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Research on singular Sturm–Liouville spectral problems with a weighted function

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Abstract

As early as 1910, Weyl gave a classification of the singular Sturm–Liouville equation, and divided it into the Limit Point Case and the Limit Circle Case at infinity. This led to the study of singular Sturm–Liouville spectrum theory. With the development of applications, the importance of singular Sturm–Liouville problems with a weighted function becomes more and more significant. This paper focuses on the study of singular Sturm–Liouville problems with a weighted function. Finally, an example of singular Sturm–Liouville problems with a weighted function is given.

Keywords: Singular Sturm–Liouville problems; Limiting point and limiting circle; Weyl function; Weighted function

1 Introduction

The Sturm–Liouville problems originated in the early 19th century by solving the heat-conduction equation in partial differential equations, obtained by the method of separation of variables, and later found a wide range of applications in mathematics and physics. For example, the first eigenvalue of the regular Sturm–Liouville problems represents the first energy level in quantum mechanics and quantum chemistry (cf. [2, 10, 11, 14]), and has been applied to the calculation of electron-cloud density, which is a powerful tool for understanding and explaining quantum phenomena. Since the mathematical models of many important issues in practical problems are defined on infinite intervals or on finite intervals with singularities at the endpoints of the coefficient functions, the singularity problem has been of interest to scholars of mathematics and physics. As the spectrum of the singular problem becomes more complicated, not only the pure point spectrum appears in the regular case, but also the absolutely continuous spectrum and the singular continuous spectrum. This leads to the fact that the spectral decomposition theorem for the regular case is no longer applicable (cf. [3, 20, 25]), and therefore more research on the spectral aspects of the singular problem is needed.

As early as 1910, Weyl gave a classification of the singular Sturm–Liouville equation by using the circle-set method, which divides it into the limiting point type and the limiting circle type at the infinity point. This led to the study of the singular Sturm–Liouville spectral theory. In 1937, Saks [15] proved De la Vallée Poussin's theorem using the Lesbgue decomposition of measures. In 1943, Loomis [9] proved Fatou's Lemma using the Poisson–

Stieltjes integral. On the basis of these measure theories the Lebesgue decomposition of the spectral measures was performed to complete the classification of the spectrum of differential operators. In 1975, Levitan and Sargsjan [8] used the Lesbgue decomposition of the spectral measure to classify the spectrum into: the absolute continuous spectrum, the singular continuous spectrum, and the pure point spectrum. Regarding the absolute continuous spectrum, in 1957 Aronsajn [1] used the $m(\lambda)$ function to prove that the absolute continuous spectrum of the Sturm–Liouville spectral problem is invariant under a rank-one perturbation. In 1986, Simon and Wolff [18] gave equivalence conditions for the Borel transform of the measure and the spectral decomposition. In 1989, Simon and Spencer [17] proved that the potential function is a High Barrier function, i.e., the potential function tends to infinity, when the corresponding Sturm–Liouville differential operator has no absolutely continuous spectrum. The singular continuous spectrum is more complicated, in 1995, Simon [16] proved the existence of a purely singular continuous spectrum for the general operator, that is, there is an interval in which there is no point spectrum and no absolute continuous spectrum, but only a singular continuous spectrum.

With the deeper and deeper study of practical problems, the importance of singular Sturm–Liouville problems (cf. [4–7, 19]) with a weighted function becomes more and more significant as the solution space expands from the L^2 space to the L^2_w space with a weighted function (cf. [12, 13, 22–24]) and has more practical applications. This paper focuses on refining the definition of spectral measures for singular Sturm–Liouville problems with a weighted function. This paper finds several differences for the case of singular Sturm–Liouville problems with a weighted function based on the analysis of the spectral problem of general singular Sturm–Liouville problems. Finally, an example of singular Sturm–Liouville problems with a weighted function is given, and its expansion theorem and the expression of the support set of spectral measure are proved using the method of this paper.

Following this section, for extending the regular Sturm–Liouville boundary value problem to the singular problem, some preliminaries will be given in Sect. 2. In Sect. 3, Wely–Titchmarsh functions are introduced and the classification of the Limit Circle Case and the Limit Point Case is derived. In Sect. 4, some criteria of the Limit Point Case will be obtained. In Sect. 5, an example of the singular Sturm–Liouville problems with a weighted function is studied.

