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Blow-up of solution to semilinear wave equations with strong damping and scattering damping

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Abstract

This paper is devoted to investigating the initial boundary value problem for a semilinear wave equation with strong damping and scattering damping on an exterior domain. By introducing suitable multipliers and applying the test-function technique together with an iteration method, we derive the blow-up dynamics and an upper-bound lifespan estimate of the solution to the problem with power-type nonlinearity $|u|^p$, derivative-type nonlinearity $|u_t|^p$, and combined type nonlinearities $|u_t|^p + |u|^q$ in the scattering case, respectively. The novelty of the present paper is that we establish the upper-bound lifespan estimate of the solution to the problem with strong damping and scattering damping, which are associated with the well-known Strauss exponent and Glassey exponent.

Keywords: Semilinear wave equation; Strong damping; Scattering damping; Iteration technique; Lifespan estimate

1 Introduction

In the present paper, we mainly consider the following initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t + \frac{\mu}{(1+t)^\beta} u_t = f(u, u_t), & x \in \Omega^c, t > 0, \\ u(x,0) = \varepsilon u_0(x), \qquad u_t(x,0) = \varepsilon u_1(x), & x \in \Omega^c, \\ u|_{\partial\Omega^c} = 0, & t > 0, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $\mu > 0$, $\beta \geq 1$, $f(u, u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$ for $1 < p, q < \infty$, $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$, $\Omega^c = \mathbb{R}^n \backslash B_1(0)$ $(n \geq 1)$. The initial data $u_0(x) \in H^1(\Omega^c)$, $u_1(x) \in L^2(\Omega^c)$ are compactly supported functions and $\operatorname{supp}(u_0(x), u_1(x)) \subset B_R(0) \cap \Omega^c$, where $B_R(0) = \{x||x| \leq R\}$, $R > 2$. $\varepsilon \in (0,1)$ is a small parameter.

In recent years, many researchers are interested in exploring blow-up results and lifespan estimates of solutions to the semilinear wave equation. The following classical Cauchy problems without damping

$$\begin{cases} u_t - \Delta u = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n \end{cases} \tag{1.2}$$

and

$$\begin{cases} u_{tt} - \Delta u = f(u,u_t), & & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & x \in \mathbb{R}^n \end{cases} \tag{1.3}$$

have been studied extensively (see [1–17]). It is worth noting that problem (1.2) admits the Fujita exponent $p_F(n) = 1 + \frac{2}{n}$. Fujita [3] shows that the solution to problem (1.2) blows up in finite time when $1 < p < p_F(n)$, while there exists a global solution when $p > p_F(n)$. Xu and Su [14] discuss the asymptotic behavior of the solution to problem (1.2) with strong damping on the exterior domain. Global existence and the blow-up dynamics of solution are verified under certain assumptions on the initial value. The problem (1.3) with $f(u,u_t) = |u|^p$ asserts the Strauss exponent $p_S(n)$. If $n = 1$, $p_S(n) = \infty$. If $n \geq 2$, $p_S(n)$ is the positive root of the quadratic equation

$$r(p,n) = -(n-1)p^2 + (n+1)p + 2 = 0.$$

John [7] establishes the blow-up result of solution in the case $1 < p < p_S(3)$ and the existence of a global solution in the case $p > p_S(3)$. Zhou and Han [17] consider the blow-up dynamics and the upper-bound lifespan estimate of the solution to the variable-coefficient wave equation with $f(u,u_t) = |u|^p$ by applying the test-function method and the Kato lemma when $n \geq 3$. Nonexistence of a global solution to problem (1.3) with $f(u,u_t) = |u|^p$ in the critical case $p = p_S(n)(n \geq 4)$ is demonstrated by taking advantage of the Kato lemma (see [15]). Han [4, 5] investigates the nonexistence of a global solution and the upper-bound lifespan estimate of the solution to the variable-coefficient wave equation with $f(u,u_t) = |u|^p$ by using the Kato lemma when $n = 1, 2$. Wakasa and Yordanov [13] verify the sharp lifespan estimate of the solution in the critical case $p = p_S(n)(n \geq 2)$. The strategy of proof is based on the test-function method and an iteration technique. Lai et al. [8] acquire local well-posedness and an upper-bound lifespan estimate of the solution to problem (1.3) with $f(u,u_t) = |u|^p(1 < p \leq p_S(2))$ on an asymptotically Euclidean exterior domain. The Cauchy problem (1.3) with $f(u,u_t) = |u_t|^p$ affirms the Glassey exponent $p_G(n)$. When $n = 1$, $p_G(n) = \infty$. When $n \geq 2$, $p_G(n) = \frac{n+1}{n-1}$. Zhou [16] illustrates the blow-up phenomenon and the upper-bound lifespan estimate of the solution to problem (1.3) with $f(u,u_t) = |u_t|^p$. Zhou and Han [17] establish the upper-bound lifespan estimate of the solution to the variable-coefficient wave equation with $f(u,u_t) = |u_t|^p$ by deriving an ordinary differential inequality. Han and Zhou [6] employ the Kato lemma to present the blow-up dynamics for problem (1.3) with $f(u,u_t) = |u_t|^p + |u|^q$.

Let us turn our attention to the following problem with a damping term

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = 0, & & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

The behaviors of the solution are divided into four cases according to the decay rate β. When $\beta \in (-\infty, -1)$, the damping term is overdamping, which means that the solution dose not decay to zero when $t \to \infty$. When $\beta \in [-1, 1)$, the damping term is effective and the solution behaves like that of the heat equation. When $\beta = 1$, the damping term is scale invariant by imposing the scaling $u(x,t) = u(\lambda x, \lambda(1+t) - 1)(\lambda > 0)$. When $\beta \in (1, \infty)$, the solution scatters to the free-wave equation when $t \to \infty$. This is the scattering case. The corresponding nonlinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{(1+t)^\beta} u_t = f(u, u_t), & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = \varepsilon u_0(x), \qquad u_t(x,0) = \varepsilon u_1(x), & x \in \mathbb{R}^n \end{cases} \tag{1.5}$$

causes extensive attention (see detailed illustrations in [18–30]), where $f(u, u_t) = |u|^p$, $|u_t|^p, |u_t|^p + |u|^q$, respectively. In the scattering case $\beta > 1$, Lai et al. [24] derive the blow-up result and the upper-bound lifespan estimate of the solution to problem (1.5) with $f(u, u_t) = |u|^p$ by employing an appropriate multiplier and an iteration method in the subcritical case $1 < p < p_S(n)$. Wakasa and Yordanov [30] demonstrate the upper-bound lifespan estimate of the solution to the variable-coefficient wave equation with $f(u, u_t) = |u|^p$ in the critical case $p = p_S(n)(n \geq 2)$. The existence of the global solution in the super-critical case $p > p_S(n)(n = 3, 4)$ is investigated (see [27]). Lai et al. [25] deduce the upper-bound lifespan estimate of solution to problem (1.5) with $f(u, u_t) = |u_t|^p$ by introducing some multipliers and constructing the Riccati equation in the scattering and scale-invariant cases, respectively. The upper-bound lifespan estimate of the solution to problem (1.5) with $f(u, u_t) = |u_t|^p + |u|^q$ is discussed by utilizing an iteration approach (see [26]). Ming et al. [28] investigate the upper-bound lifespan estimates of the solutions to the coupled system of semilinear wave equations with $|v_t|^{p_1} + |v|^{q_1}$, $|u_t|^{p_2} + |u|^{q_2}$ in the subcritical and critical cases. The methods applied in the proofs are the functional method and an iteration technique. In the scale-invariant case $\beta = 1$, Imai et al. [20] establish the lifespan estimate of the solution to problem (1.5) with $f(u, u_t) = |u|^p$ when $\mu = 2$ and $1 < p \leq p_F(2)$. Wakasa [29] and Kato et al. [22] verify the blow-up and lifespan estimate of the solution in the subcritical and critical cases $1 < p \leq p_F(1)$ when $\mu = 2$. Kato et al. [21] discuss the blow-up dynamics and lifespan estimate of the solution to problem (1.5) with $\mu = 2$, $\beta = 1$, $f(u, u_t) = |u|^p(1 < p \leq p_S(5))$ in three space dimensions. Moreover, it is shown that the existence of the global solution without a spherically symmetric assumption is obtained when $p > p_S(5)$. Lai et al. [23] consider the Cauchy problem with scale-invariant damping $\frac{\mu_1}{1+t} u_t$ and mass term $\frac{\mu_2}{(1+t)^2} u$. Formation of a singularity of the solution is derived by imposing certain assumptions on the initial values. Chen [31] illustrates blow-up phenomena and the upper-bound lifespan estimate of the solution to problem (1.5) with $\beta = 1$ and $f(u, u_t) = |u_t|^p, |u_t|^p + |u|^q$, respectively. The key tool used in the proofs is the test-function technique. We refer the readers to the works in [32–36] for more details.

