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Existence of positive solutions for a singular third-order two-point boundary value problem on the half-line

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Abstract

In this paper, we consider the following singular third-order two-point boundary value problem on the half-line of the form

$$\begin{cases} x''' + \phi(t)f(t,x,x',x'') = 0, & 0 < t < +\infty, \\ x(0) = 0, \quad x'(0) = a_1, & x'(+\infty) = b_1, \end{cases}$$

where $\phi \in C[0,+\infty)$, $f \in C([0,+\infty) \times (0,+\infty) \times \mathbb{R}^2, \mathbb{R})$ may be singular at $x = 0$, and a_1, b_1 are positive constants. Using the Leray–Schauder nonlinear alternative and the diagonalization method together with the truncation function technique, we obtain the existence and qualitative properties of positive solutions for the problem. As applications, an example is given to illustrate our result.

MSC: 34B15; 34B16; 34B18; 34B40

Keywords: Singular; Leray–Schauder nonlinear alternative; Diagonalization method; Half-line

1 Introduction

In this paper, we study the existence and qualitative properties of positive solutions to the singular third-order two-point boundary value problem on the half-line of the form

$$\begin{cases} x''' + \phi(t)f(t,x,x',x'') = 0, & 0 < t < +\infty, \\ x(0) = 0, \quad x'(0) = a_1, & x'(+\infty) = b_1, \end{cases} \tag{1.1}$$

where $\phi \in C[0,+\infty)$ with $\phi(t) > 0$ for $t \in (0,+\infty)$, $f \in C([0,+\infty) \times (0,+\infty) \times \mathbb{R}^2, \mathbb{R})$ may be singular at $x = 0$, and a_1, b_1 are positive constants with $a_1 < b_1$.

Third-order differential equations on an infinite interval arise from many physical phenomena, such as free convection problems in boundary layer theory, and the draining or coating fluid flow problems [7, 17, 19, 20]. Hence the third-order boundary value problems on the infinite interval have been extensively studied. For more details on non-singular third-order boundary value problems on the infinite interval, see, for instance, [1, 3, 4, 8, 9, 13, 15, 18, 21, 22] and the references therein. For singular third-order boundary value problems on the infinite interval, we refer the reader to [2, 5–7, 10, 12, 14, 16, 19, 20] and the references therein.

In recent years, Benbaziz and Djebali [5] considered the following singular third-order multi-point boundary value problem on the half-line:

$$\begin{cases} x'''(t) + f(t, x(t), x'(t)) = 0, \quad t > 0, \\ x(0) = \sum_{i=1}^{n_1} \alpha_i x(\xi_i), \qquad x'(0) = \sum_{i=1}^{n_1} \beta_i x(\eta_i), \qquad u''(+\infty) = 0, \end{cases} \tag{1.2}$$

where $\alpha_i \geq 0$ $(i = 1, 2, \dots, n_1)$ with $\sum_{i=1}^{n_1} \alpha_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{n_1} < +\infty$, $\beta_i \geq 0$ $(i = 1, 2, \dots, n_2)$ with $\sum_{i=1}^{n_2} \beta_i < 1$, $0 < \eta_1 < \eta_2 < \cdots < \xi_{n_2} < +\infty$. The nonlinearity $f \in C((0, +\infty) \times [0, +\infty) \times [0, +\infty), [0, +\infty))$ satisfies upper and lower-homogeneity conditions in the space variables x, y and may be singular at time variable $t = 0$. The authors presented sufficient conditions which guarantee the existence of positive solutions to problem (1.2) by using the Krasnosel'skii fixed point theorem on cone compression and expansion of norm type. In [6], Benmezaï and Sedkaoui considered the following singular third-order two-point boundary value problem on the half-line:

$$\begin{cases} x'''(t) - \kappa^2 x'(t) + \phi(t) f(t, x(t), x'(t)) = 0, \quad t > 0, \\ x(0) = 0, \qquad x'(0) = 0, \qquad x'(+\infty) = 0, \end{cases} \tag{1.3}$$

where κ is a positive constant, $\phi \in L^1(0, +\infty)$ is nonnegative and does not vanish identically on $(0, +\infty)$, the function $f : \mathbb{R}^+ \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}^+$ is continuous and may be singular at the space variable and its derivative. They provided sufficient conditions for the existence of a positive solution to problem (1.3) by employing the Krasnosel'skii fixed point theorem on cone compression and expansion of norm type.

