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Direct and inverse problems for nonlocal heat equation with boundary conditions of periodic type

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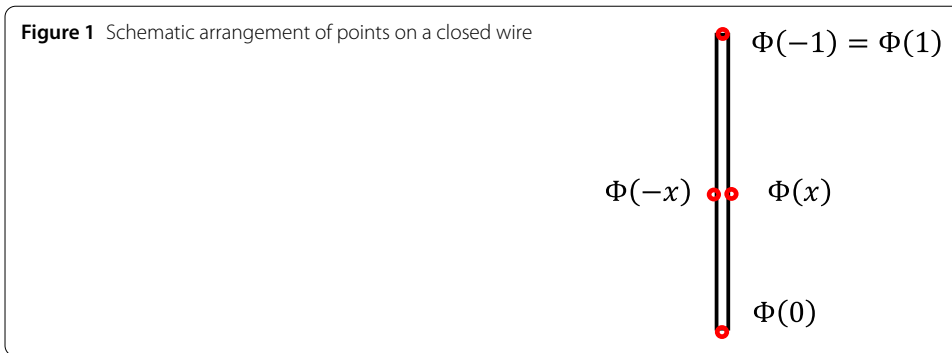available at the end of the article

Abstract

A mathematical model of the process of heat diffusion in a closed metal wire is considered. This wire is wrapped around a thin sheet of insulation material. We assume that the insulation is slightly permeable. Because of this, the temperature at the point of the wire on one side of the insulation influences the diffusion process in the wire on the other side of the insulation. Thus, the standard heat equation will change and an extra term with involution will be added. When modeling of this process there arises an initial-boundary value problem for a one-dimensional heat equation with involution and with a boundary condition of periodic type with respect to a spatial variable. We prove the well-posedness of the formulated problem in the class of strong generalized solutions. The use of the method of separation of variables leads to a spectral problem for an ordinary differential operator with involution at the highest derivative. All eigenfunctions of the problem are constructed. In the case when all eigenvalues of the problem are simple, the system of eigenfunctions does not form an unconditional basis. A criterion when this spectral problem can have an infinite number of multiple eigenvalues is proved. Corresponding root subspaces consist of one eigenfunction and one associated function. We prove that the system of root functions forms an unconditional basis and can be used for constructing a solution of the heat conduction problem by the method of separation of variables. We also consider an inverse problem. This is the problem on restoring (simultaneously with solving) of an unknown stationary source of external influence with respect to an additionally known final state. The existence of a unique solution of this inverse problem and its stability with respect to initial and final data are proved.

Keywords: Heat equation with involution; Nonlocal heat equation; Initial-boundary value problem; Boundary condition of periodic type; Eigenfunctions; Associated functions; Inverse problem

1 Introduction and statement of the problem

A mathematical model of the process of heat diffusion in a closed metal wire is considered. This wire is wrapped around a thin sheet of insulation material. We assume that the insulation is slightly permeable. Because of this, the temperature at the point of the wire on one side of the insulation influences the diffusion process in the wire

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Figure 1 Schematic arrangement of points on a closed wire

on the other side of the insulation. Thus, the standard heat equation will change and an extra term with involution will be added. When modeling this process there arises an initial-boundary value problem for a one-dimensional heat equation with involution and with a boundary condition of periodic type with respect to a spatial variable.

In the past few years, many studies have been devoted to problems for differential operators with involutional argument deviation. Problems with involution in the lowest terms of the equation and problems containing the argument deviation under the highest derivatives of the equation have been considered. Unlike many previous studies, in our paper we use nonlocal boundary conditions with respect to a spatial variable.

Let us consider a problem of modeling the thermal diffusion process. The related problem was described in the work of Cabada and Tojo [1]. They investigated a case describing a concrete situation in physics. Consider a closed metal wire (length 2) which is wrapped around a thin sheet of insulation material, as shown in Fig. 1.

Let $x = 0$ denote the position that is the lowest of the wire. The wire goes around the insulation up to the left to the point −1 and to the right to the point 1. The wire is closed, so the points −1 and 1 coincide.

The layer of insulation is assumed to be slightly permeable. Because of this, the temperature from one side of the insulation influences the diffusion process on the other side. Therefore, the classic heat equation changes with the addition of an additional term $\varepsilon \frac{\partial^2 \Phi}{\partial x^2}(-x, t)$ to $\frac{\partial^2 \Phi}{\partial x^2}(x, t)$ (where $|\varepsilon| < 1$). Here, $\Phi(x, t)$ is the temperature at the point x of the wire at time t.

We will consider the process of heat diffusion that is described by a heat equation. Thus, this process is described by the nonlocal heat equation

$$\Phi_t(x, t) - \Phi_{xx}(x, t) + \varepsilon \Phi_{xx}(-x, t) = F(x, t) \tag{1.1}$$

in the domain $\Omega = \{(x, t) : -1 < x < 1, 0 < t < T\}$. Here, $F(x, t)$ is the influence of an external source; $t = 0$ is an initial time point and $t = T$ is a final one.

The initial temperature distribution of $\Phi(x, t)$ in the wire is considered known:

$$\Phi(x, 0) = \phi(x), \quad x \in [-1, 1]. \tag{1.2}$$

As the wire is closed, it follows that the temperature at the ends of wire is the same:

$$\Phi(-1, t) = \Phi(1, t), \quad t \in [0, T]. \tag{1.3}$$

Consider the problem when an additional external thermal effect occurs at the junction of the ends of the wire. We consider the process in which the temperature flux at one end at each time t is proportional to the rate of change of the average temperature over the entire wire. We describe such an external influence with the help of the formula:

$$\Phi_x(1, t) = \gamma \frac{d}{dt} \int_{-1}^{1} \Phi(\xi, t) \, d\xi, \quad t \in [0, T]; \tag{1.4}$$

here, $\gamma \neq 0$ is a proportionality coefficient.

Hence, the process under study is reduced to the following problem: *Find a solution $\Phi(x, t)$ of the nonlocal heat equation* (1.1) *to the initial condition* (1.2), *the periodic boundary condition* (1.3), *and condition* (1.4).

Note that in [2], instead of condition (1.4), an inverse problem of recovering the heat-conduction process from nonlocal data

$$\Phi(-1, t) = \gamma \frac{d}{dt} \int_{-1}^{1} \Phi(\xi, t) \, d\xi, \quad t \in [0, T]; \tag{1.5}$$

was considered. In that work, when using condition (1.5), the arising spectral problem has only a simple spectrum. In contrast, in the case under consideration when using condition (1.4), the spectrum of the problem can be double.

Note that the result of the work [2] was developed in [3] for the case of a nonlocal heat equation with a time-fractional derivative:

$$t^{-\beta} D_t^\alpha \Phi(x, t) - \Phi_{xx}(x, t) + \varepsilon \Phi_{xx}(-x, t) = F(x).$$

Here, the derivative D_t^α that is defined as

$$D_t^\alpha \varphi(t) = I^{1-\alpha} \left[\frac{d}{dt} \varphi(t) \right], \quad 0 < \alpha < 1, t \in [0, T],$$

is a Caputo derivative for a differentiable function that is built on the Riemann–Liouville fractional integral

$$I^{1-\alpha} \left[\varphi(t) \right] = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\varphi(s)}{(t - s)^\alpha} \, ds, \quad 0 < \alpha < 1, t \in [0, T].$$

2 Reduction of the problem

As is easily seen, condition (1.4) is nonlocal. In this condition, there is the integral along inner lines of the domain. We transform this condition, using the idea of Samarskii. In consideration of equation (1.1) from (1.4), we obtain

$$\Phi_x(1, t) = \gamma \int_{-1}^{1} \left\{ \Phi_{\xi\xi}(\xi, t) - \varepsilon \Phi_{\xi\xi}(-\xi, t) + F(\xi, t) \right\} d\xi, \quad t \in [0, T].$$

Hence,

$$\Phi_x(1,t) = \gamma(1-\varepsilon)\big[\Phi_x(1,t) - \Phi_x(-1,t)\big] + \gamma \int_{-1}^{1} F(\xi,t)\,d\xi, \quad t \in [0,T].$$

Then, we have

$$\Phi_x(-1,t) + a\Phi_x(1,t) = \gamma_1 \int_{-1}^{1} F(\xi,t)\,d\xi, \quad t \in [0,T], \tag{2.1}$$

where the notations $a = \frac{1}{\gamma(1-\varepsilon)} - 1$ and $\gamma_1 = \frac{1}{(1-\varepsilon)}$ are used.

Remark 2.1 As will be shown below (in Sect. 6), if $a = 1$, then for $F \equiv 0$ the problem with the zero initial conditions (1.2) and the boundary conditions (1.3), (2.1) has an infinite number of linearly independent solutions. That is, for $a = 1$ the problem is not Noetherian. Therefore, throughout what follows we consider $a \neq 1$.

