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On global strong solutions to the 3D MHD flows with density-temperature-dependent viscosities

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Abstract

In this paper, we establish the global existence of strong solutions for the 3D viscous, compressible, and heat conducting magnetohydrodynamic (MHD) flows with density-temperature-dependent viscosities in a bounded domain. We essentially show that for the initial boundary value problem with initial density allowed to vanish, the strong solution exists globally under some suitable small conditions. As a byproduct, we obtain the nonlinear exponential stability of the solution.

MSC: 35B40; 35B45; 76N10

Keywords: Compressible magnetohydrodynamic system; Full compressible Navier–Stokes system; Global strong solutions; Density-temperature-dependent viscosities; Vacuum

1 Introduction

In this paper, we are concerned with the following 3D viscous, compressible, and heat conducting magnetohydrodynamic (MHD) equation, which is a combination of the compressible Navier–Stokes equation of fluid dynamics and Maxwell equation of electromagnetism (see [15, 18]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(2\mu \mathcal{D}(\mathbf{u})) - \nabla(\lambda \operatorname{div} \mathbf{u}) + \nabla P = (\operatorname{curl} \mathbf{B}) \times \mathbf{B}, \\ c_v[(\rho\theta)_t + \operatorname{div}(\rho \mathbf{u}\theta)] - \kappa \Delta\theta + P \operatorname{div} \mathbf{u} = 2\mu |\mathcal{D}(\mathbf{u})|^2 + \lambda(\operatorname{div} \mathbf{u})^2 + \nu |\operatorname{curl} \mathbf{B}|^2, \\ \mathbf{B}_t - \operatorname{curl}(\mathbf{u} \times \mathbf{B}) = \nu \Delta \mathbf{B}, \quad \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ is the time, $x \in \Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\rho, \mathbf{u} = (u^1, u^2, u^3)^{\text{tr}}, \theta, \mathbf{B} = (b^1, b^2, b^3)^{\text{tr}}$, and $P = R\rho\theta$ $(R > 0)$ denote the fluid density, velocity, absolute temperature, magnetic field, and pressure, respectively, and $\mathcal{D}(\mathbf{u})$ is the deformation tensor given by

$$\mathcal{D}(\mathbf{u}) = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\text{tr}}\right].$$

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The viscosity coefficients $\mu = \mu(\rho,\theta) \in C^1(\mathbb{R}^2)$ and $\lambda = \lambda(\rho,\theta) \in C^1(\mathbb{R}^2)$ satisfy the physical restrictions

$$\mu(\rho,\theta) > 0, \quad 2\mu(\rho,\theta) + 3\lambda(\rho,\theta) \geq 0. \tag{1.2}$$

The positive constants c_ν, κ, and ν are the heat capacity, heat conductivity, and magnetic diffusion coefficient, respectively.

As initial and boundary conditions, we consider

$$(\rho, \mathbf{u}, \theta, \mathbf{B})|_{t=0} = (\rho_0, \mathbf{u}_0, \theta_0, \mathbf{B}_0)(x), \quad x \in \Omega, \tag{1.3}$$

and

$$(\mathbf{u}, \theta, \mathbf{B})(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1.4}$$

The compressible MHD system (1.1) describes the relationship between the compressible Navier–Stokes equation of fluid dynamics and Maxwell equation of electromagnetism, which has been studied by many papers [3, 4, 9, 11, 12] and the references therein. When there is no electromagnetic effect, that is, $\mathbf{B} = 0$, system (1.1) reduces to the full compressible Navier–Stokes system

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(2\mu\mathcal{D}(\mathbf{u})) - \nabla(\lambda \operatorname{div}\mathbf{u}) + \nabla P = 0, \\ c_\nu[(\rho\theta)_t + \operatorname{div}(\rho\mathbf{u}\theta)] - \kappa\Delta\theta + P\operatorname{div}\mathbf{u} = 2\mu|\mathcal{D}(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2. \end{cases} \tag{1.5}$$

Because of the important physical phenomenon and mathematical challenges of the full compressible Navier–Stokes system (1.5), there is a wide literature investigating the complexity and rich phenomena of system (1.5). In the case of strictly positive initial density and temperature, Nash [22] and Serrin [23] obtained the local existence and uniqueness of classical solutions, respectively. Matsumura and Nishida [19–21] first obtained the global classical solutions when the initial data are close to a nonvacuum equilibrium in $H^s(\mathbb{R}^3)$. Valli and Zajaczkowski [24] established the existence and stability of the periodic solution in a bounded domain. Hoff [7, 8] proved the global existence of weak solutions when the initial data are discontinuous.

When the initial density can contain vacuum, Feireisl [6] investigated the full compressible Navier–Stokes equations with temperature-dependent heat conductivity coefficient and obtained the existence of "variational" weak solutions for large initial data with vacuum. Bresch and Desjardins [1] studied the Cauchy problem of system (1.5) with density-dependent viscosities and obtained the global stability of weak solutions. Recently, Yu and Zhang [25] considered the three-dimensional full compressible Navier–Stokes equations with density-temperature-dependent viscosities and proved the existence of global strong solutions in a bounded domain in $\mathbb{R}^3$.

For the compressible MHD system (1.1), Chen and Wang in [2] investigated the nonlinear MHD equations with general initial data and obtained the global existence and uniqueness of solutions with large initial data. Hu and Wang [10] investigated the compactness of weak solution of 3D full compressible MHD equations with density-dependent-heat conductivity and the magnetic coefficient with vacuum. Later, Huang and Li [13] studied the mechanism of blowup and structure of possible singularities of strong solutions to system (1.1) and obtained a blowup criterion, which is analogous to the well-known Serrin blowup criterion for the Cauchy problem and the initial boundary value one of system (1.1). Due to the physical importance, complexity, and mathematical challenges, our main aim in this paper is to investigate the global existence of strong solutions to the 3D MHD flows with density-temperature-dependent viscosities in a bounded domain.

Before stating our main result, we define q by

$$\frac{1}{3+\beta} = \frac{1}{12} + \frac{1}{q} \tag{1.6}$$

for some $0 < \beta \leq 1$, and thus $q \in (4, 6]$. For simplicity, we denote

$$\big\|(f,g)\big\|_{L^p} \triangleq \|f\|_{L^p} + \|g\|_{L^p}.$$

Now we are in a position to formulate our main results.

Theorem 1.1 *Assume that the initial data* $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{B}_0)$ *satisfy*

$$0 \leq \rho_0 \leq \bar{\rho}, \qquad \rho_0 \in W^{1,q}, \qquad \theta_0 \geq 0, \qquad (\mathbf{u}_0, \theta_0, \mathbf{B}_0) \in H_0^1 \cap H^2 \tag{1.7}$$

for some positive constant $\bar{\rho}$ *and that the following compatibility conditions hold*:

$$\begin{cases} -\operatorname{div}(2\mu(\rho_0,\theta_0)\mathcal{D}(\mathbf{u}_0)) - \nabla(\lambda(\rho_0,\theta_0)\operatorname{div}\mathbf{u}_0) - \nabla P(\rho_0,\theta_0) - (\operatorname{curl}\mathbf{B}_0) \times \mathbf{B}_0 \\ \quad = \rho_0^{1/2} g_1, \\ -\kappa\Delta\theta_0 + 2\mu(\rho_0,\theta_0)|\mathcal{D}(\mathbf{u}_0)|^2 + \lambda(\rho_0,\theta_0)(\operatorname{div}\mathbf{u}_0)^2 + \nu|\operatorname{curl}\mathbf{B}_0|^2 = \rho_0^{1/2} g_2 \end{cases} \tag{1.8}$$

for some $g_1, g_2 \in L^2$. *Then there exists a positive constant* ε, *depending only on* Ω, g_1, g_2, κ, ν, R, c_v, *and* ρ_0, *such that if*

$$C_0 \triangleq \big\|(\nabla\mathbf{u}_0, \nabla\theta_0, \nabla\mathbf{B}_0)\big\|_{L^2}^2 \leq \varepsilon,$$

then the initial boundary value problem (1.1)–(1.4) *has a global strong solution* $(\rho, \mathbf{u}, \theta, \mathbf{B})$ *on* $[0,1] \times [0,\infty)$ *satisfying*

$$\begin{cases} 0 \leq \rho(x,t) \leq 2\bar{\rho}, \qquad \theta(x,t) \geq 0, \qquad (x,t) \in \Omega \times [0,\infty), \\ \rho \in C([0,\infty); W^{1,q}), \qquad \rho_t \in C([0,\infty); L^q), \\ (\mathbf{u}, \theta, \mathbf{B}) \in C([0,\infty); H_0^1 \cap H^2) \cap L^2(0,\infty; W^{2,q}), \\ (\rho^{1/2}\mathbf{u}_t, \rho^{1/2}\theta_t, \mathbf{B}_t) \in L^\infty(0,\infty; L^2), \qquad (\mathbf{u}_t, \theta_t, \mathbf{B}_t) \in L^2(0,\infty; H_0^1). \end{cases}$$

Moreover, for any $t \geq 0$, *we have that*

$$\big\|\big(\rho^{1/2}\mathbf{u}, \rho^{1/2}\theta, \mathbf{B}\big)(t)\big\|_{L^2} \leq Ce^{-Ct}$$

and

$$\left\|\left(\rho^{1/2}\mathbf{u},\rho^{1/2}\theta,\mathbf{B}\right)(t)\right\|_{L^2}+\left\|(\nabla\mathbf{u},\nabla\theta,\nabla\mathbf{B})(t)\right\|_{H^1}\le Ct^{-1}$$

with positive constant C depending only on $\Omega,\rho_0,\kappa,\nu,R,c_v,g_1$, and g_2.

Now we make some comments on the analysis of this paper. Note that for the Cauchy problem with constant viscosities satisfying (1.7)–(1.8), the local existence of strong solutions to the compressible MHD equations (1.1) with large initial data has been recently established [5]. Thus, to extend the strong solutions globally in time, we need global a priori estimates on smooth solutions for $(\rho,\mathbf{u},\theta,\mathbf{B})$. Some of the main new difficulties are due to the appearance of the density-temperature-dependent viscosities and the bounded domain. It turns out that the key issue of this paper is to derive the time-uniform upper bounds for the gradient of the density to bound $\|\nabla^2\mathbf{u}\|_{L^q}$ and $\|\nabla^2\mathbf{B}\|_{L^q}$. We start with the a priori hypothesis on $\|\nabla\rho\|_{L^q}$ and initial layer analysis and succeed in deriving an estimate of $\|\nabla\mathbf{u}_t\|_{L^1(0,T;L^2)}$ and time-weighted estimates on the gradient of $\mathbf{u}_t$, θ_t, and $\mathbf{B}_t$. Another difficulty caused by the bounded domain can be overcome by the energy method.

The rest of the paper is organized as follows. In Sect. 2, we establish estimates of the global strong solutions, which are independent of time t, to the initial boundary value problem (1.1)–(1.4). With the help of global (uniform) estimates at hand, in Sect. 3, we prove Theorem 1.1. In Sect. 4, we give some declarations of this paper.

2 Preliminaries

In this section, we recall some known facts and elementary inequalities. Before stating the results, we denote

$$\int f\,dx=\int_\Omega f\,dx.$$

We first begin with the following local existence result of the initial-boundary value problem (1.1)–(1.4), which is obtained on a small time interval in [5].

Proposition 2.1 *Assume that the initial data $(\rho_0\ge 0,\mathbf{u}_0,\theta_0,\mathbf{B}_0)$ satisfy (1.7) and (1.8). Then there exist a small time $T_*>0$ and a strong solution $(\rho,\mathbf{u},\theta,\mathbf{B})$ to the initial boundary value problem (1.1)–(1.4) on $\Omega\times(0,T_*]$.*

Next, we give the Korn inequality, which can be found in [14].

Lemma 2.1 *Let condition (1.2) be satisfied. Assume that Ω is a smooth bounded domain, and let $\mathbf{u}\in H^1$, $\mu(x),\lambda(x)\in C(\bar\Omega)$. Then there exist a positive constant $\underline{\mu}\triangleq\underline{\mu}(\Omega,\mu,\lambda)$ such that*

$$\int\left(2\mu\left|\mathcal{D}(\mathbf{u})\right|^2+\lambda(\operatorname{div}\mathbf{u})^2\right)dx\ge\underline{\mu}\int|\nabla\mathbf{u}|^2\,dx.$$

Consider the elliptic system

$$\begin{cases}\sum_{j=1}^3\sum_{\alpha,\beta=1}^3 D_\alpha(G_{ij}^{\alpha\beta}(x)D_\beta u^j)=f_i, & i=1,2,3,\\ \mathbf{u}=0, \quad x\in\partial\Omega,\end{cases}\tag{2.1}$$

with smooth bounded domain $\Omega \subset \mathbb{R}^3$, $\mathbf{u} = (u^1, u^2, u^3)$, and

$$G_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \geq \sigma|\xi|^2 \quad \forall \xi \in \mathbb{R}^3, \sigma > 0, A_{ij}^{\alpha\beta}(x) \in L^\infty.$$

Lemma 2.2 ([17]) *For $p \in [2,\infty)$, assume that $|\nabla G_{ij}^{\alpha\beta}(x)||\nabla \mathbf{u}|, f_i \in L^p$. Then the solution $\mathbf{u}$ of system (2.1) satisfies $\mathbf{u} \in W^{2,p}$ with*

$$\|\nabla^2 \mathbf{u}\|_{L^p} \leq C\left(\left\||\nabla G_{ij}^{\alpha\beta}(x)||\nabla \mathbf{u}|\right\|_{L^p} + \|f_i\|_{L^p}\right).$$

Lemma 2.3 (Gagliardo–Nirenberg [16]) *For $p \in [2,6], q \in (1,\infty)$, and $r \in (3,\infty)$, there exists a generic constant $C > 0$, depending only on q, r, and Ω, such that*

(i) if $f \in H_0^1$ or $f \in H^1, \bar{f} = 0$, and $g \in W_0^{1,r}$ or $g \in W^{1,r}, \bar{g} = 0$, then

$$\|f\|_{L^p} \leq C\|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2p}}, \tag{2.2}$$

$$\|g\|_{L^\infty} \leq C\|g\|_{L^q}^{\frac{q(r-3)}{3r+q(r-3)}} \|\nabla g\|_{L^r}^{\frac{3r}{3r+q(r-3)}}, \tag{2.3}$$

where $\bar{f} \triangleq \frac{1}{|\Omega|}\int_\Omega f\,dx$ and $\bar{g} \triangleq \frac{1}{|\Omega|}\int_\Omega g\,dx$.