Many scholars focus on studying the Cauchy problem for the wave equation with a strong damping term

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = f(u, u_t), & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.6}$$

D'Abbicco and Reissig [37] prove the blow-up of the solution to problem (1.6) with $f(u, u_t) = |u|^p$ when $1 < p < 1 + \frac{2}{n}$. In addition, the existence of a global solution is obtained when $p > 1 + \frac{3}{n-1} (n \geq 2)$. Ikehata and Inoue [38] demonstrate the global existence of the solution to problem (1.6) with $f(u, u_t) = |u|^p$ on the exterior domain in two space dimensions when $p > 6$. Fino [39] considers the initial boundary value problem for wave equation with $f(u, u_t) = |u|^p$ and small initial values. The blow-up dynamics of the solution to the problem is established by applying the Kato lemma. Fino [40] employs the test-function technique to present the blow-up result of the solution to problem (1.6) with $f(u, u_t) = |u|^p$ on an exterior domain. Lian and Xu [41] investigate the existence of a global solution to the initial boundary value problem of the wave equation with weak and strong dampings and $f(u, u_t) = u \ln |u|$ by utilizing the contraction mapping principle. Yang and Zhou [42] study the existence of the global solution to the fractional Kirchhoff wave equation with structural damping or strong damping on an exterior domain. Chen and Fino [43] discuss the formation of the singularity of the solution to the initial boundary value problem (1.6) with $f(u, u_t) = |u_t|^p$ and $f(u, u_t) = |u_t|^p + |u|^q$ by taking advantage of the test-function method. However, the upper bound of the lifespan estimate of solution has not been derived in [39, 40, 43].

Motivated by the previous works in [24–26, 40, 43], we are concerned with the blow-up dynamics and the upper-bound lifespan estimate of the solution to problem (1.1) on an exterior domain. The nonlinear terms are presented in the form of $f(u, u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$, respectively. The method utilized in this paper is different from the Kato lemma employed in [4–6, 17], where the semilinear wave equations without damping are discussed. We observe that Lai et al. [24–26] consider the nonexistence of a global solution and the upper-bound lifespan estimate of the solution to the Cauchy problem with scattering damping in n space dimensions by applying an iteration approach. The results derived in [24–26] are extended to the case on an exterior domain. Fino et al. [40, 43] illustrate the blow-up of the solution to the initial boundary value problem with strong damping by making use of the test-function technique. Concerning that the upper-bound lifespan estimate of the solution to the semilinear wave equation with strong damping has not been established yet in [40, 43], we fill this gap by introducing appropriate multipliers and utilizing the test-function technique together with an iteration method. It is worth mentioning that the interaction between the strong damping term and the scattering damping term on the blow-up of the solution is analyzed. The main new contribution is that we provide an upper-bound lifespan estimate of the solution to the problem with two types of damping terms if the exponents in nonlinear terms and initial values satisfy some conditions. To the best of our knowledge, the results in Theorems 1.1–1.7 are new for problem (1.1).

The main results in this paper are stated by the following theorems.

Theorem 1.1 *Let $n \geq 3$, $1 < p < p_S(n)$, $\mu > 0$ and $\beta > 1$. Assume that $(u_0, u_1) \in H^1(\Omega^c) \times L^2(\Omega^c)$ are nonnegative functions and u_0 does not vanish identically. If a solution u to problem (1.1) with $f(u, u_t) = |u|^p$ satisfies $\operatorname{supp} u \subset \{(x, t) \in \Omega^c \times [0, T) | |x| \leq t + R\}$, then u blows up in finite time. Moreover, there exists a constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, \mu, \beta, R) > 0$ such that the lifespan estimate of solution $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq C \varepsilon^{-\frac{2p(p-1)}{r(p,n)}}, \tag{1.7}$$

where $0 < \varepsilon \leq \varepsilon_0$, $C > 0$ is a constant independent of ε.

Theorem 1.2 *Let $n = 2$, $1 < p < p_S(2)$, $\mu > 0$, and $\beta > 1$. Assume that the initial values satisfy the same conditions in Theorem 1.1. If*

$$\int_{\Omega^c} u_1(x)\phi_0(x)\,dx \neq 0,$$

then (1.7) is replaced by

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{p-1}{3-p}}, & 1 < p < 2, \\ C\varepsilon^{-\frac{2p(p-1)}{r(p,2)}}, & 2 \leq p < p_S(2). \end{cases}$$

Theorem 1.3 *Let $n = 1$, $\mu > 0$ and $\beta > 1$. Assume that the initial values satisfy the same conditions in Theorem 1.1. If*

$$\int_{\Omega^c} u_1(x)\phi_0(x)\,dx \neq 0,$$

then (1.7) is replaced by

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{p-1}{3-p}}, & 1 < p < 2, \\ C\varepsilon^{-\frac{p(p-1)}{2}}, & 2 \leq p < +\infty. \end{cases}$$

Theorem 1.4 *Let $n \geq 1$, $\mu > 0$ and $\beta > 1$. Assume that $(u_0, u_1) \in H^1(\Omega^c) \times L^2(\Omega^c)$ are nonnegative functions and u_1 does not vanish identically. If a solution u to problem (1.1) with $f(u, u_t) = |u_t|^p$ satisfies $\operatorname{supp} u \subset \{(x,t) \in \Omega^c \times [0,T) \mid |x| \leq t + R\}$, then u blows up in finite time. Moreover, the lifespan estimate of solution $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{2(p-1)}{2-(n-1)(p-1)}}, & 1 < p < p_G(n), \\ \exp(C\varepsilon^{-(p-1)}), & p = p_G(n), \end{cases}$$

where $C > 0$ is a constant independent of ε.

Theorem 1.5 *Let $n \geq 1$, $\mu > 0$, and $\beta = 1$. Assume that the initial values satisfy the same conditions in Theorem 1.4. The lifespan estimate of solution $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{2(p-1)}{2-(n+2\mu-1)(p-1)}}, & 1 < p < p_G(n+2\mu), \\ \exp(C\varepsilon^{-(p-1)}), & p = p_G(n+2\mu), \end{cases} \tag{1.8}$$

where $C > 0$ is a constant independent of ε.

Theorem 1.6 *Let $n \geq 1$, $\mu > 0$ and $\beta > 1$. Assume that $(u_0, u_1) \in H^1(\Omega^c) \times L^2(\Omega^c)$ are nonnegative functions and u_1 does not vanish identically. Suppose that a solution u to problem (1.1) with $f(u, u_t) = |u_t|^p + |u|^q$ satisfies $\operatorname{supp} u \subset \{(x,t) \in \Omega^c \times [0,T) \mid |x| \leq t + R\}$. If $p > 1$ and*

$$\begin{cases} 1 < q < \min\{1 + \frac{4}{(n-1)p-2}, \frac{2n}{n-2}\}, & n \geq 2, \\ 1 < q, & n = 1, \end{cases} \tag{1.9}$$

then u blows up in finite time. Moreover, the lifespan estimate of solution $T(\varepsilon)$ satisfies

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{2p(q-1)}{2q+2-(n-1)p(q-1)}}, & n \geq 2, \\ C\varepsilon^{-\frac{p(q-1)}{2}}, & n = 1, \end{cases}$$

where $C > 0$ is a constant independent of ε.

