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
J_3 &\le \frac{2\chi(p-1)\|n^0\|_{L^\infty(\Omega)}}{\theta_0^2}\big\|\hat{n}^{\frac{p}{2}-1}\big\|_{L^4(\Omega)}\|\hat{c}\|_{L^8(\Omega)}\big\|\nabla c^\kappa\big\|_{L^8(\Omega)}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)} \\
&\le \frac{p-1}{4p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + \frac{4\chi^2p(p-1)\|n^0\|_{L^\infty(\Omega)}^2}{\theta_0^4}\big\|\hat{n}^{\frac{p}{2}-1}\big\|_{L^4(\Omega)}^2\big\|\nabla c^\kappa\big\|_{L^8(\Omega)}^2\|\hat{c}\|_{L^8(\Omega)}^2 \\
&\le \frac{p-1}{4p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 \\
&\quad+c_5\Big(\big\|\hat{n}^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{1}{p-2}}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\frac{p-3}{p-2}} + \big\|\hat{n}^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}\Big)^{\frac{2(p-2)}{p}}\Big(\|\hat{c}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\hat{c}\|_{L^2(\Omega)}^{\frac{3}{2}} + \|\hat{c}\|_{L^2(\Omega)}^2\Big) \\
&\le \frac{p-1}{2p}\big\|\nabla\hat{n}^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + c_6\Big(\|\hat{n}\|_{L^2(\Omega)}^{\frac{p}{3}}\Big(\|\hat{c}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\hat{c}\|_{L^2(\Omega)}^{\frac{3}{2}} + \|\hat{c}\|_{L^2(\Omega)}^2\Big)^{\frac{p}{3}} \\
&\quad+\|\hat{n}\|_{L^2(\Omega)}^{p-2}\Big(\|\hat{c}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\hat{c}\|_{L^2(\Omega)}^{\frac{3}{2}} + \|\hat{c}\|_{L^2(\Omega)}^2\Big)\Big)
\end{aligned}
\tag{3.22}
$$

$$\leq \frac{p-1}{2p}\left\|\nabla \hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + c_7\kappa^p$$

as well as

$$\begin{aligned}
J_4 &\leq \frac{2\chi(p-1)\|n^0\|_{L^\infty(\Omega)}}{\theta_0}\left\|\hat{n}^{\frac{p}{2}-1}\right\|_{L^{\frac{2p}{p-2}}(\Omega)}\|\nabla\hat{c}\|_{L^p(\Omega)}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)} \\
&\leq \frac{p-1}{4p}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \frac{4\chi^2p(p-1)\|n^0\|_{L^\infty(\Omega)}^2}{\theta_0^2}\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p-2)}{p}}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{4}{p}} \\
&\leq \frac{p-1}{4p}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \frac{p}{4}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + c_8\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 \\
&\leq \frac{p-1}{4p}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \frac{p}{4}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 \\
&\quad + c_8C_{GN}\left(\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{2}{p}}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{p-2}{p}} + \left\|\hat{n}^{\frac{p}{2}}\right\|_{L^{\frac{4}{p}}(\Omega)}\right)^2 \\
&\leq \frac{p-1}{4p}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \frac{p}{4}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 \\
&\quad + 2c_8C_{GN}\left(\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p-2)}{p}} + \left\|\hat{n}^{\frac{p}{2}}\right\|_{L^{\frac{4}{p}}(\Omega)}^2\right) \\
&\leq \frac{p-1}{2p}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \frac{p}{4}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + c_9\|\hat{n}\|_{L^2(\Omega)}^p \\
&\leq \frac{p-1}{2p}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \frac{p}{4}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + c_{10}\kappa^p \quad \text{for all } p > 3.
\end{aligned} \tag{3.23}$$