Let us set $\gamma_2 = \frac{\gamma_1}{2(1-a)}$ and

$$u(x,t) = \Phi(x,t) - \gamma_2 x^2 \int_{-1}^{1} F(\xi,t)\,d\xi.$$

Then, for the new function $u(x,t)$, we obtain the following problem: *In the domain* $\Omega = \{(x,t) : -1 < x < 1, 0 < t < T\}$ *find a solution* $u(x,t)$ *of the nonlocal heat equation*

$$\mathbb{L}u \equiv u_t(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x,t), \tag{2.2}$$

satisfying one initial condition

$$u(x,0) = \varphi(x), \quad x \in [-1,1], \tag{2.3}$$

and the boundary condition

$$\begin{cases} u_x(-1,t) + au_x(1,t) = 0, \\ u(-1,t) - u(1,t) = 0, \end{cases} \quad t \in [0,T], \tag{2.4}$$

where $\varphi(x)$ *and* $f(x,t)$ *are given sufficiently smooth functions;* $a \neq 1$ *is a real number; and* ε *is a nonzero real number such that* $|\varepsilon| < 1$.

To formulate this problem we have used the following notations:

$$f(x,t) = F(x,t) - \gamma_2 x^2 \int_{-1}^{1} F_t(\xi,t)\,d\xi - 2\gamma_2(1-\varepsilon) \int_{-1}^{1} F(\xi,t)\,d\xi, \tag{2.5}$$

$$\varphi(x) = \phi(x) - \gamma_2 x^2 \int_{-1}^{1} F(\xi,0)\,d\xi. \tag{2.6}$$

3 Summary of related papers

Note that initial-boundary value problems for the nonlocal heat equation (2.4) were repeatedly investigated earlier. In [4] the authors considered inverse problems on recovering

the right-hand side of the nonlocal heat equation

$$u_t(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x),$$

with the Dirichlet boundary conditions $u(-1,t) = u(1,t) = 0$ and the Neumann boundary conditions $u_x(-1,t) = u_x(1,t) = 0$.

In [5] inverse problems on recovering the right-hand side of the nonlocal heat equation with a time-fractional derivative

$$D_t^\alpha u(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x), \quad 0 < \alpha < 1$$

were considered. The authors used Dirichlet boundary conditions, Neumann boundary conditions, the periodic boundary conditions $u(-1,t) = u(1,t)$, $u_x(-1,t) = u_x(1,t)$, and the antiperiodic boundary conditions $u(-1,t) = -u(1,t)$, $u_x(-1,t) = -u_x(1,t)$ as boundary conditions with respect to a spatial variable.

In [6] an inverse problem with the Dirichlet boundary conditions $u(-1,t) = u(1,t) = 0$ for the time fractional evolution equations with an involution perturbation

$$D_t^\alpha \left\{ u(x,t) - u(x,0) - tu_t(x,0) \right\} - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x), \quad 0 < \alpha < 1$$

was considered.

References [4] and [6] contain a rather extensive bibliography on problems for differential operators with involution. A huge number of articles and a sizeable number of books either from the theoretical point of view or for practical considerations were devoted to differential evolution equations with time delays. Note that first Carleman (equations with shift (involution)) and then Przewoerska-Rolewicz, Aftabizadeh, Andreev, Burlutskayaa, Khromov, Gupta, Viner, Kirane, and Torebek paid great attention to differential equations with operations with respect to a spatial variable; the recent works of Kaliev, Sadybekov, Sarsenbi, Ashyralyev, Khromov, Kritskov, and Shkalikov were devoted to spectral problems and inverse problems for equations with involutions. In contrast to the previous works we consider problems with more complex boundary conditions. In this case, we obtain more complex spectral properties (the presence of double eigenvalues and associated vectors).

Also, note the recent work [7] in which the operator of the form $L = JP + Q$ was studied, where J is an involution operator in the space $L_2(a,b)$. In this work P and Q are ordinary differential operators of order n and m generated by s-pieces of boundary conditions (where $s = \max\{n, m\}$) on a bounded interval $[a, b]$. The authors announced theorems on the unconditional basis property and completeness of the root functions of the operator L depending on the type of boundary conditions.

We also note a few more works ([8–35]) that are close to the topic of our investigation.

4 Spectral problem

The use of the Fourier method for solving problem (2.2)–(2.4) leads to a spectral problem for the operator $\mathcal{L}$ given by the differential expression

$$\mathcal{L}y(x) \equiv -y''(x) + \varepsilon y''(-x) = \lambda y(x), \quad -1 < x < 1, \tag{4.1}$$

and the boundary conditions of periodic type

$$\begin{cases} U_1(y) \equiv y'(-1) + ay'(1) = 0, \\ U_2(y) \equiv y(-1) - y(1) = 0, \end{cases} \tag{4.2}$$

where λ is a spectral parameter.

As is known, spectral problems for equation (4.1) were first considered in [25, 26]. The authors considered cases of the Dirichlet and Neumann boundary conditions ($a = -1$). They singled out cases of the boundary conditions when the system of root vectors forms a Riesz basis in L_2. Here, we consider a case $a \neq -1$. Until now, no one has investigated this case.

Spectral problems related to our topic were considered in the works [27–32]. In [27] a problem with the nonlocal conditions

$$y(-1) = 0, \qquad y'(-1) = y'(1)$$

for equation (4.1) was studied. The authors proved that if $r = \sqrt{(1 - \varepsilon)/(1 + \varepsilon)}$ is irrational, then the system of eigenfunctions is complete and minimal in $L_2(-1, 1)$, but is not an unconditional basis. For rational r, it was proved that the system of root functions of the problem (for special choices of the associated functions) is the unconditional basis in $L_2(-1, 1)$. A similar result was proved in [28] for the case of the space $L_p(-1, 1)$.

In [31] a problem for equation (4.1) with the nonlocal boundary conditions

$$y(-1) = \beta y(1), \qquad y'(-1) = y'(1)$$

was investigated for the case of the space $L_2(-1, 1)$ and in [32] for the space $L_p(-1, 1)$. In these papers it was also shown that the multiplicity of eigenvalues depends on the rationality or irrationality of the number r.

Since for equation (4.1) the spectral theory of boundary value problems is not yet fully formed, then each separate case of boundary conditions must be considered separately. The spectral problems with the nonlocal conditions (4.2) have not been previously considered. In this connection, we note the works [3–24] in which close problems related with spectral properties of nonlocal problems were considered.

5 General solution of equation (4.1)

To construct a general solution of equation (4.1), consider the Cauchy problem with data at the interior point

$$\mathcal{L}y(x) \equiv -y''(x) + \varepsilon y''(-x) - \lambda y(x) = f(x), \quad -1 < x < 1, \tag{5.1}$$

$$y(0) = A, \qquad y'(0) = B, \tag{5.2}$$

with arbitrary constants A and B. Here, $f(x) \in C[-1, 1]$.

By direct calculation it is easy to show that this problem (5.1) to (5.2) is equivalent to the integral equation

$$y(x) + \lambda \int_{-x}^{x} k(x, t) y(t) \, dt = A + Bx - \int_{-x}^{x} k(x, t) f(t) \, dt, \tag{5.3}$$

with the integral operator

$$\int_{-x}^{x} k(x,t)\varphi(t)\,dt \equiv \frac{1}{1-\alpha^2}\left\{\alpha\int_{-x}^{0}(x+t)\varphi(t)\,dt + \int_{0}^{x}(x-t)\varphi(t)\,dt\right\}.$$

Let us show that the integral equation (5.3) has a unique solution. For this, we introduce a new function

$$Y(x) = y(x)e^{-\mu|x|},$$

where $\mu > 0$ is a positive parameter that we will choose below.

Then, for $Y(x)$ we obtain the integral equation

$$Y(x) + \lambda \int_{-x}^{x} k_1(x,t)Y(t)\,dt = \psi(x), \tag{5.4}$$

where it is indicated

$$k_1(x,t) = k(x,t)e^{-\mu[|x|-|t|]},$$

$$\psi(x) = (A + Bx)e^{-\mu|x|} - \int_{-x}^{x} k_1(x,t)f_1(t)\,dt,$$

$$f_1(x) = f(x)e^{-\mu|x|}.$$

Let I_λ denote the integral operator in the left-hand side of (5.4). Estimating its norm in $L_2(-1,1)$, we have

$$\|I_\lambda\| \leq \frac{|\lambda|}{1-\alpha^2}\frac{\sqrt{2\mu - 1 + e^{-2\mu}}}{\mu}.$$

Hence, it is easy to see that for any λ we always can choose a positive number $\mu > 0$ such that the operator norm will be less than one: $\|I_\lambda\| \leq \delta < 1$. Therefore, with this choice of μ, equation (5.4) has the unique solution $Y(x) \in L_2(-1,1)$.

That is why equation (5.3) has the unique solution $y(x) \in L_2(-1,1)$.

It is easily seen that

$$A + Bx - \int_{-x}^{x} k(x,t)f(t)\,dt \in C^2[-1,1]$$

for $f(x) \in C[-1,1]$.

It is also easy to see that if $y(x) \in L_2(-1,1)$, then

$$\int_{-x}^{x} k(x,t)y(t)\,dt \in C[-1,1].$$

Therefore, from equation (5.3) we obtain that $y(x) \in C[-1,1]$ for $f(x) \in C[-1,1]$.

Further, since $y(x) \in C[-1,1]$, then it is easy to see that

$$\int_{-x}^{x} k(x,t)y(t)\,dt \in C^2[-1,1].$$

Therefore, from equation (5.3) we obtain that $y(x) \in C^2[-1,1]$ for $f(x) \in C[-1,1]$.

Thus, the following lemma is proved.