(*ii*) if $f \in H^1$, then

$$\|f\|_{L^p} \leq C\|f\|_{L^2}^{\frac{6-p}{2p}} \|f\|_{H^1}^{\frac{3p-6}{2p}}. \tag{2.4}$$

3 A priori estimates

In this section, we establish the uniform a priori estimates of solutions to the initial boundary value problem (1.1)–(1.4) to extend the local strong solution guaranteed by Proposition 2.1. Then we assume that $(\rho, \mathbf{u}, \theta, \mathbf{B})$ is a smooth solution to (1.1)–(1.4) on $\Omega \times (0,T)$ for some positive time $T > 0$ with smooth initial data $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{B}_0)$ satisfying (1.7) and (1.8). Define

$$A_1(T) \triangleq \sup_{0\leq t\leq T} \left\|(\nabla \mathbf{u}, \nabla \mathbf{B})\right\|_{L^2}^2 + \int_0^T \left\|\left(\rho^{1/2}\mathbf{u}_t, \nabla^2 \mathbf{B}, \mathbf{B}_t\right)\right\|_{L^2}^2 dt,$$

$$A_2(T) \triangleq \sup_{0\leq t\leq T} \|\nabla\theta\|_{L^2}^2 + \int_0^T \left\|\rho^{1/2}\theta_t\right\|_{L^2}^2 dt,$$

$$A_3(T) \triangleq \sup_{0\leq t\leq T} \left\|\left(\rho^{1/2}\mathbf{u}_t, \mathbf{B}_t, \nabla^2 \mathbf{B}\right)\right\|_{L^2}^2 + \int_0^T \left\|(\nabla \mathbf{u}_t, \nabla \mathbf{B}_t)\right\|_{L^2}^2 dt,$$

$$A_4(T) \triangleq \sup_{0\leq t\leq T} \left\|\rho^{1/2}\theta_t\right\|_{L^2}^2 + \int_0^T \|\nabla\theta_t\|_{L^2}^2 dt,$$

$$A_5(T) \triangleq \sup_{0\leq t\leq T} \|\nabla\rho\|_{L^q},$$

$$\begin{aligned} A_6(T) \triangleq \sup_{0\leq t\leq T} \big(&\|\nabla \mathbf{u}\|_{L^2}^{1/2}\|\nabla \mathbf{u}\|_{H^1}^{1/2} + \|\nabla \mathbf{u}\|_{L^2}^{1/4}\|\nabla \mathbf{u}\|_{H^1}^{3/2} \\ &+ \|\nabla \mathbf{B}\|_{L^2}^{1/2}\|\nabla \mathbf{B}\|_{H^1}^{1/2} + \|\nabla \mathbf{B}\|_{L^2}^{1/4}\|\nabla \mathbf{B}\|_{H^1}^{3/2}\big), \end{aligned}$$

$$A_7(T) \triangleq \sup_{0\leq t\leq T} \left\|\nabla G_{ij}^{\alpha\beta}(\rho,\theta)\right\|_{L^q}^{\frac{q}{q-3}} \|\nabla \mathbf{u}\|_{L^2}^{\frac{1}{2}}, \quad \left(\text{see } G_{ij}^{\alpha\beta} \text{ in } (2.1)\right),$$

and

$$K_1 \triangleq \left\|\left(\rho^{1/2}\mathbf{u}_t, \mathbf{B}_t\right)\right\|_{L^2}^2|_{t=0}, \qquad K_2 \triangleq \left\|\rho^{1/2}\theta_t\right\|_{L^2}^2|_{t=0}.$$

We have the following key a priori estimates on $(\rho, \mathbf{u}, \theta, \mathbf{B})$.

Proposition 3.1 *For a constant* $\bar{\rho} > 0$ *and* q *satisfying* (1.6), *assume that* $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{B}_0)$ *satisfies* (1.7) *and* (1.8). *Let* $(\rho, \mathbf{u}, \theta, \mathbf{B})$ *be a smooth solution of* (1.1)–(1.4) *on* $\Omega \times (0, T]$ *satisfying*

$$\begin{cases} 0 \le \rho(x,t) \le 3\bar{\rho} \quad \text{for all } (x,t) \in \Omega \times [0,T], \\ A_1(T) \le 2C_0^{1/4}, \qquad A_2(T) \le 2C_0^{1/2}, \qquad A_3(T) \le 3K_1, \\ A_4(T) \le 3K_2, \qquad A_5(T) \le 4\|\nabla\rho_0\|_{L^q}, \qquad A_6(T) + A_7(T) \le 2. \end{cases} \tag{3.1}$$

Then there exists a constant $\varepsilon > 0$ *such that*

$$\begin{cases} 0 \le \rho(x,t) \le 2\bar{\rho} \quad \text{for all } (x,t) \in \Omega \times [0,T], \\ A_1(T) \le C_0^{1/4}, \qquad A_2(T) \le C_0^{1/2}, \qquad A_3(T) \le 2K_1, \\ A_4(T) \le 2K_2, \qquad A_5(T) \le 3\|\nabla\rho_0\|_{L^q}, \qquad A_6(T) + A_7(T) \le 1, \end{cases}$$

provided that $C_0 \le \varepsilon$.

Proof The proof of Proposition 3.1 will be done by a series of lemmas below. □

Throughout this paper, we denote by C and C_i $(i = 1, 2, \dots)$ generic positive constants, which may depend on $\Omega, \bar{\rho}, \|\rho_0\|_{L^1}, \kappa, \nu, R, c_\nu, g_1$, and g_2.

We start with the following uniform estimates for $(\rho, \mathbf{u}, \theta, \mathbf{B})$ under conditions (3.1).

Lemma 3.1 *Under condition* (3.1), *we have*

$$\sup_{t\in[0,T]} \left(\|\nabla\theta\|_{H^1} + \|\theta\|_{L^\infty} + \|\nabla\theta\|_{L^6}\right) \le C. \tag{3.2}$$

Proof Equation $(1.1)_3$, together with (3.1), (2.2), and (2.4), yields that

$$\begin{aligned} \left\|\nabla^2\theta\right\|_{L^2} &\le C\left(\|\rho\theta_t\|_{L^2} + \|\rho\mathbf{u}\cdot\nabla\theta\|_{L^2} + \|\rho\theta \operatorname{div}\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^4}^2 + \|\nabla\mathbf{B}\|_{L^4}^2\right) \\ &\le C\left(\left\|\rho^{1/2}\theta_t\right\|_{L^2} + \|\mathbf{u}\|_{L^6}\|\nabla\theta\|_{L^3} + \|\theta\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + \|\nabla\mathbf{u}\|_{L^4}^2 + \|\nabla\mathbf{B}\|_{L^4}^2\right) \\ &\le C\left(\left\|\rho^{1/2}\theta_t\right\|_{L^2} + A_1^{1/2}(T)\|\nabla\theta\|_{L^2}^{1/2}\|\nabla\theta\|_{H^1}^{1/2} + A_6(T)\|\nabla\theta\|_{L^2}\right) \\ &\quad + CA_6(T)\left(\|\nabla\mathbf{u}\|_{L^2}^{1/4} + \|\nabla\mathbf{B}\|_{L^2}^{1/4}\right) \\ &\le \frac{1}{2}\left\|\nabla^2\theta\right\|_{L^2} + C\left(\left\|\rho^{1/2}\theta_t\right\|_{L^2} + \|\nabla\theta\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{1/4} + \|\nabla\mathbf{B}\|_{L^2}^{1/4}\right), \end{aligned}$$

and thus

$$\|\nabla\theta\|_{H^1} \le C\left(\left\|\rho^{1/2}\theta_t\right\|_{L^2} + \|\nabla\theta\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{1/4} + \|\nabla\mathbf{B}\|_{L^2}^{1/4}\right). \tag{3.3}$$

On the other hand, from the Sobolev and Poincaré inequalities it follows that

$$\|\theta\|_{L^\infty} \le C\|\nabla\theta\|_{L^6} \le C\|\nabla\theta\|_{H^1},$$

which, together with (3.3), leads to (3.2). The proof of the lemma is therefore completed. □

Lemma 3.2 *Under condition* (3.1),

$$\sup_{t\in[0,T]}\big(\|\nabla\rho\|_{L^r} + \big\|(\nabla\mathbf{u},\nabla\mathbf{B})\big\|_{H^1}\big) \le C, \quad r\in[2,q], \tag{3.4}$$

where q is defined as in (1.6).

Proof Thanks to the bounded domain Ω, we obtain

$$\|\nabla\rho\|_{L^r} \le C\|\nabla\rho\|_{L^q} \le CA_5(T) \le C \quad \text{for } r\in[2,q].$$

On the other hand, it follows from Lemma 2.2, (2.2), (2.4), and (3.1) that

$$\begin{aligned}
\big\|\nabla^2\mathbf{u}\big\|_{L^2} &\le C\big(\|\rho\mathbf{u}_t\|_{L^2} + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2} + \big\|\nabla G_{ij}^{\alpha\beta}(\rho,\theta)\nabla\mathbf{u}\big\|_{L^2}\big)\\
&\quad + C\big(\|\nabla P\|_{L^2} + \big\|(\operatorname{curl}\mathbf{B})\times\mathbf{B}\big\|_{L^2}\big)\\
&\le C\big(\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2} + \|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + \big\|\nabla G_{ij}^{\alpha\beta}(\rho,\theta)\big\|_{L^q}\|\nabla\mathbf{u}\|_{L^{2q/(q-2)}}\big)\\
&\quad + C\big(\|\theta\|_{L^6}\|\nabla\rho\|_{L^3} + \|\nabla\theta\|_{L^2} + \|\mathbf{B}\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\big)\\
&\le \frac{1}{2}\big\|\nabla^2\mathbf{u}\big\|_{L^2} + \frac{1}{4}\big\|\nabla^2\mathbf{B}\big\|_{L^2} + C\big(\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^3 + \|\nabla\mathbf{B}\|_{L^2}^3\big)\\
&\quad + C\big\|\nabla G_{ij}^{\alpha\beta}(\rho,\theta)\big\|_{L^q}^{q/(q-3)}\|\nabla\mathbf{u}\|_{L^2} + C\big(1 + \|\nabla\rho\|_{L^3}\big)\|\nabla\theta\|_{L^2}\\
&\le \frac{1}{2}\big\|\nabla^2\mathbf{u}\big\|_{L^2} + \frac{1}{4}\big\|\nabla^2\mathbf{B}\big\|_{L^2} + C\big[\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2} + A_1^{5/4}(T)\big(\|\nabla\mathbf{u}\|_{L^2}^{1/2} + \|\nabla\mathbf{B}\|_{L^2}^{1/2}\big)\big]\\
&\quad + CA_7(T)\|\nabla\mathbf{u}\|_{L^2}^{1/2} + C\|\nabla\theta\|_{L^2} + CA_5(T)\|\nabla\theta\|_{L^2},
\end{aligned}$$

and thus

$$\begin{aligned}
\big\|\nabla^2\mathbf{u}\big\|_{L^2} &\le \frac{1}{2}\big\|\nabla^2\mathbf{B}\big\|_{L^2} + C\big(\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{1/2} + \|\nabla\mathbf{B}\|_{L^2}^{1/2} + \|\nabla\theta\|_{L^2}\big)\\
&\le C\big(A_3^{1/2}(T) + A_2^{1/2}(T) + A_1^{1/4}(T)\big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\big\|\nabla^2\mathbf{B}\big\|_{L^2} &\le C\big(\|\mathbf{B}_t\|_{L^2} + \|\mathbf{u}\|_{L^6}\|\nabla\mathbf{B}\|_{L^3} + \|\mathbf{B}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\big)\\
&\le \frac{1}{2}\big\|\nabla^2\mathbf{B}\big\|_{L^2} + C\big(\|\mathbf{B}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{B}\|_{L^2}\big)\\
&\le \frac{1}{2}\big\|\nabla^2\mathbf{B}\big\|_{L^2} + C\|\mathbf{B}_t\|_{L^2} + C\|\nabla\mathbf{u}\|_{L^2}^{1/2},
\end{aligned}$$

and thus

$$\begin{aligned}\|\nabla^2\mathbf{u}\|_{L^2}+\|\nabla^2\mathbf{B}\|_{L^2}&\le C\big(\|(\rho^{1/2}\mathbf{u}_t,\mathbf{B}_t)\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}^{1/2}+\|\nabla\mathbf{B}\|_{L^2}^{1/2}+\|\nabla\theta\|_{L^2}\big)\\&\le C\big(A_3^{1/2}(T)+A_2^{1/2}(T)+A_1^{1/4}(T)\big),\end{aligned}\tag{3.5}$$

which gives (3.4). The proof of the lemma is therefore completed. □

Lemma 3.3 *There exist a constant* $\varepsilon_1>0$ *such that*

$$A_6(T)+A_7(T)\le 1,$$

provided that $C_0\le\varepsilon_1$.

