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General decay for weak viscoelastic equation of Kirchhoff type containing Balakrishnan–Taylor damping with nonlinear delay and acoustic boundary conditions

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Abstract

In this paper, we consider the general decay of solutions for the weak viscoelastic equation of Kirchhoff type containing Balakrishnan–Taylor damping with nonlinear delay and acoustic boundary conditions. By using suitable energy and Lyapunov functionals, we prove the general decay for the energy, which depends on the behavior of both σ and k.

MSC: 35L70; 35B40; 35B35

Keywords: Weak viscoelastic equation; Balakrishnan–Taylor damping; Acoustic boundary conditions; Nonlinear delay; General decay rate

1 Introduction

The objective of this work is to study the general decay of solutions for the weak viscoelastic equation of Kirchhoff type containing Balakrishnan–Taylor damping with nonlinear delay and acoustic boundary conditions

$$|w_t|^\rho w_{tt} - \big(a_0 + b_0\|\nabla w\|^2 + b_1(\nabla w, \nabla w_t)\big)\Delta w - \Delta w_{tt} + \sigma(t)\int_0^t k(t-s)\Delta w(s)\,ds$$
$$= |w|^{p-2}w \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$w = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \tag{1.2}$$

$$\big(a_0 + b_0\|\nabla w\|^2 + b_1(\nabla w, \nabla w_t)\big)\frac{\partial w}{\partial \nu} + \frac{\partial w_{tt}}{\partial \nu} - \sigma(t)\int_0^t k(t-s)\frac{\partial w(s)}{\partial \nu}\,ds$$
$$+ \mu_1\big|w_t(x,t)\big|^{q-1}w_t(x,t) + \mu_2\big|w_t(x,t-\tau)\big|^{q-1}w_t(x,t-\tau)$$
$$= m(x)u_t \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{1.3}$$

$$w_t + g(x)u_t + h(x)u = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{1.4}$$

$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x) \quad \text{in } \Omega, \tag{1.5}$$

$$u(x,0) = u_0(x) \quad \text{on } \Gamma_1, \tag{1.6}$$

$$w_t(x,t-\tau) = f_0(x,t-\tau) \quad \text{on } \Gamma_1, 0 < t < \tau, \tag{1.7}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint and ν is the unit outward normal to Γ. w_0, w_1, u_0, and f_0 are given functions. All the parameters $a_0, b_0, b_1, \rho, p, q, \mu_1$, and μ_2 are positive constants, the functions $m, g, h : \Gamma_1 \to \mathbb{R}$ are essentially bounded. Moreover, k represents the kernel of the memory term and $\tau > 0$ represents the time delay.

The equation (1.1) with $b_0 = b_1 = 0$ and $a_0 = \sigma(t) = 1$,

$$|w_t|^\rho w_{tt} - \Delta w - \Delta w_{tt} + \int_0^t k(t-s)\Delta w(s)\,ds$$
$$= |w|^{p-2}w \quad \text{in } \Omega \times \mathbb{R}^+, w = 0 \text{ on } \Gamma \tag{1.8}$$

has been studied by Messaoudi and Tatar [16]. The case of $\rho = 1$ and $b_1 = \sigma(t) = 0$ in the absence of the dispersion term, the equation (1.1) reduces to the well-known Kirchhoff equation that has been introduced in [8] in order to describe the nonlinear vibrations of an elastic string.

The model with Balakrishnan–Taylor damping ($b_1 > 0$) and $k = 0$, was initially proposed by Balakrishnan and Taylor in [2]. Several authors have studied the asymptotic behavior of the solution for the nonlinear viscoelastic Kirchhoff equations with Balakrishnan–Taylor damping (see [17, 22, 24] and references and therein). Recently, Al-Gharabli *et al.* [1] considered the following Balakrishnan–Taylor viscoelastic equation with a logarithmic source term

$$|w_t|^\rho w_{tt} - \big(a_0 + b_0\|\nabla w\|^2 + b_1(\nabla w, \nabla w_t)\big)\Delta w - \Delta w_{tt} + \int_0^t k(t-s)\Delta w(s)\,ds + h(w_t)$$
$$= kw\ln|w| \quad \text{in } \Omega \times \mathbb{R}^+, w = 0 \text{ on } \Gamma. \tag{1.9}$$

They proved the general decay rates, using the multiplier method and some properties of the convex functions. Lian and Xu [11] investigated the problem (1.9) with weak and strong damping terms and $\rho = b_0 = b_1 = k = 0$.

For $\sigma(t) > 0$, Messaoudi [15] studied the following viscoelastic wave equation

$$w_{tt} - \Delta w + \sigma(t)\int_0^t k(t-s)\Delta w(s)\,ds = 0 \quad \text{in } \Omega \times \mathbb{R}^+.$$

The author obtained the general decay result that depends on the behavior of both σ and k. For other related works, we refer the readers to [3, 13, 14].

Since most phenomena naturally depend not only on the present state but also on some past occurrences, in recent years, there has been published much work concerning the wave equation with delay effects that often appear in many practical problems [18–21]. Feng and Li [7] proved the general energy decay for a viscoelastic Kirchhoff plate equation with a time delay. Lee *et al.* [9] showed the general energy decay of solutions for system (1.1)–(1.7) with $\sigma(t) = 1$ and $q = 1$.

Motivated by previous work, we study the general energy decay of solutions for the system (1.1)–(1.7) that depends on the behavior of the potential σ and the relaxation function

k satisfying the suitable conditions. The acoustic boundary condition (1.4) and the coupled impenetrability boundary condition (1.3) were proposed by Beale and Rosencrans [5]. For physical application of acoustic boundary conditions, we refer to [4, 6]. The stability of various models with acoustic boundary conditions has been discussed by many researchers [10, 12, 14, 23]. The outline of this paper is as follows. In Sect. 2, we present some preparations and hypotheses for our main result. In Sect. 3, we obtain the general energy decay of the system (1.1)–(1.7) by using the energy-perturbation method.

2 Preliminary

In this section, we present some material that we shall use in order to prove our result. We denote by

$$V = \left\{ w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_0 \right\}.$$

The Poincaré inequality holds in V, i.e., there exists a constant C_* such that

$$\|w\|_r \le C_* \|\nabla w\|, \quad 2 \le r \le \frac{2n}{n-2}, \forall w \in V, \tag{2.1}$$

and there exists a constant $\tilde{C}_*$ such that

$$\|w\|_{r,\Gamma_1} \le \tilde{C}_* \|\nabla w\|, \quad \forall w \in V. \tag{2.2}$$

For our study of problem (1.1)–(1.7), we will need the following assumptions.