Theorem 1.7 *Let $n \geq 2$, $\mu > 0$, and $\beta > 1$. Assume that the initial values satisfy the same conditions in Theorem 1.6. If $p > \frac{2n}{n-1}$ and $1 < q < \frac{n+1}{n-1}$, then a solution u to problem (1.1) with $f(u, u_t) = |u_t|^p + |u|^q$ blows up in finite time. Moreover, the lifespan estimate of solution $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq C\varepsilon^{-\frac{q-1}{q+1-n(q-1)}},$$

where $C > 0$ is a constant independent of ε.

Remark 1.1 The restriction $q < \frac{2n}{n-2}$ is necessary to guarantee the integrability of the non-linear term $|u|^q$ in Theorem 1.6.

2 Preliminaries

In order to prove the main results, we present several related lemmas and the definition of a weak solution.

Lemma 2.1 ([4, 5, 17]) *There exists a function $\phi_0(x) \in C^2(\Omega^c)$ that satisfies*

$$\begin{cases} \Delta\phi_0(x) = 0, & x \in \Omega^c, n \geq 1, \\ \phi_0|_{\partial\Omega^c} = 0. \end{cases}$$

In the case $n \geq 3$, $\phi_0(x) \to 1$ as $|x| \to \infty$, $0 < \phi_0(x) < 1$ for all $x \in \Omega^c$. In the case $n = 2$, $\phi_0(x) \to \infty$ as $|x| \to \infty$, $0 < \phi_0(x) \leq C \ln|x|$ for all $x \in \Omega^c$. In the case $n = 1$, $\phi_0(x) \to \infty$ as $|x| \to \infty$, $C_1 x \leq \phi_0(x) \leq C_2 x$ for all $x \in \Omega^c$. Here, C, C_1, C_2 are positive constants.

Lemma 2.2 ([39]) *There exists a function $\varphi_1(x) \in C^2(\Omega^c)$ satisfying the following boundary value problem*

$$\begin{cases} \Delta\varphi_1(x) = \frac{1}{2}\varphi_1(x), & x \in \Omega^c, n \geq 1, \\ \varphi_1|_{\partial\Omega^c} = 0, \\ |x| \to \infty, & \varphi_1(x) \to \int_{\mathbb{S}^{n-1}} e^{x\cdot\omega}\, d\omega. \end{cases}$$

Moreover, $0 < \varphi_1(x) < C(1 + |x|)^{-(n-1)/2} e^{|x|}$ for all $x \in \Omega^c$, where $C > 0$ is a constant.

We define $\psi_1(x,t) = e^{-t}\varphi_1(x)$. Direct calculation shows

$$(\psi_1)_t = -\psi_1, \qquad (\psi_1)_{tt} = \psi_1, \qquad \Delta\psi_1 = \frac{1}{2}\psi_1. \tag{2.1}$$

Lemma 2.3 ([17]) *Let $n \geq 1$, $p > 1$. It holds that*

$$\int_{\Omega^c \cap \{|x| \leq t+R\}} \left(\phi_0(x)\right)^{-\frac{1}{p-1}} \left(\psi_1(x,t)\right)^{p'} \, dx \leq C(t+R)^{n-1-\frac{(n-1)p'}{2}},$$

where $p' = \frac{p}{p-1}$, C is a positive constant.

Lemma 2.4 ([17]) *Let $n \geq 1$, $p > 1$. For all $t \geq 0$, it holds that*

$$\int_{\Omega^c \cap \{|x| \leq t+R\}} \psi_1(x,t) \, dx \leq C(t+R)^{(n-1)/2},$$

where C is a positive constant.

Definition 2.1 Assume that $u \in C([0,T), H^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c))$. $u \in L^p_{\text{loc}}([0,T) \times \Omega^c)$ when the nonlinear term is $f(u,u_t) = |u|^p$. $u_t \in L^p_{\text{loc}}([0,T) \times \Omega^c)$ when the nonlinear term is $f(u,u_t) = |u_t|^p$. $u_t \in L^p_{\text{loc}}([0,T) \times \Omega^c)$ and $u \in L^q_{\text{loc}}([0,T) \times \Omega^c)$ when the nonlinear term is $f(u,u_t) = |u_t|^p + |u|^q$. It holds that

$$\begin{aligned}
&\int_{\Omega^c} u_t(x,t)\phi(x,t) \, dx - \int_{\Omega^c} \varepsilon u_1(x)\phi(x,0) \, dx \\
&\quad - \int_0^t \int_{\Omega^c} \left\{ u_s(x,s)\phi_s(x,s) + u(x,s)\Delta\phi(x,s) + u_s(x,s)\Delta\phi(x,s) \right\} dx \, ds \\
&\quad + \int_0^t \int_{\Omega^c} \frac{\mu}{(1+s)^\beta} u_s(x,s)\phi(x,s) \, dx \, ds \\
&= \int_0^t \int_{\Omega^c} f(u,u_t)\phi(x,s) \, dx \, ds,
\end{aligned} \tag{2.2}$$

where $u(x,0) = \varepsilon u_0(x)$, $\phi \in C_0^\infty([0,T) \times \Omega^c)$.

Employing the integration by parts in (2.2) and letting $t \to T$, we obtain

$$\begin{aligned}
&\int_0^T \int_{\Omega^c} u(x,s) \left\{ \phi_{ss}(x,s) - \Delta\phi(x,s) + \Delta\phi_s(x,s) - \left(\frac{\mu\phi(x,s)}{(1+s)^\beta} \right)_s \right\} dx \, ds \\
&= \int_{\Omega^c} \varepsilon u_1(x)\phi(x,0) \, dx - \int_{\Omega^c} \varepsilon u_0(x)\phi_t(x,0) \, dx - \int_{\Omega^c} \varepsilon u_0(x)\Delta\phi(x,0) \, dx \\
&\quad + \int_{\Omega^c} \varepsilon u_0(x)\mu\phi(x,0) \, dx + \int_0^T \int_{\Omega^c} f(u,u_t)\phi(x,s) \, dx \, ds.
\end{aligned}$$

We present a multiplier

$$m(t) = \exp\left(\mu \frac{(1+t)^{1-\beta}}{1-\beta} \right). \tag{2.3}$$

In the case $\beta > 1$, we have

$$m'(t) = \frac{\mu}{(1+t)^\beta} m(t), \quad 0 < m(0) \leq m(t) \leq 1, t \geq 0. \tag{2.4}$$

3 Proof of Theorem 1.1

We set

$$\begin{cases} F_0(t) = \int_{\Omega^c} u(x,t)\phi_0(x)\,dx, \\ F_1(t) = \int_{\Omega^c} u(x,t)\psi_1(x,t)\,dx, \\ F_2(t) = \int_{\Omega^c} u_t(x,t)\psi_1(x,t)\,dx. \end{cases}$$

Lemma 3.1 *Let $n \geq 1$. Under the same assumptions in Theorem 1.1, for all $t \geq 0$, it holds that*

$$F_1(t) \geq \frac{1}{3}m(0)\varepsilon \int_{\Omega^c} u_0(x)\varphi_1(x)\,dx > 0.$$

Proof of Lemma 3.1 Employing (2.2), we deduce

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega^c} u_t(x,t)\phi(x,t)\,dx + \int_{\Omega^c} \frac{\mu}{(1+t)^\beta} u_t(x,t)\phi(x,t)\,dx \\ &\quad - \int_{\Omega^c} \bigl\{u_t(x,t)\phi_t(x,t) + u(x,t)\Delta\phi(x,t) + u_t(x,t)\Delta\phi(x,t)\bigr\}\,dx \\ &= \int_{\Omega^c} f(u,u_t)\phi(x,t)\,dx. \end{aligned} \tag{3.1}$$

