Siva Ranjani et al. *Boundary Value Problems* (2022) 2022:49
https://doi.org/10.1186/s13661-022-01628-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

# Analysis of stochastic neutral fractional functional differential equations

Alagesan Siva Ranjani[1], Murugan Suvinthra[1], Krishnan Balachandran[2] and Yong-Ki Ma[3*]

*Correspondence:
ykma@kongju.ac.kr
[3]Department of Applied Mathematics, Kongju National University, Chungcheongnam-do 32588, Republic of Korea
Full list of author information is available at the end of the article

**Abstract**

This work deals with the large deviation principle which studies the decay of probabilities of certain kind of extremely rare events. We consider stochastic neutral fractional functional differential equation with multiplicative noise and show large deviation principle for its solution processes in a suitable Polish space. The existence and uniqueness results are presented using the Picard iterative method, which is indeed essential for further analysis. The establishment of Freidlin–Wentzell type large deviation principle is solely based on the variational representation developed by Budhiraja and Dupuis in which the weak convergence technique is used to show the sufficient condition. Examples are provided to emphasize the theory.

**MSC:** 34A08; 34K40; 60F10; 60H10

**Keywords:** Fractional differential equations; Neutral functional differential equations; Large deviations; Stochastic differential equations

## 1 Introduction

The theory of differential equations with memory has recently been studied intensively. Many researchers have given special interest to the study of dynamical systems that rely on the past and present state as well as on the derivatives of the past state of the system. The neutral differential equations are often used to describe such systems which have applications in various physical scenarios like heat conduction material with memory, lossless transmission connection, vibrating masses attached to an elastic bar, and collision theory. These equations are encountered when memory argument arises in the derivative of state variable as well as independent variable. Several works on stochastic neutral type model were discussed by Mao [25–27, 43]. The solutions of perturbed neutral stochastic differential equation and its appropriate unperturbed system were compared in [14]. To know more on neutral type stochastic differential equations, one can refer [13, 15, 22, 35].

Fractional calculus has gained a noticeable popularity in recent years. It allows one to define the derivatives and integrals to an arbitrary order. Employing fractional order operator would give a better outcome while framing the dynamical systems. Modeling a real world problem would be even more realistic when a fractional order system is incorporated with some randomness and will be an extension of a deterministic system. The advancement of fractional calculus inspired many research works to explore the solutions of differential equations driven by fractional operators. The existence and uniqueness of solution of differential equation is critical in order to validate the model, which then allows for further exploration of numerous additional problems. Umamaheshwari et al. [39, 40] examined the existence and uniqueness of solutions of some fractional stochastic differential equations using the Picard–Lindelöf successive approximation scheme. Several studies on existence, uniqueness, qualitative and quantitative analysis of stochastic fractional differential equations have been handled by many authors [9, 24]. In this work, we focus on the analysis of the large deviation principle for solution processes of stochastic neutral fractional functional differential equation with multiplicative noise. The monographs [1, 2] justify this choice of fractional system.

The large deviation principle (LDP) deals with the asymptotic behavior of rare events. Those highly improbable events may have a huge impact during their occurrence, and so the study of this theory is indeed essential. Though there were works on large deviation theory by mathematicians and economists, Varadhan [41] was the one who formulated the large deviation theory in his work initially. A fundamental work on large deviation principle was done by Freidlin and Wentzell [12]. Moreover, the results to stochastic evolution equations have been established intensively in recent decades, and we refer to [30] for the extensions to an infinite dimensional system under the global Lipschitz condition, that is, the measure obtained for LDP by Freidlin and Wentzell has been extended to the measures obtained by the stochastic evolution equations with non-additive perturbations. The Freidlin–Wentzell large deviation principle was established for the stochastic evolution equations with small multiplicative noise in [20]. Ren and Zhang [33] proved the LDP of Freidlin–Wentzell type for multivalued stochastic differential equations with monotone drifts. The study of large deviations for stochastic system with memory has also gained much attention, see, for example, [17, 21, 28, 29, 34]. Subsequently, many authors have attempted to establish the results of large deviation theory under less restrictive conditions. An extensive collection of LDP for various kinds of stochastic differential equations can be found in [11, 18, 19, 38, 42] and the references therein.

In this work, we use the weak convergence approach to study the large deviations for stochastic neutral fractional functional differential equation. Specifically, we use the variational representation formula for functionals of infinite dimensional Brownian motion established by Budhiraja and Dupuis [5]. The use of this weak convergence approach allows one to frame the assumptions needed for the large deviation principle to hold. Using this approach, Ren and Zhang [32] studied the large deviations for homeomorphism flows of non-Lipschitz SDEs. By using this weak convergence approach, the LDP for stochastic fractional integrodifferential equations was studied in [38]. For more information on this approach, one may refer [4, 6, 8, 36].

The organization of this work is as follows: In Sect. 2, the system for which the large deviation principle is to be proved is described. We present some basic definitions from the theory of large deviations and fractional calculus. Also, the solution representation of the fractional system using the Mittag-Leffler function is derived. Section 3 provides the existence and uniqueness of the considered system using the Picard iterative scheme. Section 4 discusses the sufficient conditions to be satisfied in order to estimate the LDP for the stochastic neutral fractional functional differential equation with multiplicative noise using the weak convergence technique. Section 5 includes an example to illustrate large deviations for the considered system.

## 2 Problem formulation

Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a complete filtered probability space with a family $\{\mathscr{F}_t, t \in [0,T]\}$ of increasing sub-$\sigma$-algebras called filtration. The filtration is stated as right continuous if $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ for all $t \in [0,T]$. Let $\mathbb{X}$ and $\mathbb{H}$ be separable Hilbert spaces. Let $\mathrm{L}(\mathbb{X})$ be the space of all bounded linear operators and $W(t)$ be an $\mathbb{H}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. Furthermore, consider the Hilbert space $\mathscr{H}_0 = Q^{1/2}\mathbb{H}$ with the inner product $(X,Y)_0 = (Q^{-1/2}X, Q^{-1/2}Y)$ for all $X, Y \in \mathscr{H}_0$, and the corresponding norm is denoted by $\|\cdot\|_0$. Let $\mathrm{L}_Q$ be the space of all Hilbert–Schmidt operators from $\mathscr{H}_0$ to $\mathbb{X}$. Also, we denote the expectation with respect to probability $\mathcal{P}$ by $\mathbb{E}$. Consider the stochastic neutral fractional functional differential equation of the form

$$\begin{cases} {}^C D^\alpha (x(t) - g(t,x_t)) = Ax(t) + f(t,x_t) + \sigma(t,x_t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}, & t \in [0,T], \\ x(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $1/2 < \alpha \leq 1$. The solution $x(t,\omega)$, $t \in [-\tau, T]$, $\omega \in \Omega$, represented as $x(t) : \Omega \to \mathbb{X}$, takes values in a real separable Hilbert space $\mathbb{X}$. We represent $x_t : \Omega \to C_\tau$, $t \in [0,T]$ by defining $x_t = \{x(t+\theta) : \theta \in [-\tau, 0]\}$, the past history of the state regarded as a $C_\tau$-valued stochastic process where $C_\tau = C([-\tau,0];\mathbb{X})$ is the space of all continuous functions $\varphi$ from $[-\tau, T]$ to $\mathbb{X}$ with the supremum norm $\|\varphi\|_{C_\tau}^2 = \sup\{\|\varphi(\theta)\|_{\mathbb{X}}^2 : -\tau \leq \theta \leq 0\}$, $0 < \tau < \infty$. Further, the initial condition $x_0 = \phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ is considered to be a continuous function. Also $g : [0,T] \times C_\tau \to \mathbb{X}$ is continuous and $A$ is a bounded linear operator from $\mathbb{X}$ to $\mathbb{X}$. The coefficients $f : [0,T] \times C_\tau \to \mathbb{X}$ and $\sigma : [0,T] \times C_\tau \to \mathrm{L}_Q(\mathscr{H}_0; \mathbb{X})$ denote the Borel measurable functions which are continuous and satisfy the Lipschitz condition: for all $x_1, x_2 \in C_\tau$ and $t \in [0,T]$, there exist $L_1, L_2 > 0$ such that

$$\left\|f(t,x_1) - f(t,x_2)\right\|_{\mathbb{X}} \leq L_1\left(\|x_1 - x_2\|_{C_\tau}\right), \tag{2.2}$$

$$\left\|\sigma(t,x_1) - \sigma(t,x_2)\right\|_{\mathrm{L}_Q} \leq L_2\left(\|x_1 - x_2\|_{C_\tau}\right). \tag{2.3}$$

Also, $f$ and $\sigma$ satisfy the linear growth condition: for all $x \in C_\tau$ and $t \in [0,T]$, there exist positive constants $L_3, L_4 > 0$ such that

$$\left\|f(t,x)\right\|_{\mathbb{X}}^2 \leq L_3\left(1 + \|x\|_{C_\tau}^2\right), \tag{2.4}$$

$$\left\|\sigma(t,x)\right\|_{\mathrm{L}_Q}^2 \leq L_4\left(1 + \|x\|_{C_\tau}^2\right). \tag{2.5}$$

We impose some hypothesis on the continuous function $g$ as follows: Assume there is a constant $\gamma > 0$ such that, for all $x \in C_\tau$ and $t \in [0,T]$,

$$\left\|g(t,x)\right\|_{\mathbb{X}}^2 \leq \gamma^2\left(1 + \|x\|_{C_\tau}^2\right). \tag{2.6}$$

Also, let the function $g$ be a contraction, that is, there exists a constant $\eta \in (0,1)$ such that, for all $x_1, x_2 \in C_\tau$ and $t \in [0,T]$,

$$\left\|g(t,x_1) - g(t,x_2)\right\|_{\mathbb{X}} \leq \eta\|x_1 - x_2\|_{C_\tau}. \tag{2.7}$$

We now present some well-known standard definitions in fractional calculus that are used frequently in establishing our results. For $\alpha > 0$, with $n-1 < \alpha < n$ and $n \in \mathbb{N}$, we state the following well-known definitions.

**Definition 2.1** (Caputo fractional derivative [16]) The Riemann–Liouville fractional integral of a function $f$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \tag{2.8}$$

and the Caputo derivative of $f$ is ${}^C D^{\alpha} f(t) = I^{n-\alpha} f^{(n)}(t)$, that is,

$${}^C D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \tag{2.9}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$.

**Definition 2.2** (Mittag-Leffler function (see [3])) A two-parameter family of Mittag-Leffler operator functions for the bounded linear operator $A$ is defined as

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta > 0. \tag{2.10}$$

In particular, for $\beta = 1$, the one-parameter Mittag-Leffler function is

$$E_{\alpha,1}(A) = E_{\alpha}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha + 1)}. \tag{2.11}$$

For simplicity, the bounds of Mittag-Leffler functions with one and two parameters when acting on the bounded linear operator $A$ of (2.1) are represented as follows:

$$S_1 = \sup_{t \in [0,T]} \left\| E_{\alpha}\left(At^{\alpha}\right) \right\|_{\mathrm{L}(\mathbb{X})}, \qquad S_2 = \sup_{t \in [0,T]} \left\| E_{\alpha,\alpha}\left(At^{\alpha}\right) \right\|_{\mathrm{L}(\mathbb{X})}. \tag{2.12}$$

Our next intention is to find a solution representation of (2.1) based on the approach followed in [23]. In order to find the solution representation, we need the following hypothesis.

(H1) The operator $A \in \mathrm{L}(\mathbb{X})$ commutes with the fractional integral operator $I^{\alpha}$ on $\mathbb{X}$ and $\|A\|^2_{\mathrm{L}(\mathbb{X})} < \frac{(2\alpha-1)(\Gamma(\alpha))^2}{T^{2\alpha}}$.