(H1) The constants ρ and q satisfy

$$0 < \rho, q \le \frac{2}{n-2} \quad \text{if } n \ge 3, \qquad \rho, q > 0 \quad \text{if } n = 1, 2, \tag{2.3}$$

and p satisfies

$$0 < p \le \frac{4}{n-2} \quad \text{if } n \ge 3, \qquad p > 2 \quad \text{if } n = 1, 2. \tag{2.4}$$

For the relaxation function k and potential σ, as in [15], we assume that

(H2) $k, \sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing differentiable functions such that k is a C^2 function and σ is a C^1 function satisfying

$$\begin{aligned} &k(0) > 0, \quad \int_0^\infty k(s)\,ds = k_0 < \infty, \qquad \sigma(t) > 0, \\ &a_0 - \sigma(t) \int_0^t k(s)\,ds \ge l > 0, \quad \forall t \ge 0, \end{aligned} \tag{2.5}$$

$$\left(\sigma(t) \int_0^t k(s)\,ds \right)' \ge 0, \quad \forall t \in [0, t_0], \tag{2.6}$$

where l and t_0 are suitable positive constants. There exists a nonincreasing differentiable function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\zeta(t) > 0, \quad k'(t) \le -\zeta(t) k(t), \qquad \forall t \ge 0, \quad \lim_{t\to\infty} \frac{-\sigma'(t)}{\zeta(t)\sigma(t)} = 0. \tag{2.7}$$

(H3) There exist three positive constants m_1, g_1, and h_1 such that

$$m_1 \le m(x), \qquad g_1 \le g(x), \qquad h_1 \le h(x), \quad x \in \Gamma_1. \tag{2.8}$$

(H4) We assume that the constants μ_1 and μ_2 satisfy $\mu_2 < \mu_1$.

Remark 2.1 ([15]) 1. Note that (2.7) implies that $\lim_{t\to\infty} \frac{-\sigma'(t)}{\sigma(t)} = 0$.
2. Examples of functions k and σ satisfying (H2) are

$$\sigma(t) = \frac{1}{1+t}, \qquad k(t) = ae^{-b(1+t)^c}, \quad 0 < c \le 1,$$

for $a, b > 0$, to be chosen properly.

As in [19], let us introduce the function

$$z(x,\delta,t) = w_t(x, t-\tau\delta), \quad x \in \Omega, \delta \in (0,1), \forall t > 0. \tag{2.9}$$

Then, problem (1.1)–(1.7) is equivalent to

$$\begin{cases} |w_t|^\rho w_{tt} - (a_0 + b_0\|\nabla w\|^2 + b_1(\nabla w, \nabla w_t))\Delta w - \Delta w_{tt} + \sigma(t)\int_0^t k(t-s)\Delta w(s)\,ds \\ \quad = |w|^{p-2}w \quad \text{in } \Omega \times \mathbb{R}^+, \\ w = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \\ (a_0 + b_0\|\nabla w\|^2 + b_1(\nabla w, \nabla w_t))\frac{\partial w}{\partial \nu} + \frac{\partial w_{tt}}{\partial \nu} - \sigma(t)\int_0^t k(t-s)\frac{\partial w(s)}{\partial \nu}\,ds \\ \qquad + \mu_1|w_t(x,t)|^{q-1}w_t(x,t) + \mu_2|z(x,1,t)|^{q-1}z(x,1,t) \\ \quad = m(x)u_t \quad \text{on } \Gamma_1 \times (0,1) \times \mathbb{R}^+, \\ \tau z_t(x,\delta,t) + z_\delta(x,\delta,t) = 0 \quad \text{on } \Gamma_1 \times (0,1) \times \mathbb{R}^+, \\ w_t + g(x)u_t + h(x)u = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \\ w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x) \quad \text{in } \Omega, \\ u(x,0) = u_0(x) \quad \text{on } \Gamma_1, \\ z(x,\delta,0) = f_0(x,-\tau\delta) \quad \text{on } \Gamma_1 \times (0,1). \end{cases} \tag{2.10}$$

By combining with the argument of [5], we now state the local existence result of problem (2.10), which can be obtained.

Theorem 2.1 *Suppose that* (H1)–(H4) *hold and that* $(w_0, w_1) \in (H^2(\Omega) \cap V) \times V$, $u_0 \in L^2(\Gamma_1)$ *and* $f_0 \in L^2(\Gamma_1 \times (0,1))$. *Then, for any* $T > 0$, *there exists a unique solution* (w, u, z) *of problem* (2.10) *on* $[0, T]$ *such that*

$$\begin{aligned} &w \in L^\infty\bigl(0,T;H^2(\Omega)\cap V\bigr), \qquad w_t \in L^\infty(0,T;V) \cap L^{q+1}\bigl(\Gamma_1 \times (0,T)\bigr), \\ &m^{1/2}u \in L^\infty\bigl(0,T;L^2(\Gamma_1)\bigr), \qquad m^{1/2}u_t \in L^2\bigl(0,T;L^2(\Gamma_1)\bigr). \end{aligned}$$

3 Main result

In this section, we state and show our main result. For this purpose, we define

$$J(t) = \frac{1}{2}\left(a_0 - \sigma(t)\int_0^t k(s)\,ds\right)\bigl\|\nabla w(t)\bigr\|^2 + \frac{b_0}{4}\bigl\|\nabla w(t)\bigr\|^4$$

$$+\frac{1}{2}\left\|\nabla w_t(t)\right\|^2+\frac{1}{2}\sigma(t)(k\circ\nabla w)(t)$$
$$+\frac{\xi}{2}\int_{\Gamma_1}\int_0^1\left|z(x,\delta,t)\right|^{q+1}d\delta\,d\Gamma+\frac{1}{2}\int_{\Gamma_1}h(x)m(x)u^2(t)\,d\Gamma-\frac{1}{p}\left\|w(t)\right\|_p^p, \tag{3.1}$$

and

$$\begin{aligned}I(t)={}&\left(a_0-\sigma(t)\int_0^t k(s)\,ds\right)\left\|\nabla w(t)\right\|^2+\frac{b_0}{2}\left\|\nabla w(t)\right\|^4\\&+\left\|\nabla w_t(t)\right\|^2+\sigma(t)(k\circ\nabla w)(t)\\&+\xi\int_{\Gamma_1}\int_0^1\left|z(x,\delta,t)\right|^{q+1}d\delta\,d\Gamma+\int_{\Gamma_1}h(x)m(x)u^2(t)\,d\Gamma-\left\|w(t)\right\|_p^p,\end{aligned} \tag{3.2}$$

where $(k\circ w)(t)=\int_0^t k(t-s)\|w(t)-w(s)\|^2\,ds$. From direct calculation, we find that

$$\begin{aligned}&\sigma(t)(k*w,w_t)\\&\quad=-\frac{\sigma(t)}{2}k(t)\left\|w(t)\right\|^2-\frac{d}{dt}\left[\frac{\sigma(t)}{2}(k\circ w)(t)-\frac{\sigma(t)}{2}\left(\int_0^t k(s)\,ds\right)\left\|w(t)\right\|^2\right]\\&\qquad+\frac{\sigma(t)}{2}\left(k'\circ w\right)(t)+\frac{\sigma'(t)}{2}(k\circ w)(t)-\frac{\sigma'(t)}{2}\left(\int_0^t k(s)\,ds\right)\left\|w(t)\right\|^2,\end{aligned} \tag{3.3}$$

and

$$(k*w,w)\le\left(\int_0^t k(s)\,ds\right)\left\|w(t)\right\|^2+\frac{1}{4}(k\circ w)(t), \tag{3.4}$$

where $(k*w)(t)=\int_0^t k(t-s)w(s)\,ds$.