Multiplying (3.1) by $m(t)$ and integrating over $(0,t)$ yields

$$\begin{aligned} &m(t)\int_{\Omega^c} u_t(x,t)\phi(x,t)\,dx - m(0)\int_{\Omega^c} \varepsilon u_1(x)\phi(x,0)\,dx \\ &\quad - \int_0^t \int_{\Omega^c} m(s)\bigl\{u_s(x,s)\phi_s(x,s) + u(x,s)\Delta\phi(x,s) + u_s(x,s)\Delta\phi(x,s)\bigr\}\,dx\,ds \\ &= \int_0^t \int_{\Omega^c} m(s)f(u,u_t)\phi(x,s)\,dx\,ds. \end{aligned}$$

Replacing $\phi(x,t)$ by $\psi_1(x,t)$, we come to

$$\begin{aligned} m(t)\left(F_1'(t) + \frac{3}{2}F_1(t)\right) &= m(0)\int_{\Omega^c} \varepsilon\left(u_1(x) + \frac{1}{2}u_0(x)\right)\varphi_1(x)\,dx \\ &\quad + \frac{1}{2}\int_0^t \int_{\Omega^c} \frac{\mu}{(1+s)^\beta} m(s)u(x,s)\psi_1(x,s)\,dx\,ds \\ &\quad + \int_0^t \int_{\Omega^c} m(s)f(u,u_t)\psi_1(x,s)\,dx\,ds. \end{aligned}$$

It follows that

$$\begin{aligned} F_1'(t) + \frac{3}{2}F_1(t) &\geq m(0)\int_{\Omega^c} \varepsilon\left(u_1(x) + \frac{1}{2}u_0(x)\right)\varphi_1(x)\,dx \\ &\quad + \frac{1}{2m(t)}\int_0^t \frac{\mu}{(1+s)^\beta} m(s)F_1(s)\,ds. \end{aligned} \tag{3.2}$$

From straightforward computations, we observe

$$\begin{aligned} e^{\frac{3}{2}t}F_1(t) &\geq m(0)\int_{\Omega^c} \varepsilon u_0(x)\varphi_1(x)\,dx \\ &\quad + m(0)\frac{2}{3}\big(e^{\frac{3}{2}t}-1\big)\int_{\Omega^c} \varepsilon\left(u_1(x)+\frac{1}{2}u_0(x)\right)\varphi_1(x)\,dx \\ &\geq \frac{1}{3}m(0)e^{\frac{3}{2}t}\int_{\Omega^c} \varepsilon u_0(x)\varphi_1(x)\,dx, \end{aligned}$$

which results in

$$F_1(t) \geq \frac{1}{3}m(0)\varepsilon\int_{\Omega^c} u_0(x)\varphi_1(x)\,dx > 0.$$

This completes the proof of Lemma 3.1. □

Proof of Theorem 1.1 Letting $\phi(x,t) = \phi_0(x)$ in (2.2) with $f(u,u_t) = |u|^p$ and applying Lemma 2.1, we derive

$$F_0''(t) + \frac{\mu}{(1+t)^\beta}F_0'(t) = \int_{\Omega^c} |u|^p\phi_0(x)\,dx. \tag{3.3}$$

Multiplying (3.3) by $m(t)$, we acquire

$$\big[m(t)F_0'(t)\big]' = m(t)\int_{\Omega^c} |u|^p\phi_0(x)\,dx. \tag{3.4}$$

Integrating (3.4) over the interval $(0,t)$ and employing (2.4) yields

$$F_0'(t) \geq m(t)F_0'(t) \geq m(0)\int_0^t\int_{\Omega^c} |u|^p\phi_0(x)\,dx\,dt. \tag{3.5}$$

Applying the Holder inequality and Lemma 2.1 gives rise to

$$\begin{aligned} \int_{\Omega^c} |u|^p\phi_0(x)\,dx &\geq \frac{|\int_{\Omega^c} u\phi_0(x)\,dx|^p}{(\int_{\Omega^c\cap\{|x|\leq t+R\}}\phi_0(x)\,dx)^{p-1}} \\ &= C_3(t+R)^{-n(p-1)}\big|F_0(t)\big|^p. \end{aligned} \tag{3.6}$$

Inserting (3.6) into (3.5), we deduce

$$F_0(t) \geq C_4\int_0^t\int_0^s (r+R)^{-n(p-1)}\big|F_0(r)\big|^p\,dr\,ds. \tag{3.7}$$

Taking advantage of the Holder inequality and Lemma 2.3, we come to

$$\begin{aligned} \int_{\Omega^c} |u|^p\phi_0(x)\,dx &\geq \frac{|\int_{\Omega^c} u\psi_1\,dx|^p}{(\int_{\Omega^c\cap\{|x|\leq t+R\}}(\phi_0)^{-\frac{1}{p-1}}(\psi_1)^{\frac{p}{p-1}})^{p-1}} \\ &\geq C_5(t+R)^{(n-1)(1-p/2)}\big|F_1(t)\big|^p. \end{aligned} \tag{3.8}$$

Combining (3.5) and (3.8) with Lemma 3.1, we conclude

$$F_0(t) \geq C_6\varepsilon^p \int_0^t \int_0^s (r+R)^{(n-1)(1-p/2)}\,dr\,ds \geq \frac{C_6}{n(n+1)}\varepsilon^p (t+R)^{-(n-1)p/2} t^{n+1}, \tag{3.9}$$

where $C_6 = C_5 m(0)(\frac{1}{3}m(0)\int_{\Omega^c} u_0(x)\varphi_1(x)\,dx)^p > 0$.

Suppose that

$$F_0(t) \geq D_j (t+R)^{-a_j} t^{b_j}, \quad t \geq 0, j \in \mathbb{N}^* \tag{3.10}$$

with $D_1 = \frac{C_6}{n(n+1)}\varepsilon^p$, $a_1 = \frac{(n-1)p}{2}$, $b_1 = n+1$. Inserting (3.10) into (3.7) leads to

$$F_0(t) \geq D_{j+1}(t+R)^{-a_{j+1}} t^{b_{j+1}},$$

where

$$D_{j+1} \geq \frac{C_4 D_j^p}{(b_j p + 2)^2}, \quad a_{j+1} = a_j p + n(p-1), b_{j+1} = b_j p + 2.$$

Direct calculation indicates

$$a_j = p^{j-1}\big((n-1)p/2 + n\big) - n,$$
$$b_j = p^{j-1}\big(n+1+2/(p-1)\big) - 2/(p-1),$$
$$D_j \geq \frac{C_7 D_{j-1}^p}{p^{2(j-1)}},$$

where $C_7 = \frac{C_4}{(n+1+2/(p-1))^2}$. A straightforward calculation shows

$$D_j \geq \exp\big(p^{j-1}\big(\log D_1 - S_p(\infty)\big)\big),$$

where $S_p(\infty) = \lim_{j\to\infty}\sum_{k=1}^{j-1}\frac{2k\log p - \log C_7}{p^k}$. From (3.10), we acquire

$$F_0(t) \geq (t+R)^n t^{-\frac{2}{p-1}} \exp\big(p^{j-1}J(t)\big), \tag{3.11}$$

where $J(t) = -((n-1)p/2+n)\log(t+R) + (n+1+2/(p-1))\log t + \log D_1 - S_p(\infty)$. When $t \geq R$, we come to

$$J(t) \geq \log\big(t^{\frac{\gamma(p,n)}{2(p-1)}} D_1\big) - C_8,$$

where $C_8 = ((n-1)p/2+n)\log 2 + S_p(\infty) > 0$. It turns out that $J(t) > 1$ when $t > C_9\varepsilon^{-\frac{2p(p-1)}{r(p,n)}}$. It is deduced from (3.11) that $F_0(t) \to \infty$ as $j \to \infty$. Consequently, we arrive at the lifespan estimate

$$T(\varepsilon) \leq C\varepsilon^{-\frac{2p(p-1)}{r(p,n)}}.$$

The proof of Theorem 1.1 is finished. □

4 Proofs of Theorems 1.2 and 1.3

4.1 Proof of Theorem 1.2

Taking advantage of Lemma 2.1, we obtain

$$\begin{aligned}\int_{\Omega^c\cap\{|x|\le t+R\}}\phi_0(x)\,dx &\le \int_{\{|x|\le t+R\}\setminus B_1(0)}\ln|x|\,dx\\ &\le C_{10}(t+R)^2\ln(t+R).\end{aligned}$$