Proof From (3.1), (3.2), and (3.4) we obtain

$$\begin{aligned}A_6(T)+A_7(T)&\le C\big(\|\nabla\rho\|_{L^q}+\|\nabla\theta\|_{L^q}\big)^{q/(q-3)}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\\&\quad+\big(\|\nabla\mathbf{u}\|_{L^2}^{1/4}\|\nabla\mathbf{u}\|_{H^1}^{1/2}+\|\nabla\mathbf{u}\|_{H^1}^{3/2}\big)\|\nabla\mathbf{u}\|_{L^2}^{1/4}\\&\quad+\big(\|\nabla\mathbf{B}\|_{L^2}^{1/4}\|\nabla\mathbf{B}\|_{H^1}^{1/2}+\|\nabla\mathbf{B}\|_{H^1}^{3/2}\big)\|\nabla\mathbf{B}\|_{L^2}^{1/4}\\&\le C_1C_0^{1/32}\le 1,\end{aligned}$$

provided that $C_0\le\varepsilon_1\triangleq\min\{1,C_1^{-32}\}$. The proof of the lemma is therefore completed. □

Lemma 3.4 *Let* $(\rho,\mathbf{u},\theta,\mathbf{B})$ *be a smooth solution of* (1.1)–(1.4) *satisfying* (3.1). *Then there exist positive constants C and* ε_2, *both depending only on* $\kappa,\nu,R,c_v,\underline{\mu},\bar\rho,\Omega,g_1$, *and* g_2 *such that*

$$\sup_{t\in[0,T]}t^i\|(\rho^{1/2}\mathbf{u},\rho^{1/2}\theta,\mathbf{B})\|_{L^2}^2+\int_0^T t^i\|(\nabla\mathbf{u},\nabla\theta,\nabla\mathbf{B})\|_{L^2}^2\,dt\le CC_0\tag{3.6}$$

for $i=0,1,\dots,32$, *provided that* $C_0\le\varepsilon_2$.

Proof Multiplying $(1.1)_2$ by $\mathbf{u}$ in L^2, from (2.2) and (3.4) we have that

$$\begin{aligned}&\frac12\frac{d}{dt}\|\rho^{1/2}\mathbf{u}\|_{L^2}^2+\int\big[2\mu(\rho,\theta)|\mathcal{D}(\mathbf{u})|^2+\lambda(\rho,\theta)(\operatorname{div}\mathbf{u})^2\big]\,dx\\&\quad=\int R\rho\theta\operatorname{div}\mathbf{u}\,dx-\int\mathbf{B}\cdot\nabla\mathbf{u}\cdot\mathbf{B}\,dx+\frac12\int|\mathbf{B}|^2\operatorname{div}\mathbf{u}\,dx\\&\quad\le\frac12\underline{\mu}\|\nabla\mathbf{u}\|_{L^2}^2+C\|\rho\|_{L^3}^2\|\theta\|_{L^6}^2+C\|\mathbf{B}\|_{L^3}^2\|\mathbf{B}\|_{L^6}^2\\&\quad\le\frac12\underline{\mu}\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2+C\|\nabla\mathbf{B}\|_{L^2}^2,\end{aligned}$$

and thus by Lemma 2.1 we have

$$\frac{d}{dt}\|\rho^{1/2}\mathbf{u}\|_{L^2}^2+\underline{\mu}\|\nabla\mathbf{u}\|_{L^2}^2\le C_2\|\nabla\theta\|_{L^2}^2+C_3\|\nabla\mathbf{B}\|_{L^2}^2.\tag{3.7}$$

Multiplying $(1.1)_3$ by θ in L^2 and using the facts that $\mu(\rho,\theta),\lambda(\rho,\theta)\in C^1(\mathbb{R}^2)$, we have from (2.2), (3.1), and (3.4) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\rho^{1/2}\theta\right\|_{L^2}^2+\kappa\|\nabla\theta\|_{L^2}^2\\
&\quad=-\int R\rho\theta^2\operatorname{div}\mathbf{u}\,dx+\nu\int|\operatorname{curl}\mathbf{B}|^2\theta\,dx\\
&\qquad+\int\left[2\mu(\rho,\theta)\left|\mathcal{D}(\mathbf{u})\right|^2+\lambda(\rho,\theta)(\operatorname{div}\mathbf{u})^2\right]\theta\,dx\\
&\quad\le C\|\nabla\mathbf{u}\|_{L^2}\|\rho\|_{L^6}\|\theta\|_{L^6}^2+C\|\theta\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\|\nabla\mathbf{B}\|_{L^2}+C\|\theta\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\\
&\quad\le C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^2}\|\nabla\mathbf{B}\|_{L^2}^{3/2}+C\|\nabla\theta\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{3/2}\\
&\quad\le\left(C\|\nabla\mathbf{u}\|_{L^2}+\frac{\kappa}{4}\right)\|\nabla\theta\|_{L^2}^2+C\|\nabla\mathbf{B}\|_{L^2}^3+C\|\nabla\mathbf{u}\|_{L^2}^3\\
&\quad\le\left(CA_1^{1/2}(T)+\frac{\kappa}{4}\right)\|\nabla\theta\|_{L^2}^2+CA_1^{1/2}(T)\|\nabla\mathbf{B}\|_{L^2}^2+CA_1^{1/2}(T)\|\nabla\mathbf{u}\|_{L^2}^2\\
&\quad\le\left(C_4C_0^{1/8}+\frac{\kappa}{4}\right)\|\nabla\theta\|_{L^2}^2+CC_0^{1/8}\|\nabla\mathbf{B}\|_{L^2}^2+CC_0^{1/8}\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned}$$

Choosing $C_0\le\varepsilon_{2,1}\triangleq\min\{\varepsilon_1,(\frac{\kappa}{4C_4})^8\}$, we obtain

$$\frac{d}{dt}\left\|\rho^{1/2}\theta\right\|_{L^2}^2+\kappa\|\nabla\theta\|_{L^2}^2\le CC_0^{1/8}\|\nabla\mathbf{B}\|_{L^2}^2+CC_0^{1/8}\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.8}$$

Multiplying $(1.1)_4$ by $\mathbf{B}$ in L^2, we obtain from (2.2), (3.1), and (3.4) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{B}\|_{L^2}^2+\nu\|\nabla\mathbf{B}\|_{L^2}^2&=-\frac{1}{2}\int\mathbf{B}^2\operatorname{div}\mathbf{u}\,dx+\int\mathbf{B}\cdot\nabla\mathbf{u}\cdot\mathbf{B}\,dx\\
&\le C\|\nabla\mathbf{B}\|_{L^2}\|\mathbf{B}\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\\
&\le C\|\nabla\mathbf{B}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}\\
&\le CA_1^{1/2}(T)\|\nabla\mathbf{B}\|_{L^2}^2\\
&\le C_5C_0^{1/8}\|\nabla\mathbf{B}\|_{L^2}^2,
\end{aligned}$$

where we have used the Poincaré inequality and the estimates

$$\|\mathbf{B}\|_{L^3}\le C\|\mathbf{B}\|_{L^2}^{1/2}\|\nabla\mathbf{B}\|_{L^2}^{1/2}\le C\|\nabla\mathbf{B}\|_{L^2}.$$

Thus

$$\frac{d}{dt}\|\mathbf{B}\|_{L^2}^2+\nu\|\nabla\mathbf{B}\|_{L^2}^2\le 0, \tag{3.9}$$

provided that $C_0\le\varepsilon_{2,2}\triangleq\min\{1,(\frac{\nu}{2C_5})^8\}$.

Calculating (3.7)+$\frac{2C_2}{\kappa}\times$(3.8)+$\frac{2C_3}{\nu}\times$(3.9) yields

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\rho^{1/2}\mathbf{u}\right\|_{L^2}^2+\frac{2C_2}{\kappa}\left\|\rho^{1/2}\theta\right\|_{L^2}^2+\frac{2C_3}{\nu}\|\mathbf{B}\|_{L^2}^2\right)+\underline{\mu}\|\nabla\mathbf{u}\|_{L^2}^2+C_2\|\nabla\theta\|_{L^2}^2+C_3\|\nabla\mathbf{B}\|_{L^2}^2\\
&\quad\le C_6C_0^{1/8}\|\nabla\mathbf{B}\|_{L^2}^2+C_7C_0^{1/8}\|\nabla\mathbf{u}\|_{L^2}^2,
\end{aligned}$$

and thus

$$\frac{d}{dt}\left\|\left(\rho^{1/2}\mathbf{u},\rho^{1/2}\theta,\mathbf{B}\right)\right\|_{L^2}^2+\left\|(\nabla\mathbf{u},\nabla\theta,\nabla\mathbf{B})\right\|_{L^2}^2\le 0, \tag{3.10}$$

provided that $C_0\le\varepsilon_2\triangleq\min\{\varepsilon_{2,1},\varepsilon_{2,2},(\frac{C_3}{2C_6})^8,(\frac{\mu}{2C_7})^8\}$.

Integrating (3.10) over $[0,T]$ and using the Poincaré inequality, we get

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\|\left(\rho^{1/2}\mathbf{u},\rho^{1/2}\theta,\mathbf{B}\right)\right\|_{L^2}^2+\int_0^T\left\|(\nabla\mathbf{u},\nabla\theta,\nabla\mathbf{B})\right\|_{L^2}^2\,dt\\
&\quad\le C\left\|(\nabla\mathbf{u}_0,\nabla\theta_0,\nabla\mathbf{B}_0)\right\|_{L^2}^2\\
&\quad\le CC_0.
\end{aligned}\tag{3.11}$$

Multiplying (3.10) by t, integrating the result over $[0,T]$, and using the Poincaré inequality again, we have

$$\begin{aligned}
&\sup_{t\in[0,T]}t\left\|\left(\rho^{1/2}\mathbf{u},\rho^{1/2}\theta,\mathbf{B}\right)\right\|_{L^2}^2+\int_0^T t\left\|(\nabla\mathbf{u},\nabla\theta,\nabla\mathbf{B})\right\|_{L^2}^2\,dt\\
&\quad\le C\left\|(\nabla\mathbf{u},\nabla\theta,\nabla\mathbf{B})\right\|_{L^2}^2\\
&\quad\le CC_0,
\end{aligned}\tag{3.12}$$

which, together with (3.11), leads to (3.6) for $i=0,1$. Similarly to the proof of (3.11) and (3.12), we can obtain (3.6) for $i=3,4,\dots,32$. The proof of the lemma is therefore completed. □

Before stating the following lemma, we define

$$\sigma(T)\triangleq\min\{1,T\}.$$

Then we be establish a uniform upper bound for $A_1(T)$.

Lemma 3.5 *Let $(\rho,\mathbf{u},\theta,\mathbf{B})$ be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant ε_3, depending only on $\kappa,\nu,R,c_v,\underline{\mu},\bar{\rho},\Omega,g_1$, and g_2, such that*

$$\sup_{t\in[0,T]}\left\|(\nabla\mathbf{u},\nabla\mathbf{B})\right\|_{L^2}^2+\int_0^T\left\|\left(\rho^{1/2}\mathbf{u}_t,\mathbf{B}_t,\nabla^2\mathbf{B}\right)\right\|_{L^2}^2\,dt\le C_0^{1/4} \tag{3.13}$$

and

$$\sup_{t\in[0,T]}t^i\left\|(\nabla\mathbf{u},\nabla\mathbf{B})\right\|_{L^2}^2+\int_0^T t^i\left\|\left(\rho^{1/2}\mathbf{u}_t,\mathbf{B}_t,\nabla^2\mathbf{B}\right)\right\|_{L^2}^2\,dt\le CC_0^{5/16} \tag{3.14}$$

for $i=1,2,\dots,8$, provided that $C_0\le\varepsilon_3$.