It is worthy to note that none of the nonlinearity f in works concerned with the singular third-order boundary value problems on the half-line we mentioned above involves the variables x''. Up to now, we have not found the works that studied the fully nonlinear case of which f contains explicitly t and every derivative of x up to order two.

Motivated and inspired by the above works and [2], in this paper we present sufficient conditions for the existence of positive solutions to problem (1.1) and study the qualitative properties of positive solutions. Our main tool is the Leray–Schauder nonlinear alternative and the diagonalization method together with the truncation function technique.

The rest of this paper is organized as follows. In Sect. 2, we first discuss the existence of positive solutions for singular third-order boundary value problems on the finite interval by the Leray–Schauder nonlinear alternative, and then we investigate the existence of positive solutions to problem (1.1) by using the diagonalization method together with the truncation function technique. In Sect. 3, as application, we give an example to illustrate our result.

2 Main results

At first, we present some lemmas, which will be useful in the proof of our main results.

Lemma 2.1 ([11]) *Assume that Ω is a relatively open subset of a convex set C in a Banach space E. Let $T : \overline{\Omega} \to C$ be a compact map and $p \in \Omega$. Then either*

(1) *T has a fixed point in $\overline{\Omega}$; or*
(2) *there are $x \in \partial\Omega$ and $\lambda \in (0,1)$ such that $x = (1-\lambda)p + \lambda Tx$.*

Lemma 2.2 *Let $b > 0$ and suppose that $f : [0,b] \times \mathbb{R}^3 \to \mathbb{R}$ and $\phi : [0,b] \to \mathbb{R}$ are continuous. In addition, assume that there is a constant $M > \max\{a_0 + \frac{b}{2}(a_1 + b_1), b_1\}$, independent of λ, with*

$$\|x\| = \max\Big\{ \sup_{t\in[0,b]} \big|x(t)\big|, \sup_{t\in[0,b]} \big|x'(t)\big|, \sup_{t\in[0,b]} \big|x''(t)\big| \Big\} \neq M$$

for any solution $x \in C^2[0,b] \cap C^3(0,b)$ to

$$\begin{cases} x''' + \lambda\phi(t)f(t,x,x',x'') = 0, \quad 0 < t < b, \\ x(0) = a_0 \geq 0, \qquad x'(0) = a_1 \geq 0, \qquad x'(b) = b_1 > a_1 \end{cases} \tag{2.1$_\lambda$}$$

for each $\lambda \in (0,1)$. Then problem $(2.1)_1$ has at least one solution $x \in C^2[0,b] \cap C^3(0,b)$ with $\|x\| \leq M$.

Proof Consider the Banach space $E = C^2[0,b]$ with the norm

$$\|x\| = \max\big\{ \|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty \big\}, \quad x \in C^2[0,b].$$

Take the convex subset $C = E$ and the open set $\Omega = \{x \in C : \|x\| < M\}$. Let us define the operator $T : \overline{\Omega} \to E$ by

$$(Tx)(t) = a_0 + a_1 t + \frac{b_1 - a_1}{2b} t^2 + \int_0^b G(t,s)\phi(s)f\big(s,x(s),x'(s),x''(s)\big)\,\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} -\frac{1}{2b}t^2 s + ts - \frac{1}{2}s^2, & 0 \leq s \leq t \leq b; \\ \frac{1}{2}t^2 - \frac{1}{2b}t^2 s, & 0 \leq t \leq s \leq b. \end{cases}$$

By the Arzelà–Ascoli theorem, we can easily prove that T is a compact operator, and $x \in C^2[0,b] \cap C^3(0,b)$ is the solution to problem $(2.1)_1$ if and only if $x \in C^2[0,b]$ is a fixed point of operator T. Let $\omega(t) = a_0 + a_1 t + \frac{b_1 - a_1}{2b}t^2$. Then $\omega \in C$ with $\|\omega\| = \max\{a_0 + \frac{b}{2}(b_1 + a_1), b_1\} < M$. Hence $\omega \in \Omega$. Noticing that the solvability of problem $(2.1)_\lambda$ is equivalent to the solvability of the operator equation $x = (1-\lambda)\omega + \lambda Tx$, it follows from the assumption that

$$x \neq (1-\lambda)\omega + \lambda Tx, \quad x \in \partial\Omega, \lambda \in (0,1).$$

