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A new type of solutions for a nonlinear Schrödinger system with $\chi^{(2)}$ nonlinearities

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Abstract

We are concerned with the question of constructing a new type of solution to the problem with $\chi^{(2)}$ nonlinearities

$$\begin{cases} -\Delta u + P(x)u = \alpha uv, & \text{in } \mathbb{R}^N, \\ -\Delta v + Q(x)v = \frac{\alpha}{2}u^2 + \beta v^2, & \text{in } \mathbb{R}^N, \end{cases} \quad (0.1)$$

where $P(x) = P(|x|)$ and $Q(x) = Q(|x|)$ are positive bounded radial potentials, $3 \leq N < 6$, $\alpha > 0$ and $\alpha > \beta$. Assuming that the potentials $P(x)$ and $Q(x)$ satisfy certain conditions, the existence of a new type of solutions is proved.

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Keywords: Lyapunov–Schmidt reduction; $\chi^{(2)}$ nonlinearities; New type of solutions

1 Introduction

This paper deals with the questions of the existence of a new type of solution of the following systems of coupled elliptic equations with $\chi^{(2)}$ nonlinearities

$$\begin{cases} -\Delta u + P(x)u = \alpha uv, & \text{in } \mathbb{R}^N, \\ -\Delta v + Q(x)v = \frac{\alpha}{2}u^2 + \beta v^2, & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $P(x) = P(|x|)$ and $Q(x) = Q(|x|)$ are positive bounded radial potentials, $3 \leq N < 6$, $\alpha > 0$ and $\alpha > \beta$.

The system (1.1) has strongly attracted researchers' attention and has been extensively studied because it arises from nonlinear optical theory. In the nonlinear optical theory, the following cubic nonlinear Schrödinger equation

$$i \frac{\partial \phi}{\partial z} + r \nabla^2 \phi + \chi |\phi|^2 \phi = 0 \quad (1.2)$$

is the basic equation that can be used to describe the formation and propagation of optical solutions in Kerr-type materials [6, 15]. Here, the slowly varying envelope of electric field ϕ represents the relative strength, the real-valued parameter r is concerned with the sign

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of dispersion/diffraction, and χ represents the nonlinearity. z describes the propagation distance coordinate. The Laplacian operator ∇^2 can either be $\frac{\partial^2}{\partial \tau^2}$ for temporal solitons, where τ is the normalized retarded time, or $\nabla^2 = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, where $x = (x_1, \dots, x_N)$ is in the direction orthogonal to z . Solitary wave solutions to (1.2) and its generations have been proved in [2, 14].

As the optical material has a $\chi^{(2)}$ (i.e., quadratic) nonlinear response instead of a conventional $\chi^{(3)}$ material for which the problem (1.2) is based on (see [3, 4]), the nonlinear optical effects such as Second Harmonic Generation were discovered. As is known, the $\chi^{(3)}$ nonlinear Schrödinger system is well studied and has been explored by many authors in recent years, one can refer to [1, 7, 9, 10, 12, 13, 18] and the references therein. For the $\chi^{(2)}$ nonlinear Schrödinger system, in the case $\alpha = 1$, $\beta = 0$, $N = 1$, Zhao, Zhao and Shi in [22] proved the existence of a ground-state solution for (1.1). Very recently, using the finitely dimensional reduction method, Wang and Zhou in [17] studied the existence of infinitely many nonradial positive synchronized solutions of the system (1.1) under radial potentials satisfying some algebraic decay. Using the same method in [19], Yang and Zhou proved the existence of a single peak solution for (1.1). For more results, we refer the readers to [16, 20, 21] and the references therein.

In this paper, we construct a new type of solutions for the $\chi^{(2)}$ nonlinear Schrödinger system (1.1). The new type of solutions that were first introduced by Duan and Musso recently in [8] have polygonal symmetry in the (x_1, x_2) -plane, even symmetry in the x_3 direction, and radial symmetry in other variables. Due to the $\chi^{(2)}$ nonlinearity that appears in our paper, we need to improve some estimates and give precise computing techniques. To the best of our knowledge, there is no result on such a question in the current the literature.

First, we will give some notations.

