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Existence and multiplicity of solutions for $p(x)$-Laplacian problem with Steklov boundary condition

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Abstract

We study the existence and multiplicity of weak solutions for an elliptic problem involving $p(x)$-Laplacian operator under Steklov boundary condition. The approach is based on variational methods.

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Keywords: Variational method; Steklov boundary condition; $p(x)$-Laplacian operator

1 Introduction

Steklov conditions are considered a more "realistic" description of the interactions at the boundary of a physical system. For example, the heat flow through the surface of a body generally depends on the value of the temperature at the surface itself (see [2, 4, 11, 14] and the references therein for some kinds of Steklov problems).

Recently, Afrouzi et al. [1] studied the existence of multiple solutions of the following Steklov problem involving $p(x)$-Laplacian operator:

$$\begin{cases} -\Delta_{p(x)}u = a(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded smooth domain, λ is a positive parameter, $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with a growth condition, and $a \in L^\infty(\Omega)$. Also, the existence of at least one positive radial solution belonging to the space $W_0^{1,p(x)}(B) \cap L_a^{q(x)}(B) \cap L_b^{r(x)}(B)$ for the problem

$$\begin{cases} -\Delta_{p(x)}u + R(x)u^{p(x)-2}u = a(x)|u|^{q(x)-2}u - b(x)|u|^{r(x)-2}u, & x \in B, \\ u > 0, & x \in B, \\ u = 0, & x \in \partial B \end{cases}$$

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has been proved [21], where B is the unit ball centered at the origin in $\mathbb{R}^N, N \geq 3, p, q, r \in C_+(B)$, R is a positive radial function that satisfies the suitable conditions and

$$a(x) = \theta\big(|x|\big) \quad \text{and} \quad b(x) = \xi\big(|x|\big),$$

in which $\theta, \xi \in L^\infty(0,1)$ such that θ is a positive nonconstant radially nondecreasing function and ξ is a nonnegative radially nonincreasing function (see [17–20, 22–24] and the references therein).

Motivated by their works, here we are interested in finding enough conditions for the existence and multiplicity of weak solutions to the following Steklov $p(x)$-Laplacian problem:

$$\begin{cases} -\Delta_{p(x)}u + c(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain, for $p \in C(\overline{\Omega})$, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplace operator, $c \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega c(x) > 0$. $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function with the following conditions:

$$(F0) \quad \big|f(x,s)\big| \leq a|s|^{\gamma(x)-1}$$

for $(x,s) \in \Omega \times \mathbb{R}$, where a is a positive constant and $\gamma \in C(\Omega)$ such that

$$\gamma(x) \leq p(x), \quad x \in \Omega,$$

and

$$(F1) \quad f(x,s)s \leq 0$$

for $(x,s) \in \Omega \times \mathbb{R}$. And $g : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function with the following growth condition:

$$(G0) \quad \big|g(x,s)\big| \leq b_1(x) + b_2|s|^{\beta(x)-1}$$

for all $(x,s) \in \partial\Omega \times \mathbb{R}$, where $b_1 \in L^{\frac{\beta(x)}{\beta(x)-1}}(\partial\Omega)$, $b_2 \geq 0$ is a constant, $\beta : \overline{\Omega} \to \mathbb{R}$ such that $\beta \in C(\partial\Omega)$ and

$$1 < \beta^- := \inf_{x\in\overline{\Omega}} \beta(x) \leq \beta(x) \leq \beta^+ := \sup_{x\in\overline{\Omega}} \beta(x) < p^-.$$

We recall that $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function if $x \mapsto f(x,\xi)$ is measurable for all $\xi \in \mathbb{R}$ and $\xi \mapsto f(x,\xi)$ is continuous for a.e. $x \in \Omega$.

The definition of the weak solution of problem (1.1) is as follows.

Definition 1.1 We say that the function $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) if

$$\int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u \nabla v + c(x)|u|^{p(x)-2}uv\big)\,dx - \int_\Omega f(x,u)v\,dx = \lambda \int_{\partial\Omega} g(x,u)v\,d\sigma$$

is true for all $v \in W^{1,p(x)}(\Omega)$.

Remark 1.1 An interested reader may study the problem in the Orlicz–Sobolev spaces or on the Heisenberg groups (see [12, 13, 25, 26] and the references therein for details of these spaces).

One of the main results of this paper is as follows.