Hence from Lemma 2.1, T has a fixed point on $\overline{\Omega}$, and thus problem $(2.1)_1$ has at least one solution $x \in \overline{\Omega}$. This completes the proof of the lemma. □

We now discuss the solvability of problem (1.1) by using the diagonalization method together with the truncation function technique.

Theorem 2.1 *Suppose that $f : [0,+\infty) \times (0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\phi \in C[0,+\infty)$ with $\phi(t) > 0$ for $t \in (0,+\infty)$, and $\phi(t)$ is nondecreasing on $(0,+\infty)$. In addition, assume that the following conditions hold*:

(i) $zf(t,x,y,z) \geq 0$ *for any* $(t,x,y,z) \in [0,+\infty) \times (0,+\infty) \times \mathbb{R}^2$;

(ii) *There exist a function $h \in C([0,+\infty),(0,+\infty))$ and a nondecreasing function $g \in C((0,+\infty),(0,+\infty))$ such that*

$$f(t,x,y,z) \leq g(x)h(z), \quad (t,x,y,z) \in [0,+\infty) \times (0,+\infty) \times [a_1,b_1] \times [0,+\infty),$$

where

$$\int_0^{+\infty} \frac{\mathrm{d}u}{h(u)} > \int_0^{b_1-a_1} \frac{\mathrm{d}u}{h(u)} + \frac{\phi(1)}{a_1} \int_0^{1+b_1} g(u)\,\mathrm{d}u;$$

(iii) *For any constant $K > 0$, there exist a continuous function $\Psi_K(t)$ defined on $[0,+\infty)$, which is positive and nondecreasing on $(0,+\infty)$, and a constant $r \in [1,2)$ such that*

$$f(t,x,y,z) \geq \Psi_K(t)z^r, \quad (t,x,y,z) \in [0,+\infty) \times (0,+\infty) \times [a_1,b_1] \times [0,K].$$

Then problem (1.1) *has a convex and monotonically increasing positive solution $x \in C^2[0,+\infty) \cap C^3(0,+\infty)$.*

Proof We shall complete the proof in two steps.

Step 1. We show that, for each $n \in \mathbb{N}$, the singular boundary value problem on the finite interval

$$\begin{cases} x''' + \phi(t)f(t,x,x',x'') = 0, \quad 0 < t < n, \\ x(0) = 0, \qquad x'(0) = a_1, \qquad x'(n) = b_1 \end{cases} \tag{2.2}$$

has a solution $x \in C^2[0,n] \cap C^3(0,n]$. For this, consider the nonsingular problem

$$\begin{cases} x''' + \phi(t)f(t,x,x',x'') = 0, \quad 0 < t < n, \\ x(0) = \frac{1}{m}, \qquad x'(0) = a_1, \qquad x'(n) = b_1, \end{cases} \tag{2.3}$$

where $m \in \mathbb{N}$. In order to prove that problem (2.3) has a solution, consider the modified problems

$$\begin{cases} x''' + \lambda\phi(t)f^\star(t,x,x',x'') = 0, \quad 0 < t < n, \\ x(0) = \frac{1}{m}, \qquad x'(0) = a_1, \qquad x'(n) = b_1, \end{cases} \tag{2.4$_\lambda$}$$

where $\lambda \in (0,1)$ and

$$f^\star(t,x,y,z) = \begin{cases} f(t,x,y,z), & x \geq \frac{1}{m}; \\ f(t,\frac{1}{m},y,z), & x \leq \frac{1}{m}. \end{cases}$$

Obviously, $f^\star \in C([0,+\infty)\times\mathbb{R}^3,\mathbb{R})$ and

$$zf^\star(t,x,y,z)\geq 0,\quad (t,x,y,z)\in[0,+\infty)\times(0,+\infty)\times\mathbb{R}^2.$$