**Lemma 2.1** ([37]) *Suppose that $A$ is a linear bounded operator defined on $\mathbb{X}$, and assume that $\|A\|_{\mathrm{L}(\mathbb{X})} < 1$. Then $(I-A)^{-1}$ is linear bounded and*

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

*The convergence of the above series is in the operator norm and $\|(I-A)^{-1}\|_{\mathrm{L}(\mathbb{X})} \leq (1 - \|A\|_{\mathrm{L}(\mathbb{X})})^{-1}$.*

We now validate the inequality $\|I^{\alpha}A\|_{\mathrm{L}(\mathbb{X})} < 1$. Then, by the above lemma, we could reach the conclusion: $(I - I^{\alpha}A)^{-1}$ is bounded and linear. Let $x \in \mathbb{X}$; we have

$$\begin{aligned}
\mathbb{E}\big[\big\|(I^{\alpha}A)x\big\|^2_{C([-\tau,T];L^2(\Omega,\mathbb{X}))}\big] &\leq \frac{T}{(\Gamma(\alpha))^2}\mathbb{E}\Big[\sup_{t\in[0,T]}\int_0^t (t-s)^{2\alpha-2}\big\|Ax(s)\big\|^2_{\mathbb{X}}\,\mathrm{d}s\Big] \\
&\leq \frac{T^{2\alpha}}{(2\alpha-1)(\Gamma(\alpha))^2}\mathbb{E}\Big[\sup_{t\in[0,T]}\big\|Ax(t)\big\|^2_{\mathbb{X}}\Big] \\
&< \mathbb{E}\|x\|^2_{C([-\tau,T];L^2(\Omega,\mathbb{X}))}
\end{aligned}$$

by (H1), and hence we get the required inequality. Operating by $I^{\alpha}$ on both sides of (2.1), we have

$$x(t) = x(0) + g(t,x_t) - g(0,x_0) + I^{\alpha}Ax(t) + I^{\alpha}f(t,x_t) + I^{\alpha}\sigma(t,x_t)\frac{\mathrm{d}W(t)}{\mathrm{d}t},$$

and so

$$x(t) = \big(I - I^{\alpha}A\big)^{-1}\bigg\{\phi(0) + g(t,x_t) - g(0,\phi) + I^{\alpha}f(t,x_t) + I^{\alpha}\sigma(t,x_t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\bigg\}.$$

By means of Lemma 2.1, we obtain

$$\begin{aligned}
x(t) &= \sum_{k=0}^{\infty}\big(I^{\alpha}A\big)^k\bigg\{\phi(0) + g(t,x_t) - g(0,\phi) + I^{\alpha}f(t,x_t) + I^{\alpha}\sigma(t,x_t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\bigg\} \\
&= \sum_{k=0}^{\infty} I^{k\alpha}A^k\big[\phi(0) - g(0,\phi)\big] + g(t,x_t) + \sum_{k=1}^{\infty} I^{k\alpha}A^k g(t,x_t) \\
&\quad + \sum_{k=0}^{\infty} I^{k\alpha+\alpha}A^k\bigg\{f(t,x_t) + \sigma(t,x_t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\bigg\} \\
&= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha)}\int_0^t (t-s)^{k\alpha-1}A^k\big[\phi(0) - g(0,\phi)\big]\,\mathrm{d}s + g(t,x_t) \\
&\quad + \sum_{k=1}^{\infty}\frac{1}{\Gamma(k\alpha)}\int_0^t (t-s)^{k\alpha-1}A^k g(s,x_s)\,\mathrm{d}s \\
&\quad + \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+\alpha)}\int_0^t (t-s)^{k\alpha+\alpha-1}A^k f(s,x_s)\,\mathrm{d}s \\
&\quad + \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+\alpha)}\int_0^t (t-s)^{k\alpha+\alpha-1}A^k \sigma(s,x_s)\,\mathrm{d}W(s), \\
x(t) &= \sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}\big[\phi(0) - g(0,\phi)\big] + g(t,x_t) \\
&\quad + \int_0^t A(t-s)^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)}g(s,x_s)\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)}f(s,x_s)\,\mathrm{d}s
\end{aligned}$$

$$+\int_0^t (t-s)^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)}\sigma(s,x_s)\,\mathrm{d}W(s).$$

The solution representation is

$$\begin{aligned}x(t) &= E_\alpha\bigl(At^\alpha\bigr)\bigl[\phi(0)-g(0,\phi)\bigr]+g(t,x_t)+\int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)g(s,x_s)\,\mathrm{d}s\\&\quad+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)f(s,x_s)\,\mathrm{d}s\\&\quad+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)\sigma(s,x_s)\,\mathrm{d}W(s).\end{aligned}$$

Since

$$\mathbb{E}\int_0^t \bigl\|(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)\sigma(s,x_s)\bigr\|^2_{\mathrm{L_Q}}\,\mathrm{d}s<\infty, \tag{2.13}$$

we can say that the stochastic integral is well defined by (H1) and the Hilbert–Schmidt operator (see, Prato and Zabczyk [31]).

Next, we present some standard definitions and implications from the theory of large deviations. Let $\{x^\epsilon\}$ be a family of random variables defined on a Polish space $\mathcal{X}$. The large deviation theory is concerned with events $\mathcal{A}$ for which probability $\mathcal{P}(x^\epsilon\in\mathcal{A})$ converges to zero exponentially fast as $\epsilon\to 0$. The rate of exponential decay of such probability is expressed in terms of rate function.

**Definition 2.3** (Rate function [7]) A function $\mathcal{I}$ from $\mathcal{X}$ to $[0,+\infty]$ is called

- a rate function if $\mathcal{I}$ is lower semi-continuous, which means that the level sets $\{h\in\mathcal{X}:\mathcal{I}(h)\le k\}$ are closed for any $k\ge 0$.
- a good rate function if, for each $k<\infty$, the level set is compact.

**Definition 2.4** (Large deviation principle) Let $\mathcal{I}$ be a rate function. For each Borel subset $K$ of $\mathcal{X}$, the family $\{x^\epsilon(t),\epsilon>0\}$ is said to satisfy the LDP with rate function $\mathcal{I}$ if the following conditions hold:

(i) Large deviation upper bound:

$$\limsup_{\epsilon\to 0}\epsilon\log\mathcal{P}\bigl(x^\epsilon\in K\bigr)\le -\mathcal{I}(K)\quad\text{for each } K \text{ closed.}$$

(ii) Large deviation lower bound:

$$\liminf_{\epsilon\to 0}\epsilon\log\mathcal{P}\bigl(x^\epsilon\in K\bigr)\ge -\mathcal{I}(K)\quad\text{for each } K \text{ open.}$$

**Definition 2.5** (Laplace principle) The family $\{x^\epsilon(t),\epsilon>0\}$ is said to satisfy the Laplace principle with rate function $\mathcal{I}$ if for each bounded continuous real-valued function $h$ defined on $\mathcal{X}$

$$\lim_{\epsilon\to 0}\epsilon\log E\biggl\{\exp\biggl[-\frac{1}{\epsilon}h\bigl(x^\epsilon\bigr)\biggr]\biggr\}=-\inf_{f\in\mathcal{X}}\bigl\{h(f)+\mathcal{I}(f)\bigr\}.$$

The main result of the theory of large deviations in a Polish space is the equivalence between the Laplace principle and the large deviation principle. For a proof, refer to [[8], Sect. 1.2].

**Theorem 2.1** *The family* $\{x^\epsilon\}$ *satisfies the Laplace principle with a good rate function* $\mathcal{I}(\cdot)$ *on* $\mathcal{X}$ *if and only if* $\{x^\epsilon\}$ *satisfies the large deviation principle with the same rate function* $\mathcal{I}(\cdot)$.

**Definition 2.6** (Convergence in distribution) A sequence of random variables $x_1, x_2, \ldots$ converges in distribution to a random variable $x$, shown by $x_n \xrightarrow{d} x$, if

$$\lim_{n\to\infty} F_{x_n}(x) = F_x(x) \tag{2.14}$$

for all $x$ at which $F_x(x)$ is continuous.

Note that the convergence in distribution is the weakest convergence amongst all the other convergence types, and thus convergence in probability implies convergence in distribution. Next, we collect the following famous inequalities which will be applied in proving the main results.

**Theorem 2.2** (Gronwall's inequality [26]) *Let* $T > 0$ *and* $c \geq 0$. *Let* $u(\cdot)$ *be a Borel measurable bounded nonnegative function on* $[0, T]$, *and let* $v(\cdot)$ *be a nonnegative integrable function on* $[0, T]$. *If*

$$u(t) \leq c + \int_0^t v(s)u(s)\,ds \quad \textit{for all } 0 \leq t \leq T,$$

*then*

$$u(t) \leq c\exp\biggl(\int_0^t v(s)\,ds\biggr) \quad \textit{for all } 0 \leq t \leq T.$$

**Theorem 2.3** (Holder's inequality) *Assume* $\Upsilon$ *to be a domain in* $\mathbb{R}^n$. *Also let* $1 < p < \infty$ *and* $p'$ *denote the conjugate exponent defined by*

$$p' = \frac{p}{p-1}, \quad \textit{that is, } \frac{1}{p} + \frac{1}{p'} = 1,$$

*which also satisfies* $1 < p' < \infty$. *If* $u \in L^p(\Upsilon)$ *and* $v \in L^{p'}(\Upsilon)$, *then* $uv \in L^1(\Upsilon)$ *and*

$$\int_\Upsilon \bigl|u(x)v(x)\bigr|\,\mathrm{d}x \leq \biggl(\int_\Upsilon \bigl|u(x)\bigr|^p\,\mathrm{d}x\biggr)^{1/p}\biggl(\int_\Upsilon \bigl|v(x)\bigr|^{p'}\,\mathrm{d}x\biggr)^{1/p'}.$$

*The equality holds if and only if* $|u(x)|^p$ *and* $|v(x)|^{p'}$ *are proportional a.e. in* $\Upsilon$.

**Lemma 2.2** ([26]) *For any* $a, b \geq 0$ *and* $0 < \gamma < 1$, *we have*

$$(a+b)^2 \leq \frac{a^2}{\gamma} + \frac{b^2}{1-\gamma}. \tag{2.15}$$

The following inequality is the generalization of Doob's martingale inequality, which will be useful in our proofs to bound the stochastic integrals.

**Theorem 2.4** ([26]) *Let $p \geq 2$. Let $v \in L^p(\Omega \times [0,T];\mathbb{R})$ such that*

$$\mathbb{E}\int_0^T \left|v(s)\right|^p \,\mathrm{d}s < \infty.$$

*Then*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t v(s)\,\mathrm{d}W(s)\right|^p\right) \leq \left(\frac{p^3}{2(p-1)}\right)^{p/2} T^{\frac{p-2}{2}}\mathbb{E}\int_0^T \left|v(s)\right|^p \,\mathrm{d}s. \tag{2.16}$$

**Theorem 2.5** (Arzela–Ascoli compactness criterion [10]) *Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of real-valued functions defined on $\mathbb{R}^n$ such that*

$$\left|f_k(x)\right| \leq M \quad \left(k = 1,2,\dots,x \in \mathbb{R}^n\right) \tag{2.17}$$

*for some constant $M$, and $\{f_k\}_{k=1}^{\infty}$ are uniformly equicontinuous. Then there exist a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ and a continuous function $f$ such that $f_{k_j} \to f$ uniformly on compact subsets of $\mathbb{R}^n$.*

## 3 Existence and uniqueness of solutions

The next lemma points us in the direction of establishing the solution's existence and uniqueness.