For J_5 and J_6, using Lagrange's mean value theorem, Hölder's inequality, Young's inequality, and the Gagliardo–Nirenberg inequality, we see that

$$\begin{aligned}
J_5 + J_6 &\leq \left(\mu_1 + \lambda\mu_2\max\left\{\left\|n^\kappa\right\|_{L^\infty(\Omega)}^{\lambda-1}, \left\|n^0\right\|_{L^\infty(\Omega)}^{\lambda-1}\right\}\right)\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 \\
&\leq c_{11}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p-2)}{p}}\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}} + c_{11}\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^{\frac{4}{p}}(\Omega)}^2 \\
&\leq \frac{p-1}{p}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + c_{12}\kappa^p,
\end{aligned} \tag{3.24}$$

where c_{11} and c_{12} are constants.

Substituting (3.20)–(3.24) into (3.19), we see that there exists a positive constant c_{13} such that

$$\frac{d}{dt}\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \frac{p-1}{p}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 \leq \frac{p}{4}\left\|\nabla\hat{c}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + c_{13}\kappa^p. \tag{3.25}$$

We apply ∇ to equation $(3.1)_2$ and then multiply the resulting equation by $|\nabla\hat{c}|^{p-2}\nabla\hat{c}$ to deduce that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 &= \int_\Omega |\nabla\hat{c}|^{p-2}\nabla\hat{c}\cdot\nabla\Delta\hat{c} - \int_\Omega|\nabla\hat{c}|^p + \int_\Omega|\nabla\hat{c}|^{p-2}\nabla\hat{c}\cdot\nabla\hat{n} \\
&\quad - \int_\Omega|\nabla\hat{c}|^{p-2}\nabla\hat{c}\cdot\nabla\left(\boldsymbol{u}^\kappa\cdot\nabla\hat{c}\right) - \int_\Omega|\nabla\hat{c}|^{p-2}\nabla\hat{c}\cdot\nabla\left(\hat{\boldsymbol{u}}\cdot\nabla c^0\right)
\end{aligned}$$

$$\begin{aligned}
&= \int_\Omega |\nabla\hat{c}|^{p-2}\nabla\hat{c}\cdot\nabla\Delta\hat{c} - \int_\Omega |\nabla\hat{c}|^p + \int_\Omega |\nabla\hat{c}|^{p-2}\nabla\hat{c}\cdot\nabla\hat{n} \\
&\quad - \int_\Omega |\nabla\hat{c}|^{p-2}\nabla\hat{c}\cdot\nabla\boldsymbol{u}^\kappa\cdot\nabla\hat{c} + \int_\Omega \hat{\boldsymbol{u}}\cdot\nabla c^0\nabla\cdot\left(|\nabla\hat{c}|^{p-2}\nabla\hat{c}\right) \\
&:= K_1 - \int_\Omega |\nabla\hat{c}|^p + K_2 + K_3 + K_4.
\end{aligned} \tag{3.26}$$

Using the pointwise identity $\Delta|\nabla\hat{c}|^2 = 2\nabla\hat{c}\cdot\nabla\Delta\hat{c} + 2|D^2\hat{c}|^2$, we infer

$$|\nabla\hat{c}|^{p-2}(\nabla\hat{c}\cdot\nabla\Delta\hat{c}) = \frac{1}{2}\Delta|\nabla\hat{c}|^2|\nabla\hat{c}|^{p-2} - \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2}.$$

Since $\partial_\nu\hat{c} = 0$ on $\partial\Omega$ along with the convexity of Ω ensures that $\frac{\partial|\nabla c|^2}{\partial\nu} \le 0$ on $\partial\Omega$ ([21], Lemma I.1, p.350), we have