Now, we denote the modified energy functional $E(t)$ associated with problem (2.10) by

$$\begin{aligned}E(t)={}&\frac{1}{\rho+2}\left\|w_t(t)\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left(a_0-\sigma(t)\int_0^t k(s)\,ds\right)\left\|\nabla w(t)\right\|^2\\&+\frac{b_0}{4}\left\|\nabla w(t)\right\|^4+\frac{1}{2}\left\|\nabla w_t(t)\right\|^2\\&+\frac{1}{2}\sigma(t)(k\circ\nabla w)(t)+\frac{\xi}{2}\int_{\Gamma_1}\int_0^1\left|z(x,\delta,t)\right|^{q+1}d\delta\,d\Gamma\\&+\frac{1}{2}\int_{\Gamma_1}h(x)m(x)u^2(t)\,d\Gamma-\frac{1}{p}\left\|w(t)\right\|_p^p\\={}&\frac{1}{\rho+2}\left\|w_t(t)\right\|_{\rho+2}^{\rho+2}+J(t),\end{aligned} \tag{3.5}$$

where ξ is a positive constant such that

$$\frac{2\tau\mu_2 q}{q+1}<\xi<\frac{2\tau\mu_1(q+1)-2\tau\mu_2}{q+1}. \tag{3.6}$$

Note that this choice of ξ is possible from assumption (H4).

Lemma 3.1 *Assume that* (H2) *and* (H4) *hold. Then, for the solution of problem* (2.10), *the energy functional* $E(t)$ *satisfies*

$$\begin{aligned} E'(t) \le{} & -C_1\|w_t(t)\|_{q+1,\Gamma_1}^{q+1} - C_2\int_{\Gamma_1}|z(x,1,t)|^{q+1}\,d\Gamma - b_1\left(\frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|^2\right)^2 \\ & - \frac{\sigma(t)k(t)}{2}\|\nabla w(t)\|^2 - \frac{\sigma'(t)}{2}\left(\int_0^t k(s)\,ds\right)\|\nabla w(t)\|^2 + \frac{\sigma'(t)}{2}(k\circ\nabla w)(t) \\ & + \frac{\sigma(t)}{2}(k'\circ\nabla w)(t) - \int_{\Gamma_1} m(x)g(x)u_t^2(t)\,d\Gamma \le 0, \quad \forall t\in[0,t_0], \end{aligned} \tag{3.7}$$

where C_1 *and* C_2 *are some positive constants.*

Proof Multiplying in the first equation of (2.10) by w_t and integrating over Ω, using (3.3), we have

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{\rho+2}\|w_t(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(a_0 - \sigma(t)\int_0^t k(s)\,ds\right)\|\nabla w(t)\|^2 \right. \\ & \quad + \frac{b_0}{4}\|\nabla w(t)\|^4 + \frac{1}{2}\|\nabla w_t(t)\|^2 \\ & \quad \left. + \frac{1}{2}\sigma(t)(k\circ\nabla w)(t) - \frac{1}{p}\|w(t)\|_p^p + \frac{1}{2}\int_{\Gamma_1} h(x)m(x)u^2(t)\,d\Gamma\right] \\ & = -\mu_1\|w_t(t)\|_{q+1,\Gamma_1}^{q+1} - \mu_2\int_{\Gamma_1}|z(x,1,t)|^{q-1}z(x,1,t)w_t(t)\,d\Gamma - b_1\left(\frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|^2\right)^2 \\ & \quad - \frac{\sigma(t)}{2}k(t)\|\nabla w(t)\|^2 - \frac{\sigma'(t)}{2}\left(\int_0^t k(s)\,ds\right)\|\nabla w(t)\|^2 + \frac{\sigma'(t)}{2}(k\circ\nabla w)(t) \\ & \quad + \frac{\sigma(t)}{2}(k'\circ\nabla w)(t) - \int_{\Gamma_1} m(x)g(x)u_t^2(t)\,d\Gamma. \end{aligned} \tag{3.8}$$

Multiplying the equation in the fourth equation of (2.10) by $\xi|z|^{q-1}z$ and integrating the result over $\Gamma_1\times(0,1)$, we obtain

$$\begin{aligned} & \frac{\xi}{2}\frac{d}{dt}\int_{\Gamma_1}\int_0^1 |z(x,\delta,t)|^{q+1}\,d\delta\,d\Gamma \\ & = -\frac{\xi}{2\tau}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial\delta}|z(x,\delta,t)|^{q+1}\,d\delta\,d\Gamma \\ & = -\frac{\xi}{2\tau}\int_{\Gamma_1}|z(x,1,t)|^{q+1}\,d\Gamma + \frac{\xi}{2\tau}\int_{\Gamma_1}|w_t(t)|^{q+1}\,d\Gamma. \end{aligned} \tag{3.9}$$

By using Young's inequality, we obtain

$$\begin{aligned} & \left|\mu_2\int_{\Gamma_1}|z(x,1,t)|^{q-1}z(x,1,t)w_t(t)\Gamma\right| \\ & \le \frac{\mu_2 q}{q+1}\int_{\Gamma_1}|z(x,1,t)|^{q+1}\,d\Gamma + \frac{\mu_2}{q+1}\int_{\Gamma_1}|w_t(t)|^{q+1}\,d\Gamma. \end{aligned} \tag{3.10}$$

Thus, from (3.8)–(3.10) and the definition of $E(t)$, we have

$$\begin{aligned} E'(t) \le & -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{q+1}\right)\|w_t(t)\|_{q+1,\Gamma_1}^{q+1} - \left(\frac{\xi}{2\tau} - \frac{\mu_2 q}{q+1}\right)\int_{\Gamma_1} |z(x,1,t)|^{q+1}\,d\Gamma \\ & - b_1\left(\frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|^2\right)^2 - \frac{\sigma(t)}{2}k(t)\|\nabla w(t)\|^2 - \frac{\sigma'(t)}{2}\left(\int_0^t k(s)\,ds\right)\|\nabla w(t)\|^2 \\ & + \frac{\sigma'(t)}{2}(k\circ\nabla w)(t) + \frac{\sigma(t)}{2}(k'\circ\nabla w)(t) - \int_{\Gamma_1} m(x)g(x)u_t^2(t)\,d\Gamma. \end{aligned}$$

Using (3.6), we take $C_1 = \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{q+1} > 0$ and $C_2 = \frac{\xi}{2\tau} - \frac{\mu_2 q}{q+1} > 0$. From (2.6), we obtain the desired inequality (3.7). □

Lemma 3.2 *Suppose that* (H1) *and* (H2) *hold. Let* (w,u,z) *be the solution of problem* (2.10). *Assume that* $I(0) > 0$ *and*

$$\alpha = \frac{C_*^p}{l}\left(\frac{2pE(0)}{l(p-2)}\right)^{\frac{p-2}{2}} < 1. \tag{3.11}$$

Then, $I(t) > 0$ *for* $t \in [0,T]$, *where* $I(t)$ *is defined in* (3.2).