Thus, we have

$$\int_{\Omega^c}|u|^p\phi_0(x)\,dx \ge C_{11}(t+R)^{-2(p-1)}\bigl|F_0(t)\bigr|^p, \tag{4.1}$$

where $C_{11} = C_{10}^{-(p-1)}(\ln(T+R))^{-(p-1)} > 0$. Inserting (4.1) into (3.5), we deduce

$$F_0(t) \ge C_{11}m(0)\int_0^t\int_0^s (r+R)^{-2(p-1)}\bigl|F_0(r)\bigr|^p\,dr\,ds. \tag{4.2}$$

Similar to the proof of Theorem 1.1, we achieve

$$F_0(t) \ge \frac{C_6}{6}\varepsilon^p(t+R)^{-\frac{p}{2}}t^3. \tag{4.3}$$

Assume that

$$F_0(t) > \bar{D}_j(t+R)^{-\bar{a}_j}t^{\bar{b}_j}, \quad t\ge 0, j\in\mathbb{N}^* \tag{4.4}$$

with $\bar{D}_1 = \frac{C_6}{6}\varepsilon^p$, $\bar{a}_1 = \frac{p}{2}$, $\bar{b}_1 = 3$. Inserting (4.4) into (4.2) yields

$$F_0(t) \ge \frac{C_{11}m(0)\bar{D}_j^p}{(\bar{b}_jp+2)^2}(t+R)^{-\bar{a}_jp-2(p-1)}t^{\bar{b}_jp+2}.$$

Direct computation gives rise to

$$F_0(t) \ge (t+R)^2t^{-\frac{2}{p-1}}\exp\bigl(p^{j-1}\bar{J}(t)\bigr),$$

where $\bar{J}(t) \ge \log(\bar{D}_1t^{\frac{-p^2+3p+2}{2(p-1)}}) - C_{12}$. Consequently, we come to the lifespan estimate

$$T(\varepsilon) \le C\varepsilon^{-\frac{2p(p-1)}{r(p,2)}}. \tag{4.5}$$

On the other hand, thanks to the conditions $u_1(x)\ge 0$, $\int_{\Omega^c}u_1(x)\phi_0(x)\,dx \ne 0$, we derive $\int_{\Omega^c}u_1(x)\phi_0(x)\,dx > 0$. Applying (3.4) and (2.4) leads to

$$F_0'(t) \ge m(0)F_0'(0) = C_{13}\varepsilon,$$

where $C_{13} = m(0)\int_{\Omega^c}u_1(x)\phi_0(x)\,dx > 0$. It follows that

$$F_0(t) \ge C_{13}\varepsilon + \int_{\Omega^c}\varepsilon u_0(x)\phi_0(x)\,dx \ge C_{14}\varepsilon(1+t), \tag{4.6}$$

where $C_{14} = \min\{C_{13}, \int_{\Omega^c} u_0(x)\phi_0(x)\,dx\} > 0$. Inserting (4.6) into (4.2), we arrive at

$$F_0(t) \ge C_{15}\varepsilon^p \int_0^t \int_0^s (r+R)^{-(p-2)}\,dr\,ds \\ \ge \frac{C_{15}\varepsilon^p}{6}(t+R)^{-(p-1)}t^3, \tag{4.7}$$

where $C_{15} = C_{11}m(0)C_{14}^p > 0$. Assume that

$$F_0(t) \ge \bar{D}_j(t+R)^{-\bar{a}_j}t^{\bar{b}_j}, \quad t \ge 0, j \in \mathbb{N}^* \tag{4.8}$$

with $\bar{D}_1 = \frac{C_{15}\varepsilon^p}{6}$, $\bar{a}_1 = p-1$, $\bar{b}_1 = 3$. Combining (4.8) with (4.2), we acquire

$$F_0(t) \ge \bar{D}_{j+1}(t+R)^{-\bar{a}_{j+1}}t^{\bar{b}_{j+1}},$$

where

$$\bar{D}_{j+1} > \frac{C_{11}m(0)\bar{D}_j^p}{(\bar{b}_j p + 2)^2}, \qquad \bar{a}_{j+1} = p\bar{a}_j + 2(p-1), \bar{b}_{j+1} = p\bar{b}_j + 2.$$

Straightforward calculation shows

$$\bar{a}_j = p^{j-1}(p+1) - 2, \qquad \bar{b}_j = p^{j-1}\left(3 + \frac{2}{p-1}\right) - \frac{2}{p-1},$$
$$\bar{D}_j \ge \frac{C_{16}\bar{D}_{j-1}^p}{p^{2(j-1)}} \ge \exp\bigl(p^{j-1}\bigl(\log \bar{D}_1 - \bar{S}_p(\infty)\bigr)\bigr),$$

where $\bar{S}_p(\infty) = \lim_{j\to\infty}\sum_{k=1}^{j-1}\frac{2k\log p - \log C_{16}}{p^k}$. Thus, taking advantage of (4.8) generates

$$F_0(t) \ge (t+R)^2 t^{-\frac{2}{p-1}} \exp\bigl(p^{j-1}\bar{J}(t)\bigr),$$

where $\bar{J}(t) \ge \log(\bar{D}_1 t^{\frac{p(3-p)}{p-1}}) - C_{17}$ with $t \ge R$. When $1 < p < 2$ and $t \ge C\varepsilon^{-\frac{p-1}{3-p}}$, we have $J(t) > 1$. Therefore, we derive the lifespan estimate

$$T(\varepsilon) \le C\varepsilon^{-\frac{p-1}{3-p}}. \tag{4.9}$$

It is easy to check that (4.9) is stronger than (4.5) when $1 < p < 2$, which is equivalent to

$$\frac{p-1}{3-p} < \frac{2p(p-1)}{r(p,2)}.$$

This completes the proof of Theorem 1.2.

4.2 Proof of Theorem 1.3

Utilizing Lemma 2.1, we obtain

$$\int_{\Omega^c\cap\{|x|\le t+R\}} \phi_0(x)\,dx \le \int_1^{t+R} C_2 x\,dx \le \frac{C_2}{2}(t+R)^2.$$

Thus, we arrive at

$$\int_{\Omega^c} |u|^p \phi_0(x)\,dx \geq C_{18}(t+R)^{-2(p-1)}\bigl|F_0(t)\bigr|^p. \tag{4.10}$$

According to (3.5) and (4.10), we conclude

$$F_0(t) \geq C_{18}m(0)\int_0^t \int_0^s (r+R)^{-2(p-1)}\bigl|F_0(r)\bigr|^p\,dr\,ds. \tag{4.11}$$

Similar to the discussion in Theorem 1.1, we derive

$$F_0(t) \geq \frac{C_6}{2}\varepsilon^p t^2.$$

Assume that

$$F_0(t) > \bar{D}_j(t+R)^{-\bar{a}_j}t^{\bar{b}_j}, \quad t \geq 0, j \in \mathbb{N}^*$$

with $\bar{D}_1 = \frac{C_6}{2}\varepsilon^p$, $\bar{a}_1 = 0$, $\bar{b}_1 = 2$. Direct calculation gives rise to

$$\bar{J}(t) \geq \log\bigl(\bar{D}_1 t^{\frac{2}{p-1}}\bigr) - C_{19}.$$

As a result, we come to the lifespan estimate

$$T(\varepsilon) \leq C\varepsilon^{-\frac{p(p-1)}{2}}. \tag{4.12}$$

On the other hand, similar to the proof of Theorem 1.2, we have

$$F_0(t) \geq C_{14}\varepsilon(t+1). \tag{4.13}$$

Inserting (4.13) into (4.11), we deduce

$$F_0(t) \geq C_{20}\varepsilon^p(t+R)^{-(p-4)}.$$

Assume that

$$F_0(t) > \bar{D}_j(t+R)^{-\bar{a}_j}t^{\bar{b}_j}, \quad t \geq 0, j \in \mathbb{N}^* \tag{4.14}$$

with $\bar{D}_1 = C_{20}\varepsilon^p$, $\bar{a}_1 = p-4$, $\bar{b}_1 = 0$. A series of calculations show $J(t) \geq \log(D_1 t^{\frac{p(3-p)}{p-1}}) - C_{21}$. Consequently, we obtain the lifespan estimate

$$T(\varepsilon) \leq C\varepsilon^{-\frac{p-1}{3-p}}. \tag{4.15}$$

We deduce that (4.15) is stronger than (4.12) when $1 < p < 2$, which is equivalent to

$$\frac{p-1}{3-p} < \frac{p(p-1)}{2}.$$

This finishes the proof of Theorem 1.3.