Proof We multiply $(1.1)_2$ by $\mathbf{u}_t$ and integrate the result over Ω:

$$\begin{aligned}
&\frac{d}{dt}\int\left[\mu(\rho,\theta)\left|\mathcal{D}(\mathbf{u})\right|^2+\frac{\lambda(\rho,\theta)}{2}(\operatorname{div}\mathbf{u})^2\right]dx+\left\|\rho^{1/2}\mathbf{u}_t\right\|_{L^2}^2\\
&=\int\left[\mu_t(\rho,\theta)\left|\mathcal{D}(\mathbf{u})\right|^2+\frac{\lambda_t(\rho,\theta)}{2}(\operatorname{div}\mathbf{u})^2\right]dx-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,dx\\
&\quad+\int R\rho\theta\operatorname{div}\mathbf{u}_t\,dx-\int\mathbf{B}\cdot\nabla\mathbf{u}_t\cdot\mathbf{B}\,dx+\int\mathbf{B}^2\operatorname{div}\mathbf{u}_t\,dx\\
&\triangleq\sum_{j=1}^4 N_j.
\end{aligned}\tag{3.15}$$

The right-hand side terms of (3.15) can be estimated as follows. By (3.4) and the fact that $\mu(\rho,\theta),\lambda(\rho,\theta)\in C^1(\mathbb{R}^2)$ we have that

$$\begin{aligned}
&\left|\int\left[\mu_\rho\rho_t\left|\mathcal{D}(\mathbf{u})\right|^2+\frac{\lambda_\rho\rho_t}{2}(\operatorname{div}\mathbf{u})^2\right]dx\right|\\
&\leq C\int\left(|\nabla\rho||\mathbf{u}||\nabla\mathbf{u}|^2+|\rho||\nabla\mathbf{u}|^3\right)dx\\
&\leq\|\nabla\rho\|_{L^3}\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^4}^2+C\|\nabla\mathbf{u}\|_{L^3}^3\\
&\leq C\|\nabla\mathbf{u}\|_{L^2}^{3/2}\|\nabla\mathbf{u}\|_{H^1}^{3/2}\leq C\|\nabla\mathbf{u}\|_{L^2}^{3/2}
\end{aligned}$$

and

$$\begin{aligned}
\left|\int\left[\mu_\theta\theta_t\left|\mathcal{D}(\mathbf{u})\right|^2+\frac{\lambda_\theta\theta_t}{2}(\operatorname{div}\mathbf{u})^2\right]dx\right|&\leq\|\theta_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\\
&\leq C\|\nabla\theta_t\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}^{1/2}\|\nabla\mathbf{u}\|_{L^2}^{1/2},
\end{aligned}$$

and thus

$$\begin{aligned}
N_1&\leq C\|\nabla\mathbf{u}\|_{L^2}^{3/2}+C\|\nabla\theta_t\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}^{1/2}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\\
&\leq CA_1^{1/4}(T)\|\nabla\mathbf{u}\|_{L^2}+C\|\nabla\theta_t\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}^{1/2}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\\
&\leq C\|\nabla\mathbf{u}\|_{L^2}+C\|\nabla\theta_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{H^1}^{1/2}.
\end{aligned}$$

By (3.4) and the Poincaré inequality we have

$$\begin{aligned}
N_2&=-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,dx\\
&\leq\frac{1}{2}\left\|\rho^{1/2}\mathbf{u}_t\right\|_{L^2}^2+C\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^3}^2\\
&\leq\frac{1}{2}\left\|\rho^{1/2}\mathbf{u}_t\right\|_{L^2}^2+CA_1(T)\|\nabla\mathbf{u}\|_{L^2}\\
&\leq\frac{1}{2}\left\|\rho^{1/2}\mathbf{u}_t\right\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2},\\
N_3&=\int R\rho\theta\operatorname{div}\mathbf{u}_t\,dx\leq C\|\theta\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\leq C\|\nabla\theta\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned} N_4 &= -\int \mathbf{B}\cdot\nabla\mathbf{u}_t\cdot\mathbf{B}\,dx + \int \mathbf{B}^2 \operatorname{div}\mathbf{u}_t\,dx \\ &\le C\|\mathbf{B}\|_{L^6}\|\mathbf{B}\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2} \\ &\le C\|\nabla\mathbf{B}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}. \end{aligned}$$

Plugging N_j ($j = 1,2,3,4$) into (3.15) yields

$$\begin{aligned} &\frac{d}{dt}\int\Big[\mu(\rho,\theta)\big|\mathcal{D}(\mathbf{u})\big|^2 + \frac{\lambda(\rho,\theta)}{2}(\operatorname{div}\mathbf{u})^2\Big]\,dx + \big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2}^2 \\ &\quad\le C\|\nabla\mathbf{u}\|_{L^2} + C\|\nabla\theta_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{H^1}^{1/2} \\ &\qquad + C\|\nabla\theta\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2} + C\|\nabla\mathbf{B}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}. \end{aligned} \tag{3.16}$$

Multiplying $(1.1)_4$ by $\mathbf{B}_t$ in L^2, we have from (2.3) and the Poincaré inequality that

$$\begin{aligned} &\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{B}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 \\ &\quad= -\int \mathbf{u}\cdot\nabla\mathbf{B}\cdot\mathbf{B}_t\,dx + \int \mathbf{B}\cdot\nabla\mathbf{u}\cdot\mathbf{B}_t\,dx - \int \mathbf{B}\cdot\mathbf{B}_t \operatorname{div}\mathbf{u}\,dx \\ &\quad\le C\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\|\mathbf{B}_t\|_{L^2} + C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{B}\|_{L^\infty}\|\mathbf{B}_t\|_{L^2} \\ &\quad\le \frac{1}{2}\|\mathbf{B}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{B}\|_{H^1}^2, \end{aligned}$$

and thus from (3.1) and (3.4) it follows that

$$\frac{d}{dt}\|\nabla\mathbf{B}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 \le C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{B}\|_{H^1}^2 \le C\|\nabla\mathbf{u}\|_{L^2}. \tag{3.17}$$

On the other hand, from $(1.1)_4$ it follows that

$$\begin{aligned} \big\|\nabla^2\mathbf{B}\big\|_{L^2}^2 &\le C\big(\|\mathbf{B}_t\|_{L^2}^2 + \|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{B}\|_{L^3}^2 + \|\mathbf{B}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2\big) \\ &\le \frac{1}{2}\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2 + C\big(\|\mathbf{B}_t\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^4\|\nabla\mathbf{B}\|_{L^2}^2\big) \\ &\le \frac{1}{2}\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2 + C\|\mathbf{B}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}. \end{aligned} \tag{3.18}$$

Combining (3.16), (3.17), and (3.18), we have

$$\begin{aligned} &\frac{d}{dt}\int\Big[\mu(\rho,\theta)\big|\mathcal{D}(\mathbf{u})\big|^2 + \frac{\lambda(\rho,\theta)}{2}(\operatorname{div}\mathbf{u})^2 + |\nabla\mathbf{B}|^2\Big]\,dx + \big\|\big(\rho^{1/2}\mathbf{u}_t,\mathbf{B}_t,\nabla^2\mathbf{B}\big)\big\|_{L^2}^2 \\ &\quad\le C\|\nabla\mathbf{u}\|_{L^2} + C\|\nabla\theta_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{H^1}^{1/2} \\ &\qquad + C\|\nabla\theta\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2} + C\|\nabla\mathbf{B}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}. \end{aligned} \tag{3.19}$$

The Hölder inequality, together with (3.1) and (3.6), yields that for $i = 1, 2, \ldots, 8$,

$$\begin{aligned}\int_0^T t^i \|\nabla \mathbf{u}\|_{L^2}\,dt &\le \left(\int_0^{\sigma(T)} t^{2i}\|\nabla \mathbf{u}\|_{L^2}^2\,dt\right)^{1/2}\left(\int_0^{\sigma(T)} dt\right)^{1/2} \\ &\quad + \left(\int_{\sigma(T)}^T t^{2i+2}\|\nabla \mathbf{u}\|_{L^2}^2\,dt\right)^{1/2}\left(\int_{\sigma(T)}^T t^{-2}\,dt\right)^{1/2} \\ &\le C C_0^{1/2}.\end{aligned} \tag{3.20}$$

By (3.1), (3.5), (3.6), and (3.20) we have that

$$\begin{aligned}&\int_0^T t^i \|\nabla \theta_t\|_{L^2}\|\nabla \mathbf{u}\|_{L^2}^{1/2}\|\nabla \mathbf{u}\|_{H^1}^{1/2}\,dt \\ &\le \left(\int_0^T \|\nabla \theta_t\|_{L^2}^2\,dt\right)^{1/2}\left(\int_0^T \|\nabla \mathbf{u}\|_{H^1}^2\,dt\right)^{1/4}\left(\int_T^T t^{4i}\|\nabla \mathbf{u}\|_{L^2}^2\,dt\right)^{1/4} \\ &\le C A_4^{1/2}(T) C_0^{1/4} \\ &\quad \times \left[\int_0^T \left(\left\|\nabla^2 \mathbf{B}\right\|_{L^2}^2 + \left\|\rho^{1/2}\mathbf{u}_t\right\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{B}\|_{L^2} + \|\nabla \theta\|_{L^2}^2\right)dt\right]^{1/4} \\ &\le C C_0^{5/16}\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\int_0^T t^i \|\nabla \theta\|_{L^2}\|\nabla \mathbf{u}_t\|_{L^2}\,dt &\le \left(\int_0^T t^{2i}\|\nabla \theta\|_{L^2}^2\,dt\right)^{1/2}\left(\int_0^T \|\nabla \mathbf{u}_t\|_{L^2}^2\,dt\right)^{1/2} \\ &\le C A_3^{1/2}(T) C_0^{1/2} \\ &\le C C_0^{1/2}.\end{aligned} \tag{3.22}$$

Integrating (3.19) over $[0, T]$ and using Lemma 2.1, (3.1), (3.6), and (3.20)–(3.22) for $i = 0$, we obtain

$$\begin{aligned}&\sup_{t\in[0,T]} \left\|(\nabla \mathbf{u}, \nabla \mathbf{B})\right\|_{L^2}^2\,dx + \int_0^T \left\|\left(\rho^{1/2}\mathbf{u}_t, \mathbf{B}_t, \nabla^2 \mathbf{B}\right)\right\|_{L^2}^2\,dt \\ &\le C\left\|(\nabla \mathbf{u}_0, \nabla \mathbf{B}_0)\right\|_{L^2}^2 + C C_0^{1/2} + C C_0^{5/16} \\ &\le C_8 C_0^{5/16} \le C_0^{1/4},\end{aligned} \tag{3.23}$$

provided that $C_0 \le \varepsilon_3 \triangleq \min\{\varepsilon_2, C_8^{-16}\}$. Then we immediately get (3.13).

Next, multiplying (3.19) by t and integrating the result over $[0, T]$, it follows from (3.6) and (3.20)–(3.22) that

$$\sup_{t\in[0,T]} t\left\|(\nabla \mathbf{u}, \nabla \mathbf{B})\right\|_{L^2}^2\,dx + \int_0^T t\left\|\left(\rho^{1/2}\mathbf{u}_t, \mathbf{B}_t, \nabla^2 \mathbf{B}\right)\right\|_{L^2}^2\,dt \le C C_0^{5/16}. \tag{3.24}$$

Similarly to the proof of (3.23) and (3.24), we can obtain (3.14) for $i = 2, 3, \ldots, 8$. The proof of the lemma is therefore completed. □

Lemma 3.6 *Let $(\rho,\mathbf{u},\theta,\mathbf{B})$ be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant ε_4, depending only on $\kappa,\nu,R,c_\nu,\underline{\mu},\bar{\rho},\Omega,g_1$, and g_2, such that*

$$\sup_{t\in[0,T]}\big\|\big(\rho^{1/2}\mathbf{u}_t,\mathbf{B}_t,\nabla^2\mathbf{B}\big)\big\|_{L^2}^2+\int_0^T\big\|(\nabla\mathbf{u}_t,\nabla\mathbf{B}_t)\big\|_{L^2}^2\,dt\le 2K_1, \tag{3.25}$$

provided that $C_0\le\varepsilon_4$.

Proof By $(1.1)_1$ and $(1.1)_2$ we have that

$$\begin{aligned}&\rho\mathbf{u}_{tt}+\rho\mathbf{u}\cdot\nabla\mathbf{u}_t-\operatorname{div}\big(2\mu\mathcal{D}(\mathbf{u}_t)\big)-\nabla(\lambda\operatorname{div}\mathbf{u}_t)\\&\quad=\operatorname{div}(\rho\mathbf{u})(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})-\rho\mathbf{u}_t\cdot\nabla\mathbf{u}+\operatorname{div}\big[(2\mu_\rho\rho_t+2\mu_\theta\theta_t)\mathcal{D}(\mathbf{u})\big]\\&\qquad+\nabla\big[(\lambda_\rho\rho_t+\lambda_\theta\theta_t)\operatorname{div}\mathbf{u}\big]-\nabla P_t+\mathbf{B}\cdot\nabla\mathbf{B}_t+\mathbf{B}_t\cdot\nabla\mathbf{B}-\nabla\mathbf{B}_t\cdot\mathbf{B}-\nabla\mathbf{B}\cdot\mathbf{B}_t.\end{aligned} \tag{3.26}$$

Multiplying (3.26) by $\mathbf{u}_t$ and integrating the resulting equation over $[0,1]$, we get

$$\begin{aligned}&\frac12\frac{d}{dt}\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2}^2+\int\big[2\mu(\rho,\theta)\big|\mathcal{D}(\mathbf{u}_t)\big|^2+\lambda(\rho,\theta)(\operatorname{div}\mathbf{u}_t)^2\big]\,dx\\&\quad=\int\operatorname{div}(\rho\mathbf{u})(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_t\,dx-\int\rho\mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,dx+\int P_t\operatorname{div}\mathbf{u}_t\,dx\\&\qquad-\int(\mu_\rho\rho_t+\mu_\theta\theta_t)\big[\big|\mathcal{D}(\mathbf{u})\big|^2\big]_t\,dx-\frac12\int(\lambda_\rho\rho_t+\lambda_\theta\theta_t)\big[(\operatorname{div}\mathbf{u})^2\big]_t\,dx\\&\qquad+\int(\mathbf{B}\cdot\nabla\mathbf{B}_t+\mathbf{B}_t\cdot\nabla\mathbf{B}-\nabla\mathbf{B}_t\cdot\mathbf{B}-\nabla\mathbf{B}\cdot\mathbf{B}_t)\cdot\mathbf{u}_t\,dx\\&\quad\triangleq\sum_{j=1}^6 M_j.\end{aligned} \tag{3.27}$$