Proof Since $I(0) > 0$ and continuity of $w(t)$, then there exists $t_1 < T$ such that

$$I(t) \ge 0, \quad \forall t \in [0,t_1]. \tag{3.12}$$

From (2.5), (3.1), (3.2), and (3.12), we obtain

$$\begin{aligned} J(t) &= \frac{p-2}{2p}\left[\left(a_0 - \sigma(t)\int_0^t k(s)\,ds\right)\|\nabla w(t)\|^2 + \frac{b_0}{2}\|\nabla w(t)\|^4 \right. \\ &\quad + \|\nabla w_t(t)\|^2 + \sigma(t)(k\circ\nabla w)(t) \\ &\quad \left. + \xi\int_{\Gamma_1}\int_0^1 |z(x,\delta,t)|^{q+1}\,d\delta\,d\Gamma + \int_{\Gamma_1} h(x)m(x)u^2(t)\,d\Gamma\right] + \frac{1}{p}I(t) \\ &\ge \frac{p-2}{2p}l\|\nabla w(t)\|^2, \quad \forall t \in [0,t_1]. \end{aligned} \tag{3.13}$$

Using (3.5), (3.7), and (3.13), we obtain

$$l\|\nabla w(t)\|^2 \le \frac{2p}{p-2}J(t) \le \frac{2p}{p-2}E(t) \le \frac{2p}{p-2}E(0), \quad \forall t \in [0,T^*], \tag{3.14}$$

where $T^* = \min\{t_0, t_1\}$. Applying (2.1), (2.5), (3.11), and (3.14), we have

$$\begin{aligned} \|w(t)\|_p^p &\le C_*^p\|\nabla w(t)\|^p \\ &\le \alpha l\|\nabla w(t)\|^2 \le \left(a_0 - \sigma(t)\int_0^t k(s)\,ds\right)\|\nabla w(t)\|^2, \quad \forall t \in [0,T^*]. \end{aligned}$$

Consequently, we arrive at

$$I(t) = \left(a_0 - \sigma(t)\int_0^t k(s)\,ds\right)\|\nabla w(t)\|^2 + \frac{b_0}{2}\|\nabla w(t)\|^4 + \|\nabla w_t(t)\|^2 + \sigma(t)(k\circ\nabla w)(t)$$

$$+ \xi \int_{\Gamma_1} \int_0^1 \left|z(x,\delta,t)\right|^{q+1} d\delta \, d\Gamma + \int_{\Gamma_1} h(x)m(x)u^2(t)\, d\Gamma - \left\|w(t)\right\|_p^p > 0, \quad \forall t \in \left[0, T^*\right].$$

By repeating this procedure, and using the fact that

$$\lim_{t\to T^*} \frac{C_*^p}{l} \left(\frac{2pE(t)}{l(p-2)}\right)^{\frac{p-2}{2}} \le \alpha < 1,$$

T^* is extended to T. Thus, the proof is complete. □

We state the global existence result, which can be obtained by the arguments of [9, 22, 24].

Theorem 3.1 *Suppose that* (H1)–(H4) *hold. Let* $(w_0, w_1) \in (H^2(\Omega) \cap V) \times V$, $u_0 \in L^2(\Gamma_1)$, $f_0 \in L^2(\Gamma_1 \times (0,1))$. *If* $I(0) > 0$ *and satisfy* (3.11), *then the solution* (w,u,z) *of* (2.10) *is bounded and global in time.*

Now, we will establish the general decay property of the solution for problem (2.10) in the case $\mu_2 < \mu_1$. For this purpose, we define the functional

$$\Xi(t) = ME(t) + \varepsilon\sigma(t)\Phi_1(t) + \sigma(t)\Phi_2(t), \tag{3.15}$$

where M and ε are positive constants that will be specified later and

$$\Phi_1(t) = \frac{1}{\rho+1}\int_\Omega \left|w_t(t)\right|^\rho w_t(t)w(t)\, dx + \frac{b_1}{4}\left\|\nabla w(t)\right\|^4 + \int_\Omega \nabla w_t(t)\nabla w(t)\, dx + \int_{\Gamma_1} m(x)w(t)u(t)\, d\Gamma + \frac{1}{2}\int_{\Gamma_1} m(x)g(x)u^2(t)\, d\Gamma, \tag{3.16}$$

and

$$\Phi_2(t) = -\frac{1}{\rho+1}\int_\Omega \left|w_t(t)\right|^\rho w_t(t)\int_0^t k(t-s)\bigl(w(t)-w(s)\bigr)\, ds\, dx - \int_\Omega \nabla w_t(t)\int_0^t k(t-s)\bigl(\nabla w(t) - \nabla w(s)\bigr)\, ds\, dx. \tag{3.17}$$

Before we show our main result, we need the following lemmas.

Lemma 3.3 *Let* $w \in L^\infty([0,T]; H_0^1(\Omega))$, *then we have*

$$\int_\Omega \left(\sigma(t)\int_0^t k(t-s)\bigl(w(t)-w(s)\bigr)\, ds\right)^{\rho+2} dx \le (a_0 - l)^{\rho+1}\alpha_1\sigma(t)(k \circ \nabla w)(t), \tag{3.18}$$

where $\alpha_1 = C_*^{\rho+2}(\frac{2pE(0)}{l(p-2)})^{\frac{\rho}{2}}$.