Let $x\in C^3[0,n]$ be any solution of $(2.4)_\lambda$. We now assert that

$$x''(t)\geq 0,\quad t\in[0,n],\qquad x''(0)>0. \tag{2.5}$$

Indeed, for any $\eta\in[0,n)$ and $\eta<t\leq n$, we have

$$-\bigl(x''(t)\bigr)^2+\bigl(x''(\eta)\bigr)^2=2\lambda\int_\eta^t \phi(s)x''(s)f^\star\bigl(s,x(s),x'(s),x''(s)\bigr)\,\mathrm{d}s\geq 0,$$

and so

$$\bigl(x''(t)\bigr)^2\leq\bigl(x''(\eta)\bigr)^2,\quad t\in[\eta,n]. \tag{2.6}$$

If $x''(0)=0$, then (2.6) with $\eta=0$ implies $x''(t)=0$ for $t\in[0,n]$, which contradicts $x'(0)=a_1<b_1=x'(n)$. Thus $x''(0)\neq 0$. Now we have two cases to consider:

Case 1. $x''(t)\neq 0$ for $t\in[0,n)$. If $x''(t)<0$ for $t\in[0,n)$, then $a_1=x'(0)>x'(n)=b_1$, which is a contradiction. Thus $x''(t)>0$ for $t\in[0,n)$.

Case 2. There exists $\delta\in(0,n)$ with $x''(t)\neq 0$ for $t\in[0,\delta)$ and $x''(\delta)=0$. Now (2.6) with $\eta=\delta$ implies $x''(t)=0$ for $t\in[\delta,n]$, and so $x'(t)\equiv b_1$ on $[\delta,n]$. If $x''(t)<0$ for $t\in[0,\delta)$, then $b_1=x'(\delta)<x'(0)=a_1$, which is a contradiction. Hence $x''(t)>0$ for $t\in[0,\delta)$.

In summary, (2.5) is true. Hence $x'''(t)\leq 0$ for $t\in[0,n]$. Meanwhile,

$$a_1\leq x'(t)\leq b_1,\quad t\in[0,n], \tag{2.7}$$

and thus

$$\frac{1}{m}\leq a_1t+\frac{1}{m}\leq x(t)\leq nb_1+1,\quad t\in[0,n]. \tag{2.8}$$

Obviously, $\max_{t\in[0,n]}x''(t)=x''(0)$. In addition, according to the differential mean value theorem, there exists $\xi\in(0,1)$ such that

$$0\leq x''(\xi)=x'(1)-x'(0)\leq b_1-a_1.$$

Now, from (2.5), (2.7), (2.8) and assumption (ii), we have

$$-x'''(t)=\lambda\phi(t)f^\star\bigl(t,x(t),x'(t),x''(t)\bigr)\leq\phi(t)g\bigl(x(t)\bigr)h\bigl(x''(t)\bigr),\quad t\in(0,n),$$

that is,

$$\frac{-x'''(t)}{h(x''(t))}\leq\phi(t)g\bigl(x(t)\bigr),\quad t\in(0,n).$$

Integrating from 0 to ξ, we obtain

$$\begin{aligned}\int_{x''(\xi)}^{x''(0)} \frac{\mathrm{d}u}{h(u)} &\leq \phi(1)\int_0^{\xi} g\bigl(x(s)\bigr)\,\mathrm{d}s = \phi(1)\int_0^{\xi} \frac{g(x(s))}{x'(s)}x'(s)\,\mathrm{d}s \\ &\leq \frac{\phi(1)}{a_1}\int_{x(0)}^{x(\xi)} g(u)\,\mathrm{d}u \leq \frac{\phi(1)}{a_1}\int_0^{1+b_1} g(u)\,\mathrm{d}u.\end{aligned}$$

Consequently,

$$\int_0^{x''(0)} \frac{\mathrm{d}u}{h(u)} \leq \int_0^{b_1-a_1} \frac{\mathrm{d}u}{h(u)} + \frac{\phi(1)}{a_1}\int_0^{1+b_1} g(u)\,\mathrm{d}u. \tag{2.9}$$