Now we estimate M_j $(j=1,2,\dots,6)$. It follows from (2.2)–(2.4) and (3.4) that

$$\begin{aligned}M_1&\le C\int|\rho\operatorname{div}\mathbf{u}+\mathbf{u}\cdot\nabla\rho|\big(|\mathbf{u}_t|^2+|\mathbf{u}||\nabla\mathbf{u}||\mathbf{u}_t|\big)\,dx\\&\le C\big(\|\rho\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}+\|\mathbf{u}\|_{L^6}\|\nabla\rho\|_{L^2}\big)\|\mathbf{u}_t\|_{L^6}^2\\&\quad+C\big(\|\rho\|_{L^6}\|\nabla\mathbf{u}\|_{L^6}+\|\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^3}\big)\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^6}\\&\le C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{H^1}^{3/2}\|\nabla\mathbf{u}\|_{L^2}^{3/2}\|\nabla\mathbf{u}_t\|_{L^2}\\&\le CA_1^{1/2}(T)\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2\\&\le C_9C_0^{1/8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2,\\M_2&\le\left|\int\rho\mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\,dx\right|\le C\|\rho\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^6}^2\\&\le C_{10}C_0^{1/8}\|\nabla\mathbf{u}_t\|_{L^2}^2,\end{aligned}$$

and

$$
\begin{aligned}
M_3 &\le \left|\int P_t \operatorname{div}\mathbf{u}_t\,dx\right| = R\left|\int (\rho_t\theta + \rho\theta_t)\operatorname{div}\mathbf{u}_t\,dx\right| \\
&\le C\int \big(|\rho\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla\rho||\theta||\nabla\mathbf{u}_t| + \rho|\theta_t||\nabla\mathbf{u}_t|\big)\,dx \\
&\le C\big(\|\rho\|_{L^6}\|\nabla\mathbf{u}\|_{L^6} + \|\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^3}\big)\|\theta\|_{L^6}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\quad + C\big\|\rho^{1/2}\theta_t\big\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\le \frac{1}{8}\underline{\mu}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\big\|\rho^{1/2}\theta_t\big\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2.
\end{aligned}
$$

By $(1.1)_1$ and (3.4) we have

$$
\begin{aligned}
M_4 &\le \left|\int (\mu_\rho\rho_t + \mu_\theta\theta_t)\big[|\mathcal{D}(\mathbf{u})|^2\big]_t\,dx\right| \\
&\le C\int \big[|\mu_\rho||\rho\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla\rho||\nabla\mathbf{u}||\nabla\mathbf{u}_t| + |\mu_\theta||\theta_t||\nabla\mathbf{u}||\nabla\mathbf{u}_t|\big]\,dx \\
&\le C\big(\|\rho\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^3} + \|\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^3}\big)\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\quad + C\|\theta_t\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\le C\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}_t\|_{L^2} + C\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\theta_t\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\le \frac{1}{8}\underline{\mu}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2} + C\|\nabla\mathbf{u}\|\|\nabla\theta_t\|_{L^2}^2, \\
M_5 &\le \left|\int (\lambda_\rho\rho_t + \lambda_\theta\theta_t)\big[(\operatorname{div}\mathbf{u})^2\big]_t\,dx\right| \\
&\le \frac{1}{8}\underline{\mu}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2} + C\|\nabla\mathbf{u}\|\|\nabla\theta_t\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
M_6 &\le \left|\int (\mathbf{B}\cdot\nabla\mathbf{B}_t + \mathbf{B}_t\cdot\nabla\mathbf{B} - \nabla\mathbf{B}_t\cdot\mathbf{B} - \nabla\mathbf{B}\cdot\mathbf{B}_t)\cdot\mathbf{u}_t\,dx\right| \\
&\le C\|\mathbf{B}\|_{L^3}\|\nabla\mathbf{B}_t\|_{L^2}\|\mathbf{u}_t\|_{L^6} + C\|\mathbf{B}\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{B}_t\|_{L^6} \\
&\le \frac{1}{8}\underline{\mu}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{B}_t\|_{L^2}^2.
\end{aligned}
$$

Substituting M_1–M_6 into (3.27) and using (3.1) and Lemma 2.1, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2}^2 + \|\nabla\mathbf{u}_t\|_{L^2}^2 \\
&\quad\le C_{11}\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2} + C\big\|\rho^{1/2}\theta_t\big\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + CC_0^{1/8}\|\nabla\theta_t\|_{L^2}^2,
\end{aligned}
\tag{3.28}
$$

provided that $C_0 \le \varepsilon_{4,1} \triangleq \min\{1, (\frac{\underline{\mu}}{8C_9})^8, (\frac{\underline{\mu}}{8C_{10}})^8\}$.

Differentiating $(1.1)_4$ with respect to t yields

$$
\mathbf{B}_{tt} - \nu\Delta\mathbf{B}_t = -\mathbf{u}_t\cdot\nabla\mathbf{B} - \mathbf{u}\cdot\nabla\mathbf{B}_t + \mathbf{B}_t\cdot\nabla\mathbf{u} + \mathbf{B}\cdot\nabla\mathbf{u}_t - \mathbf{B}_t\operatorname{div}\mathbf{u} - \mathbf{B}\operatorname{div}\mathbf{u}_t. \tag{3.29}
$$

Multiplying (3.29) by $\mathbf{B}_t$ in L^2 and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{B}_t\|_{L^2}^2+\nu\|\nabla\mathbf{B}_t\|_{L^2}^2\\
&\quad\le C\bigl(\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{B}\|_{L^3}\|\mathbf{B}_t\|_{L^6}+\|\mathbf{u}_t\|_{L^6}\|\mathbf{B}\|_{L^3}\|\nabla\mathbf{B}_t\|_{L^2}\bigr)\\
&\qquad+C\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{B}_t\|_{L^2}\|\mathbf{B}_t\|_{L^3}+C\|\mathbf{B}_t\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{B}_t\|_{L^3}\\
&\quad\le\frac{\nu}{4}\|\nabla\mathbf{B}_t\|_{L^2}^2+C\bigl(\|\mathbf{B}\|_{L^3}^2\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\mathbf{B}_t\|_{L^3}^2\|\nabla\mathbf{u}\|_{L^2}^2\bigr)\\
&\quad\le\left(\frac{\nu}{4}+C\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla\mathbf{B}_t\|_{L^2}^2+C\|\nabla\mathbf{B}\|_{L^2}^2\|\nabla\mathbf{u}_t\|_{L^2}^2\\
&\quad\le\left(\frac{\nu}{4}+C_{12}C_0^{1/4}\right)\|\nabla\mathbf{B}_t\|_{L^2}^2+CC_0^{1/4}\|\nabla\mathbf{u}_t\|_{L^2}^2,
\end{aligned}$$

where we have used (2.2) and the Poincaré inequality. Thus

$$\frac{d}{dt}\|\mathbf{B}_t\|_{L^2}^2+\|\nabla\mathbf{B}_t\|_{L^2}^2\le C_{13}C_0^{1/4}\|\nabla\mathbf{u}_t\|_{L^2}^2, \tag{3.30}$$

provided that $C_0\le\varepsilon_{4,2}\triangleq\min\{1,(\frac{\nu}{4C_{12}})^4\}$.

Calculating (3.28)+(3.30)$\times 2C_{11}$ gives

$$\begin{aligned}
&\frac{d}{dt}\bigl(\bigl\|\rho^{1/2}\mathbf{u}_t\bigr\|_{L^2}^2+\|\mathbf{B}_t\|_{L^2}^2\bigr)+\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\nabla\mathbf{B}_t\|_{L^2}^2\\
&\quad\le C\|\nabla\mathbf{u}\|_{L^2}+C\bigl\|\rho^{1/2}\theta_t\bigr\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2+CC_0^{1/8}\|\nabla\theta_t\|_{L^2}^2,
\end{aligned}\tag{3.31}$$

provided that $C_0\le\varepsilon_{4,3}\triangleq\min\{1,(\frac{1}{4C_{11}C_{13}})^4\}$. Integrating (3.31) over $[0,T]$ and using (3.18), we have from (3.1), (3.6), and (3.20) that

$$\begin{aligned}
&\sup_{t\in[0,T]}\bigl(\bigl\|\rho^{1/2}\mathbf{u}_t\bigr\|_{L^2}^2+\|\mathbf{B}_t\|_{L^2}^2+\bigl\|\nabla^2\mathbf{B}\bigr\|_{L^2}^2\bigr)+\int_0^T\bigl(\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\nabla\mathbf{B}_t\|_{L^2}^2\bigr)\,dt\\
&\quad\le K_1+C\|\nabla\mathbf{u}\|_{L^2}+CC_0^{1/8}A_4(T)+CA_2(T)+CC_0^{1/2}\\
&\quad\le K_1+CA_1^{1/2}(T)+CC_0^{1/8}A_4(T)+CA_2(T)+CC_0^{1/2}\\
&\quad\le K_1+C_{14}C_0^{1/8}\\
&\quad\le 2K_1,
\end{aligned}$$

provided that $C_0\le\varepsilon_4\triangleq\min\{\varepsilon_{4,1},\varepsilon_{4,2},\varepsilon_{4,3},(\frac{K_1}{C_{14}})^8\}$. Thus we immediately obtain (3.25). The proof of the lemma is therefore completed. □

Lemma 3.7 *Let* $(\rho,\mathbf{u},\theta,\mathbf{B})$ *be a smooth solution of* (1.1)–(1.4) *satisfying* (3.1). *Then there exists a positive constant* ε_5, *depending only on* $\kappa,\nu,R,c_\nu,\underline{\mu},\bar{\rho},\Omega,g_1$, *and* g_2, *such that*

$$\sup_{t\in[0,T]}\|\nabla\theta\|_{L^2}^2+\int_0^T\bigl\|\rho^{1/2}\theta_t\bigr\|_{L^2}^2\,dt\le C_0^{1/2}\tag{3.32}$$

and

$$\sup_{t\in[0,T]} t^2\|\nabla\theta\|_{L^2}^2 + \int_0^T t^2\|\rho^{1/2}\theta_t\|_{L^2}^2\,dt \le CC_0^{9/16}, \tag{3.33}$$

provided that $C_0 \le \varepsilon_5$.

Proof We multiply $(1.1)_3$ by θ_t in L^2. Then from (2.2)–(2.4), (3.1), (3.2), and (3.4) it follows that

$$\begin{aligned}
&\frac{\kappa}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + c_v\|\rho^{1/2}\theta_t\|_{L^2}^2 \\
&\quad= \int\bigl[-c_v\rho\mathbf{u}\cdot\nabla\theta - P\operatorname{div}\mathbf{u} + 2\mu|\mathcal{D}(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2 + \nu|\operatorname{curl}\mathbf{B}|^2\bigr]\theta_t\,dx \\
&\quad\le \frac{c_v}{2}\|\rho^{1/2}\theta_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{H^1}^2\|\nabla\theta\|_{L^2}^2 + C\|\nabla\theta\|_{H^1}^2\|\nabla\mathbf{u}\|_{L^2}^2 \\
&\qquad + C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3}\|\theta_t\|_{L^6} + C\|\nabla\mathbf{B}\|_{L^2}\|\nabla\mathbf{B}\|_{L^3}\|\theta_t\|_{L^6} \\
&\quad\le \frac{c_v}{2}\|\rho^{1/2}\theta_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2 + CA_1^{1/4}(T)\bigl(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{B}\|_{L^2}\bigr)\|\nabla\theta_t\|_{L^2},
\end{aligned}$$

and thus

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \|\rho^{1/2}\theta_t\|_{L^2}^2 \\
&\quad\le C\|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2 + CC_0^{1/16}\bigl(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{B}\|_{L^2}\bigr)\|\nabla\theta_t\|_{L^2}.
\end{aligned} \tag{3.34}$$

Integrating (3.34) over $[0,T]$, from (3.1) and (3.6) we have that

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\nabla\theta\|_{L^2}^2 + \int_0^T\|\rho^{1/2}\theta_t\|_{L^2}^2\,dt \\
&\quad\le C_0 + CC_0 + CC_0^{9/16}A_4^{1/2}(T) \\
&\quad\le C_{15}C_0 + C_{16}C_0^{9/16}A_4^{1/2}(T) \le C_0^{1/2},
\end{aligned}$$

provided that $C_0 \le \varepsilon_{5,1} \triangleq \min\{\varepsilon_4, \frac{C_0^{1/2}}{2C_{15}}, (\frac{C_0^{1/2}}{2C_{16}K_2^{1/2}})^{16/9}\}$, which leads to (3.32).

Next, multiplying (3.34) by t^2 and integrating the result over $[0,T]$, from (3.1) and (3.6) we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} t^2\|\nabla\theta\|_{L^2}^2 + \int_0^T t^2\|\rho^{1/2}\theta_t\|_{L^2}^2\,dt \\
&\quad\le C_{17}C_0 + CC_0^{9/16}A_4^{1/2}(T) \le CC_0^{9/16},
\end{aligned}$$

provided that $C_0 \le \varepsilon_{5,2} \triangleq \min\{1, \frac{C_0^{9/16}}{2C_{17}}\}$. Thus we immediately obtain (3.33). Choosing $C_0 \le \varepsilon_5 \triangleq \min\{\varepsilon_{5,1}, \varepsilon_{5,2}\}$, we complete the proof of the lemma. □

Lemma 3.8 *Let $(\rho, \mathbf{u}, \theta, \mathbf{B})$ be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant ε_6, depending only on $\kappa, \nu, R, c_v, \underline{\mu}$, $\bar{\rho}$, Ω, g_1, and g_2, such that*

$$\sup_{t\in[0,T]} t^2\left\|\left(\rho^{1/2}\mathbf{u}_t, \mathbf{B}_t, \nabla^2\mathbf{B}\right)\right\|_{L^2}^2 + \int_0^T t^2\left\|(\nabla\mathbf{u}_t, \nabla\mathbf{B}_t)\right\|_{L^2}^2 dt \le CC_0^{5/16}, \tag{3.35}$$

provided that $C_0 \le \varepsilon_6$.