2 Preliminary

In this paper, we will extend the regular Sturm–Liouville boundary value problem to the singular problem. Consider the regular Sturm–Liouville problem with separable boundary conditions

$$\tau y := \frac{1}{w}\left(-\left(py'\right)' + qy\right) = \lambda y, \quad y = y(x), x \in (0,b), b < \infty, \tag{1}$$

$$\cos\alpha y(0,\lambda) - \sin\alpha py'(0,\lambda) = 0, \tag{2}$$

$$\cos\beta y(b,\lambda) - \sin\beta py'(b,\lambda) = 0, \tag{3}$$

where $\alpha, \beta \in [0,\pi)$, $1/p, q, w \in L^1_{\text{loc}}[0,\infty)$, $p, w > 0$ a. e. and

$$\int_0^\infty 1/p + |q| + w = \infty.$$

Take $y(x,\lambda)$ as satisfying the Cauchy problem,

$$\frac{1}{w}\big(-(py')'+qy\big)=\lambda y,\quad y=y(x),$$
$$y(0,\lambda)=\sin\alpha,\qquad py'(0,\lambda)=\cos\alpha,$$

which is the solution of equation (1). Denote $\lambda_{n,b}$ as the nth eigenvalue of the regular problem (1), (2), and (3). Then, the corresponding eigenfunction is $y_{n,b}(x):=y(x,\lambda_{n,b})$, which satisfies the right boundary condition (3), $\cos\beta y_{n,b}(b)-\sin\beta py'_{n,b}(b)=0$. Denote

$$a_{n,b}^2:=\int_0^b y^2(x,\lambda_{n,b})w(x)\,\mathrm{d}x.$$

By the Parseval Identity, for any $f\in L_w^2(0,b)$, we have

$$\int_0^b f^2(x)w(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{a_{n,b}^2}\left(\int_0^b f(x)y_{n,b}(x)w(x)\,\mathrm{d}x\right)^2,$$

where

$$L_w^2(0,b):=\left\{f:\int_0^b f^2(x)w(x)\,\mathrm{d}x<\infty\right\}.$$

Now, we introduce the monotone nondecreasing function or measure $\rho_b(\lambda)$,

$$\rho_b(\lambda):=\begin{cases}-\sum_{\lambda<\lambda_{n,b}\le 0}\frac{1}{a_{n,b}^2}, & \text{as } \lambda\le 0,\\ \sum_{0<\lambda_{n,b}\le\lambda}\frac{1}{a_{n,b}^2}, & \text{as } \lambda>0.\end{cases}$$

By definition, $\rho_b(0)=0$. Then, the Parseval Identity can be rewritten as,

$$\int_0^b f^2(x)w(x)\,\mathrm{d}x=\int_{-\infty}^{\infty}F^2(\lambda)\,\mathrm{d}\rho_b(\lambda),\tag{4}$$

where

$$F(\lambda)=\int_0^b f(x)y(x,\lambda)w(x)\,\mathrm{d}x.$$

The above equation is called the generalized Fourier transform of $f(x)$. In the following, let $b\to\infty$, and we will prove that the Parseval Identity still holds.

Lemma 2.1 *For any positive integer N, there exists a positive constant $A(N,w)$, such that*

$$\rho_b(N)-\rho(-N)=\sum_{-N<\lambda_{n,b}\le N}\frac{1}{a_{n,b}^2}<A(N,w),\tag{5}$$

where $A(N,w)$ only depends on N and w, and is independent of b.

Proof Let

$$c_h := \int_0^h w(x)\,\mathrm{d}x, \quad h > 0.$$

Since $w(x) \in L^1_{\mathrm{loc}}[0,\infty)$ and $w(x) > 0$, a.e., it follows that $c_h > 0$, for any $h > 0$.

In the case $\sin\alpha \neq 0$, there exists sufficiently small positive numbers h, such that $|y(t,\lambda)| > |\sin\alpha|/\sqrt{2}$, for any $t \in [0,h]$ and $\lambda \in [-N,N]$. This fact leads to

$$\left(\frac{1}{h}\int_0^h y(x,\lambda)w(x)\,\mathrm{d}x\right)^2 > \frac{c_h^2}{2h^2}\sin^2\alpha. \tag{6}$$

Define function $f_h(x)$

$$f_h(x) := \begin{cases} 1/h & 0 \le x < h, \\ 0 & x > h. \end{cases}$$

Using (6) and the Parseval Identity (4), we can obtain that

$$\begin{aligned} \frac{c_h}{h^2} &= \int_0^h f_h^2(x)w(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\left(\frac{1}{h}\int_0^h y(x,\lambda)w(x)\,\mathrm{d}x\right)^2 \mathrm{d}\rho_b(\lambda) \\ &\ge \int_{-N}^{N}\left(\frac{1}{h}\int_0^h y(x,\lambda)w(x)\,\mathrm{d}x\right)^2 \mathrm{d}\rho_b(\lambda) \\ &> \frac{c_h^2}{2h^2}\sin^2\alpha\int_{-N}^{N}\mathrm{d}\rho_b(\lambda) \\ &= \frac{c_h^2}{2h^2}\sin^2\alpha\big[\rho_b(N) - \rho_b(-N)\big]. \end{aligned} \tag{7}$$

Hence,

$$\big[\rho_b(N) - \rho_b(-N)\big] < \frac{2}{c_h\sin^2\alpha} =: A(N,w).$$

Note that c_h only depends on N and w, and is independent of b. Thus, (5) has been proved.