Lemma 5.1 *For any values of the parameter λ, of the constants A and B and for any function $f(x) \in C[-1,1]$ the Cauchy problem* (5.1) *to* (5.2) *has the unique solution $y(x) \in C^2[-1,1]$.*

As follows from this lemma, the general solution of equation (4.1) is two-parameter. As fundamental solutions we choose two functions $c(x,\lambda) \in C^2[-1,1]$ and $s(x,\lambda) \in C^2[-1,1]$ that are solutions of equation (4.1) and satisfy the Cauchy conditions:

$$c(0,\lambda) = s'(0,\lambda) = 1, \qquad c'(0,\lambda) = s(0,\lambda) = 0.$$

The existence of such solutions is ensured by Lemma 5.1.

By direct calculation it is easy to obtain these solutions explicitly:

$$c(x,\lambda) = \cos(\mu_1 x), \qquad s(x,\lambda) = \frac{1}{\mu_2}\sin(\mu_2 x), \qquad \mu_1 = \sqrt{\frac{\lambda}{1-\varepsilon}}, \qquad \mu_2 = \sqrt{\frac{\lambda}{1+\varepsilon}}.$$

It is also easy to verify that the chosen solutions have the following symmetry properties:

$$c(-x,\lambda) = c(x,\lambda), \qquad s(-x,\lambda) = -s(x,\lambda), \quad -1 \leq x \leq 1. \tag{5.5}$$

Thus, the general solution of equation (4.1) has the form:

$$y(x,\lambda) = C_1 c(x,\lambda) + C_2 s(x,\lambda) \tag{5.6}$$

with arbitrary constants C_1 and C_2.

6 Eigenvalues of problem (4.1) to (4.2)

First, it is easy to see that $\lambda = 0$ is an eigenvalue of problem (4.1) to (4.2). The corresponding eigenfunction has the form:

$$y_0(x) = 1.$$

Consider a case $\lambda \neq 0$. Satisfying the general solution (5.6) of equation (4.1) to the boundary conditions (4.2), we obtain the linear system

$$\begin{cases} C_1 U_1(c(x,\lambda)) + C_2 U_1(s(x,\lambda)) = 0, \\ C_1 U_2(c(x,\lambda)) + C_2 U_2(s(x,\lambda)) = 0. \end{cases} \tag{6.1}$$

Its determinant will be the characteristic determinant of the spectral problem (4.1) to (4.2):

$$\triangle(\lambda) \equiv \begin{vmatrix} U_1(c(x,\lambda)) & U_1(s(x,\lambda)) \\ U_2(c(x,\lambda)) & U_2(s(x,\lambda)) \end{vmatrix} = 0.$$

Therefore, taking into account the symmetry conditions (5.5), we calculate

$$\triangle(\lambda) \equiv 2(1-a)c'(1,\lambda)s(1,\lambda) = 0. \tag{6.2}$$

First, from (6.2) we obtain that for $a = 1$ each number λ is the eigenvalue of problem (4.1) to (4.2), regardless of the value of ε. In this case, system (6.1) has the form

$$
\begin{cases}
C_2 s'(1, \lambda) = 0, \\
C_2 s(1, \lambda) = 0.
\end{cases}
$$

Since $|s'(1, \lambda)| + |s(1, \lambda)| > 0$, then it follows that $C_2 = 0$.

Thus, the following lemma is proved.

Lemma 6.1 *For $a = 1$ each number λ is the eigenvalue of problem (4.1) to (4.2). The corresponding eigenfunctions have the form*

$$
y(x, \lambda) = \cos(\mu_1 x), \qquad \mu_1 = \sqrt{\frac{\lambda}{1 - \varepsilon}}. \tag{6.3}
$$

Now consider the case when $a \neq 1$. Then, from (6.2) we obtain $c'(1, \lambda)s(1, \lambda) = 0$. Therefore, taking into account the explicit form of fundamental solutions, we have

$$
\sin(\mu_1) \sin(\mu_2) = 0.
$$

Thus, problem (4.1) to (4.2) has two series of the eigenvalues

$$
\begin{aligned}
\lambda_k^{(1)} &= (1 - \varepsilon)(k\pi)^2, \quad k = 0, 1, 2, \ldots, \\
\lambda_n^{(2)} &= (1 + \varepsilon)(n\pi)^2, \quad n = 1, 2, \ldots.
\end{aligned} \tag{6.4}
$$

Lemma 6.2 *Problem (4.1) to (4.2) has multiple eigenvalues if and only if the number $r = \sqrt{(1 - \varepsilon)/(1 + \varepsilon)}$ is rational.*

Proof Indeed, suppose that any two eigenvalues from different series coincide:

$$
\lambda_k^{(1)} = \lambda_n^{(2)}.
$$

This is equivalent to the equality

$$
(1 - \varepsilon)(k\pi)^2 = (1 + \varepsilon)(n\pi)^2.
$$

That is, the coincidence of eigenvalues is possible if and only if for some $k_0, n_0 \in N$ $r = n_0/k_0$ holds. That is, only if the value r is rational. □

7 Spectral problem for irrational numbers *r*

Let r be an irrational number. Then, by virtue of Lemma 6.2, all eigenvalues of problem (4.1) to (4.2) are simple and are given by the formulas (6.4). By direct calculation from (6.1) we obtain that

$$
\begin{aligned}
y_k^{(1)}(x) &= \cos(k\pi x), \\
y_n^{(2)}(x) &= (1 + a)r \cos(n\pi) \cos\left(\frac{n\pi x}{r}\right) + (a - 1)\sin\left(\frac{n\pi}{r}\right) \sin(n\pi x),
\end{aligned} \tag{7.1}
$$

where $k = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$, correspond to these eigenvalues.

Lemma 7.1 *The system of functions* (7.1) *is complete and minimal in* $L_2(-1, 1)$.

Proof Consider an arbitrary function $f(x)$ orthogonal to system (7.1). Since it is orthogonal to all functions $y_k^{(1)}(x)$, $k = 0, 1, 2, \ldots$, we have

$$0 = \int_{-1}^{1} f(x)\cos(k\pi x)\,dx = \int_{0}^{1} \{f(x) + f(-x)\}\cos(k\pi x)\,dx.$$

However, the system $\{\cos(k\pi x), k = 0, 1, 2, \ldots\}$ forms a basis in $L_2(0, 1)$. Therefore, $f(x) + f(-x) = 0$ holds almost everywhere on the interval $(-1, 1)$. That is, this function is odd.

Therefore, from the orthogonality of $f(x)$ to all functions $y_n^{(2)}(x)$, $n = 1, 2, \ldots$ we obtain

$$0 = \int_{-1}^{1} f(x)y_n^{(2)}(x)\,dx = (a - 1)\sin\left(\frac{n\pi}{r}\right)\int_{-1}^{1} f(x)\sin(n\pi x)\,dx.$$

Since $a \neq 1$ and the number r is irrational, from this we have

$$0 = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{0}^{1} \{f(x) - f(-x)\}\sin(n\pi x)\,dx.$$

However, the system $\{\sin(n\pi x), n = 1, 2, \ldots\}$ forms a basis in $L_2(0, 1)$. Therefore, $f(x) - f(-x) = 0$ holds almost everywhere on the interval $(-1, 1)$. That is, this function is even.

Thus, the function $f(x)$ turns out to be simultaneously even and odd almost everywhere on the interval $(-1, 1)$. Consequently, $f(x) = 0$ holds almost everywhere on the interval $(-1, 1)$. This proves the completeness of the system of functions (7.1) in $L_2(-1, 1)$.

Since the system under consideration (7.1) is a system of eigenfunctions of a linear operator, then it has a biorthogonal system consisting of eigenfunctions of an adjoint operator. We will not dwell here on a specific form of this system and the adjoint operator. However, from the existence of the biorthogonal system follows the minimality of the system of functions (7.1) in $L_2(-1, 1)$. Thus, the Lemma is proved.

Now let us show that despite the fact that the system of functions (7.1) is complete and minimal in $L_2(-1, 1)$, it does not form an unconditional basis. For this, we use the necessary condition for the basis property from [36].   □

Lemma 7.2 ([36], Th. 3.135, s. 219) *Let* $\{u_j\}$ *be a closed and minimal system in a Hilbert space* H. *If the system* $\{u_j\}$ *is an unconditional basis in* H, *then the strict inequality holds*

$$\limsup_{j\to\infty}\left|\left\langle \frac{u_j}{\|u_j\|}, \frac{u_{j+1}}{\|u_{j+1}\|}\right\rangle\right| < 1, \tag{7.2}$$

where $\langle\cdot, \cdot\rangle$ *is the inner products in* H.

By virtue of this lemma, for the unconditional basis property in $L_2(-1, 1)$ of the system of functions (7.1), it is necessary to satisfy the strict inequality

$$\limsup_{j\to\infty}\left|\left\langle \frac{y_{k_j}^{(1)}}{\|y_{k_j}^{(1)}\|}, \frac{y_{n_j}^{(2)}}{\|y_{n_j}^{(2)}\|}\right\rangle\right| < 1 \tag{7.3}$$

for all possible infinitely increasing subsequences k_j and n_j.