Proof Multiplying (3.31) by t^2 and integrating the resulting equation over $[0, T]$, from (3.6), (3.14), (3.18), (3.20), and (3.33) it follows that

$$\begin{aligned}
&\sup_{t\in[0,T]} t^2\left(\left\|\rho^{1/2}\mathbf{u}_t\right\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right) + \int_0^T t^2\left(\|\nabla\mathbf{u}_t\|_{L^2}^2 + \|\nabla\mathbf{B}_t\|_{L^2}^2\right) dt \\
&\quad\le C\int_0^T t\left(\left\|\rho^{1/2}\mathbf{u}_t\right\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right) dt \\
&\qquad + C\int_0^T t^2\left(\|\nabla\mathbf{u}\|_{L^2} + \left\|\rho^{1/2}\theta_t\right\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\theta_t\|_{L^2}^2\right) dt \\
&\quad\le CC_0^{5/16} + CC_0^{1/2} + CC_0^{9/16} + CC_0 \\
&\quad\le CC_0^{5/16},
\end{aligned}$$

provided that $C_0 \le \varepsilon_6 \triangleq \min\{1, \varepsilon_5\}$. Thus we immediately obtain (3.35). The proof of the lemma is therefore completed. □

Lemma 3.9 *Let $(\rho, \mathbf{u}, \theta, \mathbf{B})$ be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant ε_7, depending only on $\kappa, \nu, R, c_v, \underline{\mu}$, $\bar{\rho}$, Ω, g_1, and g_2, such that*

$$\sup_{t\in[0,T]} \left\|\rho^{1/2}\theta_t\right\|_{L^2}^2 + \int_0^T \|\nabla\theta_t\|_{L^2}^2 dt \le 2K_2, \tag{3.36}$$

provided that $C_0 \le \varepsilon_7$.

Proof By $(1.1)_1$ and $(1.1)_3$ we have that

$$\begin{aligned}
&c_v[\rho\theta_{tt} + \rho\mathbf{u}\cdot\nabla\theta_t] - \kappa\Delta\theta_t \\
&\quad= \operatorname{div}(\rho\mathbf{u})(c_v\theta_t + c_v\mathbf{u}\cdot\nabla\theta + R\theta\operatorname{div}\mathbf{u}) - \rho(c_v\mathbf{u}_t\cdot\nabla\theta + R\theta_t\operatorname{div}\mathbf{u}) - R\rho\theta\operatorname{div}\mathbf{u}_t \\
&\qquad + (2\mu_\rho\rho_t + 2\mu_\theta\theta_t)\left|\mathcal{D}(\mathbf{u})\right|^2 + 4\mu(\rho,\theta)\mathcal{D}(\mathbf{u}):\mathcal{D}(\mathbf{u}_t) \\
&\qquad + (\lambda_\rho\rho_t + \lambda_\theta\theta_t)(\operatorname{div}\mathbf{u})^2 + 2\lambda(\rho,\theta)\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{u}_t + 2\nu\operatorname{curl}\mathbf{B}\cdot\operatorname{curl}\mathbf{B}_t.
\end{aligned} \tag{3.37}$$

Multiplying (3.37) by θ_t and integrating the resulting equation on Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\rho^{1/2}\theta_t\right\|_{L^2}^2 + \kappa\|\nabla\theta_t\|_{L^2}^2 \\
&\quad= \int\left[\operatorname{div}(\rho\mathbf{u})(c_v\theta_t + c_v\mathbf{u}\cdot\nabla\theta + R\theta\operatorname{div}\mathbf{u})\right]\theta_t\, dx
\end{aligned}$$

$$
\begin{aligned}
&-\int\big[\rho(c_v\mathbf{u}_t\cdot\nabla\theta+R\theta_t\operatorname{div}\mathbf{u})\big]\theta_t\,dx-\int R\rho\theta\operatorname{div}\mathbf{u}_t\theta_t\,dx\\
&+\int 4\mu(\rho,\theta)\mathcal{D}(\mathbf{u}):\mathcal{D}(\mathbf{u}_t)\theta_t\,dx\\
&+\int(2\mu_\rho\rho_t+2\mu_\theta\theta_t)\big|\mathcal{D}(\mathbf{u})\big|^2\theta_t\,dx+\int(\lambda_\rho\rho_t+\lambda_\theta\theta_t)(\operatorname{div}\mathbf{u})^2\theta_t\,dx\\
&+2\int\lambda(\rho,\theta)\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{u}_t\theta_t\,dx+2\int\nu\operatorname{curl}\mathbf{B}\cdot\operatorname{curl}\mathbf{B}_t\theta_t\,dx\triangleq\sum_{i=1}^{6}I_i.
\end{aligned}
\tag{3.38}
$$

The right-hand side of (3.38) can be estimated as follows. By (2.2)–(2.4), (3.1), (3.2), and (3.4) we have

$$
\begin{aligned}
I_1&=\int\big[(\mathbf{u}\cdot\nabla\rho+\rho\operatorname{div}\mathbf{u})(c_v\theta_t+c_v\mathbf{u}\cdot\nabla\theta+R\theta\operatorname{div}\mathbf{u})\big]\theta_t\,dx\\
&\le C\big(\|\mathbf{u}\|_{L^6}\|\nabla\rho\|_{L^2}+\|\rho\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\big)\|\theta_t\|_{L^6}^2+C\|\mathbf{u}\|_{L^\infty}^2\|\nabla\rho\|_{L^3}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^6}\\
&\quad+C\|\nabla\mathbf{u}\|_{L^6}\|\mathbf{u}\|_{L^6}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^6}+C\|\nabla\mathbf{u}\|_{L^6}\|\theta\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\theta_t\|_{L^6}\\
&\quad+C\|\mathbf{u}\|_{L^\infty}\|\theta\|_{L^\infty}\|\nabla\rho\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\|\theta_t\|_{L^6}\\
&\le C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^2}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{H^1}^2\big(\|\theta\|_{L^\infty}+\|\nabla\rho\|_{L^3}+1\big)\|\nabla\theta\|_{L^2}\|\nabla\theta_t\|_{L^2}\\
&\le\left[CC_0^{1/8}+\frac{\kappa}{16}\right]\|\nabla\theta_t\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2,\\
I_2&=-\int\big[\rho(c_v\mathbf{u}_t\cdot\nabla\theta+R\theta_t\operatorname{div}\mathbf{u})\big]\theta_t\,dx\\
&\le C\|\rho\|_{L^6}\|\mathbf{u}_t\|_{L^6}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^6}+C\|\rho\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\theta_t\|_{L^6}^2\\
&\le C\|\nabla\rho\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\|\nabla\theta\|_{L^2}\|\nabla\theta_t\|_{L^2}+C\|\nabla\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\theta_t\|_{L^2}^2\\
&\le\left[CC_0^{1/8}+\frac{\kappa}{16}\right]\|\nabla\theta_t\|_{L^2}^2+CC_0^{1/8}\|\nabla\mathbf{u}_t\|_{L^2}^2,\\
I_3&=-R\int\rho\theta\operatorname{div}\mathbf{u}_t\theta_t\,dx\\
&\le\|\rho\|_{L^6}\|\theta\|_{L^6}\|\nabla\mathbf{u}_t\|_{L^2}\|\theta_t\|_{L^6}\\
&\le\|\nabla\rho\|_{L^2}\|\nabla\theta\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\\
&\le CC_0^{1/8}\|\nabla\theta_t\|_{L^2}^2+CC_0^{1/8}\|\nabla\mathbf{u}_t\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
I_4&=\int 4\mu(\rho,\theta)\mathcal{D}(\mathbf{u}):\mathcal{D}(\mathbf{u}_t)\theta_t\,dx+\int(2\mu_\rho\rho_t+2\mu_\theta\theta_t)\big|\mathcal{D}(\mathbf{u})\big|^2\theta_t\,dx\\
&\le C\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2}\|\theta_t\|_{L^6}+C\|\nabla\mathbf{u}\|_{L^3}^2\|\theta_t\|_{L^6}^2\\
&\quad+\big(\|\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\big)\|\nabla\mathbf{u}\|_{L^6}^2\|\theta_t\|_{L^6}\\
&\le C\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{H^1}^{1/2}\|\nabla\mathbf{u}_t\|_{L^2}\|\nabla\theta_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\|\nabla\theta_t\|_{L^2}^2\\
&\quad+\big(\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{L^6}^{1/2}+\|\nabla\mathbf{u}\|_{L^2}\big)\|\nabla\mathbf{u}\|_{L^6}^2\|\nabla\theta_t\|_{L^2}\\
&\le\frac{\kappa}{8}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\theta_t\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2
\end{aligned}
$$

$$\le \left[CC_0^{1/8} + \frac{\kappa}{16}\right]\|\nabla\theta_t\|_{L^2}^2 + CC_0^{1/8}\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^2}^2.$$

Similarly,

$$\begin{aligned} I_5 &= \int (\lambda_\rho \rho_t + \lambda_\theta \theta_t)(\operatorname{div}\mathbf{u})^2\theta_t\,dx + 2\int \lambda(\rho,\theta)\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{u}_t\theta_t\,dx \\ &\le \left[CC_0^{1/8} + \frac{\kappa}{16}\right]\|\nabla\theta_t\|_{L^2}^2 + CC_0^{1/8}\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} I_6 &= 2\int \nu \operatorname{curl}\mathbf{B}\cdot\operatorname{curl}\mathbf{B}_t\theta_t\,dx \le C\|\nabla\mathbf{B}\|_{L^3}\|\nabla\mathbf{B}_t\|_{L^2}\|\theta_t\|_{L^6} \\ &\le C\|\nabla\mathbf{B}\|_{L^2}^{1/2}\|\nabla\mathbf{B}\|_{H^1}^{1/2}\|\nabla\mathbf{B}_t\|_{L^2}\|\nabla\theta_t\|_{L^2} \\ &\le C\|\nabla\mathbf{B}\|_{L^2}\|\nabla\theta_t\|_{L^2}^2 + C\|\nabla\mathbf{B}\|_{L^2}\|\nabla\mathbf{B}_t\|_{L^2}^2 \\ &\le CC_0^{1/8}\|\nabla\theta_t\|_{L^2}^2 + CC_0^{1/8}\|\nabla\mathbf{B}_t\|_{L^2}^2. \end{aligned}$$

Substituting I_1–I_6 into (3.38), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\rho^{1/2}\theta_t\right\|_{L^2}^2 + \left[\frac{3\kappa}{4} - C_{18}C_0^{1/8}\right]\|\nabla\theta_t\|_{L^2}^2 \\ &\quad \le C\|\nabla\theta\|_{L^2}^2 + CC_0^{1/8}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2 + CC_0^{1/8}\|\nabla\mathbf{B}_t\|_{L^2}^2, \end{aligned}$$

and thus

$$\frac{d}{dt}\left\|\rho^{1/2}\theta_t\right\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^2 \le C\left\|(\nabla\theta,\nabla\mathbf{u})\right\|_{L^2}^2 + CC_0^{1/8}\left\|(\nabla\mathbf{u}_t,\nabla\mathbf{B}_t)\right\|_{L^2}^2, \tag{3.39}$$

provided that $C_0 \le \varepsilon_{7,1} \triangleq \min\{1, (\frac{\kappa}{4C_{18}})^2\}$.

Integrating (3.39) over $[0,T]$ gives

$$\sup_{t\in[0,T]}\left\|\rho^{1/2}\theta_t\right\|_{L^2}^2 + \int_0^T \|\nabla\theta_t\|_{L^2}^2\,dt \le K_2 + C_{19}C_0 + C_{20}C_0^{1/8}K_1 \le 2K_2,$$

provided that $C_0 \le \varepsilon_7 \triangleq \min\{\varepsilon_{7,1}, \frac{K_2}{2C_{19}}, (\frac{K_2}{2C_{20}K_1})^8\}$. Thus we immediately obtain (3.36). The proof of the lemma is therefore completed. □

Lemma 3.10 *Let* $(\rho,\mathbf{u},\theta,\mathbf{B})$ *be a smooth solution of* (1.1)–(1.4) *satisfying* (3.1). *Then there exists a positive constant* ε_8, *depending only on* $\kappa,\nu,R,c_v,\underline{\mu}$, $\bar{\rho}$, Ω, g_1, *and* g_2, *such that*

$$\sup_{(x,t)\in\Omega\times[0,T]}\rho(x,t) \le 2\bar{\rho}, \tag{3.40}$$

provided that $C_0 \le \varepsilon_8$.