Theorem 1.1 *Let $f, g : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ be Carathéodory functions satisfying $(F0), (F1)$ and $(G0)$, respectively. Assume that there exists $d > 0$ such that*

$$\int_{\partial\Omega} \max_{|t|\leq\xi} G(x,t)\,d\sigma \leq \int_{\partial\Omega} G(x,d)\,d\sigma \tag{1.2}$$

with $\xi := (\frac{p^+}{k}(\frac{K}{p^-} + M)\,d^{\hat{p}})^{\frac{1}{\hat{p}}}$. Then, for each

$$\lambda \in \Lambda := \left]\frac{\Phi(d)}{\Psi(d)}, \frac{(\frac{K}{p^-} + M)\,d^{\hat{p}}}{\int_{\partial\Omega} \max_{|t|\leq\xi} G(x,t)\,d\sigma}\right[, \tag{1.3}$$

problem (1.1) admits at least one nontrivial weak solution.

Subsequently, by Theorem 4.1 and Theorem 4.2, we present the existence of two and three weak solutions of problem (1.1), respectively.

The rest of the paper is organized as follows: In Sect. 2, some preliminaries and basic facts are recalled and the function space is introduced. Also some critical point theorems are recalled, and we use them for the main results. In Sect. 3, the existence of at least one weak solution for problem (1.1) is proved. Finally, in Sect. 4 the existence of multiple weak solutions for problem (1.1) is proved.

2 Function spaces and critical point theorems

We suppose that $p \in C(\overline{\Omega})$ satisfies the following condition:

$$N < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < \infty. \tag{2.1}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}$$

with the Luxemburg norm

$$|u|_{p(x)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, where $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)}$$

holds true. Also, for $u \in L^{p(x)}(\partial\Omega)$, we put

$$|u|_{p(x),\partial} := \int_{\partial\Omega} |u|^{p(x)}\, d\sigma.$$

Following the authors of paper [21], for any $\kappa > 0$, we put

$$\kappa^{\check{r}} := \begin{cases} \kappa^{r^+} & \kappa < 1, \\ \kappa^{r^-} & \kappa \geq 1; \end{cases}$$

and

$$\kappa^{\hat{r}} := \begin{cases} \kappa^{r^-} & \kappa < 1, \\ \kappa^{r^+} & \kappa \geq 1; \end{cases}$$

for $r \in C_+(\Omega)$. The following proposition is well known in Lebesgue spaces with variational exponent (for instance, see [15, Proposition 2.7]).

Proposition 2.1 *For each $u \in L^{p(x)}(\Omega)$, we have*

$$|u|_{p(x)}^{\check{p}} \leq \int_\Omega \left|u(x)\right|^{p(x)} dx \leq |u|_{p(x)}^{\hat{p}}.$$

We denote the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

As pointed out in [10, 16], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$, and since $p^- > N$, $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, $W^{1,p(x)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. So, in particular, there exists a positive constant $m > 0$ such that

$$\|u\|_{C^0(\overline{\Omega})} \leq m\|u\|_{p(x)}$$

for each $u \in W^{1,p(x)}(\Omega)$. When Ω is convex, an explicit upper bound for the constant m (see [8]) is as follows:

$$m \leq 2^{\frac{p^- -1}{p^-}} \max\left\{\left(\frac{1}{\|c\|_1}\right)^{\frac{1}{p^-}}, \frac{d}{N^{\frac{1}{p^-}}}\left(\frac{p^- - 1}{p^- - N}|\Omega|\right)^{\frac{p^- -1}{p^-}} \frac{\|c\|_\infty}{\|c\|_1}\right\}\left(1 + |\Omega|\right),$$

where $d := \operatorname{diam}(\Omega)$, $|\Omega|$ is the Lebesgue measure of Ω,

$$\|c\|_1 := \int_\Omega c(x)\, dx \quad \text{and} \quad \|c\|_\infty := \sup_{x\in\Omega} c(x).$$

It is well known that, in view of (2.1), both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable reflexive and uniformly convex Banach spaces [10].

Remark 2.1 For $u \in W^{1,p(x)}(\Omega)$, there exist $k, K > 0$ such that

$$k\|u\|_{p(x)}^{\check{p}} \leq \int_\Omega \left(|\nabla u|^{p(x)} + c(x)|u|^{p(x)}\right) dx \leq K\|u\|_{p(x)}^{\hat{p}}.$$

Proof Since $\operatorname{ess\,inf}_\Omega c > 0$, so there exists $0 < \delta < 1$ such that $\delta < c(x)$. Using Proposition 2.1 and the hypothesis $c \in L^\infty(\Omega)$, we gain

$$\delta|u|_{p(x)}^{\check{p}} \leq \int_\Omega c(x)\left|u(x)\right|^{p(x)} dx \leq \|c\|_\infty |u|_{p(x)}^{\hat{p}}$$

and

$$\delta|\nabla u|_{p(x)}^{\check{p}} \leq |\nabla u|_{p(x)}^{\check{p}} \leq \int_\Omega \left|\nabla u(x)\right|^{p(x)} dx \leq |\nabla u|_{p(x)}^{\hat{p}}.$$