Proof From (2.1), (2.5), (3.14), and Hölder's inequality, we obtain

$$\int_\Omega \left(\sigma(t)\int_0^t k(t-s)\bigl(w(t)-w(s)\bigr)\, ds\right)^{\rho+2} dx$$

$$\leq \int_\Omega \left(\sigma(t)\int_0^t k(t-s)\,ds\right)^{\rho+1}\left(\sigma(t)\int_0^t k(t-s)\left|w(t)-w(s)\right|^{\rho+2}ds\right)dx$$
$$\leq (a_0-l)^{\rho+1}C_*^{\rho+2}\left(\frac{2pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}}\sigma(t)(k\circ\nabla w)(t). \qquad \square$$

Lemma 3.4 *Let* (w,u,z) *be the solution of* (2.10) *and suppose that* (H1)–(H3) *hold, then there exist two positive constants* β_1 *and* β_2 *such that*

$$\beta_1 E(t) \leq \Xi(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \tag{3.19}$$

Proof Using (2.1), (2.2), (2.8), (3.14), and Young's inequality, we obtain

$$\left|\frac{1}{\rho+1}\int_\Omega \left|w_t(t)\right|^\rho w_t(t)w(t)\,dx\right|$$
$$\leq \frac{1}{\rho+2}\left\|w_t(t)\right\|_{\rho+2}^{\rho+2}+\frac{\alpha_1}{(\rho+2)(\rho+1)}\left\|\nabla w(t)\right\|^2, \tag{3.20}$$
$$\left|\int_\Omega \nabla w_t(t)\nabla w(t)\,dx\right| \leq \frac{1}{2}\left\|\nabla w_t(t)\right\|^2+\frac{1}{2}\left\|\nabla w(t)\right\|^2, \tag{3.21}$$
$$\left|\int_{\Gamma_1} m(x)w(t)u(t)\,d\Gamma\right| \leq \frac{\|m\|_\infty}{2h_1}\int_{\Gamma_1} h(x)m(x)u^2(t)\,d\Gamma+\frac{\tilde{C}_*^2}{2}\left\|\nabla w(t)\right\|^2. \tag{3.22}$$

Similarly, using (2.1), (2.5), (3.18), and Young's inequality, we see that

$$\left|\frac{1}{\rho+1}\int_\Omega \sigma(t)\left|w_t(t)\right|^\rho w_t(t)\int_0^t k(t-s)\big(w(t)-w(s)\big)\,ds\,dx\right|$$
$$\leq \frac{\sigma(t)}{\rho+2}\left\|w_t(t)\right\|_{\rho+2}^{\rho+2}+\frac{(a_0-l)^{\rho+1}\alpha_1}{(\rho+2)(\rho+1)}\sigma(t)(k\circ\nabla w)(t), \tag{3.23}$$

and

$$\left|-\int_\Omega \sigma(t)\nabla w_t(t)\int_0^t k(t-s)\big(\nabla w(t)-\nabla w(s)\big)\,ds\,dx\right|$$
$$\leq \frac{\sigma(t)}{2}\left\|\nabla w_t(t)\right\|^2+\frac{k_0}{2}\sigma(t)(k\circ\nabla w)(t). \tag{3.24}$$

Combining (3.15)–(3.17), (3.20)–(3.24), and using (H2), we obtain

$$\left|\Xi(t)-ME(t)\right|$$
$$\leq \varepsilon\sigma(t)\left|\Phi_1(t)\right|+\sigma(t)\left|\Phi_2(t)\right|$$
$$\leq \frac{\sigma(t)(\varepsilon+1)}{\rho+2}\left\|w_t(t)\right\|_{\rho+2}^{\rho+2}+\frac{\sigma(t)(\varepsilon+1)}{2}\left\|\nabla w_t(t)\right\|^2$$
$$+\varepsilon\sigma(t)\left(\frac{\alpha_1}{(\rho+2)(\rho+1)}+\frac{\tilde{C}_*^2}{2}+\frac{1}{2}\right)\left\|\nabla w(t)\right\|^2$$
$$+\left(\frac{(a_0-l)^{\rho+1}\alpha_1}{(\rho+2)(\rho+1)}+\frac{k_0}{2}\right)\alpha(t)(k\circ\nabla w)(t)$$

$$+\varepsilon\sigma(t)\left(\frac{b_1}{4}\|\nabla w(t)\|^4+\frac{\|m\|_\infty+\|g\|_\infty}{2h_1}\int_{\Gamma_1}h(x)m(x)u^2(t)\,d\Gamma\right)$$
$$\le CE(t),$$

where C is some positive constant. Choosing $M>0$ sufficiently large and ε small, we obtain (3.19). □

The following theorem is our main result.

Theorem 3.2 *Suppose that* (H1)–(H4) *and* (3.6) *hold. If* $(w_0,w_1)\in(H^2(\Omega)\cap V)\times V$, $u_0\in L^2(\Gamma_1)$, $f_0\in L^2(\Gamma_1\times(0,1))$ *and satisfying* (3.11). *Then, for any* $t>t_0^*$, *there exist positive constants* K *and* κ *such that the energy of the solution for problem* (2.10) *satisfies*

$$E(t)=Ke^{-\kappa\int_{t_0^*}^t\sigma(s)\zeta(s)\,ds},\quad \forall t\ge t_0^*. \tag{3.25}$$

Proof From Lemma 3.4, it suffices to prove that we obtain the estimate of $\Xi(t)$. For this purpose, first we estimate $\Phi_1'(t)$. It follows from (2.10) and (3.16) that

$$\begin{aligned}\Phi_1'(t)={}&-\left(a_0+b_0\|\nabla w(t)\|^2\right)\int_\Omega|\nabla w(t)|^2\,dx+\int_\Omega\nabla w(t)\sigma(t)\int_0^t k(t-s)\nabla w(s)\,ds\,dx\\&-\mu_1\int_{\Gamma_1}|w_t(t)|^{q-1}w_t(t)w(t)\,d\Gamma-\mu_2\int_{\Gamma_1}|z(x,1,t)|^{q-1}z(x,1,t)w(t)\,d\Gamma\\&+\frac{1}{\rho+1}\|w_t(t)\|_{\rho+2}^{\rho+2}\\&+2\int_{\Gamma_1}m(x)w(t)u_t(t)\,d\Gamma-\int_{\Gamma_1}h(x)m(x)u^2(t)\,d\Gamma+\int_\Omega|\nabla w_t(t)|^2\,dx\\&+\int_\Omega|w(t)|^p\,dx.\end{aligned} \tag{3.26}$$