Proof Lemma 2.2, together with $(1.1)_2$, (1.6), (3.1), (3.2), (3.4), and the Hölder inequality, gives

$$\left\|\nabla^2\mathbf{u}\right\|_{L^{3+\beta}} \le C\left(\|\rho\mathbf{u}_t\|_{L^{3+\beta}} + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^{3+\beta}} + \left\|(\nabla\rho + \nabla\theta)\nabla\mathbf{u}\right\|_{L^{3+\beta}}\right)$$

$$
\begin{aligned}
&+ C\big(\|\nabla P\|_{L^{3+\beta}} + \|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^{3+\beta}}\big)\\
&\le C\big[\|\rho\mathbf{u}_t\|_{L^4} + \big(\|\mathbf{u}\|_{L^q} + \|\nabla\rho\|_{L^q} + \|\nabla\theta\|_{L^q}\big)\|\nabla\mathbf{u}\|_{L^{12}}\big]\\
&\quad + C\big(\|\rho\|_{L^{12}}\|\nabla\theta\|_{L^q} + \|\nabla\rho\|_{L^q}\|\theta\|_{L^{12}} + \|\mathbf{B}\|_{L^q}\|\nabla\mathbf{B}\|_{L^{12}}\big)\\
&\le C\big(\|\rho\mathbf{u}_t\|_{L^2}^{1/4}\|\rho\mathbf{u}_t\|_{L^6}^{3/4} + \|\nabla\mathbf{u}\|_{L^{12}} + \|\nabla\theta\|_{L^q} + \|\theta\|_{L^{12}} + \|\nabla\mathbf{B}\|_{L^{12}}\big)\\
&\le C\big(\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2}^{1/4}\|\nabla\mathbf{u}_t\|_{L^2}^{3/4} + \big\||\nabla\mathbf{u}|^2\big\|_{H^1}^{1/2} + \big\||\nabla\mathbf{B}|^2\big\|_{H^1}^{1/2}\big)\\
&\quad + C\big(\|\nabla\theta\|_{H^1} + \big\|\theta^2\big\|_{H^1}^{1/2}\big)\\
&\le C\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2}^{1/4}\|\nabla\mathbf{u}_t\|_{L^2}^{3/4} + C\|\nabla\mathbf{u}\|_{L^4} + C\|\nabla\mathbf{u}\|_{L^{2(3+\beta)/(1+\beta)}}^{1/2}\big\|\nabla^2\mathbf{u}\big\|_{L^{3+\beta}}^{1/2}\\
&\quad + C\|\nabla\mathbf{B}\|_{L^4} + C\|\nabla\mathbf{B}\|_{L^{2(3+\beta)/(1+\beta)}}^{1/2}\big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}}^{1/2} + C\|\nabla\theta\|_{H^1}\\
&\le \frac{1}{2}\big\|\nabla^2\mathbf{u}\big\|_{L^{3+\beta}} + \frac{1}{2}\big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}} + C\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2}^{1/4}\|\nabla\mathbf{u}_t\|_{L^2}^{3/4}\\
&\quad + C\|\nabla\mathbf{u}\|_{H^1} + C\|\nabla\mathbf{B}\|_{H^1} + C\|\nabla\theta\|_{H^1},
\end{aligned}
$$

where we have used the estimates

$$
\|\rho\mathbf{u}_t\|_{L^{3+\beta}} \le C\bigg(\int|\rho\mathbf{u}_t|^{(3+\beta)\times\frac{4}{3+\beta}}\,dx\bigg)^{\frac{1}{3+\beta}\times\frac{3+\beta}{4}}\bigg(\int 1^{\frac{4}{1-\beta}}\,dx\bigg)^{\frac{1}{3+\beta}\times\frac{1-\beta}{4}} \le C\|\rho\mathbf{u}_t\|_{L^4}
$$

and

$$
\|\nabla\mathbf{u}\|_{L^{12}} = \big\||\nabla\mathbf{u}|^2\big\|_{L^6}^{1/2} \le C\big\||\nabla\mathbf{u}|^2\big\|_{H^1}^{1/2}.
$$

Thus

$$
\big\|\nabla^2\mathbf{u}\big\|_{L^{3+\beta}} \le \frac{1}{4}\big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}} + C\big\|(\nabla\mathbf{u},\nabla\theta,\nabla\mathbf{B})\big\|_{H^1} + C\big\|\rho^{1/2}\mathbf{u}_t\big\|_{L^2}^{1/4}\|\nabla\mathbf{u}_t\|_{L^2}^{3/4}. \tag{3.41}
$$

On the other hand, from (1.1)$_4$ and (3.4) it follows that

$$
\begin{aligned}
\big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}} &\le C\big(\|\mathbf{B}_t\|_{L^{3+\beta}} + \|\mathbf{u}\cdot\nabla\mathbf{B}\|_{L^{3+\beta}} + \|\mathbf{B}\cdot\nabla\mathbf{u}\|_{L^{3+\beta}} + \|\mathbf{B}\operatorname{div}\mathbf{u}\|_{L^{3+\beta}}\big)\\
&\le C\|\mathbf{B}_t\|_{L^4} + C\|\mathbf{u}\|_{L^q}\|\nabla\mathbf{B}\|_{L^{12}} + C\|\nabla\mathbf{u}\|_{L^q}\|\mathbf{B}\|_{L^{12}}\\
&\le C\|\mathbf{B}_t\|_{L^2}^{1/4}\|\nabla\mathbf{B}_t\|_{L^2}^{3/4} + C\|\nabla\mathbf{B}\|_{L^{12}} + C\|\mathbf{B}\|_{L^{12}}\\
&\le C\|\mathbf{B}_t\|_{L^2}^{1/4}\|\nabla\mathbf{B}_t\|_{L^2}^{3/4} + C\|\nabla\mathbf{B}\|_{H^1} + C\|\nabla\mathbf{B}\|_{L^{2(3+\beta)/(1+\beta)}}^{1/2}\big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}}^{1/2}\\
&\le \frac{1}{2}\big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}} + C\|\mathbf{B}_t\|_{L^2}^{1/4}\|\nabla\mathbf{B}_t\|_{L^2}^{3/4} + C\|\nabla\mathbf{B}\|_{H^1},
\end{aligned}
$$

and thus

$$
\big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}} \le C\|\mathbf{B}_t\|_{L^2}^{1/4}\|\nabla\mathbf{B}_t\|_{L^2}^{3/4} + C\|\nabla\mathbf{B}\|_{H^1}. \tag{3.42}
$$

Due to (3.1), (3.3), and (3.5), we have

$$
\big\|(\nabla\mathbf{u},\nabla\theta,\nabla\mathbf{B})\big\|_{H^1} \le C\big\|\big(\rho^{1/2}\mathbf{u}_t,\rho^{1/2}\theta_t,\mathbf{B}_t\big)\big\|_{L^2} + C\|\nabla\theta\|_{L^2} + C\big\|(\nabla\mathbf{u},\nabla\mathbf{B})\big\|_{L^2}^{1/4}. \tag{3.43}
$$

Combining (3.41) and (3.42) with (3.43), yields

$$\begin{aligned} &\|\nabla^2\mathbf{u}\|_{L^{3+\beta}} + \|\nabla^2\mathbf{B}\|_{L^{3+\beta}} \\ &\quad \le C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{1/4}\|\nabla\mathbf{u}_t\|_{L^2}^{3/4} + C\|\mathbf{B}_t\|_{L^2}^{1/4}\|\nabla\mathbf{B}_t\|_{L^2}^{3/4} \\ &\qquad + C\|(\rho^{1/2}\mathbf{u}_t, \rho^{1/2}\theta_t, \mathbf{B}_t)\|_{L^2} + C\|\nabla\theta\|_{L^2} + C\|(\nabla\mathbf{u}, \nabla\mathbf{B})\|_{L^2}^{1/4}. \end{aligned} \tag{3.44}$$

Integrating (3.44) over $[0, T]$ gives

$$\begin{aligned} &\int_0^T (\|\nabla^2\mathbf{u}\|_{L^{3+\beta}} + \|\nabla^2\mathbf{B}\|_{L^{3+\beta}})\,dt \\ &\quad \le C\int_0^T \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{1/4}\|\nabla\mathbf{u}_t\|_{L^2}^{3/4}\,dt + C\int_0^T \|\mathbf{B}_t\|_{L^2}^{1/4}\|\nabla\mathbf{B}_t\|_{L^2}^{3/4}\,dt \\ &\qquad + C\int_0^T \|(\rho^{1/2}\mathbf{u}_t, \rho^{1/2}\theta_t, \mathbf{B}_t)\|_{L^2}\,dt + C\int_0^T \|\nabla\theta\|_{L^2}\,dt \\ &\qquad + C\int_0^T \|(\nabla\mathbf{u}, \nabla\mathbf{B})\|_{L^2}^{1/4}\,dt \triangleq \sum_{i=1}^5 J_5. \end{aligned} \tag{3.45}$$

The right-hand side of (3.45) cab be estimated as follows. By (3.13), (3.14), and (3.25) we have

$$\begin{aligned} J_1 &\le C\left(\int_0^T \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{2/5}\,dt\right)^{5/8}\left(\int_0^T \|\nabla\mathbf{u}_t\|_{L^2}^2\,dt\right)^{3/8} \\ &\le C\left(\int_0^{\sigma(T)} \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{2/5}\,dt\right)^{5/8} + C\left(\int_{\sigma(T)}^T \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{2/5}\,dt\right)^{5/8} \\ &\le C\left(\int_0^{\sigma(T)} \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{2}\,dt\right)^{1/8}\left(\int_0^{\sigma(T)} dt\right)^{1/2} \\ &\quad + C\left(\int_{\sigma(T)}^T t^8\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{2}\,dt\right)^{1/8}\left(\int_{\sigma(T)}^T t^{-2}\,dt\right)^{1/2} \le CC_0^{1/32}. \end{aligned}$$

Similarly,

$$\begin{aligned} J_2 &\le C\left(\int_0^{\sigma(T)} \|\mathbf{B}_t\|_{L^2}^2\,dt\right)^{1/8}\left(\int_0^{\sigma(T)} dt\right)^{1/2} \\ &\quad + C\left(\int_{\sigma(T)}^T t^8\|\nabla\mathbf{B}_t\|_{L^2}^2\,dt\right)^{1/8}\left(\int_{\sigma(T)}^T t^{-2}\,dt\right)^{1/2} \le CC_0^{1/32}. \end{aligned}$$

By (3.13), (3.14), (3.32), and (3.33) we have

$$\begin{aligned} J_3 &\le C\left(\int_0^{\sigma(T)} \|(\rho^{1/2}\mathbf{u}_t, \rho^{1/2}\theta_t, \mathbf{B}_t)\|_{L^2}^2\,dt\right)^{1/2}\left(\int_0^{\sigma(T)} dt\right)^{1/2} \\ &\quad + C\left(\int_{\sigma(T)}^T t^2\|(\rho^{1/2}\mathbf{u}_t, \rho^{1/2}\theta_t, \mathbf{B}_t)\|_{L^2}^2\,dt\right)^{1/2}\left(\int_{\sigma(T)}^T t^{-2}\,dt\right)^{1/2} \\ &\le CC_0^{1/8}, \end{aligned}$$

$$\begin{aligned} J_4 &\le C\left(\int_0^{\sigma(T)} \|\nabla\theta\|_{L^2}^2\,dt\right)^{1/2}\left(\int_0^{\sigma(T)} dt\right)^{1/2} \\ &\quad + C\left(\int_{\sigma(T)}^T t^2\|\nabla\theta\|_{L^2}^2\,dt\right)^{1/2}\left(\int_{\sigma(T)}^T t^{-2}\,dt\right)^{1/2} \\ &\le CC_0^{1/2}, \end{aligned}$$

and

$$\begin{aligned} J_5 &\le C\left(\int_0^{\sigma(T)} \big\|(\nabla\mathbf{u},\nabla\mathbf{B})\big\|_{L^2}^2\,dt\right)^{1/8}\left(\int_0^{\sigma(T)} dt\right)^{7/8} \\ &\quad + C\left(\int_{\sigma(T)}^T t^8\big\|(\nabla\mathbf{u},\nabla\mathbf{B})\big\|_{L^2}^2\,dt\right)^{1/8}\left(\int_{\sigma(T)}^T t^{-8/7}\,dt\right)^{7/8} \\ &\le CC_0^{1/8}. \end{aligned}$$

Putting J_1–J_5 into (3.45), we obtain

$$\int_0^T \left(\big\|\nabla^2\mathbf{u}\big\|_{L^{3+\beta}} + \big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}}\right)dt \le CC_0^{1/32}. \tag{3.46}$$

Thus, combining (2.2)–(2.4) with (3.46), we have

$$\begin{aligned} &\int_0^T \left(\|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla\mathbf{B}\|_{L^\infty}\right)dt \\ &\quad \le C\int_0^T \left(\|\nabla\mathbf{u}\|_{L^4} + \|\nabla\mathbf{B}\|_{L^4} + \big\|\nabla^2\mathbf{u}\big\|_{L^{3+\beta}} + \big\|\nabla^2\mathbf{B}\big\|_{L^{3+\beta}}\right)dt \\ &\quad \le CC_0^{1/32}. \end{aligned} \tag{3.47}$$

By $(1.1)_1$ we have that

$$\frac{d}{dt}\rho\big(t,U(t,s,x)\big) = -\rho\big(t,U(t,s,x)\big)\operatorname{div}\mathbf{u}\big(t,U(t,s,x)\big), \tag{3.48}$$

where

$$\frac{d}{dt}\rho\big(t,U(t,s,x)\big) = \rho_t\big(t,U(t,s,x)\big) + \mathbf{u}\big(t,U(t,s,x)\big)\cdot\nabla\rho\big(t,U(t,s,x)\big),$$

and $U \in C([0,T]\times[0,T])\times\Omega$ is the solution to the initial value problem

$$\begin{cases} \frac{d}{dt}U(t,s,x) = \mathbf{u}(t,U(t,s,x)), & t\in[0,T], \\ U(t,s,x) = x, & s\in[0,T], x\in\Omega. \end{cases}$$

With the help of (3.47) and (3.48), from the Gronwall inequality we get that

$$\rho \le \bar\rho\exp\left\{\int_0^T \|\nabla\mathbf{u}\|_{L^\infty}\,dt\right\} \le \bar\rho\exp\big\{C_{21}C_0^{1/32}\big\} \le 2\bar\rho,$$

provided that $C_0 \le \varepsilon_8 \triangleq \min\{\varepsilon_7, (\frac{\ln 2}{C_{21}})^{32}\}$. Thus we immediately obtain (3.40). The proof of the lemma is therefore completed. □

Lemma 3.11 *Let $(\rho, \mathbf{u}, \theta, \mathbf{B})$ be a smooth solution of* (1.1)–(1.4) *satisfying* (3.1). *Then there exists a positive constant ε, depending only on $\kappa, \nu, R, c_v, \underline{\mu}$, $\bar{\rho}$, Ω, g_1, and g_2, such that*

$$\sup_{t\in[0,T]} \|\nabla\rho\|_{L^q} \le 3\|\nabla\rho\|_{L^q}, \tag{3.49}$$

and

$$\int_0^T \big(\|\nabla^2\mathbf{u}\|_{L^q} + \|\nabla^2\mathbf{B}\|_{L^q}\big)\,dt \le C, \tag{3.50}$$

provided that $C_0 \le \varepsilon$.