5 Proof of Theorem 1.4

Replacing $\phi(x,t)$ by $\psi_1(x,t)$ in (2.2) with $f(u,u_t) = |u_t|^p$ and utilizing (2.1), we deduce

$$\frac{d}{dt}\int_{\Omega^c} u_t\psi_1\,dx + \int_{\Omega^c}\frac{1}{2}(u_t - u)\psi_1\,dx + \int_{\Omega^c}\frac{\mu}{(1+t)^\beta}u_t\psi_1\,dx = \int_{\Omega^c}|u_t|^p\psi_1\,dx, \tag{5.1}$$

which leads to

$$\frac{d}{dt}\int_{\Omega^c}\left(u_t + \frac{1}{2}u\right)\psi_1\,dx = \int_{\Omega^c}|u_t|^p\psi_1\,dx - \int_{\Omega^c}\frac{\mu}{(1+t)^\beta}u_t\psi_1\,dx.$$

It follows that

$$\frac{d}{dt}\left(m(t)\int_{\Omega^c}\left(u_t + \frac{1}{2}u\right)\psi_1\,dx\right) = m(t)\int_{\Omega^c}|u_t|^p\psi_1\,dx + \frac{1}{2}\frac{\mu}{(1+t)^\beta}m(t)F_1(t).$$

Making use of Lemma 3.1 yields

$$\begin{aligned}
&m(t)\int_{\Omega^c}\left(u_t + \frac{1}{2}u\right)\psi_1\,dx - m(0)\int_{\Omega^c}\varepsilon\left(u_1(x) + \frac{1}{2}u_0(x)\right)\varphi_1(x)\,dx \\
&\quad \ge \int_0^t\int_{\Omega^c} m(t)|u_t|^p\psi_1\,dx\,dt.
\end{aligned} \tag{5.2}$$

On the other hand, multiplying (5.1) by $m(t)$ leads to

$$\frac{d}{dt}\left[m(t)\int_{\Omega^c}u_t\psi_1\,dx\right] + m(t)\int_{\Omega^c}\frac{1}{2}(u_t - u)\psi_1\,dx = m(t)\int_{\Omega^c}|u_t|^p\psi_1\,dx. \tag{5.3}$$

From (5.2) and (5.3), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\left[m(t)\int_{\Omega^c}u_t\psi_1\,dx\right] + \frac{3}{2}m(t)\int_{\Omega^c}u_t\psi_1\,dx \\
&\quad \ge m(0)\int_{\Omega^c}\varepsilon\left(u_1(x) + \frac{1}{2}u_0(x)\right)\varphi_1(x)\,dx + m(t)\int_{\Omega^c}|u_t|^p\psi_1\,dx \\
&\qquad + \int_0^t\int_{\Omega^c}m(t)|u_t|^p\psi_1\,dx\,dt.
\end{aligned} \tag{5.4}$$

Setting

$$G_1(t) = m(t)\int_{\Omega^c}u_t\psi_1\,dx - \frac{2}{3}m(0)\int_{\Omega^c}\varepsilon u_1(x)\varphi_1(x)\,dx - \frac{2}{3}\int_0^t\int_{\Omega^c}m(t)|u_t|^p\psi_1\,dx\,dt,$$

we have $G_1(0) = \frac{1}{3}m(0)\int_{\Omega^c}\varepsilon u_1(x)\varphi_1(x)\,dx > 0$. It is deduced from (5.4) that

$$\begin{aligned}
\frac{d}{dt}G_1(t) + \frac{3}{2}G_1(t) &\ge \frac{1}{3}m(t)\int_{\Omega^c}|u_t|^p\psi_1\,dx \\
&\quad + \frac{1}{2}m(0)\varepsilon\int_{\Omega^c}u_0(x)\varphi_1(x)\,dx \\
&\ge 0.
\end{aligned}$$

It follows that $G_1(t) \geq e^{-\frac{3}{2}t} G_1(0) > 0$. Thus, we achieve

$$\begin{aligned} m(t) \int_{\Omega^c} u_t \psi_1 \, dx \geq \frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x) \, dx \\ + \frac{2}{3} \int_0^t \int_{\Omega^c} m(t) |u_t|^p \psi_1 \, dx \, dt. \end{aligned}$$

Taking $H_1(t) = \frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x) \, dx + \frac{2}{3} \int_0^t \int_{\Omega^c} m(t) |u_t|^p \psi_1 \, dx \, dt$, we have

$$m(t) F_2(t) \geq H_1(t) \tag{5.5}$$

and $H_1(0) = \frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x) \, dx$. Utilizing the Holder inequality and Lemma 2.4 indicates

$$\begin{aligned} \int_{\Omega^c} |u_t|^p \psi_1 \, dx &\geq \frac{| \int_{\Omega^c} u_t \psi_1 \, dx|^p}{(\int_{\Omega^c \cap \{|x| \leq t+R\}} \psi_1 \, dx)^{p-1}} \\ &= C^{-(p-1)} (t+R)^{-\frac{(n-1)(p-1)}{2}} \left| F_2(t) \right|^p. \end{aligned} \tag{5.6}$$

Taking into account (2.4), (5.5), and (5.6), we achieve

$$\begin{aligned} H_1'(t) &= \frac{2}{3} m(t) \int_{\Omega^c} |u_t|^p \psi_1 \, dx \\ &\geq \frac{2}{3} C^{-(p-1)} (t+R)^{-\frac{(n-1)(p-1)}{2}} H_1^p(t). \end{aligned} \tag{5.7}$$

By the property of the Riccati equation, we derive that the solution blows up for $\frac{(n-1)(p-1)}{2} \leq 1$.

When $\frac{(n-1)(p-1)}{2} < 1$ (namely, $1 < p < p_G(n)$), solving the ordinary differential inequality (5.7), we come to

$$H(t) \geq \left(\left(\frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x) \, dx \right)^{-(p-1)} + C(t+R)^{1-(n-1)(p-1)/2} - 1 \right)^{-\frac{1}{p-1}}.$$

Therefore, we arrive at the lifespan estimate

$$T(\varepsilon) \leq C \varepsilon^{-\frac{2(p-1)}{2-(n-1)(p-1)}}.$$

When $\frac{(n-1)(p-1)}{2} = 1$ (namely, $p = p_G(n)$), direct computation gives rise to

$$H(t) \geq \left(\left(\frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x) \, dx \right)^{-(p-1)} + C \ln(t+R) \right)^{-\frac{1}{p-1}}.$$

As a consequence, we obtain the lifespan estimate

$$T(\varepsilon) \leq \exp\bigl(C \varepsilon^{-(p-1)} \bigr).$$

The proof of Theorem 1.4 is completed.

6 Proof of Theorem 1.5

We define a multiplier

$$m_1(t) = (1+t)^{\mu}. \tag{6.1}$$

It follows that $1 = m_1(0) \le m_1(t)$, $\frac{m_1'(t)}{m_1(t)} = \frac{\mu}{1+t}$.