Let

$$I(z) = \int_0^z \frac{\mathrm{d}u}{h(u)}, \quad z \in (0,+\infty).$$

Then from assumption (iii) and (2.9) we know that

$$x''(0) \leq I^{-1}\biggl(\int_0^{b_1-a_1} \frac{\mathrm{d}u}{h(u)} + \frac{\phi(1)}{a_1}\int_0^{1+b_1} g(u)\,\mathrm{d}u\biggr) =: V.$$

Notice that $x'''(t) \leq 0$ for $t \in [0,n]$, it follows from (2.5) that

$$0 \leq x''(t) \leq V, \quad t \in [0,n].$$

This together with (2.7) and (2.8) implies that

$$\|x\| < \max\{nb_1 + 1, V\} + 1 =: M.$$

Therefore from Lemma 2.2, problem $(2.4)_1$ [and consequently problem (2.3)] has a solution $v_m \in C^3[0,n]$ with

$$\begin{aligned}&a_1 t + \frac{1}{m} \leq v_m(t) \leq nb_1 + 1, \qquad a_1 \leq v_m'(t) \leq b_1, \\ &0 \leq v_m''(t) \leq V, \quad t \in [0,n].\end{aligned} \tag{2.10}$$

Therefore from (2.10) and assumption (ii) it follows that

$$-v_m'''(t) \leq \phi(t)g\bigl(v_m(t)\bigr)h\bigl(v_m''(t)\bigr) \leq \phi(t)g(a_1 t)\max_{u\in[0,V]} h(u), \quad t \in [0,n]. \tag{2.11}$$

Notice that assumption (iii) guarantees that there are a continuous function $\Psi_V(t)$, which is positive and nondecreasing on $(0,+\infty)$, and a constant $r \in [1,2)$ such that

$$f(t,x,y,z) \geq \Psi_V(t)z^r, \quad (t,x,y,z) \in [0,+\infty)\times(0,+\infty)\times[a_1,b_1]\times[0,V],$$

and so

$$-v_m'''(t) \geq \phi(t)\Psi_V(t)\bigl(v_m''(t)\bigr)^r, \quad t \in (0,n). \tag{2.12}$$

Now we assert that

$$v'_m(t) \geq \Theta_r(t), \quad t \in [0, n], \tag{2.13}$$

where

$$\Theta_r(t) = \begin{cases} b_1 - (b_1 - a_1)\exp(-\int_0^t \phi(s)\Psi_V(s)\,\mathrm{d}s), & t \in [0, +\infty), r = 1; \\ b_1 - \frac{1}{((b_1-a_1)^{\frac{1-r}{2-r}} + \frac{r-1}{2-r}\int_0^t ((2-r)\phi(s)\Psi_V(s))^{\frac{1}{2-r}}\,\mathrm{d}s)^{\frac{2-r}{r-1}}}, & t \in [0, +\infty), r \in (1, 2). \end{cases}$$

Indeed, we have two cases to consider:

Case 1. $r = 1$. Integrating (2.12) from t to n, we can obtain

$$\begin{aligned} v''_m(t) - v''_m(n) &\geq \int_t^n \phi(s)\Psi_V(s)v''_m(s)\,\mathrm{d}s \\ &\geq \phi(t)\Psi_V(t)\big(b_1 - v'_m(t)\big), \quad t \in [0, n], \end{aligned}$$

and thus

$$v''_m(t) + \phi(t)\Psi_V(t)v'_m(t) \geq b_1\phi(t)\Psi_V(t), \quad t \in [0, n].$$

Solving the above inequality, we have

$$v'_m(t) \geq b_1 - (b_1 - a_1)e^{-\int_0^t \phi(s)\Psi_V(s)\,\mathrm{d}s}, \quad t \in [0, n].$$

Case 2. $1 < r < 2$. Note that either $v''_m(t) > 0$ for $t \in [0, n)$ or there exists $\delta \in (0, n)$ such that $v''_m(t) > 0$ for $t \in [0, \delta)$ and $v''_m(t) = 0$ for $t \in [\delta, n]$. Hence, there exists $\delta \in (0, n]$ such that $v''_m(t) > 0$ for $t \in [0, \delta)$. Multiplying (2.12) by $(v''_m(t))^{1-r}$ and integrating from t to δ (note that ϕ and Ψ_V are nondecreasing on $[0, +\infty)$), we have

$$v''_m(t) \geq \big((2 - r)\phi(t)\Psi_V(t)\big(b_1 - v'_m(t)\big)\big)^{\frac{1}{2-r}}, \quad t \in [0, \delta).$$