We will estimate the right-hand side of (3.26). By using (2.2), (2.5), (2.8), (3.14), and Young's inequality, for any $\eta>0$, we have

$$\begin{aligned}&\left|\int_\Omega\nabla w(t)\sigma(t)\int_0^t k(t-s)\nabla w(s)\,ds\,dx\right|\\&\quad\le\left|\int_\Omega\nabla w(t)\sigma(t)\int_0^t k(t-s)\big(\nabla w(s)-\nabla w(t)\big)\,ds\,dx\right|\\&\qquad+\sigma(t)\int_0^t k(s)\,ds\int_\Omega|\nabla w(t)|^2\,dx\\&\quad\le(1+\eta)(a_0-l)\|\nabla w(t)\|^2+\frac{\sigma(t)}{4\eta}(k\circ\nabla w)(t),\end{aligned} \tag{3.27}$$

$$\left|\mu_1\int_{\Gamma_1}|w_t(t)|^{q-1}w_t(t)w(t)\,d\Gamma\right|\le\mu_1\eta\alpha_2\|\nabla w(t)\|^2+\mu_1C_\eta\|w_t(t)\|_{q+1,\Gamma_1}^{q+1} \tag{3.28}$$

$$\begin{aligned}&\left|\mu_2\int_{\Gamma_1}|z(x,1,t)|^{q-1}z(x,1,t)u(t)\,d\Gamma\right|\\&\quad\le\mu_2\eta\alpha_2\|\nabla w(t)\|^2+\mu_2C_\eta\|z(x,1,t)\|_{q+1,\Gamma_1}^{q+1}\end{aligned} \tag{3.29}$$

and

$$\left|2\int_{\Gamma_1} m(x)w(t)u_t(t)\,d\Gamma\right| \le \eta\tilde{C}_*^2\left\|\nabla w(t)\right\|^2 + \frac{\|m\|_\infty}{\eta g_1}\int_{\Gamma_1} m(x)g(x)u_t^2(t)\,d\Gamma, \tag{3.30}$$

where $\alpha_2 = \tilde{C}_*^{q+1}(\frac{2pE(0)}{l(p-2)})^{\frac{q-1}{2}}$. Choosing η small enough such that

$$\eta\left(a_0 - l + \tilde{C}_*^2 + \mu_1\alpha_2 + \mu_2\alpha_2\right) \le \frac{l}{2}$$

and substituting of (3.27)–(3.30) into (3.26), we obtain

$$\begin{aligned}
\Phi_1'(t) \le{}& -\frac{l}{2}\left\|\nabla w(t)\right\|^2 - b_0\left\|\nabla w(t)\right\|^4 + \frac{1}{\rho+1}\left\|w_t(t)\right\|_{\rho+2}^{\rho+2} + \left\|\nabla w_t(t)\right\|^2 + \left\|w(t)\right\|_p^p \\
&+ \frac{\sigma(t)}{4\eta}(k\circ\nabla w)(t) + \mu_1 C_\eta\left\|w_t(t)\right\|_{q+1,\Gamma_1}^{q+1} + \mu_2 C_\eta\left\|z(x,1,t)\right\|_{q+1,\Gamma_1}^{q+1} \\
&+ \frac{\|m\|_\infty}{\eta g_1}\int_{\Gamma_1} m(x)g(x)u_t^2(t)\,d\Gamma - \int_{\Gamma_1} h(x)m(x)u^2(t)\,d\Gamma.
\end{aligned} \tag{3.31}$$

Next, we would like to estimate $\Phi_2'(t)$. Taking the derivative of $\Phi_2(t)$ in (3.17) and using (2.10), we obtain

$$\begin{aligned}
\Phi_2'(t) ={}& \left(a_0 + b_0\left\|\nabla w(t)\right\|^2\right)\int_\Omega \nabla w(t)\int_0^t k(t-s)\left(\nabla w(t) - \nabla w(s)\right)ds\,dx \\
&+ b_1\int_\Omega \nabla w(t)\nabla w_t(t)\,dx\int_\Omega \nabla w(t)\int_0^t k(t-s)\left(\nabla w(t) - \nabla w(s)\right)ds\,dx \\
&- \int_\Omega \sigma(t)\int_0^t k(t-s)\nabla w(s)\,ds\int_0^t k(t-s)\left(\nabla w(t) - \nabla w(s)\right)ds\,dx \\
&- \int_\Omega \left|w(t)\right|^{p-2}w(t)\int_0^t k(t-s)\left(w(t) - w(s)\right)ds\,dx \\
&- \int_\Omega \nabla w_t(t)\int_0^t k'(t-s)\left(\nabla w(t) - \nabla w(s)\right)ds\,dx \\
&- \frac{1}{\rho+1}\int_\Omega \left|w_t(t)\right|^\rho w_t(t)\int_0^t k'(t-s)\left(w(t) - w(s)\right)ds\,dx \\
&- \int_{\Gamma_1} m(x)u_t(t)\int_0^t k(t-s)\left(w(t) - w(s)\right)ds\,d\Gamma \\
&+ \mu_2\int_{\Gamma_1}\left|z(x,1,t)\right|^{q-1}z(x,1,t)\int_0^t k(t-s)\left(w(t) - w(s)\right)ds\,d\Gamma \\
&+ \mu_1\int_{\Gamma_1}\left|w_t(x,t)\right|^{q-1}w_t(x,t)\int_0^t k(t-s)\left(w(t) - w(s)\right)ds\,d\Gamma \\
&- \left(\int_0^t k(s)\,ds\right)\left\|\nabla w_t(t)\right\|^2 - \frac{1}{\rho+1}\left(\int_0^t k(s)\,ds\right)\left\|w_t(t)\right\|_{\rho+2}^{\rho+2} \\
:={}& E_1 + E_2 + \cdots + E_9 - \left(\int_0^t k(s)\,ds\right)\left\|\nabla w_t(t)\right\|^2 \\
&- \frac{1}{\rho+1}\left(\int_0^t k(s)\,ds\right)\left\|w_t(t)\right\|_{\rho+2}^{\rho+2}.
\end{aligned} \tag{3.32}$$

Now, we will estimate the right-hand side of (3.32). By (2.1), (2.2), (2.5), (2.8), (3.7), (3.13), (3.14), and Young's inequality, for any $\gamma > 0$, we derive the following inequalities