**Lemma 3.1** *Let $x(t)$ be the solution of (2.1) for which assumptions (2.4)–(2.6), (2.12) and $(H1)$ hold. Then*

$$\mathbb{E}\Big[\sup_{-\tau\leq t\leq T}\left\|x(t)\right\|_{\mathbb{X}}^2\Big] \leq c_1 e^{\frac{4T^{2\alpha}S_2^2(\|A\|^2_{\mathrm{L}(\mathbb{X})}\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}}. \tag{3.1}$$

*Moreover, the solution $x(t)$ belongs to $C([-\tau,T];L^2(\Omega,\mathbb{X}))$.*

*Proof* Let $\tau^m$ be the stopping time defined as

$$\tau^m = T \wedge \inf\left\{t \in [0,T] : \|x_t\|_{C_\tau} \geq m\right\}$$

for any $m \geq 1$. Fix $x^m = x(t \wedge \tau^m)$ for $t \in [-\tau,T]$. Then, for $0 \leq t \leq T$, we have

$$\begin{aligned} x^m(t) &= E_\alpha\left(At^\alpha\right)\left[\phi(0) - g(0,\phi)\right] + g\left(t,x_t^m\right) \\ &\quad + \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)g\left(s,x_s^m\right)I_{[[0,\tau^m]]}(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)f\left(s,x_s^m\right)I_{[[0,\tau^m]]}(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)\sigma\left(s,x_s^m\right)I_{[[0,\tau^m]]}(s)\,\mathrm{d}W(s). \end{aligned}$$

Applying Lemma 2.2, assumptions (2.4)–(2.6), (2.12) and Doob's martingale inequality, one can derive that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le r\le t}\|x^m(r)\|_{\mathbb{X}}^2\Big]\\
&\quad\le \gamma\mathbb{E}\Big[\sup_{-\tau\le r\le t}\big(1+\|x^m(r)\|_{\mathbb{X}}^2\big)\Big]+\frac{8S_1^2(1+\gamma^2)}{1-\gamma}\|\phi\|_{C_\tau}^2\\
&\qquad+\frac{4T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}\mathbb{E}\int_0^T\sup_{-\tau\le r\le s}\big(1+\|x^m(r)\|_{\mathbb{X}}^2\big)\,\mathrm{d}s.
\end{aligned}$$

Note that

$$\mathbb{E}\Big[\sup_{-\tau\le r\le t}\|x^m(r)\|_{\mathbb{X}}^2\Big]\le\|\phi\|_{C_\tau}^2+\mathbb{E}\Big[\sup_{0\le r\le t}\|x^m(r)\|_{\mathbb{X}}^2\Big]. \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{-\tau\le r\le t}\|x^m(r)\|_{\mathbb{X}}^2\Big]\\
&\quad\le c_1+\frac{4T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}\mathbb{E}\int_0^T\sup_{-\tau\le r\le s}\big(1+\|x^m(r)\|_{\mathbb{X}}^2\big)\,\mathrm{d}s,
\end{aligned}$$

where $c_1=\frac{\gamma}{(1-\gamma)}+(\frac{1}{(1-\gamma)}+\frac{8S_1^2(1+\gamma^2)}{(1-\gamma)^2})\|\phi\|_{C_\tau}^2+\frac{4T^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}$. Finally, by means of Gronwall's inequality,

$$\mathbb{E}\Big[\sup_{-\tau\le t\le T}\|x^m(t)\|_{\mathbb{X}}^2\Big]\le c_1e^{\frac{4T^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}}.$$

Thus,

$$\mathbb{E}\Big[\sup_{-\tau\le t\le\tau^m}\|x(t)\|_{\mathbb{X}}^2\Big]\le c_1e^{\frac{4T^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}}. \tag{3.3}$$

Hence, the required inequality is obtained by letting $m\to\infty$. □

**Theorem 3.1** *Let assumptions* (2.2)–(2.7), (2.12) *and* (H1) *hold. Then there exists a unique solution* $x(t)\in C([-\tau,T];L^2(\Omega,\mathbb{X}))$ *to system* (2.1).

*Proof* Uniqueness: Let $x(t)$ and $\bar{x}(t)$ be the solutions of (2.1) with the initial data $x(t)=\phi(t)$ and $\bar{x}(t)=\phi(t)$ for $t\in[-\tau,T]$. Both the solutions belong to the solution space $C([-\tau,T];L^2(\Omega,\mathbb{X}))$ by Lemma 3.1. Note that the difference in the solutions is

$$x(t)-\bar{x}(t)=g(t,x_t)-g(t,\bar{x}_t)+\mathcal{J}(t),$$

where

$$\mathcal{J}(t)=\int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big(g(s,x_s)-g(s,\bar{x}_s)\big)\,\mathrm{d}s$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big(f(s,x_s)-f(s,\bar{x}_s)\big)\,\mathrm{d}s$$
$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big(\sigma(s,x_s)-\sigma(s,\bar{x}_s)\big)\,\mathrm{d}W(s).$$

Applying Lemmas 2.2 and (2.7), we get

$$\big\|x(t)-\bar{x}(t)\big\|_{\mathbb{X}}^2 \le \eta\|x_t-\bar{x}_t\|_{C_\tau}^2 + \frac{1}{1-\eta}\big\|\mathcal{J}(t)\big\|_{\mathbb{X}}^2.$$

Therefore,

$$\mathbb{E}\Big[\sup_{0\le u\le t}\big\|x(u)-\bar{x}(u)\big\|_{\mathbb{X}}^2\Big] \le \frac{1}{(1-\eta)^2}\mathbb{E}\Big[\sup_{0\le u\le t}\big\|\mathcal{J}(u)\big\|_{\mathbb{X}}^2\Big]. \tag{3.4}$$

And one can easily derive that

$$\mathbb{E}\Big[\sup_{0\le u\le t}\big\|\mathcal{J}(u)\big\|_{\mathbb{X}}^2\Big] \le 3\frac{T^{2\alpha-1}}{2\alpha-1}S_2^2\big(\|A\|_{\mathrm{L}(\mathbb{X})}^2\eta^2+L_1^2+4L_2^2\big)\int_0^t \mathbb{E}\Big[\sup_{0\le u\le s}\big\|x(s)-\bar{x}(s)\big\|_{\mathbb{X}}^2\Big]\,\mathrm{d}s.$$

Consequently, we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le u\le t}\big\|x(u)-\bar{x}(u)\big\|_{\mathbb{X}}^2\Big]\\ &\quad\le \frac{3T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\eta^2+L_1^2+4L_2^2)}{2\alpha-1}\int_0^t \mathbb{E}\Big[\sup_{0\le u\le s}\big\|x(u)-\bar{x}(u)\big\|_{\mathbb{X}}^2\Big]\,\mathrm{d}s.\end{aligned}$$

Gronwall's inequality implies

$$\mathbb{E}\Big[\sup_{0\le u\le t}\big\|x(u)-\bar{x}(u)\big\|_{\mathbb{X}}^2\Big] = 0. \tag{3.5}$$

Therefore, the solutions $x(t)$ and $\bar{x}(t)$ are equal for $0\le t\le T$, hence for all $-\tau\le t\le T$, almost surely.

Existence: Let us split the existence of the solution into the following two steps.

Step 1: We consider $T$ is sufficiently small so that it satisfies

$$\rho := \gamma + \frac{3T^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_1^2+4L_2^2)}{(1-\gamma)(2\alpha-1)} < 1. \tag{3.6}$$

Set $x_0^0=\phi$ and $x^0=\phi(0)$ for $0\le t\le T$. In addition, let $x_0^n=\phi$ for each $n=1,2,3,\ldots,$ and define the Picard iterations as follows:

$$\begin{aligned}x^n(t) &= E_\alpha\big(At^\alpha\big)\big[\phi(0)-g(0,\phi)\big]+g\big(t,x_t^{n-1}\big)\\ &\quad+\int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)g\big(s,x_s^{n-1}\big)\,\mathrm{d}s\\ &\quad+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)f\big(s,x_s^{n-1}\big)\,\mathrm{d}s\\ &\quad+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s,x_s^{n-1}\big)\,\mathrm{d}W(s).\end{aligned} \tag{3.7}$$

It is self-evident that $x^0(t)$ is $\mathscr{F}_t$-measurable and belongs to $C([-\tau,T];L^2(\Omega,\mathbb{X}))$. Then, by induction, it is easy to say $x^n(t)\in C([-\tau,T];L^2(\Omega,\mathbb{X}))$. Consequently, we have

$$\sup_{0\le t\le T}\mathbb{E}\bigl\{\bigl\|x^0(t)\bigr\|_{\mathbb{X}}^2\bigr\}<\infty.$$

Applying Lemma 2.2, linear growth conditions (2.4)–(2.5), (2.12), and Doob's martingale inequality on (3.7), one can derive that

$$\begin{aligned}
&\mathbb{E}\Bigl[\sup_{0\le t\le T}\bigl\|x^n(t)\bigr\|_{\mathbb{X}}^2\Bigr]\\
&\quad\le\gamma\mathbb{E}\Bigl[\sup_{0\le t\le T}\bigl(1+\bigl\|x_t^{n-1}\bigr\|_{C_\tau}^2\bigr)\Bigr]+\frac{4S_1^2}{(1-\gamma)}\bigl\|\phi(0)-g(0,\phi)\bigr\|_{\mathbb{X}}^2\\
&\qquad+4\|A\|_{\mathrm{L}(\mathbb{X})}^2\frac{T^{2\alpha-1}}{(1-\gamma)(2\alpha-1)}S_2^2\gamma^2\mathbb{E}\int_0^T\sup_{0\le s\le T}\bigl(1+\bigl\|x_s^{n-1}\bigr\|_{C_\tau}^2\bigr)\,\mathrm{d}s\\
&\qquad+4\frac{T^{2\alpha-1}}{(1-\gamma)(2\alpha-1)}S_2^2L_3\mathbb{E}\int_0^T\sup_{0\le s\le T}\bigl(1+\bigl\|x_s^{n-1}\bigr\|_{C_\tau}^2\bigr)\,\mathrm{d}s\\
&\qquad+16\frac{T^{2\alpha-1}}{(1-\gamma)(2\alpha-1)}S_2^2L_4\mathbb{E}\int_0^T\sup_{0\le s\le T}\bigl(1+\bigl\|x_s^{n-1}\bigr\|_{C_\tau}^2\bigr)\,\mathrm{d}s\\
&\quad\le\gamma\mathbb{E}\Bigl[\sup_{-\tau\le t\le T}\bigl(1+\bigl\|x^{n-1}(t)\bigr\|_{\mathbb{X}}^2\bigr)\Bigr]+\frac{8S_1^2(1+\gamma^2)}{1-\gamma}\|\phi\|_{C_\tau}^2\\
&\qquad+\frac{4T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}\mathbb{E}\int_0^T\sup_{-\tau\le s\le T}\bigl(1+\bigl\|x^{n-1}(s)\bigr\|_{\mathbb{X}}^2\bigr)\,\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Bigl[\sup_{-\tau\le t\le T}\bigl\|x^n(t)\bigr\|_{\mathbb{X}}^2\Bigr]\\
&\quad\le\gamma\mathbb{E}\Bigl[\sup_{-\tau\le t\le T}\bigl(1+\bigl\|x^{n-1}(t)\bigr\|_{\mathbb{X}}^2\bigr)\Bigr]+\biggl(1+\frac{8S_1^2(1+\gamma^2)}{1-\gamma}\biggr)\|\phi\|_{C_\tau}^2\\
&\qquad+\frac{4T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}\int_0^T\mathbb{E}\Bigl(\sup_{-\tau\le s\le T}\bigl(1+\bigl\|x^{n-1}(s)\bigr\|_{\mathbb{X}}^2\bigr)\Bigr)\,\mathrm{d}s\\
&\quad\le k_1+\frac{4T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}\int_0^T\mathbb{E}\Bigl(\sup_{-\tau\le s\le T}\bigl\|x^{n-1}(s)\bigr\|_{\mathbb{X}}^2\Bigr)\,\mathrm{d}s,
\end{aligned}$$

where $k_1=\gamma\mathbb{E}[\sup_{-\tau\le t\le T}(1+\|x^{n-1}(t)\|_{\mathbb{X}}^2)]+(1+\frac{8S_1^2(1+\gamma^2)}{1-\gamma})\|\phi\|_{C_\tau}^2+\frac{4T^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}$. For any $r\ge1$, we have

$$\begin{aligned}
\max_{1\le n\le r}\mathbb{E}\Bigl[\sup_{-\tau\le t\le T}\bigl\|x^n(t)\bigr\|_{\mathbb{X}}^2\Bigr]&\le k_1+\frac{4T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}\\
&\quad\times\int_0^T\max_{1\le n\le r}\mathbb{E}\Bigl(\sup_{-\tau\le s\le T}\bigl\|x^{n-1}(s)\bigr\|_{\mathbb{X}}^2\Bigr)\,\mathrm{d}s\\
&\le k_1+\frac{4T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^T \Big( \mathbb{E}\|x_0\|_{C_\tau}^2 + \max_{1\le n\le r} \mathbb{E}\Big( \sup_{-\tau\le s\le T} \|x^n(s)\|_{\mathbb{X}}^2 \Big)\Big)\,\mathrm{d}s \\
&\le k_2 + \frac{4T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2 + L_3 + 4L_4)}{(1-\gamma)(2\alpha-1)} \\
&\times \int_0^T \max_{1\le n\le r} \mathbb{E}\Big( \sup_{-\tau\le s\le T} \|x^n(s)\|_{\mathbb{X}}^2 \Big)\,\mathrm{d}s,
\end{aligned}
$$

where $k_2 = k_1 + \frac{4T^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}\mathbb{E}\|x_0\|_{C_\tau}^2$. Eventually, Gronwall's inequality yields

$$
\max_{1\le n\le r} \mathbb{E}\Big[ \sup_{-\tau\le t\le T} \|x^n(t)\|_{\mathbb{X}}^2 \Big] \le k_2 e^{\frac{4T^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}}.
$$