Consequently,

$$\big(b_1 - v'_m(t)\big)^{\frac{-1}{2-r}} v''_m(t) \geq \big((2 - r)\phi(t)\Psi_V(t)\big)^{\frac{1}{2-r}}, \quad t \in [0, \delta).$$

Integrating from 0 to t, we obtain

$$\big(b_1 - v'_m(t)\big)^{\frac{1-r}{2-r}} - (b_1 - a_1)^{\frac{1-r}{2-r}} \geq \frac{r-1}{2-r}\int_0^t \big((2 - r)\phi(s)\Psi_V(s)\big)^{\frac{1}{2-r}}\,\mathrm{d}s, \quad t \in [0, \delta),$$

and so

$$1 \geq \left((b_1 - a_1)^{\frac{1-r}{2-r}} + \frac{r-1}{2-r}\int_0^t \big((2 - r)\phi(s)\Psi_V(s)\big)^{\frac{1}{2-r}}\,\mathrm{d}s\right)\big(b_1 - v'_m(t)\big)^{\frac{r-1}{2-r}}, \quad t \in [0, n].$$

Therefore

$$v'_m(t) \geq b_1 - \frac{1}{((b_1 - a_1)^{\frac{1-r}{2-r}} + \frac{r-1}{2-r}\int_0^t ((2 - r)\phi(s)\Psi_V(s))^{\frac{1}{2-r}}\,\mathrm{d}s)^{\frac{2-r}{r-1}}}, \quad t \in [0, +\infty).$$

In summary, (2.13) holds.

Notice that (2.10), (2.11), (2.13) and the Arzelà–Ascoli theorem guarantee that there exist a subsequence $\mathbb{S}$ of $\mathbb{N}$ and a function $x_n \in C^2[0,n]$ such that $v_m^{(j)}(t) \to x_n^{(j)}(t)$ $(j = 0,1,2)$ uniformly on $[0,n]$ as $m \to \infty$ $(m \in \mathbb{S})$, and

$$\begin{aligned} & x_n(0) = 0, \qquad x_n'(0) = a_1, \qquad x_n'(n) = b_1, \\ & a_1 t \le x_n(t) \le nb_1 + 1, \qquad \Theta_r(t) \le x_n'(t) \le b_1, \\ & 0 \le x_n''(n) \le V, \quad t \in [0,n]. \end{aligned} \tag{2.14}$$

Also note that

$$v_m''(t) - v_m''(0) = -\int_0^t \phi(s) f\bigl(s, v_m(s), v_m'(s), v_m''(s)\bigr)\,\mathrm{d}s, \quad t \in [0,n].$$

Let $m \to \infty$ $(m \in \mathbb{S})$, then by the Lebesgue dominated convergence theorem (note (2.11) and assumption (ii)), we have

$$x_n''(t) - x_n''(0) = -\int_0^t \phi(s) f\bigl(s, x_n(s), x_n'(s), x_n''(s)\bigr)\,\mathrm{d}s, \quad t \in [0,n].$$

Consequently, $x_n \in C^2[0,n] \cap C^3(0,n]$ is a solution of (2.2) and satisfies

$$-x_n'''(t) \le \phi(t) g\bigl(x_n(t)\bigr) h\bigl(x_n''(t)\bigr) \le \phi(t) g(a_1 t) \max_{u \in [0,V]} h(u), \quad t \in (0,n).$$

Step 2. We prove the existence of solutions to problem (2.2) by using the diagonalization method. To do this, for $n \ge 1$ an integer, we let

$$u_n(t) = \begin{cases} x_n(t), & 0 \le t \le n; \\ x_n(n), & n \le t < \infty. \end{cases}$$

Then from (2.14) it follows that

$$a_1 t \le u_n(t) \le b_1 + 1, \qquad \Theta_r(t) \le u_n'(t) \le b_1, \qquad 0 \le u_n''(n) \le V, \quad t \in [0,1], n \in \mathbb{N}.$$