In the case $\sin\alpha = 0$, $|y'(0,\lambda)| = |\cos\alpha| = 1$. Hence, there exists a sufficiently small number $h > 0$, such that $y(t,\lambda) > t/\sqrt{2}$, for any $t \in [h/2,h]$ and $\lambda \in [-N,N]$. We now have

$$\begin{aligned} \left(\frac{1}{h^2}\int_0^h y(x,\lambda)w(x)\,\mathrm{d}x\right)^2 &> \left(\frac{1}{h^2}\int_{h/2}^h \frac{h}{2\sqrt{2}}w(x)\,\mathrm{d}x\right)^2 \\ &= \frac{1}{8h^2}(c_h - c_{h/2})^2. \end{aligned} \tag{8}$$

In this case, define function $f_h(x)$ as

$$f_h(x) = \begin{cases} \frac{1}{h^2} & 0 \le x < h, \\ 0 & x > h. \end{cases}$$

Similar to the case where $\sin\alpha \neq 0$, we can obtain that

$$\begin{aligned}\frac{c_h}{h^2} &= \int_0^h f_h^2(x)w(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\left(\frac{1}{h^2}\int_0^h y(x,\lambda)w(x)\,\mathrm{d}x\right)^2 \mathrm{d}\rho_b(\lambda) \\ &\geq \int_{-N}^{N}\left(\frac{1}{h^2}\int_0^h y(x,\lambda)w(x)\,\mathrm{d}x\right)^2 \mathrm{d}\rho_b(\lambda) \\ &> \frac{1}{8h^2}(c_h - c_{h/2})^2\int_{-N}^{N}\mathrm{d}\rho_b(\lambda) \\ &= \frac{1}{8h^2}(c_h - c_{h/2})^2\big[\rho_b(N) - \rho_b(-N)\big].\end{aligned}$$

Hence,

$$\big[\rho_b(N) - \rho_b(-N)\big] < \frac{8c_h}{(c_h - c_{h/2})^2} =: A(N,w).$$

Similarly, we know that c_h only depends on N and w, and is independent of b. The proposition has been proved. □

In the following proof, Helly's selection theorem is needed.

Lemma 2.2 (Helly's Selection Theorem) *Consider a nondecreasing function sequence* $\{\rho_n(\lambda), \lambda \in (-\infty,\infty) : n = 1,2,\ldots\}$. *If in any bounded interval* $[M,N]$, $\{\rho_n(\lambda), \lambda \in [M,N] : n = 1,2,\ldots\}$ *are uniformly bounded, then there exists a subsequence* $\{\rho_{n_k}(\lambda), \lambda \in (-\infty,\infty), k = 1,2,\ldots\}$ *and a nondecreasing function* $\rho(\lambda)$, *such that*

$$\lim_{k\to\infty}\rho_{n_k}(\lambda) = \rho(\lambda), \quad \text{for any } -\infty < \lambda < \infty.$$

By Helly's selection theorem, we can use the regular Sturm–Liouville problem (1) to approximate the singular problem and study the properties of the spectrum.

Theorem 2.3 *There exists a nondecreasing function* $\rho(\lambda)$, $\lambda \in (-\infty,\infty)$, *such that for any* $f(x) \in L_w^2(0,\infty)$,

$$\int_0^{\infty} f^2(x)w(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} F^2(\lambda)\,\mathrm{d}\rho(\lambda),$$

where $F(\lambda)$ *satisfies*

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\big\{F(\lambda) - F_m(\lambda)\big\}^2\,\mathrm{d}\rho(\lambda) = 0,$$

and $F_m(\lambda) := \int_0^m f(x)y(x,\lambda)w(x)\,\mathrm{d}x$.

Proof Set

$$\mathcal{D} := \big\{f \in L_w^2(0,\infty) : f, pf' \in \mathrm{AC}[0,\infty), \text{and } \tau f \in L_w^2(0,\infty)\big\},$$

where τ is a formal differential operator in (1), and $AC[0,\infty)$ denote the absolute continuous function on $(0,\infty)$.

Suppose $f_m \in \mathcal{D}$ satisfying the left boundary condition (2) and is a compactly supported function, i.e.,

$$f_m(x) = 0, \quad x \in (0,m)^c.$$

Then, the Parseval Identity tells us,

$$\int_0^m f_m^2(x)w(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} F_m^2(\lambda)\,\mathrm{d}\rho_b(\lambda), \tag{9}$$

where F_m is the Fourier transform of f_m, i.e.,

$$F_m(\lambda) = \int_0^m f_m(x)y(x,\lambda)w(x)\,\mathrm{d}x = \int_0^\infty f_m(x)y(x,\lambda)w(x)\,\mathrm{d}x.$$

Using the Green formula, we can obtain

$$\begin{aligned}
&\int_0^b f_m(x)\bigl\{\bigl(py'\bigr)'(x,\lambda) - q(x)y(x,\lambda)\bigr\}\,\mathrm{d}x \\
&\quad = \int_0^b \bigl\{\bigl(pf_m'\bigr)'(x) - q(x)f_m(x)\bigr\}y(x,\lambda)\,\mathrm{d}x.
\end{aligned} \tag{10}$$