Since $r$ is arbitrary, we find that

$$
\mathbb{E}\Big[ \sup_{-\tau\le t\le T} \|x^n(t)\|_{\mathbb{X}}^2 \Big] \le k_2 e^{\frac{4T^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_3+4L_4)}{(1-\gamma)(2\alpha-1)}}. \tag{3.8}
$$

Note that, for $0 \le t \le T$,

$$
\begin{aligned}
&\mathbb{E}\Big[ \sup_{0\le t\le T} \|x^1(t) - x^0(t)\|_{\mathbb{X}}^2 \Big] \\
&\le 2\gamma\mathbb{E}\|\phi\|_{C_\tau}^2 + \frac{3T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2 + L_1^2 + 4L_2^2)}{(1-\gamma)(2\alpha-1)}\mathbb{E}\int_0^T \big(1 + \|x^0(s)\|_{\mathbb{X}}^2\big)\,\mathrm{d}s \\
&\le 2\gamma\mathbb{E}\|\phi\|_{C_\tau}^2 + \frac{3T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2 + L_1^2 + 4L_2^2)}{(1-\gamma)(2\alpha-1)}\big(1 + \mathbb{E}\|\phi\|_{C_\tau}^2\big)T \\
&:= K
\end{aligned} \tag{3.9}
$$

for $n \ge 1$. Next, we claim that

$$
\mathbb{E}\Big[ \sup_{0\le t\le T} \|x^{n+1}(t) - x^n(t)\|_{\mathbb{X}}^2 \Big] \le K\rho^n. \tag{3.10}
$$

For any $n \ge 1$,

$$
\begin{aligned}
x^{n+1}(t) - x^n(t) = {}& g\big(t, x_t^n\big) - g\big(t, x_t^{n-1}\big) \\
&+ \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[g\big(s, x_s^n\big) - g\big(t, x_s^{n-1}\big)\big]\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[f\big(s, x_s^n\big) - f\big(s, x_s^{n-1}\big)\big]\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[\sigma\big(s, x_s^n\big) - \sigma\big(s, x_s^{n-1}\big)\big]\,\mathrm{d}W(s).
\end{aligned}
$$

Simplifying in the same way as above, we get

$$
\mathbb{E}\Big[ \sup_{0\le t\le T} \|x^{n+1}(t) - x^n(t)\|_{\mathbb{X}}^2 \Big]
$$

$$
\begin{aligned}
&\leq \gamma \mathbb{E}\Big[\sup_{0\leq t\leq T}\big\|x^n(t)-x^{n-1}(t)\big\|_{\mathbb{X}}^2\Big] \\
&\quad + \frac{3T^{2\alpha-1}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_1^2+4L_2^2)}{(1-\gamma)(2\alpha-1)}\int_0^T \mathbb{E}\Big[\sup_{0\leq s\leq T}\big\|x^n(s)-x^{n-1}(s)\big\|_{\mathbb{X}}^2\Big] \\
&\leq \rho \mathbb{E}\Big[\sup_{0\leq t\leq T}\big\|x^n(t)-x^{n-1}(t)\big\|_{\mathbb{X}}^2\Big] \\
&\leq \rho^n \mathbb{E}\Big[\sup_{0\leq t\leq T}\big\|x^1(t)-x^{0}(t)\big\|_{\mathbb{X}}^2\Big] \\
&\leq K\rho^n. 
\end{aligned}
\tag{3.11}
$$

In view of (3.11), we say that (3.10) holds for some $n \geq 0$. Thereupon, by means of Chebyshev's inequality,

$$
\mathcal{P}\Big[\sup_{0\leq t\leq T}\big\|x^{n+1}(t)-x^n(t)\big\|_{\mathbb{X}}^2 > \frac{1}{2^n}\Big] \leq \frac{1}{(1/2^n)^2}\mathbb{E}\Big[\sup_{0\leq t\leq T}\big\|x^{n+1}(t)-x^n(t)\big\|_{\mathbb{X}}^2\Big].
$$

Thus, by applying (3.11) and summing up the resultant inequalities, we get

$$
\sum_{n=0}^{\infty}\mathcal{P}\Big[\sup_{0\leq t\leq T}\big\|x^{n+1}(t)-x^n(t)\big\|_{\mathbb{X}}^2 > \frac{1}{2^n}\Big] \leq \sum_{n=0}^{\infty}K(4\rho)^n.
$$

Since the sum of series $\sum_{n=0}^{\infty}K(4\rho)^n$ is finite, using the Borel–Cantelli lemma we can conclude that $\sup_{0\leq t\leq T}\|x^{n+1}(t)-x^n(t)\|_{\mathbb{X}}^2$ converges to zero, almost surely. Thus, the Picard iterations $x^n(t)$ converge almost surely to a limit $x(t)$ on $t \in [0,T]$ defined by

$$
\begin{aligned}
\lim_{n\to\infty}\left[x^0(t)+\sum_{i=0}^{n-1}\big(x^{i+1}(t)-x^i(t)\big)\right] &= \lim_{n\to\infty}x^n(t) \\
&= x(t).
\end{aligned}
$$

From (3.7), we have

$$
\begin{aligned}
x(t) &= E_\alpha\big(At^\alpha\big)\big[\phi(0)-g(0,\phi)\big]+g(t,x_t)+\int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)g(s,x_s)\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)f(s,x_s)\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma(s,x_s)\,\mathrm{d}W(s).
\end{aligned}
\tag{3.12}
$$

Step 2: We now eliminate condition (3.6). Take $\delta > 0$ to be sufficiently small for

$$
\gamma + \frac{3\delta^{2\alpha}S_2^2(\|A\|_{\mathrm{L}(\mathbb{X})}^2\gamma^2+L_1^2+4L_2^2)}{(1-\gamma)(2\alpha-1)} < 1.
\tag{3.13}
$$

In consequence, there exists a solution on $[-\tau,\delta]$ to system (2.1) by performing step 1. Let us now consider system (2.1) on $[\delta,2\delta]$ with the initial condition $x_\delta$. Again by step 1, there exists a solution on $[\delta,2\delta]$. Subsequently, we repeat step 1 until the existence of solution on the interval $[p\delta,T]$ occurs. Hence, we conclude that there exists a solution on the entire interval $[-\tau,T]$ as desired. □

## 4 Large deviation principle

This section is concerned with large deviations for the stochastic neutral fractional functional differential equation with multiplicative noise

$$\begin{cases} {}^{C}D^{\alpha}(x^{\epsilon}(t) - g(t, x_t^{\epsilon})) = Ax^{\epsilon}(t) + f(t, x_t^{\epsilon}) + \sqrt{\epsilon}\sigma(t, x_t^{\epsilon})\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in [0, T], \\ x^{\epsilon}(t) = \phi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{4.1}$$

There exists a Borel measurable function $\mathscr{G}^{\epsilon} : C([0, T]; \mathbb{H}) \to C([-\tau, T]; \mathbb{X})$ with $x^{\epsilon}(\cdot) = \mathscr{G}^{\epsilon}(W(\cdot))$. Let us define the control space

$$\mathcal{A} = \left\{ \nu : \nu \text{ is } \mathscr{H}_0\text{-valued } \mathscr{F}_t\text{-predictable process and } \int_0^T \|\nu(s)\|_0^2 \,\mathrm{d}s < \infty \text{ a.s.} \right\}$$

and

$$\mathbb{S}_N = \left\{ \psi \in L^2\big([0, T]; \mathscr{H}_0\big) : \int_0^T \|\psi(s)\|_0^2 \,\mathrm{d}s \le N \right\} \tag{4.2}$$

for $N \in \mathbb{N}$ and where $L^2([0, T]; \mathscr{H}_0)$ is regarded as the space of all $\mathscr{H}_0$-valued square integrable functions on $[0, T]$. Moreover, $\mathbb{S}_N$ is a compact Polish space under the weak topology. Define

$$\mathcal{A}_N = \big\{ \nu \in \mathcal{A} : \nu(\omega) \in \mathbb{S}_N, \mathcal{P}\text{-a.s.} \big\}.$$

In our work, the weak convergence approach is employed to establish the Laplace principle which is equivalent to LDP in a Polish space. The following condition (A) is required to show the Laplace principle for $x^{\epsilon}$ as $\epsilon \to 0$.

(A) There exists a measurable map $\mathscr{G}^0 : C([0, T]; \mathbb{H}) \to C([-\tau, T]; \mathbb{X})$ such that the following two conditions hold:

(i) Consider $N < \infty$ and the families $\{\nu^{\epsilon} : \epsilon > 0\} \subset \mathcal{A}_N$ such that $\nu^{\epsilon}$ converges in distribution (as $\mathbb{S}_N$-valued random elements) to $\nu$. Then

$$\mathscr{G}^{\epsilon}\left( W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^{\cdot} \nu^{\epsilon}(s) \,\mathrm{d}s \right) \longrightarrow \mathscr{G}^0\left( \int_0^{\cdot} \nu(s) \,\mathrm{d}s \right)$$

in distribution as $\epsilon \to 0$.

(ii) For each $N < \infty$, the set

$$\mathbb{K}_N = \left\{ \mathscr{G}^0\left( \int_0^{\cdot} \psi(s) \,\mathrm{d}s \right) : \psi \in \mathbb{S}_N \right\}$$

is a compact subset of $C([-\tau, T]; \mathbb{X})$.

From Theorem 4.4 of [5], if $\mathscr{G}^{\epsilon}$ satisfies condition (A), then the family $x^{\epsilon} = \mathscr{G}^{\epsilon}(W(\cdot))$ satisfies the Laplace principle in $C([-\tau, T]; \mathbb{X})$ with rate function $\mathcal{I}$ defined by

$$\mathcal{I}(h) = \inf_{\nu \in L^2([0,T]; \mathscr{H}_0) : h = \mathscr{G}^0(\int_0^{\cdot} \nu(s)\,\mathrm{d}s)} \left\{ \frac{1}{2} \int_0^T \|\nu(s)\|_0^2 \,\mathrm{d}s \right\}. \tag{4.3}$$

Hence, in order to establish the Laplace principle, it is enough to show that conditions (i) and (ii) hold. Let us construct an equation associated with (4.1):

$$\begin{cases} {}^{C}D^{\alpha}[z^{\psi}(t) - g(t, z_t^{\psi})] = Az^{\psi}(t) + f(t, z_t^{\psi}) + \sigma(t, z_t^{\psi})\psi(t), & t \in [0, T], \\ z^{\psi}(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{4.4}$$

with $z^{\psi}$ as its solution and represented as

$$\begin{aligned} z^{\psi}(t) &= E_{\alpha}\big(At^{\alpha}\big)\big[\phi(0) - g(0,\phi)\big] + g\big(t, z_t^{\psi}\big) + \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^{\alpha}\big)g\big(s, z_s^{\psi}\big)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^{\alpha}\big)f\big(s, z_s^{\psi}\big)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^{\alpha}\big)\sigma\big(s, z_s^{\psi}\big)\psi(s)\,\mathrm{d}s, \end{aligned}$$

where $\psi \in L^2([0,T]; \mathscr{H}_0)$. The main result of this work is as follows.

**Theorem 4.1** *Under assumptions* (2.2)–(2.7), (2.12) *and* (H1), *the family* $\{x^{\epsilon}(t)\}$ *which is the solution to* (4.1) *satisfies the LDP* (*equivalently, the Laplace principle*) *in* $C([-\tau,T];\mathbb{X})$ *with good rate function*

$$\mathcal{I}(h) = \inf\left\{\frac{1}{2}\int_0^T \|\psi(t)\|_0^2\,\mathrm{d}t; z^{\psi} = h\right\},$$

*where* $\psi \in L^2([0,T]; \mathscr{H}_0)$, *otherwise*, $\mathcal{I}(h) = \infty$.