Let

$$\bar{y}_j = r \left(\sqrt{1-h^2} \cos \frac{2(j-1)\pi}{m}, \sqrt{1-h^2} \sin \frac{2(j-1)\pi}{m}, h, \mathbf{0} \right), \quad j = 1, \dots, m$$

and

$$\underline{y}_j = r \left(\sqrt{1-h^2} \cos \frac{2(j-1)\pi}{m}, \sqrt{1-h^2} \sin \frac{2(j-1)\pi}{m}, -h, \mathbf{0} \right), \quad j = 1, \dots, m,$$

where k is an integer and $\mathbf{0}$ is the zero vector in $\mathbb{R}^{N-3}$, $r \in [r_1 m \ln m, r_2 m \ln m]$ for some $r_2 > r_1 > 0$, $h \in [h_1 \frac{1}{m}, h_2 \frac{1}{m}]$ for some $h_2 > h_1 > 0$.

It is well known that the following equation

$$\begin{cases} -\Delta u + u = u^2, & u > 0, & \text{in } \mathbb{R}^N, \\ u(0) = \max u(x), & u \in H^1(\mathbb{R}^N), & \text{in } \mathbb{R}^N, \end{cases}$$

has a unique positive ground state W that satisfies:

$$W(x) = W(|x|), \quad \lim_{|x| \rightarrow \infty} |x|^{\frac{N-1}{2}} e^{|x|} W(x) = \lambda > 0, \quad \lim_{|x| \rightarrow \infty} \frac{W'}{W} = -1.$$

By direct computation, we know that $(U, V) = (\mu W, \gamma W)$ solves the following limit system for (1.1):

$$\begin{cases} -\Delta u + u = \alpha uv, & \text{in } \mathbb{R}^N, \\ -\Delta v + v = \frac{\alpha}{2} u^2 + \beta v^2, & \text{in } \mathbb{R}^N, \end{cases} \quad (1.3)$$

where $\mu = \frac{1}{\alpha} \sqrt{\frac{2(\alpha-\beta)}{\alpha}}$, $\gamma = \frac{1}{\alpha}$.

Remark 1.1 If (U, V) is a solution of the system (1.3), so is $(-U, V)$.

Remark 1.2 From the proposition 2.2 in [17], (U, V) is nondegenerate for system (1.3), which is important in the proof of our result.

For any function $P(x) > 0$, we define the norm of $H_P^1(\mathbb{R}^N)$ as follows

$$\|u\|_P = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + P(x) u^2 \right)^{\frac{1}{2}},$$

with the inner product

$$\langle u, v \rangle_P = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + P(x) uv).$$

We define the product space $H_P^1(\mathbb{R}^N) \times H_Q^1(\mathbb{R}^N)$ that is denoted by H with the norm

$$\|(u, v)\|^2 = \|u\|_P^2 + \|v\|_Q^2.$$

Set

$$\Omega_j := \left\{ x = (x_1, x_2, x_3, x'') \in \mathbb{R}^3 \times \mathbb{R}^{N-3} : \left\langle \frac{(x_1, x_2)}{|(x_1, x_2)|}, \left(\cos \frac{2(j-1)\pi}{m}, \sin \frac{2(j-1)\pi}{m} \right) \right\rangle_{\mathbb{R}^2} \geq \cos \frac{\pi}{m} \right\},$$

where $j = 1, 2, \dots, m$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ denote the dot product in $\mathbb{R}^2$.

We see that $\mathbb{R}^N$ can be divided into k parts: $\Omega_1, \dots, \Omega_k$. For Ω_j , we divided it into two parts:

$$\Omega_j^+ = \{x : x = (x_1, x_2, x_3, x'') \in \Omega_j, x_3 \geq 0\},$$

$$\Omega_j^- = \{x : x = (x_1, x_2, x_3, x'') \in \Omega_j, x_3 < 0\}.$$

We know the interior of $\Omega_j \cap \Omega_i$, $\Omega_j^+ \cap \Omega_j^-$ are empty sets for $i \neq j$.