Bearing in mind the following elementary inequality: for all $q > 0$, there exists $C_q > 0$ such that

$$|a + b|^q \leq C_q\left(|a|^q + |b|^q\right)$$

for all $a, b \in \mathbb{R}$, we deduce

$$\frac{\delta}{C_{\check{p}}}\|u\|_{p(x)}^{\check{p}} \leq \int_\Omega \left(|\nabla u|^{p(x)} + c(x)|u|^{p(x)}\right) dx \leq \left(1 + \|c\|_\infty\right)\|u\|_{p(x)}^{\hat{p}}.$$

It is enough to put $k = \frac{\delta}{C_{\check{p}}}, K = 1 + \|c\|_\infty$. □

Now, define $F(x,t) := \int_0^t f(x,s)\,ds$. The growth condition $(F0)$ gives the following estimate:

$$\begin{aligned}
\left|\int_\Omega F(x,u)\,dx\right| &\leq \int_\Omega \left|F(x,u)\right| dx \\
&\leq \int_\Omega \left(\int_0^u \left|f(x,s)\right| ds\right) dx \\
&\leq \frac{a}{\gamma^-}\int_\Omega |u|^{\gamma(x)}\,dx \\
&\leq \frac{a}{\gamma^-}\|u\|_{C^0(\overline{\Omega})}^{\hat{\gamma}}|\Omega| \\
&\leq M\|u\|_{p(x)}^{\hat{\gamma}},
\end{aligned} \tag{2.2}$$

where $M = \frac{a}{\gamma^-}m^{\hat{\gamma}}|\Omega|$.

The global Ambrosetti–Rabinowitz condition (AR) for $g : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is as follows: There are constants $\mu > p^+, R > 0$ such that

$$0 < \mu G(x,s) \leq s g(x,s)$$

for all $x \in \partial\Omega$ and $|s| > R$, where $G(x,t) := \int_0^t g(x,s)\,ds$.

Definition 2.1 Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I := \Phi - \Psi$ is said to verify the Palais–Smale condition cut of upper at r (in short $(P.S.)^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}} \in X$ such that

- $I(u_n)$ is bounded;
- $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$;
- $\Phi(u_n) < r$ for each $n \in \mathbb{N}$;

has a convergent subsequence.

The following is one of the main tools of the next section.

Theorem 2.1 ([6]) *Let X be a real Banach space and $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{x\in X} \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist positive constants $r \in \mathbb{R}$ and $\overline{x} \in X$ with $0 < \Phi(\overline{x}) < r$ such that*

$$\frac{\sup_{x\in\Phi^{-1}(]-\infty,r[)} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}, \tag{2.3}$$

and for each $\lambda \in \Lambda :=]\frac{\Phi(x)}{\Psi(x)}, \frac{r}{\sup_{x\in\Phi^{-1}(]-\infty,r[)} \Psi(x)}[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition, then for each $\lambda \in \Lambda$ there is $x_\lambda \in \Phi^{-1}(]0, r[)$ such that $I_\lambda(x_\lambda) \le I_\lambda(x)$ for all $x \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_\lambda) = 0$.

Another tool is the following abstract result.

Theorem 2.2 ([5]) *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{x\in\Phi^{-1}(]-\infty,r[)} \Psi(x)}\right[,$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the Palais–Smale condition and it is unbounded from below. Then, for each

$$\lambda \in \left]0, \frac{r}{\sup_{x\in\Phi^{-1}(]-\infty,r[)} \Psi(x)}\right[,$$

the functional I_λ admits two distinct critical points.

Finally, we recall the following tool, which is in a convenient form.

Theorem 2.3 ([7]) *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable whose Gâteaux derivative is compact such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

(i) $\frac{\sup_{\Phi(x)<r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;
(ii) *for each* $\lambda \in \Lambda_r :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)<r}\Psi(x)}[$, *the functional* $I_\lambda := \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

3 Existence of weak solutions

In this section we deal with the existence of one weak solution for problem (1.1). In fact, we prove the first result of the paper, Theorem 1.1, as follows.