Substituting (10) into $F_m(\lambda)$, we have

$$\begin{aligned}
F_m(\lambda) &= -\frac{1}{\lambda}\int_0^b f_m(x)\frac{1}{w(x)}\bigl\{\bigl(py'\bigr)'(x,\lambda) - q(x)y(x,\lambda)\bigr\}w(x)\,\mathrm{d}x \\
&= -\frac{1}{\lambda}\int_0^b \bigl\{\bigl(pf_m'\bigr)'(x) - q(x)f_m(x)\bigr\}y(x,\lambda)\,\mathrm{d}x.
\end{aligned} \tag{11}$$

Using the Green formula again, we obtain

$$\begin{aligned}
&\int_{|\lambda|>N} F_m^2(\lambda)\,\mathrm{d}\rho_b(\lambda) \\
&\quad \le \frac{1}{N^2}\int_{|\lambda|>N}\left\{\int_0^b f_m(x)\bigl[\bigl(py'\bigr)'(x,\lambda) - q(x)y(x,\lambda)\bigr]\,\mathrm{d}x\right\}^2\mathrm{d}\rho_b(\lambda) \\
&\quad < \frac{1}{N^2}\int_{-\infty}^{\infty}\left\{\int_0^b f_m(x)\bigl[\bigl(py'\bigr)'(x,\lambda) - q(x)y(x,\lambda)\bigr]\,\mathrm{d}x\right\}^2\mathrm{d}\rho_b(\lambda) \\
&\quad = \frac{1}{N^2}\int_{-\infty}^{\infty}\left\{\int_0^b \frac{1}{w(x)}\bigl\{\bigl(pf_m'\bigr)'(x) - q(x)f_m(x)\bigr\}y(x,\lambda)w(x)\,\mathrm{d}x\right\}^2\mathrm{d}\rho_b(\lambda) \\
&\quad = \frac{1}{N^2}\int_0^m \left\{\frac{(pf_m')'(x) - q(x)f_m(x)}{w(x)}\right\}^2 w(x)\,\mathrm{d}x.
\end{aligned}$$

By (9), we have

$$\left|\int_0^m f_m^2(x)w(x)\,\mathrm{d}x-\int_{-N}^N F_m^2(\lambda)\,\mathrm{d}\rho_b(\lambda)\right| < \frac{1}{N^2}\int_0^m\left\{\frac{(pf_m')'(x)-q(x)f_m(x)}{w(x)}\right\}^2 w(x)\,\mathrm{d}x. \tag{12}$$

By Lemma 2.1, we know that the monotone function family $\{\rho_b(\lambda),\lambda\in(-N,N)\}$ is bounded by $A(N,w)$, which only depends on N and w, and is independent of b. Then, by Helly's Selection Theorem Lemma 2.2, there exists a subsequence b_k, such that ρ_{b_k} weakly convergent to measure ρ. Hence, for any $F\in L^2(-N,N)$, we have

$$\lim_{k\to\infty}\int_{-N}^N F^2(\lambda)\,\mathrm{d}\{\rho_{b_k}(\lambda)-\rho(\lambda)\}=0. \tag{13}$$

By (13) and (12), taking the limit, we can obtain

$$\left|\int_0^m f_m^2(x)w(x)\,\mathrm{d}x-\int_{-N}^N F_m^2(\lambda)\,\mathrm{d}\rho(\lambda)\right| < \frac{1}{N^2}\int_0^m\left\{\frac{(pf_m')'(x)-q(x)f_m(x)}{w(x)}\right\}^2 w(x)\,\mathrm{d}x.$$

Moreover, let $N\to\infty$, we can obtain

$$\int_0^m f_m^2(x)w(x)\,\mathrm{d}x=\int_{-\infty}^\infty F_m^2(\lambda)\,\mathrm{d}\rho(\lambda).$$

So far, we have proved the Parseval Identity when f is a function that is compactly supported. In the following, we will prove the general case.

For any $f\in L_w^2(0,\infty)$, there exist a sequence of compactly supported functions $\{f_m(x)\}$, such that

$$\lim_{m\to\infty}\int_0^\infty\{f(x)-f_m(x)\}^2 w(x)\,\mathrm{d}x=0.$$

Hence,

$$\int_0^\infty\{f_{m_1}(x)-f_{m_2}(x)\}^2 w(x)\,\mathrm{d}x\to 0,\quad \text{as } m_1,m_2\to\infty.$$

Using the Parseval Identity of compact support functions, the Fourier transform of sequence f_m is a Cauchy sequence in $L^2_{\mathrm{d}\rho}(-\infty,\infty)$, i.e.,

$$\int_{-\infty}^\infty\{F_{m_1}(\lambda)-F_{m_2}(\lambda)\}^2\,\mathrm{d}\rho(\lambda)=\int_0^\infty\{f_{m_1}(x)-f_{m_2}(x)\}^2 w(x)\,\mathrm{d}x\to 0.$$