$$|E_1| \le \left| \int_\Omega \left(a_0 + \frac{2b_0 pE(0)}{l(p-2)} \right) \nabla w(t) \int_0^t k(t-s)\big(\nabla w(t) - \nabla w(s)\big)\,ds\,dx \right|$$
$$\le \gamma \left\| \nabla w(t) \right\|^2 + \frac{k_0}{4\gamma} \left(a_0 + \frac{2b_0 pE(0)}{l(p-2)} \right)^2 (k \circ \nabla w)(t), \tag{3.33}$$
$$|E_2| \le \gamma b_1^2 \left(\int_\Omega \nabla w(t) \nabla w_t(t)\,dx \right)^2 \left\| \nabla w(t) \right\|^2$$
$$+ \frac{1}{4\gamma} \int_\Omega \left(\int_0^t k(t-s)\big(\nabla w(t) - \nabla w(s)\big)\,ds \right)^2 dx$$
$$\le -\frac{2\gamma b_1^2 pE(0)}{l(p-2)} E'(t) + \frac{k_0}{4\gamma} (k \circ \nabla w)(t), \tag{3.34}$$
$$|E_3| \le \gamma \int_\Omega \sigma(t) \left(\int_0^t k(t-s)\big(\left| \nabla w(t) - \nabla w(s) \right| + \left| \nabla w(t) \right| \big)\,ds \right)^2 dx$$
$$+ \frac{1}{4\gamma} \int_\Omega \sigma(t) \left(\int_0^t k(t-s) \left| \nabla w(t) - \nabla w(s) \right| ds \right)^2 dx$$
$$\le \left(2\gamma + \frac{1}{4\gamma} \right)(a_0 - l)(k \circ \nabla w)(t) + 2\gamma (a_0 - l) k_0 \left\| \nabla w(t) \right\|^2, \tag{3.35}$$
$$|E_4| \le \gamma \int_\Omega \left| w(t) \right|^{2(p-1)} dx + \frac{C_*^2 k_0}{4\gamma} (k \circ \nabla w)(t)$$
$$\le \gamma \alpha_3 \left\| \nabla w(t) \right\|^2 + \frac{C_*^2 k_0}{4\gamma} (k \circ \nabla w)(t), \tag{3.36}$$
$$|E_5| \le \gamma \left\| \nabla w_t(t) \right\|^2 - \frac{k(0)}{4\gamma} \big(k' \circ \nabla w\big)(t), \tag{3.37}$$
$$|E_6| \le \frac{\gamma \alpha_4}{\rho + 1} \left\| \nabla w_t(t) \right\|^2 - \frac{k(0) C_*^2}{4\gamma(\rho+1)} \big(k' \circ \nabla w\big)(t), \tag{3.38}$$
$$|E_7| \le \frac{\gamma \|m\|_\infty}{g_1} \int_{\Gamma_1} m(x) g(x) u_t^2(t)\,d\Gamma + \frac{\tilde{C}_*^2 k_0}{4\gamma} (k \circ \nabla w)(t), \tag{3.39}$$
$$|E_8| \le \gamma \mu_2 \left\| z(x,1,t) \right\|_{q+1,\Gamma_1}^{q+1} + \mu_2 C_\gamma k_0^q \alpha_2 (k \circ \nabla w)(t), \tag{3.40}$$

and

$$|E_9| \le \gamma \mu_1 \alpha_5 \left\| \nabla w_t(t) \right\|^2 + \frac{\mu_1 \tilde{C}_*^2 k_0}{4\gamma} (k \circ \nabla w)(t), \tag{3.41}$$

where $\alpha_3 = C_*^{2(p-1)} (\frac{2pE(0)}{l(p-2)})^{p-2}$, $\alpha_4 = C_*^{2(\rho+1)} (\frac{2pE(0)}{p-2})^\rho$, and $\alpha_5 = \tilde{C}_*^{2q} (\frac{2pE(0)}{p-2})^{q-1}$. Thus, from (3.32)–(3.41), we conclude that

$$\Phi_2'(t) \le -\frac{1}{\rho+1} \left(\int_0^t k(s)\,ds \right) \left\| w_t(t) \right\|_{\rho+2}^{\rho+2}$$
$$- \left(\int_0^t k(s)\,ds - \gamma \left(1 + \mu_1 \alpha_5 + \frac{\alpha_4}{\rho+1} \right) \right) \left\| \nabla w_t(t) \right\|^2$$

$$
\begin{aligned}
&+ C_3(k \circ \nabla w)(t) - \frac{k(0)}{4\gamma}\left(1 + \frac{C_*^2}{\rho+1}\right)(k' \circ \nabla w)(t) \\
&- \frac{2\gamma b_1^2 pE(0)}{l(p-2)} E'(t) + \gamma \mu_2 \|z(x,1,t)\|_{q+1,\Gamma_1}^{q+1} \\
&+ \gamma\left(1 + 2(a_0 - l)k_0 + \alpha_3\right)\|\nabla w(t)\|^2 + \frac{\gamma \|m\|_\infty}{g_1}\int_{\Gamma_1} m(x)g(x)u_t^2(t)\,d\Gamma, \qquad (3.42)
\end{aligned}
$$

where $C_3 = \frac{1}{4\gamma}\{k_0(a_0 + \frac{2b_0pE(0)}{l(p-2)})^2 + (8\gamma^2+1)(a_0 - l) + k_0(1 + C_*^2 + \tilde{C}_*^2 + \mu_1\tilde{C}_*^2) + 4\gamma\mu_2 C_\gamma k_0^q \alpha_2\}$. Similarly to Lemma 3.4, for any $\lambda > 0$, we obtain

$$
\begin{aligned}
\sigma'(t)\Phi_1(t) \le& -\frac{\sigma'(t)}{\rho+2}\|w_t(t)\|_{\rho+2}^{\rho+2} - C_4\sigma'(t)\|\nabla w(t)\|^2 - \frac{\sigma'(t)}{2}\|\nabla w_t(t)\|^2 \\
&- \frac{\sigma'(t)b_1}{4}\|\nabla w(t)\|^4 \\
&- \frac{\sigma'(t)(\|m\|_\infty + \|g\|_\infty)}{2h_1}\int_{\Gamma_1} h(x)m(x)u^2(t)\,d\Gamma \qquad (3.43)
\end{aligned}
$$

and

$$
\sigma'(t)\Phi_2(t) \le -\frac{\lambda\sigma'(t)}{\rho+2}\|w_t(t)\|_{\rho+2}^{\rho+2} - \lambda\sigma'(t)\|\nabla w_t(t)\|^2 - C_5\sigma'(t)(k \circ \nabla w)(t), \qquad (3.44)
$$

where $C_4 = \frac{1}{2} + \frac{\tilde{C}_*^2}{2} + \frac{\alpha_1}{(\rho+2)(\rho+1)}$ and $C_5 = \frac{C_\lambda k_0{}^{\rho+1}\alpha_1}{(\rho+2)(\rho+1)} + \frac{k_0}{4\lambda}$. Since k is positive, we have, for any $t_0^* > 0$, $\int_0^t k(s)\,ds \ge \int_0^{t_0^*} k(s)\,ds := k_1 > 0$, for all $t \ge t_0^*$. Applying (3.7), (3.31), and (3.42)–(3.44), we find that for any $t \ge t_0^*$,