Calculating the norms of the eigenfunctions, we obtain

$$\left\|y_k^{(1)}\right\| = 1,$$

$$\left\|y_n^{(2)}\right\|^2 = (1 + a)^2 r^2 \left\{1 + \frac{r}{2n\pi} \sin\left(\frac{2n\pi}{r}\right)\right\} + (1 - a)^2 \sin^2\left(\frac{n\pi}{r}\right).$$

Therefore,

$$\left\|y_n^{(2)}\right\|^2 = (1 + a)^2 r^2 + (1 - a)^2 \sin^2\left(\frac{n\pi}{r}\right) + O\left(\frac{1}{n}\right) \tag{7.4}$$

for $n \to \infty$.

Calculate the inner products in $L_2(-1, 1)$:

$$\left|\langle y_k^{(1)}, y_n^{(2)}\rangle\right| = |1 + a|r \left|\int_{-1}^{1} \cos(k\pi x) \cos\left(\frac{n\pi x}{r}\right) dx\right|$$

$$= |1 + a|r \left|\frac{\sin(k - \frac{n}{r})\pi}{(k - \frac{n}{r})\pi} + O\left(\frac{1}{k + n}\right)\right|$$

for $k, n \to \infty$.

According to Dirichlet's approximation theorem (see, example, [37], Th. 1A, p. 34), for any irrational number α there exists an infinite set of irreducible fractions $\frac{p}{q}$ (where p and q are integers) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Choosing here $\alpha = \frac{1}{r}$, we obtain that there exist infinite subsequences of the natural numbers k_j and n_j such that

$$\left|\frac{1}{r} - \frac{k_j}{n_j}\right| < \frac{1}{n_j^2}.$$

For these subsequences we will have

$$\left|k_j - \frac{n_j}{r}\right| < \frac{1}{n_j}.$$

Therefore, there exists the limit

$$\lim_{j \to \infty} \frac{\sin(k_j - \frac{n_j}{r})\pi}{(k_j - \frac{n_j}{r})\pi} = 1.$$

From this we have that the limit exists

$$\lim_{j \to \infty} \left|\langle y_{k_j}^{(1)}, y_{n_j}^{(2)}\rangle\right| = |1 + a|r.$$

From (7.4) it is easily seen that the limit exists

$$\lim_{j \to \infty} \left\|y_{n_j}^{(2)}\right\| = |1 + a|r.$$

Finally, substituting everything obtained in (7.5), we obtain

$$\lim_{j \to \infty} \left| \left\langle \frac{y_{k_j}^{(1)}}{\|y_{k_j}^{(1)}\|}, \frac{y_{n_j}^{(2)}}{\|y_{n_j}^{(2)}\|} \right\rangle \right| = 1 \tag{7.5}$$

for our chosen (according to the Dirichlet's approximation theorem) infinitely increasing subsequences k_j and n_j. That is, the necessary condition of the unconditional basis property (3.3) is not satisfied.

Thus, the following lemma is proved.

Lemma 7.3 *Let r be irrational. Then, the system of eigenfunctions* (3.2) *of the spectral problem* (1.8) *to* (1.9) *is complete and minimal but does not form an unconditional basis in* $L_2(-1, 1)$.

8 Spectral problem for rational numbers *r*

Now consider the case when r is a rational number. Then, there exist natural numbers n_0 and k_0 such that $r = \frac{n_0}{k_0}$. In this case, as follows from (6.4), problem (4.1) to (4.2) has an infinite number of double eigenvalues

$$\lambda_{k_0 j}^{(1)} = \lambda_{n_0 j}^{(2)}, \quad j \in \mathbb{N}. \tag{8.1}$$

As mentioned above, the spectral problems for equation (4.1) with periodic boundary conditions ($a = -1$) were considered in [25, 26]. For periodic problems it was shown that root subspaces, consisting of two eigenfunctions, correspond to the double eigenvalues. Here, we consider the case $a \neq -1$.

For $a \neq -1$, one eigenfunction and one associated function correspond to the double eigenvalues (8.1).

By direct calculation it is easily shown that for the cases when $\frac{n}{k} \neq \frac{n_0}{k_0}$, problem (4.1) to (4.2) has the eigenfunctions

$$
\begin{aligned}
y_k^{(1)}(x) &= \cos(k\pi x), \\
y_n^{(2)}(x) &= (1 + a)r\cos(n\pi)\cos\left(\frac{n\pi x}{r}\right) + (a - 1)\sin\left(\frac{n\pi}{r}\right)\sin(n\pi x),
\end{aligned} \tag{8.2}
$$

where $k = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$, except for the cases when $k = k_0 j$, $n = n_0 j$ for some j.

For those cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), problem (4.1) to (4.2) has the eigenfunctions $y_{k_0 j}^{(1)}(x)$ and the corresponding associated functions $y_{n_0 j,1}(x)$:

$$
\begin{aligned}
y_{k_0 j}^{(1)}(x) &= \cos(k_0 j\pi x), \\
y_{n_0 j,1}(x) &= -\frac{1}{2k_0 j\pi(1 - \varepsilon)}\left\{ x\sin(k_0 j\pi x) + \frac{1 - a}{1 + a}\frac{1}{r}(-1)^{(n_0 + k_0)j}\sin(n_0 j\pi x) \right\}.
\end{aligned} \tag{8.3}
$$

Here, we mean by the associated functions (according to M.V. Keldysh) solutions of the inhomogeneous equation

$$\mathcal{L}y_{k,1}(x) \equiv -y_{k,1}''(x) + \varepsilon y_{k,1}''(-x) = \lambda_k^{(1)} y_{k,1}(x) + y_k^{(1)}(x), \quad -1 < x < 1, \tag{8.4}$$

satisfying the boundary conditions (4.2).

It is well known that the associated functions are not defined uniquely. Functions of the form

$$\widetilde{y}_{k_0 j,1}(x) = y_{k_0 j,1}(x) + C_j y_{k_0 j}^{(1)}(x)$$

for any constants C_j are also the associated functions of problem (4.1) to (4.2) corresponding to the eigenvalues $\lambda_{k_0 j}^{(1)}$ and the eigenfunctions $y_{k_0 j}^{(1)}(x)$. "The problem of choosing associated functions" is also well known. This problem is related to the fact that with one choice of the constants C_j the system can form an unconditional basis, and with other choices of these constants the system does not form an unconditional basis. To avoid this problem, we fix such a choice of associated functions by formula (8.3).

Lemma 8.1 *The system of eigen- and associated functions* (8.2) *to* (8.3) *of problem* (4.1) *to* (4.2) *is complete and minimal in* $L_2(-1,1)$.

The proof is similar to the proof of Lemma 7.1. Consider an arbitrary function $f(x)$ orthogonal to the system of functions (8.2) to (8.3). Since it is orthogonal to all functions $y_k^{(1)}(x)$, $k = 0, 1, 2, \ldots$, then, as in the proof of Lemma 4, we have that $f(x) + f(-x) = 0$ (that is, this function is even) holds almost everywhere on the interval $(-1, 1)$.

Further, from the orthogonality of $f(x)$ to all functions $y_n^{(2)}(x)$ from (8.2) we obtain that

$$\int_{-1}^{1} f(x) \sin(n\pi x)\, dx = 0 \tag{8.5}$$

for all $n = 1, 2, \ldots$, except the cases when $n = n_0 j$ for some j.

It follows from the oddness of $f(x)$ that it is orthogonal to the functions $x \sin(k_0 j \pi x)$. Therefore, from the orthogonality of $f(x)$ to all functions $y_{k_0 j,1}(x)$ from (8.3) we obtain that (8.5) holds and for the cases when $n = n_0 j$ for some j.

Since the system $\{\sin(n\pi x), n = 1, 2, \ldots\}$ forms the basis in $L_2(0, 1)$, then $f(x) - f(-x) = 0$ (that is, this function is even) holds almost everywhere on the interval $(-1, 1)$.

Thus, the function $f(x)$ turns out to be simultaneously even and odd almost everywhere on the interval $(-1, 1)$. Consequently, $f(x) = 0$ holds almost everywhere on the interval $(-1, 1)$. This proves the completeness of the system of functions (8.2) to (8.3) in $L_2(-1, 1)$.

Since the system under consideration (8.2) to (8.3) is the system of eigen- and associated functions of a linear operator, then it has a biorthogonal system consisting of eigen- and associated functions of an adjoint operator. We will not dwell here on a specific form of this system and the adjoint operator. However, from the existence of the biorthogonal system follows the minimality of the system of functions (8.2) to (8.3) in $L_2(-1, 1)$. The Lemma is proved.

Now, let us prove that system (8.2) to (8.3) forms an unconditional basis in $L_2(-1, 1)$. For this we need a biorthogonal system. It is a system of eigen- and associated functions of the adjoint problem:

$$\mathcal{L}^* v(x) \equiv -v''(x) + \varepsilon v''(-x) = \lambda v(x), \quad -1 < x < 1, \tag{8.6}$$

$$\begin{cases} V_1(v) \equiv v'(-1) - v'(1) = 0, \\ V_2(v) \equiv (a - \varepsilon)v(-1) + (1 - a\varepsilon)v(1) = 0. \end{cases} \tag{8.7}$$

Since the eigenvalues (6.4) of problem (4.1) to (4.2) are real, they are also the eigenvalues of the adjoint problem (8.6) to (8.7). The system of eigen- and associated functions of this problem can be constructed explicitly.

The eigenfunction

$$v_0(x) = \frac{1}{2} - \frac{(1+a)}{2(1-a)}r^2 x \tag{8.8}$$

corresponds to a zero eigenvalue.