Proof We apply Theorem 2.1. To this end, for each $u \in W^{1,p(x)}(\Omega)$, let the functionals

$$\Phi, \Psi : W^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$$

be defined by

$$\Phi(u) := \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + c(x)|u|^{p(x)}\right) dx - \int_\Omega F(x,u)\, dx$$

and

$$\Psi(u) := \int_{\partial\Omega} G(x,u)\, d\sigma.$$

Now, set

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$$

for $u \in W^{1,p(x)}(\Omega)$. So, weak solutions of (1.1) are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. Moreover, Φ is sequentially weakly lower semicontinuous and its inverse derivative is continuous (since it is a continuous convex functional). From condition $(F1)$ it is clear that $F(x,u) \le 0$, and thanks to Remark 2.1 and inequality (2.2), one has

$$\begin{aligned}
\frac{k}{p^+}\|u\|_{p(x)}^{\check{p}} \le \Phi(u)& \\
&= \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + c(x)|u|^{p(x)}\right) dx - \int_\Omega F(x,u)\, dx \\
&\le \frac{K}{p^-}\|u\|_{p(x)}^{\hat{p}} + M\|u\|_{p(x)}^{\hat{\gamma}} \\
&< \left(\frac{K+1}{p^-} + M\right)\|u\|_{p(x)}^{\hat{p}}.
\end{aligned}$$

Also, by standard arguments, we have that Φ is Gâteaux differentiable, and its Gâteaux derivative at the point $u \in W^{1,p(x)}(\Omega)$ is the functional $\Phi'(u)$ given by

$$\Phi'(u)v = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + c(x)|u|^{p(x)-2}uv\right) dx - \int_\Omega f(x,u)v\, dx$$

for every $v \in W^{1,p(x)}(\Omega)$. On the other hand, the functional Ψ is well defined, continuously Gâteaux differentiable with compact derivative, whose Gâteaux derivative at the point $u \in$

$W^{1,p(x)}(\Omega)$ is

$$\Psi'(u)v = \int_{\partial\Omega} g\big(x, u(x)\big)v(x)\,dx$$

for each $v \in W^{1,p(x)}(\Omega)$ [3]. The functional I_λ satisfies the $PS^{[r]}$-condition for all $r \in \mathbb{R}$. We will verify the condition of Theorem 2.1. Let w be a function defined by $w(x) := d$ for all $x \in \Omega$ and r with

$$r := \left(\frac{K+1}{p^-} + M\right) d^{\hat{p}}.$$

So,

$$\begin{aligned} 0 < \frac{k}{p^+} d^{\check{p}} \leq \Phi(w) &= \int_\Omega \frac{1}{p(x)} c(x)\, d^{p(x)}\,dx - \int_\Omega F(x,d)\,dx \\ &< \left(\frac{K+1}{p^-} + M\right) d^{\hat{p}} = r. \end{aligned}$$

If $u \in \Phi^{-1}(]-\infty, r[)$, we have $\|u\|_{p(x)} \leq \xi = (\frac{p^+}{k}(\frac{K+1}{p^-} + M)\, d^{\hat{p}})^{\frac{1}{\check{p}}}$, then

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \leq \int_{\partial\Omega} \max_{|t|\leq\xi} G(x,t)\,d\sigma,$$

and from boundedness Φ, one has

$$\frac{\Psi(w)}{\Phi(w)} > \frac{\int_{\partial\Omega} G(x,d)\,d\sigma}{(\frac{K+1}{p^-} + M)\, d^{\check{p}}}.$$

Therefore, the assumption condition of Theorem 2.1 is verified. So, for each

$$\lambda \in \Lambda \subseteq \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional I_λ has at least one nonzero critical point, which is the weak solution of problem (1.1). □

4 Multiplicity of weak solutions

In this section, we present enough conditions for having multiple solutions to problem (1.1).

Theorem 4.1 *Let $f, g : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ be Carathéodory functions such that f satisfies* $(F0)$, $(F1)$ *and g holds in the* (AR) *condition. Then, for each*

$$\lambda \in \left]0, \frac{r}{\int_{\partial\Omega} \max_{|t|\leq\xi} G(x,t)\,d\sigma}\right[,$$

where ξ is as in Theorem 1.1, problem (1.1) admits at least two distinct weak solutions.