$$
\begin{aligned}
\Xi'(t) =& ME'(t) + \varepsilon\sigma'(t)\Phi_1(t) + \varepsilon\sigma(t)\Phi_1'(t) + \sigma'(t)\Phi_2(t) + \sigma(t)\Phi_2'(t) \\
\le& -\sigma(t)\left(\frac{k_1 - \varepsilon}{\rho+1} + \frac{(\varepsilon+\lambda)\sigma'(t)}{(\rho+2)\sigma(t)}\right)\|w_t(t)\|_{\rho+2}^{\rho+2} - \sigma(t)\left(b_0\varepsilon + \frac{\varepsilon b_1\sigma'(t)}{4\sigma(t)}\right)\|\nabla w(t)\|^4 \\
&- \sigma(t)\left(\left(\frac{k(t)}{2} + \frac{\sigma'(t)}{2\sigma(t)}\int_0^t k(s)\,ds\right)M\right. \\
&\left. + \frac{\varepsilon C_4\sigma'(t)}{\sigma(t)} + \frac{\varepsilon l}{2} - \gamma\left(1 + 2(a_0 - l)k_0 + \alpha_3\right)\right)\|\nabla w(t)\|^2 \\
&- \sigma(t)\left(k_1 - \gamma\left(1 + \mu_1\alpha_5 + \frac{\alpha_4}{q+1}\right) - \varepsilon + \frac{(\varepsilon + 2\lambda)\sigma'(t)}{2\sigma(t)}\right)\|\nabla w_t(t)\|^2 \\
&+ \varepsilon\sigma(t)\|w(t)\|_p^p \\
&+ \sigma(t)\left(\frac{M\sigma'(t)}{2\sigma(t)} + \frac{\varepsilon\sigma(t)}{4\eta} - \frac{C_5\sigma'(t)}{\sigma(t)} + C_3\right)(k \circ \nabla w)(t) \\
&- \sigma(t)\left(\frac{C_1 M}{\sigma(t)} - \varepsilon\mu_1 C_\eta\right)\|w_t(t)\|_{q+1,\Gamma_1}^{q+1} \\
&- \sigma(t)\left(\frac{C_2 M}{\sigma(t)} - \varepsilon\mu_2 C_\eta - \gamma\mu_2\right)\|z(x,1,t)\|_{q+1,\Gamma_1}^{q+1} \\
&+ \sigma(t)\left(\frac{M}{2} - \frac{k(0)}{4\gamma}\left(1 + \frac{C_*^2}{\rho+1}\right)\right)(k' \circ \nabla w)(t) \\
&- \sigma(t)\left(\varepsilon + \frac{\varepsilon\sigma'(t)(\|m\|_\infty + \|g\|_\infty)}{2h_1\sigma(t)}\right)\int_{\Gamma_1} h(x)m(x)u^2(t)\,d\Gamma
\end{aligned}
$$

$$
\begin{aligned}
&-\sigma(t)\frac{2\gamma b_1^2 pE(0)}{l(p-2)}E'(t)\\
&-\sigma(t)\left(\frac{M}{\sigma(t)}-\frac{\varepsilon\|m\|_\infty}{\eta g_1}-\frac{\gamma\|m\|_\infty}{g_1}\right)\int_{\Gamma_1} m(x)g(x)u_t^2(t)\,d\Gamma. \qquad (3.45)
\end{aligned}
$$

Since $\lim_{t\to\infty}\frac{\sigma'(t)}{\sigma(t)}=0$, we choose $t_0^*>0$ sufficiently large. At this point, we pick $\varepsilon>0$ and $\gamma>0$ sufficiently small and we take M sufficiently large such that for $t\geq t_0^*$,

$$
\begin{aligned}
&M_1=\frac{k_1-\varepsilon}{\rho+1}+\frac{(\varepsilon+\lambda)\sigma'(t)}{(\rho+2)\sigma(t)}>0,\\
&M_2=\left(\frac{k(t)}{2}+\frac{\sigma'(t)}{2\sigma(t)}\int_0^t k(s)\,ds\right)M+\frac{\varepsilon C_4\sigma'(t)}{\sigma(t)}+\frac{\varepsilon l}{2}-\gamma\big(1+2(a_0-l)k_0+\alpha_3\big)>0,\\
&M_3=k_1-\gamma\left(1+\mu_1\alpha_5+\frac{\alpha_4}{q+1}\right)-\varepsilon+\frac{(\varepsilon+2\lambda)\sigma'(t)}{2\sigma(t)}>0,\\
&M_4=\frac{M\sigma'(t)}{2\sigma(t)}+\frac{\varepsilon\sigma(t)}{4\eta}-\frac{C_5\sigma'(t)}{\sigma(t)}+C_3>0,\qquad M_5=\frac{C_1M}{\sigma(t)}-\varepsilon\mu_1C_\eta>0,\\
&M_6=\frac{C_2M}{\sigma(t)}-\varepsilon\mu_2C_\eta-\gamma\mu_2>0,\qquad M_7=\frac{M}{2}-\frac{g(0)}{4\gamma}\left(1+\frac{C_*^2}{\rho+1}\right)>0,
\end{aligned}
$$

and

$$
M_8=\frac{M}{\sigma(t)}-\frac{\varepsilon\|m\|_\infty}{\eta g_1}-\frac{\gamma\|m\|_\infty}{g_1}>0.
$$

Then, for any $t\geq t_0^*$, using (3.5) and (3.45), we deduce that

$$
\Xi'(t)\leq -M_9\sigma(t)E(t)+M_{10}\sigma(t)(k\circ\nabla w)(t)-M_{11}\sigma(t)E'(t), \qquad (3.46)
$$

where M_9 and M_{10} are some positive constants and $M_{11}=\frac{2\gamma b_1^2 pE(0)}{l(p-2)}$. Multiplying (3.46) by $\zeta(t)$ and using (2.7) and (3.7), we obtain for any $t\geq t_0^*$,

$$
\begin{aligned}
\zeta(t)\Xi'(t)&\leq -M_9\sigma(t)\zeta(t)E(t)-M_{10}\sigma(t)\big(k'\circ\nabla w\big)(t)-M_{11}\sigma(t)\zeta(t)E'(t)\\
&\leq -M_9\sigma(t)\zeta(t)E(t)-\big(2M_{10}+M_{11}\sigma(t)\zeta(t)\big)E'(t). \qquad (3.47)
\end{aligned}
$$

Now, we define

$$
G(t)=\zeta(t)\Xi(t)+\big(2M_{10}+M_{11}\sigma(t)\zeta(t)\big)E(t).
$$

Using the fact that ζ and σ are nonincreasing positive functions and $\zeta'(t)\leq 0$ and $\sigma'(t)\leq 0$, (3.47) implies that

$$
G'(t)\leq -M_9\sigma(t)\zeta(t)E(t)\leq -\kappa\sigma(t)\zeta(t)G(t), \qquad (3.48)
$$

where κ is a positive constant. Integrating (3.48) between t_0^* and t gives the following estimation for the function $G(t)$

$$
G(t)\leq G\big(t_0^*\big)e^{-\kappa\int_{t_0^*}^t\sigma(s)\zeta(s)\,ds},\quad \forall t\geq t_0^*.
$$

Again, employing that $G(t)$ is equivalent to $E(t)$, we deduce

$$E(t) \le Ke^{-\kappa \int_{t_0^*}^{t} \sigma(s)\zeta(s)\,ds}, \quad \forall t \ge t_0^*,$$

where K is a positive constant. Thus, the proof of Theorem 3.2 is completed. □

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