Define

$$H_{P,s} = \left\{ u : u \in H_P^1(\mathbb{R}^N), u \text{ is even in } x_l, l = 2, 4, 5, \dots, N, \right. \\ \left. u(\sqrt{x_1^2 + x_2^2} \cos \theta, \sqrt{x_1^2 + x_2^2} \sin \theta, x_3, x'') \right.$$

$$= u \left(\sqrt{x_1^2 + x_2^2} \cos \left(\theta + \frac{2j\pi}{m} \right), \sqrt{x_1^2 + x_2^2} \sin \left(\theta + \frac{2j\pi}{m} \right), x_3, x'' \right), \quad (1.4)$$

where $\theta = \arctan \frac{x_2}{x_1}$.

We define $H_{Q,s}$ similarly.

Denote

$$U_{r,h}(x) = \sum_{j=1}^m U_{\bar{y}_j}(x) + \sum_{j=1}^m U_{\underline{y}_j}(x), \quad V_{r,h}(x) = \sum_{j=1}^m V_{\bar{y}_j}(x) + \sum_{j=1}^m V_{\underline{y}_j}(x),$$

where

$$\begin{aligned} U_{\bar{y}_j}(x) &= U(x - \bar{y}_j), & U_{\underline{y}_j}(x) &= U(x - \underline{y}_j), \\ V_{\bar{y}_j}(x) &= V(x - \bar{y}_j), & V_{\underline{y}_j}(x) &= V(x - \underline{y}_j). \end{aligned}$$

In what follows, we make some assumptions:

(P) There exist constants $a > 0$, $s > 1$ and $\theta > 0$, such that as $r \rightarrow +\infty$,

$$P = 1 + \frac{a}{r^s} + O\left(\frac{1}{r^{s+\theta}}\right).$$

(Q) There exist constants $b > 0$, $t > 1$ and $\varepsilon > 0$, such that as $r \rightarrow +\infty$,

$$Q = 1 + \frac{b}{r^t} + O\left(\frac{1}{r^{t+\varepsilon}}\right).$$

In this paper, we always assume

$$\begin{aligned} (r, h) \in \Lambda_m =: & \left[\left(\frac{\min\{s, t\}}{\pi} - \beta \right) m \ln m, \left(\frac{\min\{s, t\}}{\pi} + \beta \right) m \ln m \right] \\ & \times \left[\left(\frac{\pi(\min\{s, t\} + 2)}{\min\{s, t\}} - \alpha \right) \frac{1}{m}, \left(\frac{\pi(\min\{s, t\} + 2)}{\min\{s, t\}} + \alpha \right) \frac{1}{m} \right], \end{aligned} \quad (1.5)$$

for some $\alpha, \beta > 0$ small.

Our main result can be stated as follows:

Theorem 1.3 *Suppose that $P(x)$ and $Q(x)$ satisfy (P) and (Q), $3 \leq N < 6$, $\alpha > 0$ and $\alpha > \beta$, the parameters (r, h) satisfy (1.5). Then, there is an integer m_0 , such that for any integer $m \geq m_0$, (1.1) has a solution (u_m, v_m) of the form*

$$(u_m, v_m) = (U_{r_m, h_m}(x) + \varphi_m, V_{r_m, h_m}(x) + \psi_m),$$

where $(\varphi_m, \psi_m) \in H_P^1(\mathbb{R}^N) \times H_Q^1(\mathbb{R}^N)$, $(r_m, h_m) \in \Lambda_m$ and as $m \rightarrow +\infty$, $\|(\varphi_m, \psi_m)\| \rightarrow 0$.

Remark 1.4 The results in [22] show that $N = 6$ is a critical dimension, that is to say, the positive solutions do not exist when $N \geq 6$. At the same time, by the structure of the new type of solution in our paper, the dimension N must be greater than or equal to 3.

Remark 1.5 The solutions of the system (1.1) has a large number of bumps near infinity, which causes the energy to become very large.