Since our work is devoted to verifying condition (A), we split those proofs into the following lemmas.

**Lemma 4.1** *Define* $\mathscr{G}^0 : C([0,T];\mathbb{H}) \longrightarrow C([-\tau,T];\mathbb{X})$ *by*

$$\mathscr{G}^0(\phi) = \begin{cases} z^{\psi}, & \textit{if } \phi = \int_0^{\cdot} \psi(s)\,\mathrm{d}s \textit{ for some } \psi \in \mathbb{S}_N, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.5}$$

*Then, for each* $N < \infty$, *the set*

$$\mathbb{K}_N = \left\{\mathscr{G}^0\left(\int_0^{\cdot} \psi(s)\,\mathrm{d}s\right) : \psi \in \mathbb{S}_N\right\}$$

*is a compact subset of* $C([-\tau,T];\mathbb{X})$.

*Proof* We first assume that $\psi^n \to \psi$ weakly in $\mathbb{S}_N$ as $n \to \infty$ and note that a compact operator transforms weakly convergent sequences into strong convergent sequences. To estimate the continuity, we consider

$$\begin{aligned} z^{\psi^n}(t) - z^{\psi}(t) &= g\big(t, z_t^{\psi^n}\big) - g\big(t, z_t^{\psi}\big) \\ &\quad + \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^{\alpha}\big)\big(g\big(s, z_s^{\psi^n}\big) - g\big(s, z_s^{\psi}\big)\big)\,\mathrm{d}s \end{aligned}$$

$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[f\big(s,z_s^{\psi^n}\big)-f\big(s,z_s^{\psi}\big)\big]\,\mathrm{d}s$$
$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[\sigma\big(s,z_s^{\psi^n}\big)-\sigma\big(s,z_s^{\psi}\big)\big]\psi^n(s)\,\mathrm{d}s$$
$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s,z_s^{\psi}\big)\big[\psi^n(s)-\psi(s)\big]\,\mathrm{d}s. \tag{4.6}$$

Put

$$\zeta^n(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s,z_s^{\psi}\big)\big[\psi^n(s)-\psi(s)\big]\,\mathrm{d}s.$$

Applying Holder's inequality, (2.12), and (2.5), we get

$$\begin{aligned}\big\|\zeta^n(t)\big\|_{\mathbb{X}} &\le S_2\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{1/2}\left(\int_0^t \big\|\sigma\big(s,z_s^{\psi}\big)\big[\psi^n(s)-\psi(s)\big]\big\|_{\mathbb{X}}^2\,\mathrm{d}s\right)^{1/2}\\
&\le S_2T_\alpha L_4\left(\int_0^t \big(1+\big\|z_s^{\psi}\big\|_{C_\tau}^2\big)\big\|\psi^n(s)-\psi(s)\big\|_0^2\,\mathrm{d}s\right)^{1/2}\\
&< \infty,\end{aligned} \tag{4.7}$$

where $T_\alpha = \frac{T^{\alpha-1/2}}{\sqrt{2\alpha-1}}$. Therefore, it is uniformly bounded by some constant. Note the fact that a pointwise bounded family of continuous linear operators between Banach spaces is equicontinuous. Thus we can say that $\zeta^n(t)$ is equicontinuous. Since we assumed that $\psi^n \to \psi$ weakly in $L^2([0,T];\mathscr{H}_0)$, we have, by Arzela–Ascoli theorem, $\zeta^n(t)\to 0$ in $C([0,T];\mathbb{X})$, which implies

$$\lim_{n\to\infty}\big\|\zeta^n(t)\big\|_{\mathbb{X}} = 0. \tag{4.8}$$

Taking norm on both sides of (4.6) and applying (2.7) and (2.12), we get

$$\begin{aligned}\big\|z^{\psi^n}(t)-z^{\psi}(t)\big\|_{\mathbb{X}} \le{}& \eta\big\|z_t^{\psi^n}-z_t^{\psi}\big\|_{C_\tau}+\big\|\zeta^n(t)\big\|_{\mathbb{X}}\\
&+\eta S_2\int_0^t \|A\|_{\mathrm{L}(\mathbb{X})}(t-s)^{\alpha-1}\big\|z_s^{\psi^n}-z_s^{\psi}\big\|_{C_\tau}\,\mathrm{d}s\\
&+S_2\int_0^t (t-s)^{\alpha-1}\big\|f\big(s,z_s^{\psi^n}\big)-f\big(s,z_s^{\psi}\big)\big\|_{\mathbb{X}}\,\mathrm{d}s\\
&+S_2\int_0^t (t-s)^{\alpha-1}\big\|\sigma\big(s,z_s^{\psi^n}\big)-\sigma\big(s,z_s^{\psi}\big)\big\|_{\mathrm{L}_Q}\big\|\psi^n(s)\big\|_0\,\mathrm{d}s.\end{aligned}$$

Using the Lipschitz condition of $f$ and $\sigma$ given by (2.2) and (2.3), taking supremum over $u\in[0,t]$, and then by setting $\kappa^n(t)=\sup_{-\tau\le u\le t}\|z^{\psi^n}(u)-z^{\psi}(u)\|_{\mathbb{X}}$, one can then derive that

$$\begin{aligned}&\sup_{0\le u\le t}\big\|z^{\psi^n}(u)-z^{\psi}(u)\big\|_{\mathbb{X}}\\
&\quad\le \eta\kappa^n(t)+\big\|\zeta^n(t)\big\|_{\mathbb{X}}+\eta S_2\int_0^t \|A\|_{\mathrm{L}(\mathbb{X})}(t-s)^{\alpha-1}\kappa^n(s)\,\mathrm{d}s\\
&\qquad+L_1S_2\int_0^t (t-s)^{\alpha-1}\kappa^n(s)\,\mathrm{d}s+L_2S_2\int_0^t (t-s)^{\alpha-1}\kappa^n(s)\big\|\psi^n(s)\big\|_0\,\mathrm{d}s.\end{aligned}$$

Since by the fact $\sup_{-\tau\le u\le t}\|z^{\psi^n}(u)-z^{\psi}(u)\|_{\mathbb{X}} \le \|\phi\|_{C_\tau} + \sup_{0\le u\le t}\|z^{\psi^n}(u)-z^{\psi}(u)\|_{\mathbb{X}}$, we get

$$\begin{aligned}(1-\eta)\kappa^n(t) &\le \|\phi\|_{C_\tau} + \|\zeta^n(t)\|_{\mathbb{X}} + \eta S_2\int_0^t (t-s)^{\alpha-1}\|A\|_{\mathrm{L}(\mathbb{X})}\kappa^n(s)\,\mathrm{d}s\\ &\quad + L_1S_2\int_0^t (t-s)^{\alpha-1}\kappa^n(s)\,\mathrm{d}s + L_2S_2\int_0^t (t-s)^{\alpha-1}\kappa^n(s)\|\psi^n(s)\|_0\,\mathrm{d}s,\\ \kappa^n(t) &\le \frac{1}{(1-\eta)}\|\phi\|_{C_\tau} + \frac{1}{(1-\eta)}\|\zeta^n(t)\|_{\mathbb{X}} + \frac{\eta S_2}{(1-\eta)}\int_0^t (t-s)^{\alpha-1}\|A\|_{\mathrm{L}(\mathbb{X})}\kappa^n(s)\,\mathrm{d}s\\ &\quad + \frac{L_1S_2}{(1-\eta)}\int_0^t (t-s)^{\alpha-1}\kappa^n(s)\,\mathrm{d}s + \frac{L_2S_2}{(1-\eta)}\int_0^t (t-s)^{\alpha-1}\kappa^n(s)\|\psi^n(s)\|_0\,\mathrm{d}s.\end{aligned}$$

By the use of Gronwall's inequality, we obtain

$$\begin{aligned}\kappa^n(t) &\le \left(\frac{1}{(1-\eta)}\|\phi\|_{C_\tau} + \frac{1}{(1-\eta)}\|\zeta^n(t)\|_{\mathbb{X}}\right)\times\exp\biggl(\frac{\eta S_2}{(1-\eta)}\int_0^t (t-s)^{\alpha-1}\|A\|_{\mathrm{L}(\mathbb{X})}\,\mathrm{d}s\\ &\quad + \frac{L_1S_2}{(1-\eta)}\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s + \frac{L_2S_2}{(1-\eta)}\int_0^t (t-s)^{\alpha-1}\|\psi^n(s)\|_0\,\mathrm{d}s\biggr).\end{aligned}$$

Therefore, by applying Holder's inequality, we have

$$\begin{aligned}\kappa^n(t) &\le \left(\frac{1}{(1-\eta)}\|\phi\|_{C_\tau} + \frac{1}{(1-\eta)}\|\zeta^n(t)\|_{\mathbb{X}}\right)\times\exp\biggl(\frac{\eta S_2T^\alpha}{(1-\eta)\alpha}\|A\|_{\mathrm{L}(\mathbb{X})} + \frac{L_1S_2T^\alpha}{(1-\eta)\alpha}\\ &\quad + \frac{L_2S_2}{(1-\eta)}\left(\int_0^t (t-s)^{2\alpha-2}\,\mathrm{d}s\right)^{1/2}\left(\int_0^t \|\psi^n(s)\|_0^2\,\mathrm{d}s\right)^{1/2}\biggr).\end{aligned} \tag{4.9}$$

By some elementary simplification, we can end up with

$$\begin{aligned}\kappa^n(t) &\le \left(\frac{\|\phi\|_{C_\tau} + \|\zeta^n(t)\|_{\mathbb{X}}}{(1-\eta)}\right)\\ &\quad\times\exp\left(\frac{\eta S_2T^\alpha\|A\|_{\mathrm{L}(\mathbb{X})}}{(1-\eta)\alpha} + \frac{L_1S_2T^\alpha}{(1-\eta)\alpha} + \frac{L_2S_2\sqrt{N}}{(1-\eta)}\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{1/2}\right).\end{aligned}$$

Combining this observation with (4.8), we arrive at continuity, and thus the compactness of $\mathbb{K}_N$ is achieved. □

The following construction is needed for further analysis. Let us frame the controlled stochastic equation with some perturbation:

$$\begin{cases}{}^CD^\alpha(x^\epsilon_{\nu^\epsilon}(t) - g(t,x^\epsilon_{t,\nu^\epsilon}))\\ \quad = Ax^\epsilon_{\nu^\epsilon}(t) + f(t,x^\epsilon_{t,\nu^\epsilon}) + \sigma(t,x^\epsilon_{t,\nu^\epsilon})\nu(t) + \sqrt{\epsilon}\sigma(t,x^\epsilon_{t,\nu^\epsilon})\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t\in[0,T],\\ x^\epsilon_{\nu^\epsilon}(t) = \phi(t), \quad t\in[-\tau,0].\end{cases} \tag{4.10}$$

Then there exists a unique solution

$$\begin{aligned}x^\epsilon_{\nu^\epsilon}(t) &= E_\alpha\bigl(At^\alpha\bigr)\bigl[\phi(0) - g(0,\phi)\bigr] + g\bigl(t,x^\epsilon_{t,\nu^\epsilon}\bigr)\\ &\quad + \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)g\bigl(s,x^\epsilon_{s,\nu^\epsilon}\bigr)\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)f\big(s,x^\epsilon_{s,\nu^\epsilon}\big)\,\mathrm{d}s \\
&+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s,x^\epsilon_{s,\nu^\epsilon}\big)\nu(s)\,\mathrm{d}s \\
&+\sqrt{\epsilon}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s,x^\epsilon_{s,\nu^\epsilon}\big)\,\mathrm{d}W(s).
\end{aligned}
\tag{4.11}
$$

The following lemma is needed to estimate the weak convergence criterion.