$$\begin{aligned}
K_1 &= \int_\Omega |\nabla\hat{c}|^{p-2}(\nabla\hat{c}\cdot\nabla\Delta\hat{c}) = \frac{1}{2}\int_\Omega \Delta|\nabla\hat{c}|^2|\nabla\hat{c}|^{p-2} - \int_\Omega \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2} \\
&= \frac{1}{2}\int_{\partial\Omega} \frac{\partial|\nabla\hat{c}|^2}{\partial\nu}|\nabla\hat{c}|^{p-2} - \frac{1}{2}\int_\Omega \nabla|\nabla\hat{c}|^2\cdot\nabla|\nabla\hat{c}|^{p-2} - \int_\Omega \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2} \\
&\le -\frac{1}{2}\int_\Omega \nabla|\nabla\hat{c}|^2\cdot\nabla|\nabla\hat{c}|^{p-2} - \int_\Omega \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2} \\
&= -\frac{4(p-2)}{p^2}\left\|\nabla|\nabla\hat{c}|^{\frac{p}{2}}\right\|^2_{L^2(\Omega)} - \int_\Omega \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2}.
\end{aligned} \tag{3.27}$$

For K_2, K_3, and K_4, using Hölder's inequality, Young's inequality, the Gagliardo–Nirenberg inequality, and Lemma 3.1 provides some $c_i > 0$, $(i = 14, \dots, 19)$ such that

$$\begin{aligned}
K_2 &= \int_\Omega |\nabla\hat{c}|^{p-2}\nabla\hat{c}\cdot\nabla\hat{n} \\
&= -(p-2)\int_\Omega \hat{n}|\nabla\hat{c}|^{p-4}\left(\nabla\hat{c}\cdot D^2\hat{c}\right)\cdot\nabla\hat{c} - \int_\Omega \hat{n}|\nabla\hat{c}|^{p-2}\Delta\hat{c} \\
&\le (p-2+\sqrt{2})\int_\Omega |\hat{n}||\nabla\hat{c}|^{p-2}\left|D^2\hat{c}\right| \\
&\le \frac{1}{4}\int_\Omega \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2} + (p-2+\sqrt{2})^2\int_\Omega |\hat{n}|^2|\nabla\hat{c}|^{p-2} \\
&\le \frac{1}{4}\int_\Omega \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2} + \frac{1}{4}\int_\Omega |\nabla\hat{c}|^p + c_{14}\int_\Omega |\hat{n}|^p \\
&\le \frac{1}{4}\int_\Omega \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2} + \frac{1}{4}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|^2_{L^2(\Omega)} \\
&\quad + c_{15}\left(\left\|\hat{n}^{\frac{p}{2}}\right\|^{\frac{4}{p}}_{L^{\frac{4}{p}}(\Omega)}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|^{\frac{2(p-2)}{p}}_{L^2(\Omega)} + \left\|\hat{n}^{\frac{p}{2}}\right\|^2_{L^{\frac{4}{p}}(\Omega)}\right) \\
&\le \frac{1}{4}\int_\Omega \left|D^2\hat{c}\right|^2|\nabla\hat{c}|^{p-2} + \frac{1}{4}\left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|^2_{L^2(\Omega)} + \frac{p-1}{2p^2}\left\|\nabla\hat{n}^{\frac{p}{2}}\right\|^2_{L^2(\Omega)} + c_{16}\kappa^p
\end{aligned} \tag{3.28}$$

and

$$K_3 \le \left\||\nabla\hat{c}|^p\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}^\kappa\right\|_{L^2(\Omega)}$$

$$\begin{aligned}
&\le \big\| |\nabla\hat c|^{\frac{p}{2}}\big\|_{L^4(\Omega)}^2 \big\|\nabla \boldsymbol{u}^\kappa\big\|_{L^2(\Omega)} \\
&\le c_{17}\big\|\nabla|\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\frac{2(p-1)}{p}}\big\||\nabla\hat c|^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{2}{p}} + c_{17}\big\||\nabla\hat c|^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}^2 \\
&\le \frac{3(p-2)}{p^2}\big\|\nabla|\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + c_{18}\kappa^p
\end{aligned} \tag{3.29}$$