2 Finite-dimensional reduction

Define

$$\begin{aligned}\mathbb{E} = & \left\{ (u, v) \in H_P^1(\mathbb{R}^N) \times H_Q^1(\mathbb{R}^N), \left\langle \left(\frac{\partial U_{\bar{x}_j}}{\partial r}, \frac{\partial V_{\bar{x}_j}}{\partial r} \right), (u, v) \right\rangle = 0, \right. \\ & \left\langle \left(\frac{\partial U_{\bar{x}_j}}{\partial h}, \frac{\partial V_{\bar{x}_j}}{\partial h} \right), (u, v) \right\rangle = 0, \left\langle \left(\frac{\partial U_{x_j}}{\partial h}, \frac{\partial V_{x_j}}{\partial h} \right), (u, v) \right\rangle = 0, \\ & \left. \text{and } \left\langle \left(\frac{\partial U_{x_j}}{\partial r}, \frac{\partial V_{x_j}}{\partial r} \right), (u, v) \right\rangle = 0, j = 1, \dots, m \right\}.\end{aligned}\quad (2.1)$$

Set

$$\begin{aligned}I(u, v) = & \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + P(x)u^2 + |\nabla v|^2 + Q(x)v^2) \\ & - \frac{\alpha}{2} \int_{\mathbb{R}^N} u^2 v - \frac{\beta}{3} \int_{\mathbb{R}^N} v^3, \quad (u, v) \in H\end{aligned}\quad (2.2)$$

and

$$K(\varphi, \psi) = I(U_{r,h} + \varphi, V_{r,h} + \psi), \quad (\varphi, \psi) \in \mathbb{E}.$$

It is standard to see that $I \in C^2(H, \mathbb{R})$.

We expand the functional $K(\varphi, \psi)$ as follows

$$K(\varphi, \psi) = K(0, 0) + l(\varphi, \psi) + \frac{1}{2}L(\varphi, \psi) + R(\varphi, \psi), \quad (\varphi, \psi) \in \mathbb{E},$$

where

$$\begin{aligned}l(\varphi, \psi) = & \int_{\mathbb{R}^N} \nabla U_{r,h} \cdot \nabla \varphi + \int_{\mathbb{R}^N} P(x)U_{r,h}\varphi + \int_{\mathbb{R}^N} \nabla V_{r,h} \cdot \nabla \psi + \int_{\mathbb{R}^N} Q(x)V_{r,h}\psi \\ & - \frac{\alpha}{2} \int_{\mathbb{R}^N} U_{r,h}^2 \psi - \alpha \int_{\mathbb{R}^N} U_{r,h} V_{r,h} \varphi - \beta \int_{\mathbb{R}^N} V_{r,h}^2 \psi,\end{aligned}\quad (2.3)$$

$$\begin{aligned}L(\varphi, \psi) = & \int_{\mathbb{R}^N} |\nabla \varphi|^2 + \int_{\mathbb{R}^N} P(x)\varphi^2 + \int_{\mathbb{R}^N} |\nabla \psi|^2 + \int_{\mathbb{R}^N} Q(x)\psi^2 \\ & - 2\alpha \int_{\mathbb{R}^N} U_{r,h}\varphi\psi - \alpha \int_{\mathbb{R}^N} \varphi^2 V_{r,h} - 2\beta \int_{\mathbb{R}^N} V_{r,h}\psi^2\end{aligned}\quad (2.4)$$

and

$$\begin{aligned}R(\varphi, \psi) = & -\frac{\alpha}{2} \int_{\mathbb{R}^N} [(U_{r,h} + \varphi)^2 (V_{r,h} + \psi) - U_{r,h}^2 V_{r,h} - U_{r,h}^2 \psi - 2U_{r,h} V_{r,h} \varphi \\ & - 2U_{r,h} \varphi \psi - V_{r,h} \varphi^2] - \frac{\beta}{3} \int_{\mathbb{R}^N} [(V_{r,h} + \psi)^3 \\ & - V_{r,h}^3 - 3V_{r,h}^2 \psi - 3V_{r,h} \psi^2].\end{aligned}\quad (2.5)$$

Here, L is a linear operator from $\mathbb{E}$ to $\mathbb{E}$, that satisfies

$$\begin{aligned} \langle L(u, v), (\varphi, \psi) \rangle &= \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \varphi + P(x)u\varphi - \alpha V_{r,h}u\varphi) \\ &\quad + \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \psi + Q(x)v\psi - 2\beta V_{r,h}v\psi) \\ &\quad - \alpha \int_{\mathbb{R}^N} U_{r,h}v\varphi - \alpha \int_{\mathbb{R}^N} U_{r,h}u\psi, \quad (u, v), (\varphi, \psi) \in \mathbb{E}. \end{aligned}$$

$l(\varphi, \psi)$ is a bounded linear operator defined on $\mathbb{E}$, thus there exists $l_m \in \mathbb{E}$ such that $l(\varphi, \psi) = \langle l_m, (\varphi, \psi) \rangle$.