**Lemma 4.2** *Let assumptions* (2.2)–(2.7), (2.12) *and* ($H1$) *hold, then the solution* $x^\epsilon_{\nu^\epsilon}(t)$ *satisfies*

$$
\mathbb{E}\Big[\sup_{-\tau\le t\le T}\big\|x^\epsilon_{\nu^\epsilon}(t)\big\|^2_{\mathbb{X}}\Big]\le C. \tag{4.12}
$$

*Proof* From the solution of controlled stochastic equation (4.11), we write

$$
x^\epsilon_{\nu^\epsilon}(t) = g\big(t,x^\epsilon_{t,\nu^\epsilon}\big)+\Im(t), \tag{4.13}
$$

where

$$
\begin{aligned}
\Im(t) = {}& E_\alpha\big(At^\alpha\big)\big[\phi(0)-g(0,\phi)\big]+\int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)g\big(s,x^\epsilon_{s,\nu^\epsilon}\big)\,\mathrm{d}s \\
&+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)f\big(s,x^\epsilon_{s,\nu^\epsilon}\big)\,\mathrm{d}s \\
&+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s,x^\epsilon_{s,\nu^\epsilon}\big)\nu(s)\,\mathrm{d}s \\
&+\sqrt{\epsilon}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s,x^\epsilon_{s,\nu^\epsilon}\big)\,\mathrm{d}W(s).
\end{aligned}
\tag{4.14}
$$

Using (2.15) and then (2.6) in (4.13),

$$
\begin{aligned}
\big\|x^\epsilon_{\nu^\epsilon}(t)\big\|^2_{\mathbb{X}} &\le \frac{1}{\gamma}\big\|g\big(t,x^\epsilon_{t,\nu^\epsilon}\big)\big\|^2_{\mathbb{X}}+\frac{1}{1-\gamma}\big\|\Im(t)\big\|^2_{\mathbb{X}} \\
&\le \frac{1}{\gamma}\big[\gamma^2\big(1+\big\|x^\epsilon_{t,\nu^\epsilon}\big\|^2_{C_\tau}\big)\big]+\frac{1}{1-\gamma}\big\|\Im(t)\big\|^2_{\mathbb{X}} \\
&\le \gamma+\gamma\big\|x^\epsilon_{t,\nu^\epsilon}\big\|^2_{C_\tau}+\frac{1}{1-\gamma}\big\|\Im(t)\big\|^2_{\mathbb{X}}.
\end{aligned}
\tag{4.15}
$$

Taking square norm on both sides of (4.14) and using the elementary inequality

$$
|x_1+x_2+\cdots+x_p|^2\le p\big(|x_1|^2+|x_2|^2+\cdots+|x_p|^2\big),
$$

we would get

$$
\begin{aligned}
\big\|\Im(t)\big\|^2_{\mathbb{X}} \le {}& 5\big\|E_\alpha\big(At^\alpha\big)\big\|^2_{\mathrm{L}(\mathbb{X})}\big\|\phi(0)-g(0,\phi)\big\|^2_{\mathbb{X}} \\
&+5\bigg\|\int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)g\big(s,x^\epsilon_{s,\nu^\epsilon}\big)\,\mathrm{d}s\bigg\|^2_{\mathbb{X}}
\end{aligned}
$$

$$+5\left\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)f\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\mathrm{d}s\right\|^2_{\mathbb{X}}$$
$$+5\left\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)\sigma\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\nu(s)\,\mathrm{d}s\right\|^2_{\mathbb{X}}$$
$$+5\left\|\sqrt{\epsilon}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)\sigma\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\mathrm{d}W(s)\right\|^2_{\mathbb{X}}.$$

Making use of Holder's inequality and (2.12), we get

$$\begin{aligned}\left\|\mathfrak{J}(t)\right\|^2_{\mathbb{X}} &\le 5S_1^2\left\|\phi(0)-g(0,\phi)\right\|^2_{\mathbb{X}} + 5S_2^2\|A\|^2_{\mathrm{L}(\mathbb{X})}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left\|g\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\right\|^2_{\mathbb{X}}\mathrm{d}s\\ &\quad + 5S_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left\|f\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\right\|^2_{\mathbb{X}}\mathrm{d}s + 5S_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\left\|\sigma\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\right\|^2_{\mathrm{L}_Q}\left\|\nu(s)\right\|^2_0\mathrm{d}s\\ &\quad + 5\left\|\sqrt{\epsilon}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)\sigma\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\mathrm{d}W(s)\right\|^2_{\mathbb{X}}.\end{aligned}$$

Defining the stopping time, for every $m \ge 1$,

$$\tau^{m,\epsilon} = T \wedge \inf\left\{t\in[0,T]:\left\|x^\epsilon_{\nu^\epsilon}(t)\right\|_{\mathbb{X}} \ge m\right\}.$$

Applying the linear growth condition and taking supremum on both sides over $t\in[0,\tau^{m,\epsilon}]$, we obtain

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le\tau^{m,\epsilon}}\left\|\mathfrak{J}(t)\right\|^2_{\mathbb{X}}\Big]\\ &\le 10S_1^2\left\{\|\phi\|^2_{C_\tau} + \gamma^2\|\phi\|^2_{C_\tau}\right\}\\ &\quad + 5S_2^2\left(\gamma^2\|A\|^2_{\mathrm{L}(\mathbb{X})} + L_3 + NL_4\right)\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\Big(1+\mathbb{E}\sup_{-\tau\le s\le\tau^{m,\epsilon}}\left\|x^\epsilon_{\nu^\epsilon}(s)\right\|^2_{\mathbb{X}}\Big)\mathrm{d}s\\ &\quad + 5\epsilon\sup_{0\le t\le\tau^{m,\epsilon}}\left\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)\sigma\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\mathrm{d}W(s)\right\|^2_{\mathbb{X}}.\end{aligned}\tag{4.16}$$

Using Doob's martingale inequality and (2.12) on the stochastic integral, we get

$$\begin{aligned}&\sup_{0\le t\le\tau^{m,\epsilon}}\left\|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^\alpha\right)\sigma\left(s,x^\epsilon_{s,\nu^\epsilon}\right)\mathrm{d}W(s)\right\|^2_{\mathbb{X}}\\ &\le 4S_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\Big(\sup_{0\le t\le\tau^{m,\epsilon}}\left\|\sigma\left(t,x^\epsilon_{t,\nu^\epsilon}\right)\right\|^2_{\mathrm{L}_Q}\Big)\\ &\le 4S_2^2L_4\frac{T^{2\alpha-1}}{2\alpha-1}\Big(1+\mathbb{E}\sup_{-\tau\le t\le\tau^{m,\epsilon}}\left\|x^\epsilon_{\nu^\epsilon}(t)\right\|^2_{\mathbb{X}}\Big).\end{aligned}\tag{4.17}$$

From (4.15), one easily sees that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le\tau^{m,\epsilon}}\left\|x^\epsilon_{\nu^\epsilon}(t)\right\|^2_{\mathbb{X}}\Big] + \|\phi\|^2_{C_\tau}\\ &\le \|\phi\|^2_{C_\tau} + \gamma + \gamma\mathbb{E}\Big[\sup_{-\tau\le t\le\tau^{m,\epsilon}}\left\|x^\epsilon_{\nu^\epsilon}(t)\right\|^2_{\mathbb{X}}\Big] + \frac{1}{(1-\gamma)}\mathbb{E}\Big[\sup_{0\le t\le\tau^{m,\epsilon}}\left\|\mathfrak{J}(t)\right\|^2_{\mathbb{X}}\Big],\end{aligned}\tag{4.18}$$

$$\mathbb{E}\Big[\sup_{-\tau\leq t\leq \tau^{m,\epsilon}}\big\|x^{\epsilon}_{\nu^{\epsilon}}(t)\big\|^2_{\mathbb{X}}\Big] \leq \frac{\|\phi\|^2_{C_\tau}}{(1-\gamma)}+\frac{\gamma}{(1-\gamma)}+\frac{1}{(1-\gamma)^2}\mathbb{E}\Big[\sup_{0\leq t\leq \tau^{m,\epsilon}}\big\|\mathfrak{J}(t)\big\|^2_{\mathbb{X}}\Big].$$

Using (4.16) and (4.17) in (4.18), we get

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{-\tau\leq t\leq \tau^{m,\epsilon}}\big\|x^{\epsilon}_{\nu^{\epsilon}}(t)\big\|^2_{\mathbb{X}}\Big] \\
&\quad\leq \frac{\gamma}{(1-\gamma)}+\frac{1}{(1-\gamma)}\left(1+\frac{10S_1^2}{(1-\gamma)}\left(1+\gamma^2\right)\right)\|\phi\|^2_{C_\tau} \\
&\qquad+\frac{5S_2^2(\gamma^2\|A\|^2_{\mathrm{L}(\mathbb{X})}+L_3+NL_4)}{(1-\gamma)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)}\int_0^t\left(1+\mathbb{E}\sup_{-\tau\leq s\leq \tau^{m,\epsilon}}\big\|x^{\epsilon}_{\nu^{\epsilon}}(s)\big\|^2_{\mathbb{X}}\right)\mathrm{d}s \\
&\qquad+\frac{20\epsilon S_2^2L_4}{(1-\gamma)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)}\left(1+\mathbb{E}\sup_{-\tau\leq t\leq \tau^{m,\epsilon}}\big\|x^{\epsilon}_{\nu^{\epsilon}}(t)\big\|^2_{\mathbb{X}}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{-\tau\leq t\leq \tau^{m,\epsilon}}\big\|x^{\epsilon}_{\nu^{\epsilon}}(t)\big\|^2_{\mathbb{X}}\Big] &\leq \frac{1}{K}\left(\frac{\gamma}{(1-\gamma)}+\frac{1}{(1-\gamma)}\left(1+\frac{10S_1^2}{(1-\gamma)}\left(1+\gamma^2\right)\right)\|\phi\|^2_{C_\tau}\right. \\
&\quad+\frac{20\epsilon S_2^2L_4}{(1-\gamma)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)}+\frac{5S_2^2(\gamma^2\|A\|^2_{\mathrm{L}(\mathbb{X})}+L_3+NL_4)}{(1-\gamma)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)} \\
&\quad\left.\times\int_0^t\left(1+\mathbb{E}\sup_{-\tau\leq s\leq \tau^{m,\epsilon}}\big\|x^{\epsilon}_{\nu^{\epsilon}}(s)\big\|^2_{\mathbb{X}}\right)\mathrm{d}s\right),
\end{aligned}$$

where $K=(1-\frac{20\epsilon S_2^2L_4}{(1-\gamma)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)})$. By means of Gronwall's inequality, we finally arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{-\tau\leq t\leq \tau^{m,\epsilon}}\big\|x^{\epsilon}_{\nu^{\epsilon}}(t)\big\|^2_{\mathbb{X}}\Big] \\
&\quad\leq \left(1+\frac{1}{K}\left(\frac{\gamma}{(1-\gamma)}+\frac{1}{(1-\gamma)}\left(1+\frac{10S_1^2}{(1-\gamma)}\left(1+\gamma^2\right)\right)\|\phi\|^2_{C_\tau}\right.\right. \\
&\qquad\left.\left.+\frac{20S_2^2L_4}{(1-\gamma)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)}\right)\right)\times\exp\left(\frac{5S_2^2(\gamma^2\|A\|^2_{\mathrm{L}(\mathbb{X})}+L_3+NL_4)}{K(1-\gamma)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)}\right).
\end{aligned}$$

Now the required inequality (4.12) for the stopping processes $x^{\epsilon}_{\nu^{\epsilon}}(t)$ is estimated. Therefore, the general case follows by letting $m\to\infty$ i.e. $\tau^{m,\epsilon}\uparrow T$. Hence the solution of a controlled stochastic system is bounded by some constant. □

Next follows the weak convergence of the solutions as it is the only remaining condition needed to assert the main result.