as well as

$$\begin{aligned}
K_4 &= \int_\Omega \hat{\boldsymbol{u}}\cdot\nabla c^0\nabla\cdot\big(|\nabla\hat c|^{p-2}\nabla\hat c\big) \\
&= (p-2)\int_\Omega \hat{\boldsymbol{u}}\cdot\nabla c^0|\nabla\hat c|^{p-4}\nabla\hat c\cdot D^2\hat c\cdot\nabla\hat c + \int_\Omega \hat{\boldsymbol{u}}\cdot\nabla c^0|\nabla\hat c|^{p-2}\Delta\hat c \\
&\le (p-2+\sqrt{2})\int_\Omega |\hat{\boldsymbol{u}}|\big|\nabla c^0\big||\nabla\hat c|^{p-2}\big|D^2\hat c\big| \\
&\le \frac{1}{4}\int_\Omega \big|D^2\hat c\big||\nabla\hat c|^{p-2} + (p-2+\sqrt{2})^2\int_\Omega |\hat{\boldsymbol{u}}|^2\big|\nabla c^0\big|^2|\nabla\hat c|^{p-2} \\
&\le \frac{1}{4}\int_\Omega \big|D^2\hat c\big||\nabla\hat c|^{p-2} + \big\||\hat{\boldsymbol{u}}|^2\big\|_{L^\infty(\Omega)}\big\|\big|\nabla c^0\big|^2\big\|_{L^{\frac{p}{2}}(\Omega)}\big\||\nabla\hat c|^{p-2}\big\|_{L^{\frac{p}{p-2}}(\Omega)} \\
&\le \frac{1}{4}\int_\Omega \big|D^2\hat c\big||\nabla\hat c|^{p-2} + \frac{1}{4}\big\||\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + c_{19}\kappa^p.
\end{aligned} \tag{3.30}$$

Substituting (3.27)–(3.30) into (3.26), we see that there exists a positive constant c_{20} such that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\big\||\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + \frac{1}{2}\big\||\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + \frac{p-2}{p^2}\big\|\nabla|\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2}\int_\Omega \big|D^2\hat c\big|^2|\nabla\hat c|^{p-2} \le \frac{p-1}{2p^2}\big\|\nabla\hat n^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + c_{20}\kappa^p.
\end{aligned} \tag{3.31}$$

Combining with (3.25) and (3.31), there exists $c_{21} > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\Big(\big\|\hat n^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + \big\||\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2\Big) + \frac{p-1}{2p}\big\|\nabla\hat n^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + \frac{p}{4}\big\||\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 \\
&\quad + \frac{p-2}{p}\big\|\nabla|\nabla\hat c|^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + \frac{p}{2}\int_\Omega \big|D^2\hat c\big|^2|\nabla\hat c|^{p-2} \le c_{21}\kappa^p.
\end{aligned} \tag{3.32}$$

We may employ the Gagliardo–Nirenberg inequality and Young's inequality once again to deduce that

$$\begin{aligned}
\big\|\hat n^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 &\le C_{15}\Big(\big\|\hat n^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}}\big\|\nabla\hat n^{\frac{p}{2}}\big\|_{L^2(\Omega)}^{\frac{2(p-2)}{p}} + \big\|\hat n^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}^2\Big) \\
&\le \big\|\nabla\hat n^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + C_{16}\big\|\hat n^{\frac{p}{2}}\big\|_{L^{\frac{4}{p}}(\Omega)}^2 \\
&\le \big\|\nabla\hat n^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + C_{16}\|\hat n\|_{L^2(\Omega)}^p \\
&\le \big\|\nabla\hat n^{\frac{p}{2}}\big\|_{L^2(\Omega)}^2 + C_{17}\kappa^p,
\end{aligned} \tag{3.33}$$

where C_{15}, C_{16}, and C_{17} are positive constants.