By the Completeness of $L^2_{\mathrm{d}\rho}(-\infty,\infty)$, there exists a function $F(\lambda)\in L^2_{\mathrm{d}\rho}(-\infty,\infty)$ that satisfies

$$\int_{-\infty}^\infty\{F_m(\lambda)-F(\lambda)\}^2\,\mathrm{d}\rho(\lambda)\to 0,$$

and the Parseval Identity

$$\int_0^\infty f^2(x)w(x)\,\mathrm{d}x = \lim_{m\to\infty}\int_0^\infty f_m(x)^2 w(x)\,\mathrm{d}x$$
$$= \lim_{m\to\infty}\int_{-\infty}^\infty F_m^2(\lambda)\,\mathrm{d}\rho(\lambda) = \int_{-\infty}^\infty F^2(\lambda)\,\mathrm{d}\rho(\lambda).$$

The proposition has been proved □

In Theorem 2.3, $F = \int_0^\infty f(x)y(x,\lambda)w(x)\,\mathrm{d}x$ is called the generalized Fourier transform of f. For another $g \in L_w^2(0,\infty)$, and its generalized Fourier transform G, we have

$$\int_0^\infty \{f(x)+g(x)\}^2 w(x)\,\mathrm{d}x = \int_{-\infty}^\infty \{F(\lambda)+G(\lambda)\}^2\,\mathrm{d}\rho(\lambda),$$

and

$$\int_0^\infty \{f(x)-g(x)\}^2 w(x)\,\mathrm{d}x = \int_{-\infty}^\infty \{F(\lambda)-G(\lambda)\}^2\,\mathrm{d}\rho(\lambda).$$

Subtracting the two formulas, we can obtain

$$\int_0^\infty f(x)g(x)w(x)\,\mathrm{d}x = \int_{-\infty}^\infty F(\lambda)G(\lambda)\,\mathrm{d}\rho(\lambda), \tag{14}$$

and this identity is called the generalized Parseval Identity.

3 Wely–Titchmarsh functions

Suppose $\varphi(\cdot,\lambda)$ is the solution of equation (1), i.e., satisfies

$$-\big(p\varphi(t,\lambda)'\big)' + q\varphi(t,\lambda) = \lambda w(t)\varphi(t,\lambda), \tag{15}$$

with the Cauchy condition

$$\varphi(0,\lambda) = \cos\alpha, \qquad p(0)\varphi'(0,\lambda) = -\sin\alpha,$$

where $\alpha \in [0,\pi)$. Similarly, let $\psi(\cdot,\lambda)$ be another solution of (1) with the Cauchy condition

$$\psi(0,\lambda) = \sin\alpha, \qquad p(0)\psi'(0,\lambda) = \cos\alpha,$$

therefore $\psi(\cdot,\lambda)$ satisfies the left boundary condition (2),

$$\cos\alpha\,\psi(0,\lambda) - \sin\alpha\, p(0)\psi'(0,\lambda) = 0.$$

By the differentiable dependence of solutions on parameters, we know that ψ and φ are both analytic functions of λ and their Wronskian satisfies

$$W\big(\varphi(\lambda),\psi(\lambda)\big)(0) = \begin{vmatrix} \varphi(\lambda)(0) & \overline{\psi(\lambda)(0)} \\ p\varphi'(\lambda)(0) & \overline{p\psi'(\lambda)(0)} \end{vmatrix} = 1.$$

All solutions of equation (1) except ψ can be expressed as

$$\chi(\cdot,\lambda) = \varphi(\cdot,\lambda) + m\psi(\cdot,\lambda), \quad m \in \mathbb{C}.$$

For any $b \in (0,\infty)$ and $\beta \in [0,\pi)$, let $m = m(b,\lambda,\beta)$, such that χ satisfies the right boundary condition (3),

$$\chi(b)\cos\beta + p\chi'(b)\sin\beta = 0.$$

We can obtain the expression of $m(\lambda,b)$,

$$m(\lambda,b,\beta) = -\frac{p\varphi'(b,\lambda) + \varphi(b,\lambda)\cot\beta}{p\psi'(b,\lambda) + \psi(b,\lambda)\cot\beta}, \tag{16}$$

which is a fractional linear mapping. Hence, as β goes through $(0,\pi)$, the graph of m forms a circle $C(\lambda,b)$. After calculation, we can obtain that the center of circle $C(\lambda,b)$ is

$$m_0(\lambda,b) = \frac{\overline{\psi(b,\lambda)}p\varphi'(b,\lambda) - \varphi(b,\lambda)p\overline{\psi'(b,\lambda)}}{\overline{\psi(b,\lambda)}p\psi'(b,\lambda) - \psi(b,\lambda)p\overline{\psi'(b,\lambda)}} = -\frac{W(\varphi(\lambda),\psi(\lambda))(b)}{W(\psi(\lambda),\psi(\lambda))(b)},$$

the radius of circle $C(\lambda,b)$ is

$$r(\lambda,b) = \frac{1}{|W(\psi(\lambda),\psi(\lambda))(b)|},$$

and the inside of circle $C(\lambda,b)$ is

$$m : \frac{W(\chi,\chi)}{W(\psi(\lambda),\psi(\lambda))} < 0.$$

Now, consider two solutions of equation (1), $f(x)$ and $g(x)$, and they are satisfied if $\tau f = \lambda f$ and $\tau g = \tilde{\lambda} g$. Then, we can obtain