Lemma 6.1 *Let $n \ge 1$. Under the same conditions in Theorem 1.5, for all $t \ge 0$, it holds that*

$$F_1(t) \ge \frac{1}{3m_1(t)}\varepsilon \int_{\Omega^c} u_0(x)\varphi_1(x)\,dx \ge 0.$$

Proof of Lemma 6.1 The proof of Lemma 6.1 is similar to the proof of Lemma 3.1. By employing the multiplier $m_1(t)$, we acquire the desired result when $f(u,u_t) = |u_t|^p$. We omit the detailed proof. □

Proof of Theorem 1.5 Similar to the proof of Theorem 1.4, we have

$$\begin{aligned}
&\frac{d}{dt}\left[m_1(t)\int_{\Omega^c} u_t\psi_1\,dx\right] + \frac{3}{2}m_1(t)\int_{\Omega^c} u_t\psi_1\,dx \\
&\quad \ge \int_{\Omega^c} \varepsilon\left(u_1(x) + \frac{1}{2}u_0(x)\right)\varphi_1(x)\,dx + m_1(t)\int_{\Omega^c} |u_t|^p\psi_1\,dx \\
&\qquad + \int_0^t \int_{\Omega^c} m_1(t)|u_t|^p\psi_1\,dx\,dt.
\end{aligned}$$

We set

$$G_2(t) = m_1(t)\int_{\Omega^c} u_t\psi_1\,dx - \frac{2}{3}\int_{\Omega^c} \varepsilon u_1(x)\varphi_1(x)\,dx - \frac{2}{3}\int_0^t\int_{\Omega^c} m_1(t)|u_t|^p\psi_1\,dx\,dt.$$

It turns out that

$$m_1(t)\int_{\Omega^c} u_t\psi_1\,dx \ge \frac{2}{3}\int_{\Omega^c} \varepsilon u_1(x)\varphi_1(x)\,dx + \frac{2}{3}\int_0^t\int_{\Omega^c} m_1(t)|u_t|^p\psi_1\,dx\,dt.$$

Let

$$H_2(t) = \frac{2}{3}\varepsilon\int_{\Omega^c} u_1(x)\varphi_1(x)\,dx + \frac{2}{3}\int_0^t\int_{\Omega^c} m_1(t)|u_t|^p\psi_1\,dx\,dt.$$

It follows that $m_1(t)F_2(t) \ge H_2(t)$. We bear in mind

$$H_2(0) = \frac{2}{3}\int_{\Omega^c} \varepsilon u_1(x)\varphi_1(x)\,dx > 0.$$

Applying the Holder inequality, (6.1) and Lemma 2.4, we obtain

$$\begin{aligned}
H_2'(t) &= \frac{2}{3}\int_{\Omega^c} m_1(t)|u_t|^p\psi_1\,dx \\
&\ge \frac{2C^{-(p-1)}}{3(t+R)^{(p-1)(n+2\mu-1)/2}} H_2^p(t).
\end{aligned} \tag{6.2}$$

By solving the ordinary differential inequality (6.2), we derive the lifespan estimate (1.8) in Theorem 1.5. The proof of Theorem 1.5 is finished. □

7 Proofs of Theorems 1.6 and 1.7

7.1 Proof of Theorem 1.6

First, we present a lemma that will be utilized in the proof.

Lemma 7.1 *Let $n \geq 1$. Under the same conditions in Theorem 1.6, for all $t \geq 0$, it holds that*

$$F_2(t) \geq \frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x)\, dx.$$

Proof of Lemma 7.1 When $f(u, u_t) = |u_t|^p + |u|^q$, similar to the proof of Theorem 1.4, we deduce

$$\begin{aligned} &\frac{d}{dt}\left[m(t) \int_{\Omega^c} u_t \psi_1\, dx \right] + \frac{3}{2} m(t) \int_{\Omega^c} u_t \psi_1\, dx \\ &\quad \geq m(0) \int_{\Omega^c} \varepsilon \left(u_1(x) + \frac{1}{2} u_0(x) \right) \varphi_1(x)\, dx + m(t) \int_{\Omega^c} \left(|u_t|^p + |u|^q \right) \psi_1\, dx \\ &\qquad + \int_0^t \int_{\Omega^c} m(s) \left(|u_t|^p + |u|^q \right) \psi_1\, dx\, ds. \end{aligned}$$

We denote

$$\begin{aligned} G_3(t) = {}& m(t) \int_{\Omega^c} u_t \psi_1\, dx - \frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x)\, dx \\ & - \frac{2}{3} \int_0^t \int_{\Omega^c} m(s) |u_t|^p \psi_1\, dx\, ds. \end{aligned}$$

It is worth noting that $G_3(0) = \frac{1}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x)\, dx > 0$. An elementary calculation generates

$$\begin{aligned} \frac{d}{dt} G_3(t) + \frac{3}{2} G_3(t) &\geq \frac{1}{2} m(0) \int_{\Omega^c} \varepsilon u_0(x) \varphi_1(x)\, dx \\ &\quad + m(t) \int_{\Omega^c} \left(\frac{1}{3} |u_t|^p + |u|^q \right) \psi_1\, dx + \int_0^t \int_{\Omega^c} m(s) |u|^q \psi_1\, dx\, ds \\ &\geq 0. \end{aligned}$$

It follows that $G_3(t) \geq e^{-\frac{3}{2}t} G_3(0) > 0$, which leads to

$$\begin{aligned} m(t) \int_{\Omega^c} u_t \psi_1 \geq {}& \frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x)\, dx\, dx \\ & + \frac{2}{3} \int_0^t \int_{\Omega^c} m(s) |u_t|^p \psi_1\, dx\, ds. \end{aligned}$$

Thus, we have

$$F_2(t) \geq \frac{2}{3} m(0) \int_{\Omega^c} \varepsilon u_1(x) \varphi_1(x)\, dx.$$

This completes the proof of Lemma 7.1. □

Proof of Theorem 1.6 Taking $\phi(x,t) = \phi_0(x)$ in (2.2) with $f(u,u_t) = |u_t|^p + |u|^q$, we come to

$$F_0''(t) + \frac{\mu}{(1+t)^\beta} F_0'(t) = \int_{\Omega^c} \left(|u_t|^p + |u|^q\right)\phi_0(x)\,dx. \tag{7.1}$$

Multiplying (7.1) by $m(t)$ leads to

$$\left[m(t)F_0'(t)\right]' = m(t)\int_{\Omega^c} \left(|u_t|^p + |u|^q\right)\phi_0(x)\,dx. \tag{7.2}$$

Employing (2.4) gives rise to

$$F_0'(t) \geq m(0)\int_0^t \int_{\Omega^c} \left(|u_t|^p + |u|^q\right)\phi_0(x)\,dx\,dt. \tag{7.3}$$

Utilizing the Holder inequality and Lemma 2.3, we obtain

$$\begin{aligned}\int_{\Omega^c} |u_t|^p\phi_0(x)\,dx &\geq \frac{|\int_{\Omega^c} u_t\psi_1\,dx|^p}{(\int_{\Omega^c\cap\{|x|\leq t+R\}} (\phi_0)^{-\frac{1}{p-1}}(\psi_1)^{\frac{p}{p-1}})^{p-1}} \\ &\geq C^{-(p-1)}(t+R)^{-(n-1)(\frac{p}{2}-1)}\left|F_2(t)\right|^p.\end{aligned} \tag{7.4}$$

It is deduced from (7.3), (7.4), and Lemma 7.1 that

$$\begin{aligned}F_0'(t) &\geq m(0)\int_0^t \int_{\Omega^c} |u_t|^p\phi_0(x)\,dx\,dt. \\ &\geq C_{22}m(0)\varepsilon^p \int_0^t (t+R)^{-(n-1)(\frac{p}{2}-1)}\,dt,\end{aligned} \tag{7.5}$$

where $C_{22} = C^{-(p-1)}(\frac{2}{3}m(0)\int_{\Omega^c} u_1(x)\varphi_1(x)\,dx)^p > 0$. It follows that

$$F_0(t) > C_{23}\varepsilon^p(t+R)^{-(n-1)\frac{p}{2}}t^{n+1}, \tag{7.6}$$

where $C_{23} = \frac{C_{22}m(0)}{n(n+1)}$.