Hence, by the Arzelà–Ascoli theorem, there exist a subsequence $\mathbb{N}_1^*$ of $\mathbb{N}$ and a function $z_1(t) \in C^2[0,1]$ with $u_n^{(j)}(t) \to z_1^{(j)}(t)$ $(j = 0,1,2)$ uniformly on $[0,1]$ as $n \to \infty$ $(n \in \mathbb{N}_1^*)$. Also $a_1 t \le z_1(t) \le b_1 + 1$, $\Theta_r(t) \le z_1'(t) \le b_1$, $0 \le z_1''(t) \le V$, $t \in [0,1]$ and $z_1(0) = 0$, $z_1'(0) = a_1$. Let $\mathbb{N}_1 = \mathbb{N}_1^* \setminus \{1\}$. Also notice that

$$a_1 t \le u_n(t) \le 2b_1 + 1, \qquad \Theta_r(t) \le u_n'(t) \le b_1, \qquad 0 \le u_n''(n) \le V, \quad t \in [0,2], n \in \mathbb{N}_1,$$

the Arzelà–Ascoli theorem guarantees the existence of a subsequence $\mathbb{N}_2^*$ of $\mathbb{N}_1$ and a function $z_2(t) \in C^2[0,2]$ with $u_n^{(j)}(t) \to z_2^{(j)}(t)$ $(j = 0,1,2)$ uniformly on $[0,2]$ as $n \to \infty$ $(n \in \mathbb{N}_2^*)$. Note that $z_2(t) = z_1(t)$ on $[0,1]$ since $\mathbb{N}_2^* \subset \mathbb{N}_1$. Also, $a_1 t \le z_2(t) \le 2b_1 + 1$, $\Theta_r(t) \le z_2'(t) \le b_1$, $0 \le z_2''(t) \le V$, $t \in [0,2]$ and $z_2(0) = 0$, $z_2'(0) = a_1$. Let $\mathbb{N}_2 = \mathbb{N}_2^* \setminus \{2\}$ and proceed inductively to obtain for $k = 1,2,\ldots$ a subsequence $\mathbb{N}_k \subset \mathbb{N}$ with $\mathbb{N}_k \subset \mathbb{N}_{k-1}$ and a function $z_k(t) \in C^2[0,k]$ such that $u_n^{(j)}(t) \to z_k^{(j)}(t)$ $(j = 0,1,2)$ uniformly on $[0,k]$ as $n \to \infty$ $(n \in \mathbb{N}_k)$.

Also note that $z_k(t) = z_{k-1}(t)$ for $t \in [0, k-1]$ and $a_1 t \le z_k(t) \le kb_1 + 1$, $\Theta_r(t) \le z_k'(t) \le b_1$, $0 \le z_k''(t) \le V$, $t \in [0, k]$, $z_k(0) = 0$, $z_k'(0) = a_1$.

We now define a function $x(t)$ as follows. For any fixed $t \in [0, +\infty)$, take $k \in \mathbb{N}$ such that $k \ge t$. Let $x(t) = z_k(t)$. Then $x(t)$ is well defined on $[0, +\infty)$, and $x \in C^2[0, +\infty)$. In addition, we have

$$a_1 t \le x(t), \qquad \Theta_r(t) \le x'(t) \le b_1, \qquad 0 \le x''(t) \le V, \quad t \in [0, +\infty),$$

and

$$x(0) = 0, \qquad x'(0) = a_1.$$

Arbitrarily, take $k \in \mathbb{N}$. Note that $\forall t \in (0, k]$ and $\forall n \in \mathbb{N}_k$, we have

$$u_n''(t) - u_n''(0) = -\int_0^t \phi(s) f\bigl(s, u_n(s), u_n'(s), u_n''(s)\bigr)\,\mathrm{d}s.$$

Let $n \to \infty$ ($n \in \mathbb{N}_k$), from the Lebesgue dominated convergence theorem, it follows that

$$z_k''(t) - z_k''(0) = -\int_0^t \phi(s) f\bigl(s, z_k(s), z_k'(s), z_k''(s)\bigr)\,\mathrm{d}s,$$

that is,

$$x''(t) - x''(0) = -\int_0^t \phi(s) f\bigl(s, x(s), x'(s), x''(s)\bigr)\,\mathrm{d}s.$$