$$\begin{aligned}
&(\lambda - \tilde{\lambda})\int_0^b f(x)g(x)w(x)\,\mathrm{d}x \\
&= \int_0^b \left\{\frac{f(x)(q(x)g(x) - (pg')'(x))}{w(x)} - \frac{g(x)(q(x)f(x) - (pf')'(x))}{w(x)}\right\} w(x)\,\mathrm{d}x \\
&= -\int_0^b \left\{f(x)\big(pg'\big)'(x) - g(x)\big(pf'\big)'(x)\right\}\mathrm{d}x \\
&= W(f,g)(0) - W(f,g)(b),
\end{aligned}$$

where

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ pf'(x) & pg'(x) \end{vmatrix}.$$

In particular, let $f = \psi$, $g = \bar{\psi}$ and $\tilde{\lambda} = \bar{\lambda}$, then we have

$$2\Im m\lambda \int_0^b \left|\psi(t,\lambda)\right|^2 w(t)\,\mathrm{d}t = iW\big(\psi(\lambda),\overline{\psi(\lambda)}\big)(0) - iW\big(\psi(\lambda),\overline{\psi(\lambda)}\big)(b), \tag{17}$$

and hence we can deduce that

$$W\big(\psi(\lambda),\psi(\lambda)\big)(b) = 2i\Im m\lambda \int_0^b \big|\psi(t,\lambda)\big|^2 w(t)\,\mathrm{d}t,$$
$$W(\chi,\chi)(0) = m(\lambda,b) - m(\bar{\lambda},b),$$

and

$$W(\chi,\chi)(b) = -2i\Im m m(\lambda,b) + 2i\Im m\lambda \int_0^b \big|\chi(t)\big|^2 w(t)\,\mathrm{d}t.$$

If $\Im m\lambda \neq 0$, we have $m \in C(\lambda,b)$ is equivalent to

$$\int_0^b \big|\chi(t)\big|^2 w(t)\,\mathrm{d}t = \frac{\Im m m}{\Im m\lambda}.$$

The inside of circle $C(\lambda,b)$ is

$$\int_0^b \big|\chi(t)\big|^2 w(t)\,\mathrm{d}t < \frac{\Im m m}{\Im m\lambda}, \tag{18}$$

and the radius is

$$r(\lambda,b) = \frac{1}{2\Im m\lambda \int_0^b |\psi(t,\lambda)|^2 w(t)\,\mathrm{d}t}. \tag{19}$$

Using these facts, we can obtain the following summaries. The circles satisfy that for any $0 < b_1 < b_2 < \infty$, $C(\lambda,b_2) \subset C(\lambda,b_1)$. Hence, for any $\Im m\lambda \neq 0$, $C(\lambda,b) \to$ a circle $C(\lambda)$ or a point $m(\lambda)$, as $b \to \infty$. Furthermore, if we set $\chi = \varphi(\lambda) + m\psi(\lambda)$, then we have that

$$\int_0^b \big|\chi(t)\big|^2 w(t)\,\mathrm{d}t < \frac{\Im m m}{\Im m\lambda}$$

and letting $\lambda \to \infty$ we obtain

$$\int_0^\infty \big|\chi(t)\big|^2 w(t)\,\mathrm{d}t < \frac{\Im m m}{\Im m\lambda}. \tag{20}$$

By (1), we know that there exists at least one L_w^2 solution at $+\infty$. As $C(\lambda,b) \to$ a circle $C(\lambda)$, (19) tells us that all solutions of (15) belong to L_w^2 at $+\infty$, and at this time, (15) is called the Limit Circle Case at $+\infty$. As $C(\lambda,b) \to$ a point $m(\lambda)$, there is only one linear independence solution of (15) belonging to L_w^2 at $+\infty$, and at this time, (15) is called the Limit Point Case at $+\infty$. In the next section, we will study some judgments about the two classifications at infinity.

4 A criterion of the limit point case

Now, we consider the formal differential operator

$$ly := -\big(py'\big)' + qy = \lambda wy, \quad y = y(x), x \in (0,\infty), \tag{21}$$

with the nature domain

$$\mathcal{D} := \left\{ y, py' \in L_w^2[0,\infty) : y, py' \in \mathrm{AC}[0,\infty) \text{ and } \frac{1}{w} ly \in L_w^2[0,\infty) \right\},$$

where $y \in \mathrm{AC}[0,\infty)$ means y is an absolute continuous function on $[0,\infty)$.

The formal differential l and τ in (1) satisfy the relationship $l = w\tau$. In this section, we will use the coefficient functions p, q, and w in equation (21) to describe whether the differential equation is the Limit Point Case or the Limit Circle Case (see [21]). In the following, some preliminaries will be given.