Proof Differentiating $(1.1)_1$ with respect to x_i and multiplying the results by $q|\partial_i\rho|^{q-2}\partial_i\rho$ give

$$\begin{aligned}&\big(|\nabla\rho|^q\big)_t + \operatorname{div}\big(|\nabla\rho|^q\mathbf{u}\big) + (q-1)|\nabla\rho|^q \operatorname{div}\mathbf{u}\\&\quad + q|\nabla\rho|^{q-2}(\nabla\rho)^{\mathrm{tr}}\nabla\mathbf{u}(\nabla\rho) + q\rho|\nabla\rho|^{q-2}\nabla\rho\cdot\nabla\operatorname{div}\mathbf{u} = 0.\end{aligned} \tag{3.51}$$

Integrating (3.51) on Ω yields

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q} + C_{22}\|\nabla^2\mathbf{u}\|_{L^q}. \tag{3.52}$$

With the help of $(1.1)_2$, Lemma 2.2, (3.2)–(3.4), (3.40), we get

$$\begin{aligned}\|\nabla^2\mathbf{u}\|_{L^q} &\le C\big(\|\rho\mathbf{u}_t\|_{L^q} + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^q} + \|(\nabla\rho+\nabla\theta)\nabla\mathbf{u}\|_{L^q}\big)\\&\quad + C\big(\|\nabla P\|_{L^q} + \|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^q}\big)\\&\le C\big[\|\rho\mathbf{u}_t\|_{L^6} + \big(\|\mathbf{u}\|_{L^q} + \|\nabla\rho\|_{L^q} + \|\nabla\theta\|_{L^q}\big)\|\nabla\mathbf{u}\|_{L^\infty}\big]\\&\quad + C\big(\|\rho\|_{L^\infty}\|\nabla\theta\|_{L^q} + \|\nabla\rho\|_{L^q}\|\theta\|_{L^\infty} + \|\mathbf{B}\|_{L^q}\|\nabla\mathbf{B}\|_{L^\infty}\big)\\&\le C\big(\|\nabla\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla\theta\|_{H^1} + \|\nabla\mathbf{B}\|_{L^\infty}\big)\\&\le C\big(\|\nabla\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla\mathbf{B}\|_{L^\infty}\big)\\&\quad + C\big(\|\rho^{1/2}\theta_t\|_{L^2} + \|\nabla\theta\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{1/4} + \|\nabla\mathbf{B}\|_{L^2}^{1/4}\big).\end{aligned} \tag{3.53}$$

Thus, similarly to (3.45),

$$\begin{aligned}\int_0^T \|\nabla^2\mathbf{u}\|_{L^q}\,dt &\le C\int_0^T \|\nabla\mathbf{u}_t\|_{L^2}\,dt + \int_0^T \big(\|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla\mathbf{B}\|_{L^\infty}\big)\,dt\\&\quad + C\int_0^T \big(\|\rho^{1/2}\theta_t\|_{L^2} + \|\nabla\theta\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{1/4} + \|\nabla\mathbf{B}\|_{L^2}^{1/4}\big)\,dt\\&\le C\int_0^T \|\nabla\mathbf{u}_t\|_{L^2}\,dt + CC_0^{1/32}.\end{aligned} \tag{3.54}$$

On the other hand, from (3.25) and (3.35) it follows that

$$\begin{aligned}\int_0^T \|\nabla \mathbf{u}_t\|_{L^2}\,dt &\le C\left(\int_0^s \|\nabla \mathbf{u}_t\|_{L^2}^2\,dt\right)^{1/2}\left(\int_0^s dt\right)^{1/2} \\ &\quad + C\left(\int_s^T t^2\|\nabla \mathbf{u}_t\|_{L^2}^2\,dt\right)^{1/2}\left(\int_s^T t^{-2}\,dt\right)^{1/2} \\ &\le Cs^{1/2} + Cs^{-1/2}C_0^{5/32}\forall s\in(0,T],\end{aligned}$$

which, together with (3.52) and (3.54), yields

$$C_{22}\int_0^T \left\|\nabla^2 \mathbf{u}\right\|_{L^q}dt \le C_{23}s^{1/2} + C_{24}s^{-1/2}C_0^{5/32} + C_{25}C_0^{1/32} \quad \forall s\in(0,T].$$

Fixing $s \ll \sigma(T)$ such that $C_{23}s^{1/2} \le \frac{1}{6}\|\nabla\rho_0\|_{L^q}$, we have

$$C_{22}\int_0^T \left\|\nabla^2 \mathbf{u}\right\|_{L^q}dt \le \frac{1}{2}\|\nabla\rho_0\|_{L^q}, \tag{3.55}$$

provided that $C_0 \le \varepsilon_9 \triangleq \min\{\varepsilon_8, (\frac{s^{1/2}\|\nabla\rho_0\|_{L^q}}{6C_{24}})^{32/5}, (\frac{\|\nabla\rho_0\|_{L^q}}{6C_{25}})^{32}\}$.

Using (3.47), (3.52), and (3.55), from the Gronwall inequality we obtain that

$$\begin{aligned}\sup_{t\in[0,T]}\|\nabla\rho\|_{L^q} &\le \exp\left\{\int_0^T C\|\nabla \mathbf{u}\|_{L^\infty}\,dt\right\}\left[\|\nabla\rho_0\|_{L^q} + \int_0^T C_{22}\left\|\nabla^2\mathbf{u}\right\|_{L^q}dt\right] \\ &\le \frac{3}{2}\exp\left\{C_{26}C_0^{1/32}\right\}\|\nabla\rho_0\|_{L^q} \\ &\le 3\|\nabla\rho_0\|_{L^q},\end{aligned}$$

provided that $C_0 \le \varepsilon \triangleq \min\{\varepsilon_9, (\frac{\ln 2}{C_{26}})^{32}\}$. Thus we immediately obtain (3.49).

On the other hand, from (3.42) it follows that

$$\begin{aligned}\left\|\nabla^2\mathbf{B}\right\|_{L^q} &\le C\big(\|\mathbf{B}_t\|_{L^q} + \|\mathbf{u}\cdot\nabla\mathbf{B}\|_{L^q} + \|\mathbf{B}\cdot\nabla\mathbf{u}\|_{L^q} + \|\mathbf{B}\operatorname{div}\mathbf{u}\|_{L^q}\big) \\ &\le C\|\mathbf{B}_t\|_{L^6} + C\|\mathbf{u}\|_{L^q}\|\nabla\mathbf{B}\|_{L^\infty} + C\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{B}\|_{L^q} \\ &\le C\|\nabla\mathbf{B}_t\|_{L^2} + C\|\nabla\mathbf{B}\|_{L^\infty} + C\|\nabla\mathbf{u}\|_{L^\infty},\end{aligned}$$

which, together with (3.53), gives

$$\begin{aligned}&\left\|\nabla^2\mathbf{u}\right\|_{L^q} + \left\|\nabla^2\mathbf{B}\right\|_{L^q} \\ &\quad\le C\big(\|\nabla\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{B}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla\mathbf{B}\|_{L^\infty}\big) \\ &\qquad + C\big(\left\|\rho^{1/2}\theta_t\right\|_{L^2} + \|\nabla\theta\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{1/4} + \|\nabla\mathbf{B}\|_{L^2}^{1/4}\big).\end{aligned} \tag{3.56}$$

Similarly to (3.54), from (3.56) we can immediately obtain (3.50). The proof of the lemma is therefore completed. □

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 with the fundamental uniform-in-time estimates established in Sect. 3. By Proposition 2.1 we know that there exists a positive time $T_* > 0$

such that system (1.1)–(1.4) possesses a strong solution $(\rho, \mathbf{u}, \theta, \mathbf{B})$ in $\Omega \times (0, T_*]$. Next, using all the a priori estimates established in Sect. 3, we extend the local strong solution to the global one.

Proof of global existence. First, in view of the definitions of $A_i(T)$ $(i = 1, 2, \dots, 7)$, from (3.1) we easily deduce that

$$
\begin{aligned}
&A_1(0) \le C_0^{1/4}, \qquad A_2(0) \le C_0^{1/2}, \qquad A_3(0) \le 2K_1, \qquad 0 \le \rho \le 2\bar{\rho}, \\
&A_4(0) \le 2K_2, \qquad A_5(0) \le 3\|\nabla \rho_0\|_{L^q}, \qquad A_6(0) + A_7(0) \le 1,
\end{aligned}
$$

since $C_0 \le \varepsilon$. Thus there exists $T_1 \in (0, T_*]$ such that (3.1) holds for $T = T_1$.

Set

$$
T^* \triangleq \sup\bigl\{T \,|\, (\rho, \mathbf{u}, \theta, \mathbf{B}) \text{ is a strong solution on } [0, T]\bigr\}
$$

and

$$
T_1^* \triangleq \sup\bigl\{T \,|\, (\rho, \mathbf{u}, \theta, \mathbf{B}) \text{ is a strong solution on } [0, T] \text{ satisfying } (3.1)\bigr\}.
$$

Thus $T_1^* \ge T_1 > 0$. By Proposition 3.1 we know that

$$
T^* = T_1^*,
$$

provided that $C_0 \le \varepsilon$.

Next, similarly to the proof of [25, Sect. 4], we can claim that $T^* = \infty$. Thus the proof of the theorem is therefore complete.

Conflict of interest
The author declares that he has no conflict of interest.

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Mingyu Zhang wrote all the manuscript text. All authors reviewed the manuscript.

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References

1. Bresch, D., Desjardins, B.: On the existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids. J. Math. Pures Appl. **87**(9), 57–90 (2007)
2. Chen, G.Q., Wang, D.: Existence and continuous dependence of large solutions for the magnetohydrodynamic equations. Z. Angew. Math. Phys. **54**, 608–632 (2003)
3. Ducomet, B., Feireisl, E.: The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars. Commun. Math. Phys. **226**, 595–629 (2006)
4. Fan, J., Jiang, S., Nakamura, G.: Vanishing shear viscosity limit in the magnetohydrodynamic equations. Commun. Math. Phys. **270**, 691–708 (2007)
5. Fan, J., Yu, W.: Strong solution to the compressible magnetohydrodynamic equations with vacuum. Nonlinear Anal., Real World Appl. **10**(1), 392–409 (2009)
6. Feireisl, E.: Dynamics of Viscous Compressible Fluids. Oxford Univ. Press, Oxford (2004)
7. Hoff, D.: Strong convergence to global solutions fro multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. Arch. Ration. Mech. Anal. **132**, 1–14 (1995)
8. Hoff, D.: Discontinuous solutions of the Navier–Stokes equations for multidimensional flows of heat-conducting fluids. Arch. Ration. Mech. Anal. **139**, 303–354 (1997)
9. Hoff, D., Tsyganov, E.: Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics. Z. Angew. Math. Phys. **56**, 791–804 (2005)
10. Hu, X., Wang, D.: Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. Commun. Math. Phys. **283**, 255–284 (2008)
11. Hu, X., Wang, D.: Compactness of weak solutions to the three-dimensional compressible magnetohydrodynamic equations. J. Differ. Equ. **245**, 2176–2198 (2008)
12. Hu, X., Wang, D.: Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. Arch. Ration. Mech. Anal. **197**, 203–238 (2010)
13. Huang, X.D., Li, J.: Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier–Stokes and magnetohydrodynamic flows. Commun. Math. Phys. **324**, 147–171 (2013)
14. Ito, H.: Extended Korn's inequality and the associated best possible constant. J. Elast. **24**, 43–78 (1990)
15. Kulikovskiy, A.G., Lyubimov, G.A.: Magnetohydrodynamics. Addison-Wesley, Reading (1965)
16. Ladyzenskaja, O., Solonnikov, V., Uraltseva, N.: Linear and Quasilinear Equations of Parabolic Type. Am. Math. Soc., Providence (1968)
17. Ladyzhenskaya, O., Uraltseva, N.: Linear and Quasilinear Elliptic Equations. Academic Press, San Diego (1978)
18. Laudau, L.D., Lifshitz, E.M.: Electrodynamics of Continuous Media, 2nd edn. Pergamon, New York (1984)
19. Matsumura, A., Nishida, T.: The initial boundary value problem for the equations of motion of compressible viscous and heat conductive fluids. Proc. Jpn. Acad., Ser. A, Math. Sci. **55**, 337–342 (1979)
20. Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. **20**(1), 67–104 (1980)
21. Matsumura, A., Nishida, T.: Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. Commun. Math. Phys. **89**, 445–464 (1983)
22. Nash, J.: Le problème de Cauchy pour les équations différentielles d'un fluide général. Bull. Soc. Math. Fr. **90**, 487–497 (1962)
23. Serrin, J.: On the uniqueness of compressible fluid motion. Arch. Ration. Mech. Anal. **3**, 271–288 (1959)
24. Valli, A., Zajaczkowski, W.M.: Navier–Stokes equations for compressible fluids: global existence and qualitative properties of the solutions in the general case. Commun. Math. Phys. **103**, 259–296 (1986)
25. Yu, H.B., Zhang, P.X.: Global strong solutions to the 3D full compressible Navier–Stokes equations with density-temperature-dependent viscosities in bounded domains. J. Differ. Equ. **268**, 7286–7310 (2020)