We have the following important results that have been proved in [8].

Lemma 2.1 (Lemma 3.1, [8]) *For r, h as parameters in Λ_m and any $\eta \in (0, 1]$, there is a constant $C > 0$, such that*

$$\begin{aligned} \sum_{j=2}^m W_{\bar{y}_j}(x) &\leq C e^{-\eta \sqrt{1-h^2} r \frac{\pi}{m}} e^{-(1-\eta)|x-\bar{y}_1|}, \quad \forall x \in \Omega_1^+, \\ \sum_{j=2}^m W_{y_j}(x) &\leq C e^{-\eta \sqrt{1-h^2} r \frac{\pi}{m}} e^{-(1-\eta)|x-\bar{y}_1|}, \quad \forall x \in \Omega_1^+ \end{aligned}$$

and

$$W_{y_1}(x) \leq C e^{-\eta h r} e^{-(1-\eta)|x-\bar{y}_1|}, \quad \forall x \in \Omega_1^+.$$

By the same argument as that of Lemma 2.2 in [8], we can prove:

Lemma 2.2 *There exists a constant $C > 0$, independent of m , such that for any $(r, h) \in \Lambda_m$,*

$$\|L(u, v)\| \geq C \|(u, v)\|, \quad (u, v) \in \mathbb{E}.$$

The following important Proposition can be found in [17], we only need to use the contraction theorem to prove it. Meanwhile, the new type of solution is not weighing on the proof of this Proposition. Hence, we omit it for conciseness.

Proposition 2.3 *There exists an integer $m_0 > 0$, such that for any $m \geq m_0$, there is a unique C^1 map: $(\varphi, \psi) \in \mathbb{E} \rightarrow (\varphi(r, h), \psi(r, h)) \in H_P \times H_Q$ and*

$$\left\langle \frac{\partial K(\varphi, \psi)}{\partial(\varphi, \psi)}, (g, h) \right\rangle = 0, \quad \forall (g, h) \in \mathbb{E}.$$

Moreover, there is a positive constant C , such that

$$\|(\varphi, \psi)\| \leq C m \left(\frac{1}{r^s} + \frac{1}{r^t} + m^{\frac{1}{2}} e^{-(2-\tau) \frac{\pi \sqrt{1-h^2}}{m} r} + m^{\frac{1}{2}} e^{-(2-\tau) r h} \right), \quad (2.6)$$

where $\tau > 0$ is small enough.

Next, we will give the estimate for l_m .

Lemma 2.4 *There exist constants $C > 0$ independent of m and $\tau > 0$ sufficiently small such that*

$$\|l_m\| \leq Cm \left(\frac{1}{r^s} + \frac{1}{r^t} + m^{\frac{1}{2}} e^{-(2-\tau)\frac{\pi\sqrt{1-h^2}}{m}r} + m^{\frac{1}{2}} e^{-(2-\tau)rh} \right)$$

provided $m \geq m_0$ for some integer $m_0 > 0$.