By direct calculation it is easily shown that for those cases when $\frac{n}{k} \neq \frac{n_0}{k_0}$, problem (8.6) to (8.7) has the eigenfunctions

$$
\begin{aligned}
v_k^{(1)}(x) &= \cos(k\pi x) - \frac{1+a}{1-a}r^2\frac{(-1)^k}{\sin(rk\pi)}\sin(rk\pi x), \\
v_n^{(2)}(x) &= -\frac{1}{1-a}\frac{1}{\sin(\frac{n\pi}{r})}\sin(n\pi x),
\end{aligned}
\tag{8.9}
$$

corresponding to the eigenvalues $\lambda_k^{(1)}$ and $\lambda_n^{(2)}$, where $k = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$, except the cases when $k = k_0 j$, $n = n_0 j$ for some j.

For the cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), problem (8.6) to (8.7) has the eigenfunctions $v_{n_0 j}^{(2)}(x)$ and the associated functions corresponding to them $v_{k_0 j, 1}(x)$:

$$
\begin{aligned}
v_{n_0 j}^{(2)}(x) &= -k_0 j\pi(1-\varepsilon)\frac{1+a}{1-a}r(-1)^{(n_0+k_0)j}\sin(n_0 j\pi x), \\
v_{k_0 j, 1}(x) &= -\frac{1+a}{1-a}r^2(-1)^{(n_0+k_0)j}x\cos(n_0 j\pi x) + \cos(k_0 j\pi x).
\end{aligned}
\tag{8.10}
$$

When constructing this system of eigen- and associated functions of the adjoint problem, we have normalized the eigenfunctions so that the biorthogonality conditions

$$\big\langle y_k^{(1)}, v_k^{(1)}\big\rangle = 1, \qquad \big\langle y_n^{(2)}, v_n^{(2)}\big\rangle = 1,$$

hold for all $k = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$, except for the cases when $k = k_0 j$, $n = n_0 j$ for some j.

For the cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), we have required the fulfilment of the biorthogonality conditions

$$\big\langle y_{k_0 j}^{(1)}, v_{k_0 j, 1}\big\rangle = 1, \qquad \big\langle y_{n_0 j, 1}, v_{n_0 j}^{(2)}\big\rangle = 1.$$

Here, by $\langle \cdot, \cdot \rangle$ we denote the inner product in $L_2(-1, 1)$.

For what follows, we need to estimate the norms of the constructed eigen- and associated functions. By direct calculation we find

$$\big\| y_k^{(1)} \big\| = 1; \qquad \big\| y_n^{(2)} \big\|^2 = (1+a)^2 r^2\left\{1 + \frac{r}{2n\pi}\sin\left(\frac{2n\pi}{r}\right)\right\} + (1-a)^2\sin^2\left(\frac{n\pi}{r}\right);$$

$$\big\| v_k^{(1)} \big\|^2 = 1 + \left(\frac{1+a}{1-a}\right)^2\frac{r^2}{\sin^2(rk\pi)}; \qquad \big\| v_n^{(2)} \big\|^2 = \frac{1}{(1-a)^2}\frac{1}{\sin^2(\frac{n\pi}{r})};$$

$$\left\| y_{n_0 j}^{(1)} \right\| = 1; \qquad \left\| y_{n_0 j,1} \right\|^2 = \frac{1}{(2k_0 j\pi(1-\varepsilon))^2}\left\{ \frac{1}{3} - \frac{1}{2(k_0 j)^2} + \left(\frac{1-a}{1+a}\right)^2 \frac{1}{r^2} \right\};$$

$$\left\| v_{n_0 j}^{(2)} \right\|^2 = \left(2k_0 j\pi(1-\varepsilon)\right)^2 \left(\frac{1+a}{1-a}\right)^2 r^2;$$

$$\left\| v_{k_0 j,1} \right\|^2 = 1 + \left(\frac{1+a}{1-a}\right)^2 r^4 \left\{ \frac{1}{3} + \frac{1}{2(k_0 j)^2} \right\}.$$

Analyzing these explicit formulas, we see that only the asymptotic behavior of multipliers $\sin(\frac{n\pi}{r})$ and $\sin(rk\pi)$ is not obvious. Let us show that these multipliers are strictly separated from zero.

Lemma 8.2 *If r is a rational number*: $r = \frac{n_0}{k_0}$, *then for all values of the indices n and k, when $n \neq n_0 j$ and $k \neq k_0 j$, the inequalities hold*

$$\left| \sin\left(\frac{n\pi}{r}\right) \right| \geq \left| \sin\left(\frac{\pi}{n_0}\right) \right|, \qquad \left| \sin(rk\pi) \right| \geq \left| \sin\left(\frac{\pi}{k_0}\right) \right|. \tag{8.11}$$

The proof will be carried out by the method used in [30–32]. Since $n \neq n_0 j$, then the representation $n = n_0 j + i$ holds for some $j, i \in \mathbb{N}$, $1 \leq i \leq n_0 - 1$. Therefore, $\frac{n}{r} = k_0 j + \frac{k_0 i}{n_0}$. Since $\frac{n}{k} \neq \frac{n_0}{k_0}$, then this number $\frac{n}{r} = k_0 j + \frac{k_0 i}{n_0}$ is not an integer. Consequently, we have:

$$\left| \sin\left(\frac{n\pi}{r}\right) \right| = \left| \sin\left(\pi\left(\frac{n}{r} - k_0 j\right)\right) \right| = \left| \sin\left(\frac{\pi}{n_0} k_0 i\right) \right| \geq \left| \sin\left(\frac{\pi}{n_0}\right) \right|.$$

The second inequality from (8.11) is proved similarly. Since $k \neq k_0 j$, then the representation $k = k_0 j + i$ holds for some $j, i \in \mathbb{N}$, $1 \leq i \leq k_0 - 1$. Therefore, $rk = n_0 j + \frac{n_0 i}{k_0}$. Since $\frac{n}{k} \neq \frac{n_0}{k_0}$, this number $rk = n_0 j + \frac{n_0 i}{k_0}$ is not an integer. Hence, we have:

$$\left| \sin(rk\pi) \right| = \left| \sin\left(\pi(rk - n_0 j)\right) \right| = \left| \sin\left(\frac{\pi}{k_0} n_0 i\right) \right| \geq \left| \sin\left(\frac{\pi}{k_0}\right) \right|.$$

Lemma 8.3 *Let r be a rational number*: $r = \frac{n_0}{k_0}$. *Then, each of the systems* (8.2) *to* (8.3) *and* (8.8) *to* (8.10) *(after the normalization in $L_2(-1,1)$) satisfies a Bessel-type inequality. Both systems form an unconditional basis in $L_2(-1,1)$.*

Note that the system $\{\varphi_j\}$ has the Bessel property in a Hilbert space H, if there exists a constant $B > 0$ such that the Bessel-type inequality

$$\sum_j \left| \langle f, \varphi_j \rangle \right|^2 \leq B \|f\|^2$$

holds for all elements $f \in H$.

Proof By virtue of the above estimates of the eigen- and associated functions, in order to substantiate the Bessel property, we show the fulfilment of the Bessel property for the following three type of systems ($j \in \mathbb{N}$):

$$\cos(j\pi x), \qquad \sin(j\pi x); \tag{8.12}$$

$$\cos\left(\frac{k_0}{n_0}j\pi x\right), \qquad \sin\left(\frac{k_0}{n_0}j\pi x\right); \tag{8.13}$$

$$x\cos(j\pi x), \qquad x\sin(j\pi x). \tag{8.14}$$

Since system (8.12) is orthonormal in $L_2(-1,1)$, then it satisfies the Bessel-type inequality with constant $B = 1$. Since the multiplier x is bounded, the Bessel property of system (8.14) follows from the Bessel property of system (8.12). As a result, system (8.13) is the Bessel system by virtue of the following assertion proved in [30–32]. □

Lemma 8.4 ([30–32]) *Let $\{\gamma_j\}$ be a sequence of complex numbers such that*

$$\sup_j \left|\mathrm{Im}(\gamma_j)\right| < \infty, \qquad \sup_{t \geq 1} \sum_{j:|\,\mathrm{Re}(\gamma_j)-t|\leq 1} 1 < \infty. \tag{8.15}$$

Then, each of the systems $\{\sin(\gamma_j x)\}$ and $\{\cos(\gamma_j x)\}$ is a Bessel system in $L_2(-1,1)$.

System (8.13) satisfies condition (8.15) because

$$\mathrm{Im}(\gamma_j) = 0, \qquad \sum_{j:|\,\mathrm{Re}(\gamma_j)-t|\leq 1} 1 \leq 2m_0 + 1.$$

Now, from the well-known Bari theorem [38] it follows the unconditional basis property of the systems (8.2) to (8.3) and (8.8) to (8.10). The proof of Lemma 8.3 is complete.

Combining all the results, we formulate them together in the form of one theorem.

Theorem 8.5 *Let $a \neq -1$. Then, the spectral problem (4.1) to (4.2) has the following properties.*

★ For $a = 1$ each number λ will be an eigenvalue of problem (4.1) to (4.2). Corresponding eigenfunctions are of the form (6.3).

★ Problem (4.1) to (4.2) has double eigenvalues if and only if the number $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ is rational.

★ If r is an irrational number, then all eigenvalues of problem (4.1) to (4.2) are simple, and its system of eigenfunctions (7.1) is complete and minimal but does not form an unconditional basis in $L_2(-1,1)$.