Proof We apply Theorem 2.2. According to the (AR) condition, there exist $\mu > p^+$ and $R > 0$ such that, for all $x \in \partial\Omega$ and $|s| > R$,

$$0 < \mu G(x,s) \leq sg(x,s).$$

So, there exists $\alpha > 0$ such that

$$G(x,u) \geq \alpha |u|^{\mu}$$

for all $x \in \Omega$ and $|s| > R$. Let the functionals $\Phi, \Psi : W^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$ be defined as in the proof of Theorem 1.1. We show that I_λ is unbounded from below. Applying Remark 2.1, one has

$$\begin{aligned} I_\lambda(tu) &= (\Phi - \lambda\Psi)(tu) \\ &\leq K\frac{t^{p^+}}{p^-}\|u\|_{p(x)} - \int_\Omega F(x,tu)\,dx - \lambda\int_{\partial\Omega} G(x,tu)\,d\sigma \\ &\leq K\frac{t^{p^+}}{p^-}\|u\|_{p(x)} + Mt^{\hat{\gamma}}\|u\|_{p(x)}^{\hat{\gamma}} - \lambda\alpha t^{\mu}\int_{\partial\Omega} |u|^{\mu}\,d\sigma \end{aligned}$$

for $t > 1$. Since $\mu > p^+ \geq \hat{\gamma}$, for large t, this condition guarantees that I_λ is unbounded from below. By standard computation, the functional $I_\lambda = \Phi - \lambda\Psi$ verifies the Palais–Smale compactness condition, and so all hypotheses of Theorem 2.2 are verified. Therefore, for each

$$\lambda \in \left]0, \frac{(\frac{K+1}{p^-} + M)\,d^{\hat{p}}}{\int_{\partial\Omega} \max_{|t|\leq\xi} G(x,t)\,d\sigma}\right[,$$

I_λ admits at least two distinct critical points that are weak solutions of problem (1.1). □

The following gives suitable conditions for the existence of at least three weak solutions.

Theorem 4.2 *Let $f, g : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ be Carathéodory functions satisfying* $(F0)$, $(F1)$ *and* $(G0)$, *respectively. Assume that there exists $d > 0$ such that assumption* (1.2) *in Theorem* 1.1 *holds. Then, for each $\lambda \in \Lambda$, where Λ is given by* (1.3), *problem* (1.1) *has at least three weak solutions.*

Proof Our goal is to apply Theorem 2.3. The functionals Φ and Ψ defined in the proof of Theorem 1.1 satisfy all regularity assumptions requested in Theorem 2.3. So, our aim is to verify (i) and (ii). Put $r = \frac{k}{p^+} d^{\check{p}}$ and define $w(x) := d$ for all $x \in \overline{\Omega}$, and let us recall that $F(x,u) \leq 0$, so

$$\begin{aligned} \Phi(w) &\geq \frac{k}{p^+} d^{\check{p}} - \int_\Omega F(x,u)\,dx \\ &\geq \frac{k}{p^+} d^{\check{p}} = r > 0. \end{aligned}$$

Therefore, assumption (i) of Theorem 2.3 is satisfied. We prove that the functional I_λ is coercive for all $\lambda > 0$. We know that [9, Theorem 2.1]

$$W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{\beta(x)}(\partial\Omega),$$

so, for each $u \in W^{1,p(x)}(\Omega)$, there exists some constant $\theta > 0$ such that

$$|u|_{\beta(x),\partial} \le \theta \|u\|_{p(x)}.$$

Now, using Hölder's inequality and condition $(G0)$, for all $u \in W^{1,p(x)}(\Omega)$, one has

$$\begin{aligned}\Psi(u) &= \int_{\partial\Omega} G\big(x, u(x)\big)\, d\sigma = \int_{\partial\Omega} \left(\int_0^{u(x)} g(x,t) \right) d\sigma \\ &\le 2|b_1|_{\frac{\beta(x)}{\beta(x)-1},\partial} |u|_{\beta(x),\partial} + \frac{b_2}{\beta^-} \int_{\partial\Omega} \big|u(x)\big|^{\beta(x)}\, d\sigma \\ &\le 2\theta |b_1|_{\frac{\beta(x)}{\beta(x)-1},\partial} \|u\|_{p(x)} + \frac{b_2\theta^{\hat{\beta}}}{\beta^-} \|u\|_{p(x)}^{\hat{\beta}}.\end{aligned}$$

Using Remark 2.1 and condition $(F1)$, for every $\lambda > 0$, we deduce that

$$I_\lambda(u) \ge \frac{k}{p^+} \|u\|_{p(x)}^{\check{p}} - 2\lambda\theta |b_1|_{\frac{\beta(x)}{\beta(x)-1},\partial} \|u\|_{p(x)} - \lambda \frac{b_2\theta^{\hat{\beta}}}{\beta^-} \|u\|_{p(x)}^{\hat{\beta}},$$

since $\check{p} > \hat{\beta} > 1$, the functional I_λ is coercive. Then also condition (ii) holds. So, for each $\lambda > 0$, the functional I_λ admits at least three distinct critical points that are weak solutions of problem (1.1). □

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