Lemma 4.1 *If there exists $\lambda_0 \in \mathbb{C}$, such that all solutions of $ly = \lambda_0 wy$ belong to $L_w^2[0,\infty)$, then for any complex number $\lambda \in \mathbb{C}$, all solutions of $ly = \lambda wy$ also belong to $L_w^2[0,\infty)$.*

Proof Suppose $\varphi_0, \psi_0 \in L_w^2[0,\infty)$ are two linearly independent solutions of $ly = \lambda_0 wy$, and satisfy,

$$pW(\varphi_0, \psi_0) = 1.$$

For any complex number $\lambda \in \mathbb{C}$, let $\chi(t,\lambda)$ be any solution of $ly = \lambda wy$. Then, $\chi(t,\lambda)$ is the solution of the differential equation

$$ly - \lambda_0 wy = (\lambda - \lambda_0) wy.$$

By the variation of constant formula, the above equation can be transformed into an integral equation,

$$\chi(t,\lambda) = A\varphi_0(t) + B\psi_0(t) + (\lambda - \lambda_0) \int_c^t w(\tau)\big[\varphi_0(t)\psi_0(\tau) - \varphi_0(\tau)\psi_0(t)\big]\chi(\tau,\lambda)\,\mathrm{d}\tau, \quad (22)$$

where A and B are constants. Set

$$\|\chi\|_c^t := \left(\int_c^t |\chi|^2 w \,\mathrm{d}t \right)^{\frac{1}{2}},$$

and

$$\gamma_c = \max\big\{ \|\varphi_0\|_c^\infty, \|\psi_0\|_c^\infty \big\}.$$

By $\varphi_0, \psi_0 \in L_w^2[0,\infty)$, we know that $\gamma_c \to 0$, as $c \to \infty$. Hence, there exists $N > 0$ large enough, such that for any $c > N$, we have

$$|\lambda - \lambda_0| \gamma_c^2 \le \frac{1}{4}.$$

Using the Schwarz inequality, for any $t \ge c > N \ge 0$, we have

$$\left| \int_c^t w(\tau)\big[\varphi_0(t)\psi_0(\tau) - \varphi_0(\tau)\psi_0(t)\big]\chi(\tau,\lambda)\,\mathrm{d}\tau \right| \le \gamma_c \big[|\varphi_0(t)| + |\psi_0(t)| \big] \|\chi\|_c^t.$$

Using the above two inequalities and (22) we can obtain the next estimate

$$\|\chi\|_c^t \leq (|A| + |B|)\gamma_c + 2|\lambda - \lambda_0|\gamma_c^2 \big||\chi|\big|_c^t \leq (|A| + |B|)\gamma_c + \frac{1}{2}\|\chi\|_c^t,$$

and hence

$$\|\chi\|_c^t \leq 2(|A| + |B|)\gamma_c.$$

The right-hand side of the inequality is independent of t, therefore let $t \to \infty$, which can give $\chi \in L_w^2[0,\infty)$. □

Corollary 4.2 *If there exists $\lambda_0 \in \mathbb{C}$ such that $ly = \lambda_0 wy$ has a nontrivial solution φ, i.e., $\varphi \not\equiv 0$, satisfying $\varphi \notin L_w^2[0,\infty)$, then, for any $\lambda \in \mathbb{C}, \Im m\lambda \neq 0$, the differential equation $ly = \lambda wy$ has only one $L_w^2[0,\infty)$-solution, which is in the linearly independent sense.*

Now, we can obtain a criterion of discriminating the Limit Point Case.

Theorem 4.3 *If $q(t) \geq 0$, and $w \notin L^1[C,\infty)$, for any $C > 0$, then, (21) is the Limit Point Case at infinity.*

Proof Let $\varphi(t)$ be the solution of the equation $ly = 0$, satisfying the Cauchy condition,

$$\varphi(0) = 0, \qquad p\varphi'(0) = 1.$$

Set

$$c = \inf\{t \in [0,\infty) | p\varphi'(t) = 0\}.$$

Then, $c > 0$, and $\varphi(t) > 0$ on $(0,c)$. For the identity equation:

$$\big(p(t)\varphi'(t)\big)' = q(t)\varphi(t)$$

integrate over $[0,c]$ on both sides, i.e.,

$$p(c)\varphi'(c) = 1 + \int_0^c q(t)\varphi(t)\,\mathrm{d}t > 0,$$

which obtains a contradiction. This shows that $\varphi(t)$, $p\varphi'(t)$ is constant positive on $[0,\infty)$, Also, by $w \notin L^1[0,\infty)$, it follows that $\varphi(t) \notin L_w^2[0,\infty)$. According to the above inference, we can see that (21) belongs to the limiting point type at the infinity point. □

In particular, when the weighted function $w \equiv 1$, $w \notin L^1[0,\infty)$, Therefore, it follows from the above theorem that in this case (21) is a limiting point type at the infinity point. However, if $w \in L^1[0,\infty)$, this theorem does not necessarily hold, see the following example.

Example 4.4 Considering the formal differential operator (21), with $p \equiv 1$, $q \equiv 0$, $w(x) = 1/(x+1)^3$,

$$-y'' = \lambda \frac{1}{(x+1)^3} y, \quad \text{on } [0,\infty).$$

Note $\varphi(x) \equiv 1$ and $\psi(x) = x$ are two linear independent solutions of the equations at $\lambda = 0$, both belonging to L_w^2, so this form of differential operator is of limiting circular type at infinity.