Since $\frac{p-1}{2p} < \frac{p}{4}$, we can combine with (3.32) and (3.33) to see that

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2\right) + \frac{p-1}{2p}\left(\left\|\hat{n}^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \left\||\nabla\hat{c}|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2\right) \\ &\quad \le (c_{21} + C_{17})\kappa^p. \end{aligned} \tag{3.34}$$

Applying Gronwall's inequality to (3.34), we can complete the proof of Lemma 3.3. □

Lemma 3.4 *For any $\kappa \in (0,1)$, there exists a constant $\widetilde{C} > 0$ such that*

$$\left\|\hat{n}(\cdot,t)\right\|_{L^\infty(\Omega)} \le \widetilde{C}\kappa \quad \text{for all } t > 0.$$

Proof We can rewrite equation $(3.1)_1$ as

$$\partial_t\hat{n} - \Delta\hat{n} = \nabla\cdot f_2 + \mu_1\hat{n} + \mu_2\left[\left(n^\kappa\right)^\lambda - \left(n^0\right)^\lambda\right],$$

where we again abbreviate

$$f_2 := -\left(\boldsymbol{u}^\kappa\hat{n} + \hat{\boldsymbol{u}}n^0 + \chi\left(\frac{\hat{n}}{c^\kappa}\nabla c^\kappa - \frac{n^0\hat{c}}{c^0c^\kappa}\nabla c^\kappa + \frac{n^0\nabla\hat{c}}{c^0}\right)\right).$$

Since the initial data $\hat{n}(x,0) = 0$, using the variation-of-constants formula, we get

$$\hat{n}(\cdot,t) = \int_0^t e^{(t-s)\Delta}\left[\nabla\cdot f_2(\cdot,s) + \mu_1\hat{n} + \mu_2\left[\left(n^\kappa\right)^\lambda - \left(n^0\right)^\lambda\right]ds \quad \text{for all } t > 0. \tag{3.35}$$

We can use Lemma 2.3 and Lagrange's mean value theorem to obtain

$$\begin{aligned} \left\|\hat{n}(\cdot,t)\right\|_{L^\infty(\Omega)} \le c_{22}\int_0^t &\left[\left(1 + (t-s)^{-\frac{1}{2}-\frac{1}{p}}\right)e^{-\lambda_1 t}\left(\left\|f_2(\cdot,s)\right\|_{L^p(\Omega)}\right.\right. \\ &\left.\left. + \mu_1\|\hat{n}\|_{L^p(\Omega)} + \mu_2\lambda M^{\lambda-1}\|\hat{n}\|_{L^p(\Omega)}\right)\right]ds \quad \text{for all } t > 0. \end{aligned}$$

In view of Lemma 2.1 and Lemmas 3.2–3.3, from (3.35) we therefore obtain that with some $c_{23} > 0$ we have

$$\begin{aligned} \left\|f_2(\cdot,s)\right\|_{L^p(\Omega)} &\le \left\|\boldsymbol{u}^\kappa\right\|_{L^\infty(\Omega)}\|\hat{n}\|_{L^p(\Omega)} + \|\hat{\boldsymbol{u}}\|_{L^\infty(\Omega)}\left\|n^0\right\|_{L^p(\Omega)} + \frac{\chi}{\theta_0}\left(\|\hat{n}\|_{L^{2p}(\Omega)}\left\|\nabla c^\kappa\right\|_{L^{2p}(\Omega)}\right. \\ &\quad \left. + \left\|n^0\right\|_{L^{2p}(\Omega)}\|\nabla\hat{c}\|_{L^{2p}(\Omega)}\right) + \frac{\chi}{\theta_0^2}\left\|n^0\right\|_{L^\infty(\Omega)}\|\hat{c}\|_{L^{2p}(\Omega)}\left\|\nabla c^\kappa\right\|_{L^{2p}(\Omega)} \\ &\le c_{23}\kappa \quad \text{for all } t > 0, \end{aligned}$$

as desired. □

Proof of Theorem 1.1 We only need to use Lemmas 3.2–3.4 and combine the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p > 2$ and $D(A^\alpha) \hookrightarrow L^\infty(\Omega;\mathbb{R}^2)$ to complete the proof. □

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