★ If r is a rational number, then there exists an infinite countable subsequence of eigenvalues of problem (4.1) to (4.2) that are double. The rest of the eigenvalues of problem (4.1) to (4.2) (there are also infinite countable number of them) are simple. One eigenfunction and one associated function correspond to each double eigenvalue. The system of eigen- and associated functions (8.2) to (8.3) of problem (4.1) to (4.2) is complete and minimal in $L_2(-1,1)$. The associated functions of problem (4.1) to (4.2) can be chosen in such a special way that this special system of eigen- and associated functions forms an unconditional basis in $L_2(-1,1)$.

9 Construction of a formal solution of problem (2.2)–(2.4) by the method of separation of variables

As follows from Theorem 1, the system of root functions of the spectral problem (4.1) to (4.2) forms an unconditional basis if and only if the number $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ is rational. Therefore, using the method of separation of variables to solve problem (2.2)–(2.4) is possible only in the case when r is rational. Therefore, below we consider only such a case.

Moreover, if we construct associated functions according to another (special) formula so that the system of eigen- and associated functions becomes almost normalized, then this system will form a Riesz basis. The use of Riesz bases is more convenient for us, since it facilitates the proof of the convergence of the series obtained.

As before, we denote by $\mathbb{N}$ the set of all natural numbers. By N_1 denote the set of all nonnegative integers. By N_2 denote the set of all natural numbers except for multiples n_0: $N_2 = \{n \in \mathbb{N} : n \neq n_0 j, \forall j \in \mathbb{N}\}$.

As follows from Lemma 9, the system of functions

$$\left\{ y_k^{(1)}(x), k \in N_1; y_n^{(2)}(x), n \in N_2; j y_{n_0 j,1}(x), j \in \mathbb{N} \right\} \tag{9.1}$$

forms the Riesz basis in $L_2(-1, 1)$. Here, the associated function $y_{n_0 j,1}(x)$ enters with the multiple j so that due to this the system becomes almost normalized. As is known, if an unconditional basis is almost normalized, then it is the Riesz basis. These functions satisfy the equations

$$\mathcal{L} y_k^{(1)} = \lambda_k^{(1)} y_k^{(1)}; \qquad \mathcal{L} y_n^{(2)} = \lambda_n^{(2)} y_n^{(2)}; \qquad \mathcal{L} y_{n_0 j,1} = \lambda_{n_0 j}^{(2)} y_{n_0 j,1} + y_{k_0 j}^{(1)}$$

and the boundary conditions (4.2).

Therefore, any solution of problem (2.2) to (2.4) can be represented in the form of a biorthogonal series

$$u(x,t) = \sum_{k \in N_1} y_k^{(1)}(x) u_k^{(1)}(t) + \sum_{n \in N_2} y_n^{(2)}(x) u_n^{(2)}(t) + \sum_{j \in \mathbb{N}} \left\{ j y_{n_0 j,1}(x) - t j y_{k_0 j}^{(1)}(x) \right\} u_{n_0 j}^{(2)}(t), \tag{9.2}$$

where $u_k^{(1)}(t)$, $u_n^{(2)}(t)$ and $u_{n_0 j}^{(2)}(t)$ are still unknown functions that we need to define.

We also expand the right-hand sides of equation (2.2) and the initial condition (2.3) into the biorthogonal series

$$f(x,t) = \sum_{k \in N_1} y_k^{(1)}(x) f_k^{(1)}(t) + \sum_{n \in N_2} y_n^{(2)}(x) f_n^{(2)}(t) + \sum_{j \in \mathbb{N}} \left\{ j y_{n_0 j,1}(x) - t j y_{k_0 j}^{(1)}(x) \right\} f_{n_0 j}^{(2)}(t), \tag{9.3}$$

$$\varphi(x) = \sum_{k \in N_1} y_k^{(1)}(x) \varphi_k^{(1)} + \sum_{n \in N_2} y_n^{(2)}(x) \varphi_n^{(2)} + \sum_{j \in \mathbb{N}} j y_{n_0 j,1}(x) \varphi_{n_0 j}^{(2)}. \tag{9.4}$$

Here, $f_k^{(1)}(t), f_n^{(2)}(t)$ and $f_{n_0 j}^{(2)}(t)$ are known functions, and $\varphi_k^{(1)}$, $\varphi_n^{(2)}$, and $\varphi_{n_0 j}^{(2)}$ are known numerical coefficients.

It should be noted that such a method of constructing solutions by the method of separation of variables using multipliers of the form

$$\left\{ y_{n_0 j,1}(x) - t y_{k_0 j}^{(1)}(x) \right\} u_{n_0 j}^{(2)}(t)$$

is well known starting with the works by Ionkin [39] and Ionkin, and Moiseev [40]. This method was repeatedly used by other authors earlier. We have slightly modified and concretized this form making it convenient for constructing a solution of our problem.

Since system (9.1) is the Riesz basis, then the Bessel-type inequalities hold

$$\sum_{k \in N_1} \left\| f_k^{(1)} \right\|_{L_2(0,T)}^2 + \sum_{n \in N_2} \left\| f_n^{(2)} \right\|_{L_2(0,T)}^2 + \sum_{j \in \mathbb{N}} \left\| f_{n_0 j}^{(2)} \right\|_{L_2(0,T)}^2 \leq C \| f \|_0^2, \tag{9.5}$$

$$\sum_{k \in N_1} \left|\varphi_k^{(1)}\right|^2 + \sum_{n \in N_2} \left|\varphi_n^{(2)}\right|^2 + \sum_{j \in \mathbb{N}} \left|\varphi_{n_0 j}^{(2)}\right|^2 \leq C \|\varphi\|_{L_2(-1,1)}^2. \tag{9.6}$$

Here, by $\| \cdot \|_0$ the norm in $L_2(\Omega)$ is denoted.

Remark 9.1 If $\varphi(x) \in W_2^2(-1,1)$ and satisfies the boundary conditions (4.2), then the following estimate holds

$$\sum_{k \in N_1} \left|\lambda_k^{(1)}\right|^2 \left|\varphi_k^{(1)}\right|^2 + \sum_{n \in N_2} \left|\lambda_n^{(2)}\right|^2 \left|\varphi_n^{(2)}\right|^2 + \sum_{j \in \mathbb{N}} \left|\lambda_{n_0 j}^{(2)}\right|^2 \left|\varphi_{n_0 j}^{(2)}\right|^2 \leq C \|\varphi\|_{W_2^2(-1,1)}^2. \tag{9.7}$$

Substituting (9.2) in (2.2) to (2.4), taking into account (9.3) and (9.4), we obtain a problem for finding the unknown functions $u_k^{(1)}(t)$, $u_n^{(2)}(t)$, and $u_{n_0 j}^{(2)}(t)$:

$$\frac{d}{dt} u_k^{(1)}(t) + \lambda_k^{(1)} u_k^{(1)}(t) = f_k^{(1)}(t), \quad 0 < t < T, u_k^{(1)}(0) = \varphi_k^{(1)}, k \in N_1; \tag{9.8}$$

$$\frac{d}{dt} u_n^{(2)}(t) + \lambda_n^{(2)} u_n^{(2)}(t) = f_n^{(2)}(t), \quad 0 < t < T, u_n^{(2)}(0) = \varphi_n^{(2)}, n \in N_2; \tag{9.9}$$

$$\frac{d}{dt} u_{n_0 j}^{(2)}(t) + \lambda_{n_0 j}^{(2)} u_{n_0 j}^{(2)}(t) = f_{n_0 j}^{(2)}(t), \quad 0 < t < T, u_{n_0 j}^{(2)}(0) = \varphi_{n_0 j}^{(2)}, j \in \mathbb{N}. \tag{9.10}$$

The solutions of these problems exist, are unique, and can be written out explicitly:

$$u_k^{(1)}(t) = \varphi_k^{(1)} e^{-\lambda_k^{(1)} t} + \int_0^t e^{-\lambda_k^{(1)}(t-\tau)} f_k^{(1)}(\tau)\, d\tau, \quad k \in N_1; \tag{9.11}$$

$$u_n^{(2)}(t) = \varphi_n^{(2)} e^{-\lambda_n^{(2)} t} + \int_0^t e^{-\lambda_n^{(2)}(t-\tau)} f_n^{(2)}(\tau)\, d\tau, \quad n \in N_2; \tag{9.12}$$

$$u_{n_0 j}^{(2)}(t) = \varphi_{n_0 j}^{(2)} e^{-\lambda_{n_0 j}^{(2)} t} + \int_0^t e^{-\lambda_{n_0 j}^{(2)}(t-\tau)} f_{n_0 j}^{(2)}(\tau)\, d\tau, \quad j \in \mathbb{N}. \tag{9.13}$$

Substituting (9.11) to (9.13) in the biorthogonal series (9.2), we obtain a formal solution of problem (2.2) to (2.4).

10 Main theorem on the well-posedness of the direct problem (2.2) to (2.4)

To complete the investigation of the problem, we need (similarly to the classical Fourier method) to justify the smoothness of the formal solution obtained, that is, we need to justify the convergence of all series met.