This example shows that there is a fundamental difference between the Singular problem, for the weighted function case and the without-weighted function case, at the singularity point.

5 An example

In the following, we consider a differential operator,

$$\mathcal{L} := \begin{cases} l(y) = \frac{1}{x^{2\alpha}}(-(x^{2\alpha}y')' + (x^{2\alpha-2}(\alpha - \frac{1}{4}))y) = \lambda y, & \text{on } [1,\infty), \\ y(1) = 0. \end{cases} \tag{23}$$

Set

$$u = \sqrt[4]{pw}y, \qquad r = q/w - \frac{p'(pw)'}{16pw^2} - \frac{5w'(pw)'}{16w^3} + \frac{(pw)''}{4w^2}, \qquad t = \int_0^x \sqrt{\frac{w}{p}}\,\mathrm{d}x.$$

We can obtain that

$$l(u) := u'' - ru + \lambda u = 0,$$

where $r = \frac{\alpha^2 - \frac{1}{4}}{t^2}$.

Hence, we have transformed (23) into the next differential operator,

$$\mathcal{L} := \begin{cases} l(y) = -y'' + \frac{\alpha^2-\frac{1}{4}}{t^2}y, & \text{on } [1,\infty), \\ y(1) = 0. \end{cases} \tag{24}$$

Now, we obtain the Fourier–Bessel equation,

$$-y'' + \frac{\alpha^2 - \frac{1}{4}}{t^2}y = s^2 y\big(s^2 = \lambda\big). \tag{25}$$

Equation (25) has two linear independence solutions $\sqrt{t}J_\alpha(st)$ and $\sqrt{t}Y_\alpha(st)$, where, $J_\alpha(x)$, $Y_\alpha(x)$ are first-kind and second-kind Bessel functions, respectively,

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\alpha+1)}\left(\frac{x}{2}\right)^{2n+\alpha},$$

$$Y_\alpha(x) = \begin{cases} \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi}, & \alpha \neq \text{integer}, \\ \frac{2}{\pi}J_\alpha(x)\ln\frac{x}{2} + x^{-\alpha}\sum_{n=0}^{\infty} a_n x^n, & a_0 \neq 0, \alpha = \text{nonnegative integer}. \end{cases}$$

Furthermore, we can obtain the asymptotic expression of $J_\alpha(x)$ and $Y_\alpha(x)$. As $x \to \infty$, we have that,

$$J_\alpha(x) \to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{2}\right),$$

and

$$Y_\alpha(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{2}\right).$$

Now, we can consider (24). Suppose $\psi(t,\lambda)$, $\varphi(t,\lambda)$ are two solutions of (24) with the Cauchy condition,

$$\psi(1,\lambda) = 0, \qquad \psi'(1,\lambda) = -1;$$

and

$$\varphi(1,\lambda) = 1, \qquad \varphi'(1,\lambda) = 0.$$

We can obtain the two solutions

$$\psi(t,\lambda) = \frac{\pi}{2}\sqrt{t}\big(J_\alpha(ts)Y_\alpha(s) - Y_\alpha(ts)J_\alpha(s)\big),$$

and

$$\varphi(t,\lambda) = \frac{\pi}{2}\sqrt{ts}\big(J_\alpha(ts)Y'_\alpha(s) - Y_\alpha(ts)J'_\alpha(s)\big) + \frac{1}{2}\psi(t,\lambda).$$

We will reduce the solution of the (24) to the solution of (23). Set $\psi_1 = (pw)^{-\frac{1}{4}}\psi$, $\varphi_1 = (pw)^{-\frac{1}{4}}\varphi$, and $t = \int_0^x \sqrt{\frac{w}{p}}\,\mathrm{d}x$. Then, we can verify that

$$\psi_1(1,\lambda) = \frac{\psi}{x^\alpha}(1,\lambda) = 0, \qquad \psi'_1(1,\lambda) = \frac{\psi' x^\alpha - \psi\alpha x^{\alpha-1}}{x^{2\alpha}}(1,\lambda) = -1;$$

and

$$\varphi_1(1,\lambda) = \frac{\varphi}{x^\alpha}(1,\lambda) = 1, \qquad \varphi'_1(1,\lambda) = \frac{\varphi' x^\alpha - \varphi\alpha x^{\alpha-1}}{x^{2\alpha}}(1,\lambda) = \alpha.$$

Hence, $\psi_1(1,\lambda)$ and $\varphi_1(1,\lambda)$ are two linear independent solutions. By the asymptotic expression of $J_\alpha(x)$, as $x \to \infty$, we can obtain,

$$\psi_1 = (pw)^{-\frac{1}{4}}\sqrt{t}J_\alpha(t\sqrt{\lambda}) \notin L^2_w[1,\infty).$$

Hence, $l(y)$ is the Limit Point Case at ∞, and $\mathcal{L}$ is a self-adjoint operator.

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