Thus $x \in C^3(0, k]$ and

$$x'''(t) + \phi(t) f\bigl(t, x(t), x'(t), x''(t)\bigr) = 0, \quad t \in (0, k].$$

Since $k \in \mathbb{N}$ is arbitrary, we have $x \in C^2[0, +\infty) \cap C^3(0, +\infty)$ and

$$x'''(t) + \phi(t) f\bigl(t, x(t), x'(t), x''(t)\bigr) = 0, \quad t \in (0, +\infty).$$

Notice that $\Theta_r(t) \to b_1$ $(t \to +\infty)$ and $\Theta_r(t) \le x'(t) \le b_1$, $t \in [0, +\infty)$, then $x'(+\infty) = b_1$. In summary, $x(t)$ is a convex and monotonically increasing positive solution of problem (1.1). This completes the proof of the theorem. □

3 An example

In this section, we give an example to illustrate our main result.

Example 3.1 Consider the singular third-order two-point boundary value problems on the half-line

$$\begin{cases} x''' + t(1 + \frac{1}{\sqrt{x}})\sqrt{x''} = 0, \quad 0 < t < +\infty, \\ x(0) = 0, \qquad x'(0) = \beta, \qquad x'(+\infty) = 1, \end{cases} \tag{3.1}$$

where $\beta \in (0, 1)$.

Let $\phi(t) = t$, $t \in (0, +\infty)$, and

$$f(t,x,y,z) = \begin{cases} (1 + \frac{1}{\sqrt{x}})\sqrt{z}, & z \geq 0; \\ 0, & z < 0. \end{cases}$$

Then $f \in C([0, +\infty) \times (0, +\infty) \times \mathbb{R}^2, \mathbb{R})$, $\phi \in C[0, +\infty)$ is positive and nondecreasing on $(0, +\infty)$. Obviously, condition (i) in Theorem 2.1 is satisfied. We now check conditions (ii) and (iii) of Theorem 2.1. To do this, let

$$g(x) = 1 + \frac{1}{\sqrt{x}}, \quad x \in (0, +\infty); \qquad h(z) = 1 + \sqrt{z}, \quad z \in [0, +\infty).$$

Then $g \in C((0, +\infty)(0, +\infty))$, $h \in C([0, +\infty), (0, +\infty))$, and $\forall (t,x,y,z) \in [0, +\infty) \times (0, +\infty) \times (0, 1) \times [0, +\infty)$, we have

$$f(t,x,y,z) \leq g(x)h(z).$$

In addition, it is clear that

$$\int_0^2 g(u)\,\mathrm{d}u = \int_0^2 \frac{\sqrt{u}+1}{\sqrt{u}}\,\mathrm{d}u < 3\int_0^2 \frac{1}{\sqrt{u}}\,\mathrm{d}u < +\infty,$$

and by the Cauchy inequality we have

$$\int_0^{+\infty} \frac{\mathrm{d}u}{h(u)} = \int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{u}+1} \geq \frac{1}{\sqrt{2}}\int_1^{+\infty} \frac{\mathrm{d}s}{\sqrt{s}} = +\infty.$$

Thus condition (ii) holds.

Finally, for any constant $K > 0$, take $\Psi_K(t) \equiv 1/\sqrt{K}$ on $[0, +\infty)$ and $r = 1$. Then $\forall (t,x,y,z) \in [0, +\infty) \times (0, +\infty) \times (0, 1] \times [0, K]$, we have

$$g(t,x,y,z) \geq \Psi_K(t)z,$$

that is, condition (iii) holds.

In summary, all the conditions in Theorem 2.1 are satisfied. Therefore, problem (3.1) has at least one convex, strictly increasing positive solution.

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Author contributions
This work was carried out in collaboration between the three authors. MP designed the study and guided the research. YB and LW performed the analysis and wrote the first draft of the manuscript. YB, LW, and MP managed the analysis of the study. The three authors read and approved the final manuscript.

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