As usual, we denote by $W_2^{2,1}(\Omega)$ the Sobolev space with the norm

$$\|u\|_{2,1}^2 = \|u_{xx}\|_0^2 + \|u_t\|_0^2 + \|u\|_0^2.$$

Definition 10.1 The function $u(x,t) \in W_2^{2,1}(\Omega)$ will be called a strong generalized solution of the problem (2.2) to (2.4), if there exists a sequence of the smooth functions $u_n(x,t) \in C_2^{2,1}(\overline{\Omega})$ satisfying the boundary conditions (2.4) such that the sequences u_n, $\mathbb{L} u_n$, and $u_n(x,0)$ converge to u, f, and $\varphi(x)$ in norms of the spaces $W_2^{2,1}(\Omega)$, $L_2(\Omega)$, and $W_2^2(-1,1)$, respectively.

From expansion (9.2), on the basis of the Bessel-type inequalities, it is easy to obtain the estimate

$$\|u\|_{2,1}^2 \le C\left\{\sum_{k \in N_1}\left[\left\|\frac{d}{dt}u_k^{(1)}\right\|_{L_2(0,T)}^2 + \left|\lambda_k^{(1)}\right|^2\left\|u_k^{(1)}\right\|_{L_2(0,T)}^2\right]\right.$$
$$+ \sum_{n \in N_2}\left[\left\|\frac{d}{dt}u_n^{(2)}\right\|_{L_2(0,T)}^2 + \left|\lambda_n^{(2)}\right|^2\left\|u_n^{(2)}\right\|_{L_2(0,T)}^2\right] \tag{10.1}$$
$$\left.+ \sum_{j \in \mathbb{N}}\left[\left\|\frac{d}{dt}u_{n_0 j}^{(2)}\right\|_{L_2(0,T)}^2 + \left|\lambda_{n_0 j}^{(2)}\right|^2\left\|u_{n_0 j}^{(2)}\right\|_{L_2(0,T)}^2\right]\right\}.$$

From the explicit representation of the solutions of the problems (9.8) to (9.10) using the formulas (9.11) to (9.13), we calculate the estimates

$$\left\|\frac{d}{dt}u_k^{(1)}\right\|_{L_2(0,T)}^2 + \left|\lambda_k^{(1)}\right|^2\left\|u_k^{(1)}\right\|_{L_2(0,T)}^2 \le C\left(\left\|f_k^{(1)}\right\|_{L_2(0,T)}^2 + \left|\lambda_k^{(1)}\right|^2\left|\varphi_k^{(1)}\right|^2\right);$$
$$\left\|\frac{d}{dt}u_n^{(2)}\right\|_{L_2(0,T)}^2 + \left|\lambda_n^{(2)}\right|^2\left\|u_n^{(2)}\right\|_{L_2(0,T)}^2 \le C\left(\left\|f_n^{(2)}\right\|_{L_2(0,T)}^2 + \left|\lambda_n^{(2)}\right|^2\left|\varphi_n^{(2)}\right|^2\right); \tag{10.2}$$
$$\left\|\frac{d}{dt}u_{n_0 j}^{(2)}\right\|_{L_2(0,T)}^2 + \left|\lambda_{n_0 j}^{(2)}\right|^2\left\|u_{n_0 j}^{(2)}\right\|_{L_2(0,T)}^2 \le C\left(\left\|f_{n_0 j}^{(2)}\right\|_{L_2(0,T)}^2 + \left|\lambda_{n_0 j}^{(2)}\right|^2\left|\varphi_{n_0 j}^{(2)}\right|^2\right).$$

Substituting now (10.2) into (10.1), taking into account the inequalities (9.5) to (9.7), we obtain the estimate

$$\|u\|_{2,1}^2 \le C\left\{\|f\|_0^2 + \|\varphi\|_{W_2^2(-1,1)}^2\right\}. \tag{10.3}$$

Thus, the obtained (in the form of a series) formal solution of problem (2.2) to (2.4) belongs to the class $W_2^{2,1}(\Omega)$.

Obviously, for continuity on the interval $[0, T]$ right-hand sides of equations (9.11) to (9.13) their solutions will be continuously differentiable on $[0, T]$. Therefore, by virtue of estimate (10.3), the existence of the sequence of the smooth functions $u_n(x, t) \in C_2^{2,1}(\overline{\Omega})$ from the representation of the strong generalized solution is obvious. The uniqueness of the solution obtained is a consequence of the Riesz basis property of the system according to which the solution of the problem is expanded into the biorthogonal series (9.2).

Thus, the following main theorem on the well-posedness of the direct problem (2.2) to (2.4) is proved.

Theorem 10.2 *Let the number $r = \sqrt{(1 - \varepsilon)/(1 + \varepsilon)}$ be rational and $a \ne 1$. Then, for any function $f \in L_2(\Omega)$ and for any function $\varphi \in W_2^2(-1, 1)$ satisfying the boundary conditions (4.2), there exists a unique strong generalized solution $u \in W_2^{2,1}(\Omega)$ of problem (2.2) to (2.4). This solution can be represented as a convergent in $W_2^{2,1}(\Omega)$ biorthogonal series (9.2) and it satisfies inequality (10.3).*

11 Formulation of inverse problem

Consider now an inverse problem, that is, the problem on recovering the right-hand side of a nonlocal differential equation with respect to initial-boundary conditions and an additional overdetermination condition.

In the domain $\Omega = \{(x,t) : -1 < x < 1, 0 < t < T\}$ consider a problem on finding the right-hand side of $f(x)$ of the nonlocal heat equation

$$\mathbb{L}u \equiv u_t(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x), \tag{11.1}$$

and its solution $u(x,t)$ satisfying the initial condition

$$u(x,0) = \varphi(x), \quad x \in [-1,1], \tag{11.2}$$

the nonlocal boundary conditions of periodic type

$$\begin{cases} u_x(-1,t) + au_x(1,t) = 0, \\ u(-1,t) - u(1,t) = 0, \end{cases} \quad t \in [0,T], \tag{11.3}$$

with respect to the additional final "overdetermination condition":

$$u(x,T) = \psi(x), x \in [-1,1], \tag{11.4}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions; $a \neq 1$ is real number; and ε is a nonzero real number such that $|\varepsilon| < 1$.

A fairly large number of works are devoted to the theory of inverse problems for evolutionary equations. This theory is quite well developed today. The principal difference between the formulation of the problem under consideration is that the differential equation is nonlocal. It contains a term with involutional argument deviation in the main part. Another difference is that we consider nonlocal boundary conditions with respect to a spatial variable.

In most cases, the inverse problems are ill-posed or conditionally well-posed. However, the case under consideration is (as we will show below) well-posed in a classical sense: a solution of the problem exists, is unique and stable with respect to the input data of the problem.

Definition 11.1 A pair of functions $\{u(x,t), f(x)\}$, where $u(x,t) \in W_2^{2,1}(\Omega), f(x) \in L_2(-1,1)$ will be called a strong generalized solution of the inverse problem (11.1) to (11.4), if there exists a sequence of the smooth functions $u_n(x,t) \in C_2^{2,1}(\overline{\Omega})$ satisfying the boundary conditions (2.4) such that the sequences $u_n, \mathbb{L}u_n, u_n(x,0)$ and $u_n(x,T)$ converge to $u(x,t), f(x), \varphi(x)$, and $\psi(x)$ in norms of the spaces $W_2^{2,1}(\Omega), L_2(-1,1), W_2^2(-1,1),$ and $W_2^2(-1,1),$ respectively.

12 Construction of a formal solution of the inverse problem (11.1) to (11.4)

As in the solution of the direct problem, we use here the method of separation of variables. A solution $u(x,t)$ of the nonlocal heat equation (11.1) will be sought in the form of the biorthogonal series (9.2). We also converge the sought function $f(x)$ and the given functions $\varphi(x)$ and $\psi(x)$ into the biorthogonal series with respect to the system of eigen- and associated functions:

$$f(x) = \sum_{k \in N_1} y_k^{(1)}(x) f_k^{(1)} + \sum_{n \in N_2} y_n^{(2)}(x) f_n^{(2)} + \sum_{j \in \mathbb{N}} \left\{ j y_{n_0 j, 1}(x) - t j y_{k_0 j}^{(1)}(x) \right\} f_{n_0 j}^{(2)}, \tag{12.1}$$

$$\varphi(x) = \sum_{k \in N_1} y_k^{(1)}(x)\varphi_k^{(1)} + \sum_{n \in N_2} y_n^{(2)}(x)\varphi_n^{(2)} + \sum_{j \in \mathbb{N}} jy_{n_0j,1}(x)\varphi_{n_0j}^{(2)}, \tag{12.2}$$

$$\psi(x) = \sum_{k \in N_1} y_k^{(1)}(x)\psi_k^{(1)} + \sum_{n \in N_2} y_n^{(2)}(x)\psi_n^{(2)} + \sum_{j \in \mathbb{N}} \left\{ jy_{n_0j,1}(x) - Tjy_{k_0j}^{(1)}(x) \right\}\psi_{n_0j}^{(2)}. \tag{12.3}$$

Here, $f_k^{(1)}$, $f_n^{(2)}$, and $f_{n_0j}^{(2)}$ are unknown coefficients, and $\varphi_k^{(1)}$, $\varphi_n^{(2)}$, $\varphi_{n_0j}^{(2)}$, $\psi_k^{(1)}$, $\psi_n^{(2)}$, and $\psi_{n_0j}^{(2)}$ are known numerical coefficients.