In the case of $n \geq 3$, applying the Holder inequality and Lemma 2.1 yields

$$\begin{aligned}\int_{\Omega^c} |u|^q\phi_0(x)\,dx &\geq \frac{|\int_{\Omega^c} u\phi_0(x)\,dx|^q}{(\int_{\Omega^c\cap\{|x|\leq t+R\}} 1\,dx)^{q-1}} \\ &\geq C_{24}(t+R)^{-n(q-1)}\left|F_0(t)\right|^q.\end{aligned} \tag{7.7}$$

From (7.3) and (7.7), we achieve

$$\begin{aligned}F_0'(t) &\geq m(0)\int_0^t \int_{\Omega^c} |u|^q\phi_0(x)\,dx\,dt \\ &\geq C_{24}m(0)\int_0^t (t+R)^{-n(q-1)}\left|F_0(t)\right|^q\,dt,\end{aligned}$$

which implies

$$F_0(t) \geq C_{24} m(0) \int_0^t \int_0^s (r+R)^{-n(q-1)} \left|F_0(r)\right|^q \, dr \, ds. \tag{7.8}$$

In the case of $n = 2$, we acquire

$$\begin{aligned} \int_{\Omega^c} |u|^q \phi_0(x)\, dx &\geq \frac{|\int_{\Omega^c} u\phi_0(x)\, dx|^q}{(\int_{\{|x|\leq t+R\}\setminus B_1(0)} \ln|x|\, dx)^{q-1}} \\ &\geq C_{25}(t+R)^{-2(q-1)} \left|F_0(t)\right|^q, \end{aligned}$$

where $C_{25} = C^{-(q-1)}(\ln(T+R))^{-(q-1)} > 0$. In the case of $n = 1$, we derive

$$\begin{aligned} \int_{\Omega^c} |u|^q \phi_0(x)\, dx &\geq \frac{|\int_{\Omega^c} u\phi_0(x)\, dx|^q}{(\int_1^{t+R} C_2 x\, dx)^{q-1}} \\ &\geq C_{26}(t+R)^{-2(q-1)} \left|F_0(t)\right|^q. \end{aligned}$$

It turns out that

$$F_0(t) \geq C_{27} m(0) \int_0^t \int_0^s (r+R)^{-2(q-1)} \left|F_0(r)\right|^q \, dr \, ds. \tag{7.9}$$

Suppose that

$$F_0(t) \geq D_j (t+R)^{-a_j} t^{b_j}, \quad t \geq 0, j \in \mathbb{N}^* \tag{7.10}$$

with $D_1 = C_{23}\varepsilon^p$, $a_1 = (n-1)p$, $b_1 = n+1$.

When $n \geq 3$, inserting (7.10) into (7.8), we come to

$$F_0(t) \geq D_{j+1}(t+R)^{-a_{j+1}} t^{b_{j+1}},$$

where $D_{j+1} \geq \frac{C_{24} m(0) D_j^q}{(b_j p+2)^2}$, $a_{j+1} = a_j q + n(q-1)$, $b_{j+1} = b_j q + 2$. Therefore, we conclude

$$\begin{aligned} a_j &= q^{j-1}\big((n-1)p/2 + n\big) - n, \\ b_j &= q^{j-1}\big(n+1+2/(q-1)\big) - 2/(q-1), \\ D_j &\geq \frac{C_{28} D_{j-1}^q}{q^{2(j-1)}}, \end{aligned}$$

where $C_{28} = \frac{C_{24} m(0)}{(n+1+2/(q-1))^2}$. A straightforward calculation gives rise to

$$F_0(t) \geq (t+R)^n t^{-\frac{2}{q-1}} \exp\big(q^{j-1} J(t)\big),$$

where $J(t) \geq \log(t^{1+2/(q-1)-(n-1)p/2} D_1) - C_{29}$, $t \geq R$. As a consequence, we deduce the lifespan estimate

$$T(\varepsilon) \leq C\varepsilon^{-\frac{2p(q-1)}{2q+2-(n-1)p(q-1)}}.$$

When $n = 2$, inserting (7.10) into (7.9) leads to

$$F_0(t) \geq \frac{C_{27} m(0) D_j^q}{(b_j q + 2)^2} (t + R)^{-2(q-1) - a_j q} t^{b_j q + 2}.$$

Similar to the derivation in the case $n \geq 3$, we have $J(t) \geq \log(t^{1+2/(q-1)-p/2} D_1) - C_{30}$. Hence, we arrive at the lifespan estimate

$$T(\varepsilon) \leq C\varepsilon^{-\frac{2p(q-1)}{2q+2-p(q-1)}}.$$

When $n = 1$, inserting (7.10) into (7.9), we acquire

$$F_0(t) \geq \frac{C_{27} m(0) D_j^q}{(b_j q + 2)^2} (t + R)^{-a_j q - 2(q-1)} t^{b_j q + 2}.$$

A simple computation indicates

$$a_j = 2q^{j-1} - 2, \qquad b_j = \left(2 + \frac{2}{q-1}\right) q^{j-1} - \frac{2}{q-1}, \qquad D_j \geq \frac{C_{31} D_{j-1}^q}{q^{2(j-1)}}.$$

It is deduced from (7.10) that $F_0(t) \geq (t + R)^2 t^{-\frac{2}{q-1}} \exp(p^{j-1} J(t))$, where $J(t) \geq \log(D_1 t^{\frac{2}{q-1}}) - C_{32}$ $(t \geq R)$. Therefore, we derive the lifespan estimate

$$T(\varepsilon) \leq C\varepsilon^{-\frac{p(q-1)}{2}}.$$

The proof of Theorem 1.6 is finished. □

7.2 Proof of Theorem 1.7

Similar to the derivation in the proof of Theorem 1.6, we achieve

$$F_0(t) > C_{23} \varepsilon^p t^{n+1-(n-1)\frac{p}{2}}.$$

It holds that $n + 1 - (n-1)\frac{p}{2} < 1$ when $p > \frac{2n}{n-1}$. Taking advantage of (2.4) and (7.2) leads to

$$F_0(t) \geq C_{33} \varepsilon t, \tag{7.11}$$

where $C_{33} = m(0) \int_{\Omega^c} u_1(x) \phi_0(x)\, dx > 0$. Inserting (7.11) into (7.8) gives rise to

$$F_0(t) \geq C_{34} \varepsilon^q (t + R)^{-n(q-1)} t^{q+2},$$

where $C_{34} = \frac{C_{24} C_{33}^q m(0)}{q(q+1)} > 0$. Assume that

$$F_0(t) \geq \tilde{D}_j (t + R)^{-\tilde{a}_j} t^{\tilde{b}_j} \tag{7.12}$$

with $\tilde{D}_1 = C_{34} \varepsilon^q$, $\tilde{a}_1 = n(q-1)$, $\tilde{b}_1 = q + 2$. Combining (7.8) and (7.12), we conclude

$$F_0(t) \geq \frac{C_{24} m(0) \tilde{D}_j^q}{(q \tilde{b}_j + 2)^2} (t + R)^{-q\tilde{a}_j - n(q-1)} t^{q\tilde{b}_j + 2}.$$

It follows that

$$F_0(t) \geq (t+R)^n t^{-\frac{2}{q-1}} \exp\bigl(q^{j-1}\tilde{J}(t)\bigr),$$

where $\tilde{J}(t) \geq \log(t^{q+2+2/(q-1)-nq}\tilde{D}_1) - C_{35}$, $t \geq R$. Thus, we obtain the lifespan estimate

$$T(\varepsilon) \leq C\varepsilon^{-\frac{q-1}{q+1-n(q-1)}}.$$

The proof of Theorem 1.7 is completed.

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