**Lemma 4.3** *Let $\{\nu^{\epsilon}:\epsilon>0\}\subset\mathcal{A}_N$ for some $N<\infty$. Assume $\nu^{\epsilon}$ converge to $\nu$ in distribution as $\mathbb{S}_N$-valued random elements. Then*

$$\mathscr{G}^{\epsilon}\left(W(\cdot)+\frac{1}{\sqrt{\epsilon}}\int_0^{\cdot}\nu^{\epsilon}(s)\,\mathrm{d}s\right)\to\mathscr{G}^0\left(\int_0^{\cdot}\nu(s)\,\mathrm{d}s\right)$$

*in distribution as $\epsilon\to 0$.*

*Proof* Since we have to prove that the solution $\mathscr{G}^\epsilon$ converges to the solution $\mathscr{G}^0$, we consider

$$\begin{aligned} x^\epsilon_{\nu^\epsilon}(t) - z^\nu(t) = {} & g\big(t, x^\epsilon_{t,\nu^\epsilon}\big) - g\big(t, z^\nu_t\big) \\ & + \int_0^t A(t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[g\big(s, x^\epsilon_{s,\nu^\epsilon}\big) - g\big(s, z^\nu_s\big)\big]\,\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[f\big(s, x^\epsilon_{s,\nu^\epsilon}\big) - f\big(s, z^\nu_s\big)\big]\,\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[\sigma\big(s, x^\epsilon_{s,\nu^\epsilon}\big) - \sigma\big(s, z^\nu_s\big)\big]\nu^\epsilon(s)\,\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s, z^\nu_s\big)\big[\nu^\epsilon(s) - \nu(s)\big]\,\mathrm{d}s \\ & + \sqrt{\epsilon}\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s, x^\epsilon_{s,\nu^\epsilon}\big)\,\mathrm{d}W(s). \end{aligned} \tag{4.19}$$

Taking $\|\cdot\|^2$ on both sides of (4.19), we obtain

$$\begin{aligned} & \big\|x^\epsilon_{\nu^\epsilon}(t) - z^\nu(t)\big\|^2_{\mathbb{X}} \\ & \quad \leq \frac{1}{\eta}\big\|g\big(t, x^\epsilon_{t,\nu^\epsilon}\big) - g\big(t, z^\nu_t\big)\big\|^2_{\mathbb{X}} \\ & \qquad + \frac{1}{(1-\eta)}\big(\mathscr{I}_1(t) + \mathscr{I}_2(t) + \mathscr{I}_3(t) + \mathscr{I}_4(t) + \mathscr{I}_5(t)\big) \\ & \quad \leq \eta\big\|x^\epsilon_{t,\nu^\epsilon} - z^\nu_t\big\|^2_{C_\tau} + \frac{1}{(1-\eta)}\big(\mathscr{I}_1(t) + \mathscr{I}_2(t) + \mathscr{I}_3(t) + \mathscr{I}_4(t) + \mathscr{I}_5(t)\big), \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} \mathscr{I}_1(t) &= 5\bigg\|\int_0^t A(t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[g\big(s, x^\epsilon_{s,\nu^\epsilon}\big) - g\big(s, z^\nu_s\big)\big]\,\mathrm{d}s\bigg\|^2_{\mathbb{X}}, \\ \mathscr{I}_2(t) &= 5\bigg\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[f\big(s, x^\epsilon_{s,\nu^\epsilon}\big) - f\big(s, z^\nu_s\big)\big]\,\mathrm{d}s\bigg\|^2_{\mathbb{X}}, \\ \mathscr{I}_3(t) &= 5\bigg\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big[\sigma\big(s, x^\epsilon_{s,\nu^\epsilon}\big) - \sigma\big(s, z^\nu_s\big)\big]\nu^\epsilon(s)\,\mathrm{d}s\bigg\|^2_{\mathbb{X}}, \\ \mathscr{I}_4(t) &= 5\bigg\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s, z^\nu_s\big)\big[\nu^\epsilon(s) - \nu(s)\big]\,\mathrm{d}s\bigg\|^2_{\mathbb{X}}, \\ \mathscr{I}_5(t) &= 5\bigg\|\sqrt{\epsilon}\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s, x^\epsilon_{s,\nu^\epsilon}\big)\,\mathrm{d}W(s)\bigg\|^2_{\mathbb{X}}. \end{aligned}$$

Taking the expectation of supremum on (4.20), we have

$$\begin{aligned} & \mathbb{E}\Big[\sup_{0\leq u\leq t}\big\|x^\epsilon_{\nu^\epsilon}(u) - z^\nu(u)\big\|^2_{\mathbb{X}}\Big] \\ & \quad \leq \eta\mathbb{E}\Big[\sup_{-\tau\leq u\leq t}\big\|x^\epsilon_{\nu^\epsilon}(u) - z^\nu(u)\big\|_{\mathbb{X}}\Big] \\ & \qquad + \frac{1}{(1-\eta)}\Big\{\mathbb{E}\Big[\sup_{0\leq u\leq t}\mathscr{I}_1(u)\Big] + \mathbb{E}\Big[\sup_{0\leq u\leq t}\mathscr{I}_2(u)\Big] \end{aligned}$$

$$
\begin{aligned}
&\quad + \mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_3(u)\Big] + \mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_4(u)\Big] + \mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_5(u)\Big]\Big\}, \qquad (4.21)\\
&\mathbb{E}\Big[\sup_{-\tau\le u\le t}\big\|x^{\epsilon}_{\nu^{\epsilon}}(u)-z^{\nu}(u)\big\|^2_{\mathbb{X}}\Big]\\
&\quad \le \frac{\|\phi\|^2_{C_\tau}}{(1-\eta)} + \frac{1}{(1-\eta)^2}\Big\{\mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_1(u)\Big] + \mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_2(u)\Big]\\
&\quad + \mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_3(u)\Big] + \mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_4(u)\Big] + \mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_5(u)\Big]\Big\}.
\end{aligned}
$$

Using the bound of Mittag-Leffler function and (2.7) on $\mathscr{I}_1(t)$, we obtain

$$
\begin{aligned}
\mathscr{I}_1(t) &\le 5\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \|A\|^2_{\mathrm{L}(\mathbb{X})}\big\|E_{\alpha,\alpha}\big(A(t-s)^{\alpha}\big)\big\|^2_{\mathrm{L}(\mathbb{X})}\big\|g\big(s,x^{\epsilon}_{s,\nu^{\epsilon}}\big)-g\big(s,z^{\nu}_s\big)\big\|^2_{\mathbb{X}}\,\mathrm{d}s\\
&\le 5S_2^2\eta^2\frac{T^{2\alpha-1}}{2\alpha-1}\frac{(2\alpha-1)(\Gamma(\alpha))^2}{T^{2\alpha}}\int_0^t\big\|x^{\epsilon}_{s,\nu^{\epsilon}}-z^{\nu}_s\big\|^2_{C_\tau}\,\mathrm{d}s\\
&\le 5S_2^2\eta^2\frac{(\Gamma(\alpha))^2}{T}\int_0^t\big\|x^{\epsilon}_{s,\nu^{\epsilon}}-z^{\nu}_s\big\|^2_{C_\tau}\,\mathrm{d}s. \qquad (4.22)
\end{aligned}
$$

Next consider the integral $\mathscr{I}_2(t)$, and by means of (2.12) followed by the Lipschitz condition of $f$, we get

$$
\begin{aligned}
\mathscr{I}_2(t) &\le 5S_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\big\|f\big(s,x^{\epsilon}_{s,\nu^{\epsilon}}\big)-f\big(s,z^{\nu}_s\big)\big\|^2_{\mathbb{X}}\,\mathrm{d}s\\
&\le 5S_2^2L_1^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\big\|x^{\epsilon}_{s,\nu^{\epsilon}}-z^{\nu}_s\big\|^2_{C_\tau}\,\mathrm{d}s. \qquad (4.23)
\end{aligned}
$$

Applying (2.3) and (2.12) on $\mathscr{I}_3(t)$, we estimate

$$
\begin{aligned}
\mathscr{I}_3(t) &\le 5S_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\big\|\sigma\big(s,x^{\epsilon}_{s,\nu^{\epsilon}}\big)-\sigma\big(s,z^{\nu}_s\big)\big\|^2_{\mathrm{L}_Q}\big\|\nu^{\epsilon}(s)\big\|^2_0\,\mathrm{d}s\\
&\le 5S_2^2L_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\big\|x^{\epsilon}_{s,\nu^{\epsilon}}-z^{\nu}_s\big\|^2_{C_\tau}\big\|\nu^{\epsilon}(s)\big\|^2_0\,\mathrm{d}s. \qquad (4.24)
\end{aligned}
$$

Similarly, consider $\mathscr{I}_4(t)$, apply Holder's inequality, (2.12), and the linear growth condition of $\sigma$, it follows that

$$
\begin{aligned}
\mathscr{I}_4(t) &\le 5S_2^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\big\|\sigma\big(s,z^{\nu}_s\big)\big[\nu^{\epsilon}(s)-\nu(s)\big]\big\|^2_{\mathbb{X}}\,\mathrm{d}s\\
&< \infty. \qquad (4.25)
\end{aligned}
$$

From the above estimate, we conclude that the map $\zeta^{\epsilon}: S_N \to C([0,T];\mathbb{X})$ defined by

$$
\zeta^{\epsilon}(t) := \int_0^t E_{\alpha,\alpha}\big(A(t-s)^{\alpha}\big)\sigma\big(s,z^{\nu}_s\big)\big[\nu^{\epsilon}(s)-\nu(s)\big]\,\mathrm{d}s
$$

is bounded continuous and hence converges to 0 in distribution as $\nu^{\epsilon}\to\nu$ in distribution as $S_N$-valued random elements. Therefore, $\zeta^{\epsilon}(t)\to 0$ in $C([0,T];\mathbb{X})$. Finally, taking supremum and expectation on both sides of the stochastic integral $\mathscr{I}_5(t)$, and then to bound the stochastic integral, we apply Doob's martingale inequality

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_5(u)\Big] &= 5\mathbb{E}\left\{\sup_{0\le u\le t}\left\|\sqrt{\epsilon}\int_0^u (u-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(u-s)^{\alpha}\big)\sigma\big(s,x^{\epsilon}_{s,\nu^{\epsilon}}\big)\,\mathrm{d}W(s)\right\|^2_{\mathbb{X}}\right\}\\ &\le 5S_2^2\epsilon\frac{T^{2\alpha-1}}{2\alpha-1}4\mathbb{E}\Big(\sup_{0\le u\le T}\big\|\sigma\big(u,x^{\epsilon}_{u,\nu^{\epsilon}}\big)\big\|^2_{\mathrm{L}_Q}\Big)\\ &\le 20S_2^2\epsilon\frac{T^{2\alpha-1}}{2\alpha-1}\Big(1+\mathbb{E}\sup_{0\le u\le T}\big\|x^{\epsilon}_{u,\nu^{\epsilon}}\big\|^2_{C_\tau}\Big).\end{aligned}\tag{4.26}$$

Substituting estimates (4.22)–(4.26) in (4.21) and then putting $\kappa^{\epsilon}(t)=\mathbb{E}[\sup_{-\tau\le u\le t}\|x^{\epsilon}_{\nu^{\epsilon}}(u)-z^{\nu}(u)\|^2_{\mathbb{X}}]$, we get

$$\begin{aligned}\kappa^{\epsilon}(t) &\le \frac{\|\phi\|^2_{C_\tau}}{(1-\eta)}+\frac{5S_2^2\eta^2}{(1-\eta)^2}\frac{(\Gamma(\alpha))^2}{T}\int_0^t\kappa^{\epsilon}(s)\,\mathrm{d}s\\ &\quad+\frac{5S_2^2}{(1-\eta)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)}\int_0^t\big(L_1^2+L_2^2\big\|\nu^{\epsilon}(s)\big\|_0^2\big)\kappa^{\epsilon}(s)\,\mathrm{d}s\\ &\quad+\mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_4(u)\Big]+\frac{20S_2^2\epsilon}{(1-\eta)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)}\Big(1+\mathbb{E}\sup_{-\tau\le u\le T}\big\|x^{\epsilon}_{\nu^{\epsilon}}(u)\big\|^2_{\mathbb{X}}\Big).\end{aligned}$$

Herein, Gronwall's inequality is used to finish the proof:

$$\begin{aligned}\kappa^{\epsilon}(t) &\le \left(\frac{\|\phi\|^2_{C_\tau}}{(1-\eta)}+\mathbb{E}\Big[\sup_{0\le u\le t}\mathscr{I}_4(u)\Big]+\frac{20S_2^2\epsilon}{(1-\eta)^2}\frac{T^{2\alpha-1}}{(2\alpha-1)}\Big(1+\mathbb{E}\sup_{-\tau\le u\le T}\big\|x^{\epsilon}_{\nu^{\epsilon}}(u)\big\|^2_{\mathbb{X}}\Big)\right)\\ &\quad\times\exp\left(\frac{5S_2^2}{(1-\eta)^2}\int_0^t\left(\frac{\eta^2(\Gamma(\alpha))^2}{T}+\frac{T^{2\alpha-1}}{(2\alpha-1)}\Big[L_1+L_2\big\|\nu^{\epsilon}(s)\big\|_0^2\Big]\right)\mathrm{d}s\right).\end{aligned}\tag{4.27}$$

By applying Lemma 4.2, it follows that $x^{\epsilon}_{\nu^{\epsilon}}(t)$ converges in probability to $z^{\nu}(t)$. According to the fact that convergence in probability implies convergence in distribution or a weak convergence, we conclude that $x^{\epsilon}_{\nu^{\epsilon}}(t)$ converges weakly to $z^{\nu}(t)$ as $\epsilon\to 0$. □

Thus, the LDP for the considered system is established by ensuring the solution of a controlled stochastic system weakly converges to the solution of its controlled deterministic system.