Remark 12.1 If $\psi(x) \in W_2^4(-1, 1)$ and the functions $\psi(x)$ and $\psi''(x)$ satisfy the boundary conditions (4.2), then the estimate holds

$$\sum_{k \in N_1} \left|\lambda_k^{(1)}\right|^4 \left|\psi_k^{(1)}\right|^2 + \sum_{n \in N_2} \left|\lambda_n^{(2)}\right|^4 \left|\psi_n^{(2)}\right|^2 + \sum_{j \in \mathbb{N}} \left|\lambda_{n_0j}^{(2)}\right|^4 \left|\psi_{n_0j}^{(2)}\right|^2 \leq C\|\psi\|_{W_2^4(-1,1)}^2. \tag{12.4}$$

Substituting (9.2) in (11.1) to (11.4), taking into account (12.1) to (12.3), we obtain the following problems for finding the unknown functions $u_k^{(1)}(t)$, $u_n^{(2)}(t)$, $u_{n_0j}^{(2)}(t)$, and the unknown coefficients $f_k^{(1)}$, $f_n^{(2)}$, and $f_{n_0j}^{(2)}$:

$$\frac{d}{dt}u_k^{(1)}(t) + \lambda_k^{(1)}u_k^{(1)}(t) = f_k^{(1)}, \quad 0 < t < T, u_k^{(1)}(0) = \varphi_k^{(1)}, u_k^{(1)}(T) = \psi_k^{(1)}; \tag{12.5}$$

$$\frac{d}{dt}u_n^{(2)}(t) + \lambda_n^{(2)}u_n^{(2)}(t) = f_n^{(2)}, \quad 0 < t < T, u_n^{(2)}(0) = \varphi_n^{(2)}, u_n^{(2)}(T) = \psi_n^{(2)}; \tag{12.6}$$

$$\frac{d}{dt}u_{n_0j}^{(2)}(t) + \lambda_{n_0j}^{(2)}u_{n_0j}^{(2)}(t) = f_{n_0j}^{(2)}, \quad 0 < t < T, u_{n_0j}^{(2)}(0) = \varphi_{n_0j}^{(2)}, u_{n_0j}^{(2)}(T) = \psi_{n_0j}^{(2)}; \tag{12.7}$$

where $k \in N_1$, $n \in N_2$, $j \in \mathbb{N}$.

Note that as we have chosen the expansion into the biorthogonal series using a special formula containing summands of the form $\{jy_{n_0j,1}(x) - tjy_{k_0j}^{(1)}(x)\}$, the problems (12.5) to (12.7) obtained for the Fourier sought coefficients have turned out to be one-typed. This does not depend on the fact that they are the coefficients for the eigen- or for the associated functions.

Also note that $\lambda_0^{(1)} = 0$. Therefore, we will separately write out the solution of this problem (12.5) for $k = 0$.

As is easily seen from (6.4), all the rest of the eigenvalues of the spectral problem (4.1) to (4.2) are positive. Therefore, the solutions of each of the problems (12.5) to (12.7) exist, are unique, and can be written out explicitly:

$$u_0^{(1)}(t) = \varphi_0^{(1)} + \frac{\psi_0^{(1)} - \varphi_0^{(1)}}{T}t, \qquad f_0^{(1)} = \frac{\psi_0^{(1)} - \varphi_0^{(1)}}{T}; \tag{12.8}$$

$$u_k^{(1)}(t) = e^{-\lambda_k^{(1)}t}\varphi_k^{(1)} + \frac{1 - e^{-\lambda_k^{(1)}t}}{1 - e^{-\lambda_k^{(1)}T}}\left(\psi_k^{(1)} - e^{-\lambda_k^{(1)}T}\varphi_k^{(1)}\right),$$

$$f_k^{(1)} = \frac{\lambda_k^{(1)}}{1 - e^{-\lambda_k^{(1)}T}}\left(\psi_k^{(1)} - e^{-\lambda_k^{(1)}T}\varphi_k^{(1)}\right); \tag{12.9}$$

$$u_n^{(2)}(t) = e^{-\lambda_n^{(2)}t}\varphi_n^{(2)} + \frac{1 - e^{-\lambda_n^{(2)}t}}{1 - e^{-\lambda_n^{(2)}T}}\left(\psi_n^{(2)} - e^{-\lambda_n^{(2)}T}\varphi_n^{(2)}\right),$$

$$f_n^{(2)} = \frac{\lambda_n^{(2)}}{1 - e^{-\lambda_n^{(2)}T}}\left(\psi_n^{(2)} - e^{-\lambda_n^{(2)}T}\varphi_n^{(2)}\right); \tag{12.10}$$

$$u_{n_0j}^{(2)}(t) = e^{-\lambda_{n_0j}^{(2)}t}\varphi_{n_0j}^{(2)} + \frac{1 - e^{-\lambda_{n_0j}^{(2)}t}}{1 - e^{-\lambda_{n_0j}^{(2)}T}}\left(\psi_{n_0j}^{(2)} - e^{-\lambda_{n_0j}^{(2)}T}\varphi_{n_0j}^{(2)}\right),$$

$$f_{n_0j}^{(2)} = \frac{\lambda_{n_0j}^{(2)}}{1 - e^{-\lambda_{n_0j}^{(2)}T}}\left(\psi_{n_0j}^{(2)} - e^{-\lambda_{n_0j}^{(2)}T}\varphi_{n_0j}^{(2)}\right). \tag{12.11}$$

Now, substituting everything obtained from (12.8) to (12.11) into (9.2) and (12.1), we arrive at a formal solution of the inverse problem (11.1) to (11.4).

From (12.9) to (12.11) we easily obtain the uniform estimates

$$\left|u_k^{(1)}(t)\right| \le C\left(\left|\varphi_k^{(1)}\right| + \left|\psi_k^{(1)}\right|\right), \qquad \left|\frac{d}{dt}u_k^{(1)}(t)\right| \le C\left(\left|\varphi_k^{(1)}\right| + \left|\psi_k^{(1)}\right|\right)\left|\lambda_k^{(1)}\right|;$$

$$\left|u_n^{(2)}(t)\right| \le C\left(\left|\varphi_n^{(2)}\right| + \left|\psi_n^{(2)}\right|\right), \qquad \left|\frac{d}{dt}u_n^{(2)}(t)\right| \le C\left(\left|\varphi_n^{(2)}\right| + \left|\psi_n^{(2)}\right|\right)\left|\lambda_n^{(2)}\right|;$$

$$\left|u_{n_0j}^{(2)}(t)\right| \le C\left(\left|\varphi_{n_0j}^{(2)}\right| + \left|\psi_{n_0j}^{(2)}\right|\right), \qquad \left|\frac{d}{dt}u_{n_0j}^{(2)}(t)\right| \le C\left(\left|\varphi_{n_0j}^{(2)}\right| + \left|\psi_{n_0j}^{(2)}\right|\right)\left|\lambda_{n_0j}^{(2)}\right|;$$

$$\left|f_k^{(1)}\right| \le C\left(\left|\varphi_k^{(1)}\right| + \left|\lambda_k^{(1)}\right|\left|\psi_k^{(1)}\right|\right);$$

$$\left|f_n^{(2)}\right| \le C\left(\left|\varphi_n^{(2)}\right| + \left|\lambda_n^{(2)}\right|\left|\psi_n^{(2)}\right|\right);$$

$$\left|f_{n_0j}^{(2)}\right| \le C\left(\left|\varphi_{n_0j}^{(2)}\right| + \left|\lambda_{n_0j}^{(2)}\right|\left|\psi_{n_0j}^{(2)}\right|\right).$$

Hence, taking into account the inequalities (9.7) from Remark 9.1 and (12.4) from Remark 12.1, the estimates (10.2) and (10.3) for the solution of the direct problem, we obtain estimates for the solution $\{u(x,t), f(x)\}$ of the inverse problem (11.1) to (11.4)

$$\|u\|_{2,1}^2 + \|f\|_{L_2(-1,1)}^2 \le C\left\{\|\varphi\|_{W_2^2(-1,1)}^2 + \|\psi\|_{W_2^4(-1,1)}^2\right\}. \tag{12.12}$$

Not dwelling on the standard details of the proof, we present the final result on the well-posedness of the inverse problem (11.1) to (11.4).

Theorem 12.2 *Let the number* $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ *be rational and* $a \ne 1$. *Let* $\varphi \in W_2^2(-1,1)$, $\psi \in W_2^4(-1,1)$ *be arbitrary functions such that the functions* $\varphi(x)$, $\psi(x)$, *and* $\psi''(x)$ *satisfy the boundary conditions* (4.2). *Then, there exists a unique strong generalized solution* $\{u(x,t), f(x)\}$ *of the inverse problem* (11.1) *to* (11.4). *This solution can be represented in the form of the biorthogonal series* (9.2) *and* (12.1) *converging in* $W_2^{2,1}(\Omega)$ *and* $L_2(-1,1)$, *respectively. This solution satisfies inequality* (12.12) *and is stable (in corresponding norms) with respect to the initial and final data* $\varphi(x)$ *and* $\psi(x)$.

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
This work was carried out as a collaboration between the three authors. MS designed the study and guided the research. GD and MI performed the analysis and wrote the first draft of the manuscript. MS, GD, and MI managed the analysis of the study. All of the authors read and approved the final manuscript.

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