## 5 Example

The following examples illustrate the LDP for stochastic neutral fractional delay differential equation as it is a special and important class of stochastic neutral fractional functional differential equations.

*Example* 5.1 Consider the following equation:

$$\begin{cases}{}^{C}D^{3/5}[x(t)-0.5x(t-1)]=-\frac{1}{1+t}x(t)+\frac{\sqrt{\epsilon}}{1+t}\frac{\mathrm{d}W(t)}{\mathrm{d}t}, & t\in(0,T],\\ x(t)=\phi(t), & t\in[-1,0],\end{cases}\tag{5.1}$$

where $W(t)$ is a one-dimensional Brownian motion.

Let us take the control to be $v \in L^2([0,T];\mathbb{R})$, and so the corresponding controlled equation is

$$\begin{cases} {}^C D^{3/5}[z^v(t) - 0.5z^v(t-1)] = -\frac{1}{1+t} z^v(t) + \frac{1}{1+t} v(t), & t \in (0,T], \\ z^v(t) = \phi(t), & t \in [-1,0]. \end{cases} \tag{5.2}$$

The coefficients of equation (5.1) satisfy the hypothesis of Theorem 4.1, and so the LDP holds with the rate function $\mathcal{I} : C([0,T];\mathbb{R}) \to [0,+\infty]$ defined by

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |{}^C D^{3/5}[\varphi - 0.5\varphi(s-1)](1+s) + \varphi(s)|^2 \, \mathrm{d}s, & \text{if } \varphi \in L^2([0,T];\mathbb{R}), \\ \infty, & \text{otherwise}, \end{cases} \tag{5.3}$$

where

$$\begin{aligned} z^v(t) &= 0.5z^v(t-1) + \phi - 0.5\phi + \frac{1}{\Gamma(3/5)} \int_0^t \frac{(t-s)^{2/5}}{(1+s)} z^v(s) \, \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(3/5)} \int_0^t \frac{(t-s)^{2/5}}{(1+s)} v(s) \, \mathrm{d}s \end{aligned}$$

is the unique solution of (5.2).

*Example* 5.2 Consider the stochastic neutral fractional delay differential equation with multiplicative noise

$$\begin{cases} {}^C D^{2/3}[x(t) - 0.2x(t-\tau)] = -\frac{1}{1+t} x(t) + \sqrt{\epsilon}(3 + \sin x(t)) \frac{\mathrm{d}W(t)}{\mathrm{d}t}, & t \in (0,T], \\ x(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{5.4}$$

The rate function $\mathcal{I} : C([0,T];\mathbb{R}) \to [0,+\infty]$ is defined as

$$\mathcal{I}(\varphi) = \inf\left\{ \frac{1}{2}\int_0^T \|v(s)\|_0^2 \, \mathrm{d}s : v \in \left(L^2[0,T];\mathbb{R}\right) \text{ such that } z^v = \varphi \right\}, \tag{5.5}$$

where infimum over an empty set is taken as $\infty$ and where $z^v(t)$, the solution of the equation

$$\begin{aligned} z^v(t) &= 0.2z^v(t-\tau) + \phi(0) - 0.2\phi(-\tau) + \frac{1}{\Gamma(2/3)} \int_0^t (t-s)^{1/3} \frac{1}{1+s} z^v(s) \, \mathrm{d}s \\ &\quad + \frac{1}{\Gamma(2/3)} \int_0^t (t-s)^{1/3} \left(3 + \sin z^v(s)\right) v(s) \, \mathrm{d}s, \end{aligned} \tag{5.6}$$

is the unique solution of an appropriate controlled system of (5.4).

**Acknowledgements**
The authors would like to thank the anonymous reviewers for their valuable suggestions and comments.

**Funding**
The work of Yong-Ki Ma was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1F1A1048937).

**Availability of data and materials**
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

## Declarations

**Competing interests**
The authors declare that they have no competing interests.

**Author contributions**
All the authors have contributed equally to this paper. All authors read and approved the final manuscript.

**Author details**
[1]Department of Applied Mathematics, Bharathiar University, Coimbatore 641046, India. [2]Department of Mathematics, Bharathiar University, Coimbatore 641046, India. [3]Department of Applied Mathematics, Kongju National University, Chungcheongnam-do 32588, Republic of Korea.

## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 18 January 2022 Accepted: 24 June 2022 Published online: 12 July 2022

**References**

1. Abouagwa, M., Bantan, R.A.R., Almutiry, W., Khalaf, A.D., Elgarhy, M.: Mixed Caputo fractional neutral stochastic differential equations with impulses and variable delay. Fractal Fract. **5**, 239 (2021)
2. Ahmad, M., Zada, A., Ghaderi, M., George, R., Rezapour, S.: On the existence and stability of a neutral stochastic fractional differential system. Fractal Fract. **6**, 203 (2022)
3. Balachandran, K., Matar, M., Trujillo, J.J.: Note on controllability of linear fractional dynamical systems. J. Control Decis. **3**, 267–279 (2016)
4. Boué, M., Dupuis, P.: A variational representation for certain functionals of Brownian motion. Ann. Probab. **26**, 1641–1659 (1998)
5. Budhiraja, A., Dupuis, P.: A variational representation for positive functionals of infinite dimensional Brownian motion. Probab. Math. Stat. **20**, 39–61 (2000)
6. Budhiraja, A., Dupuis, P., Maroulas, V.: Large deviations for infinite dimensional stochastic dynamical systems. Ann. Probab. **36**, 1390–1420 (2008)
7. Dembo, A., Zeitouni, O.: Large Deviations Techniques and Applications. Springer, New York (2010)
8. Dupuis, P., Ellis, R.S.: A Weak Convergence Approach to the Theory of Large Deviations. Wiley, New York (1997)
9. El-Borai, M.M., El-Nadi, K.E.-S., Fouad, H.A.: On some fractional stochastic delay differential equations. Comput. Math. Appl. **59**, 1165–1170 (2010)
10. Evans, L.C.: Partial Differential Equations. Am. Math. Soc., Providence (1998)
11. Feng, J., Kurtz, T.G.: Large Deviations of Stochastic Processes. Am. Math. Soc., Providence (2006)
12. Freidlin, M.I., Wentzell, A.D.: Random Perturbations of Dynamical Systems. Springer, New York (1984)
13. Huang, L., Deng, F.: Razumikhin-type theorems on stability of neutral stochastic functional differential equations. IEEE Trans. Autom. Control **53**, 1718–1723 (2008)
14. Jovanović, M., Janković, S.: Neutral stochastic functional differential equations with additive perturbations. Appl. Math. Comput. **213**, 370–379 (2009)
15. Karthikeyan, S., Balachandran, K.: Controllability of nonlinear stochastic neutral impulsive systems. Nonlinear Anal. Hybrid Syst. **3**, 266–276 (2009)
16. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, New York (2006)
17. Kolmanovskii, V., Koroleva, N., Maizenberg, T., Mao, X., Matasov, A.: Neutral stochastic differential delay equations with Markovian switching. Stoch. Anal. Appl. **21**, 819–847 (2003)
18. Kuske, R., Keller, J.B.: Large deviation theory for stochastic difference equations. Eur. J. Appl. Math. **8**, 567–580 (1997)
19. Liu, H., Sun, C.: Large deviations for the 3D stochastic Navier-Stokes-Voight equations. Appl. Anal. **97**, 919–937 (2017)
20. Liu, W.: Large deviations for stochastic evolution equations with small multiplicative noise. Appl. Math. Optim. **61**, 27–56 (2009)
21. Ma, X., Xi, F.: Moderate deviations for neutral stochastic differential delay equations with jumps. Stat. Probab. Lett. **126**, 97–107 (2017)
22. Mabel Lizzy, R., Balachandran, K.: Boundary controllability of nonlinear stochastic neutral fractional dynamical systems. Int. J. Appl. Math. Comput. Sci. **28**, 123–133 (2018)
23. Mabel Lizzy, R., Balachandran, K., Trujillo, J.J.: Controllability of nonlinear stochastic neutral fractional dynamical systems. Nonlinear Anal., Model. Control **22**, 702–718 (2017)
24. Mabel Lizzy, R., Balachandran, K., Trujillo, J.J.: Controllability of nonlinear stochastic fractional neutral systems with multiple time varying delays in control. Chaos Solitons Fractals **102**, 162–167 (2017)
25. Mao, X.: Exponential stability in mean square of neutral stochastic functional differential equations. Syst. Control Lett. **26**, 245–251 (1995)
26. Mao, X.: Stochastic Differential Equations and Applications. Horwood, Chichester (1997)
27. Mao, X.: Razumikhin-type theorems on exponential stability of neutral stochastic differential equations. SIAM J. Math. Anal. **28**, 389–401 (1997)
28. Mo, C., Luo, J.: Large deviations for stochastic differential delay equations. Nonlinear Anal. **80**, 202–210 (2013)

29. Mohammed, S.A., Zhang, T.: Large deviations for stochastic systems with memory. Discrete Contin. Dyn. Syst., Ser. B **6**, 881–893 (2006)
30. Peszat, S.: Large deviation principle for stochastic evolution equations. Probab. Theory Relat. Fields **98**, 113–136 (1994)
31. Prato, G.D., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992)
32. Ren, J., Zhang, X.: Freidlin-Wentzell's large deviations for homeomorphism flows of non-Lipschitz SDEs. Bull. Sci. Math. **129**, 643–655 (2005)
33. Ren, J., Zhang, X.: Large deviations for multivalued stochastic differential equations. J. Theor. Probab. **23**, 1142–1156 (2010)
34. Ren, Y., Xia, N.: Existence, uniqueness and stability of the solutions to neutral stochastic functional differential equations with infinite delay. Appl. Math. Comput. **210**, 72–79 (2009)
35. Sathya, R., Balachandran, K.: Controllability of stochastic impulsive neutral integrodifferential systems with infinite delay. J. Nonlinear Anal. Optim. **5**, 89–101 (2014)
36. Sritharan, S.S., Sundar, P.: Large deviations for the two-dimensional Navier-Stokes equations with multiplicative noise. Stoch. Process. Appl. **116**, 1636–1659 (2006)
37. Suvinthra, M., Balachandran, K., Mabel Lizzy, R.: Large deviations for stochastic fractional integrodifferential equations. AIMS Math. **2**, 348–364 (2017)
38. Suvinthra, M., Sritharan, S.S., Balachandran, K.: Large deviations for stochastic tidal dynamics equation. Commun. Stoch. Anal. **9**, 477–502 (2015)
39. Umamaheswari, P., Balachandran, K., Annapoorani, N.: Existence of solution of stochastic fractional integrodifferential equations. Discontin. Nonlinearity Complex. **7**, 55–65 (2018)
40. Umamaheswari, P., Balachandran, K., Annapoorani, N.: Existence and stability results for Caputo fractional stochastic differential equations with Levy noise. Filomat **34**, 1739–1751 (2020)
41. Varadhan, S.R.S.: Asymptotic probabilities and differential equations. Commun. Pure Appl. Math. **19**, 261–286 (1966)
42. Varadhan, S.R.S.: Large deviations. Ann. Probab. **36**, 397–419 (2008)
43. Wu, F., Mao, X.: Numerical solutions of neutral stochastic functional differential equations. SIAM J. Numer. Anal. **46**, 1821–1841 (2008)