Proof Recall that

$$\begin{aligned} l(\varphi, \psi) &= \int_{\mathbb{R}^N} \nabla U_{r,h} \cdot \nabla \varphi + \int_{\mathbb{R}^N} P(x) U_{r,h} \varphi + \int_{\mathbb{R}^N} \nabla V_{r,h} \cdot \nabla \psi + \int_{\mathbb{R}^N} Q(x) V_{r,h} \psi \\ &\quad - \frac{\alpha}{2} \int_{\mathbb{R}^N} U_{r,h}^2 \psi - \alpha \int_{\mathbb{R}^N} U_{r,h} V_{r,h} \varphi - \beta \int_{\mathbb{R}^N} V_{r,h}^2 \psi \\ &= \alpha \sum_{j=1}^m \int_{\mathbb{R}^N} U_{y_j} V_{y_j} \varphi + \alpha \sum_{j=1}^m \int_{\mathbb{R}^N} U_{\bar{y}_j} V_{\bar{y}_j} \varphi + \sum_{j=1}^m \int_{\mathbb{R}^N} (P(x) - 1) U_{\bar{y}_j} \varphi \\ &\quad + \sum_{j=1}^m \int_{\mathbb{R}^N} (P(x) - 1) U_{y_j} \varphi + \frac{\alpha}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} U_{\bar{y}_j}^2 \psi + \beta \sum_{j=1}^m \int_{\mathbb{R}^N} V_{\bar{y}_j}^2 \psi \\ &\quad + \frac{\alpha}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} U_{y_j}^2 \psi + \beta \sum_{j=1}^m \int_{\mathbb{R}^N} V_{y_j}^2 \psi + \sum_{j=1}^m \int_{\mathbb{R}^N} (Q(x) - 1) V_{\bar{y}_j} \psi \\ &\quad + \sum_{j=1}^m \int_{\mathbb{R}^N} (Q(x) - 1) V_{y_j} \psi - \frac{\alpha}{2} \int_{\mathbb{R}^N} U_{r,h}^2 \psi - \alpha \int_{\mathbb{R}^N} U_{r,h} V_{r,h} \varphi - \beta \int_{\mathbb{R}^N} V_{r,h}^2 \psi \\ &= \sum_{j=1}^m \int_{\mathbb{R}^N} (P(x) - 1) U_{\bar{y}_j} \varphi + \sum_{j=1}^m \int_{\mathbb{R}^N} (P(x) - 1) U_{y_j} \varphi + \sum_{j=1}^m \int_{\mathbb{R}^N} (Q(x) - 1) V_{\bar{y}_j} \psi \\ &\quad + \sum_{j=1}^m \int_{\mathbb{R}^N} (Q(x) - 1) V_{y_j} \psi + \alpha \int_{\mathbb{R}^N} \left(\sum_{j=1}^m V_{y_j} U_{y_j} + \sum_{j=1}^m U_{\bar{y}_j} V_{\bar{y}_j} - U_{r,h} V_{r,h} \right) \varphi \\ &\quad + \beta \int_{\mathbb{R}^N} \left(\sum_{j=1}^m V_{\bar{y}_j}^2 + \sum_{j=1}^m V_{y_j}^2 - V_{r,h}^2 \right) \psi \\ &\quad + \frac{\alpha}{2} \int_{\mathbb{R}^N} \left(\sum_{j=1}^m U_{\bar{y}_j}^2 + \sum_{j=1}^m U_{y_j}^2 - U_{r,h}^2 \right) \psi. \end{aligned} \quad (2.7)$$

By symmetry, we have

$$\begin{aligned} &\sum_{j=1}^m \int_{\mathbb{R}^N} (P(x) - 1) U_{\bar{y}_j} \varphi + \sum_{j=1}^m \int_{\mathbb{R}^N} (P(x) - 1) U_{y_j} \varphi \\ &= 2m \int_{\mathbb{R}^N} (P(x) - 1) U_{y_1} \varphi \end{aligned}$$

$$\begin{aligned}
&\leq 2m \left(\int_{\mathbb{R}^N} (P(x + \bar{y}_1) - 1)^2 U^2 \right)^{\frac{1}{2}} \|\varphi\| \\
&\leq Cm \frac{1}{r^s} \|\varphi\|.
\end{aligned} \tag{2.8}$$

Similarly, we can obtain

$$\sum_{j=1}^m \int_{\mathbb{R}^N} (Q(x) - 1) U_{y_j} \psi + \sum_{j=1}^m \int_{\mathbb{R}^N} (Q(x) - 1) U_{y_j} \psi \leq Cm \frac{1}{r^t} \|\psi\|.$$

By using the estimates (3.14) and (3.15) in [8], we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \left(\sum_{j=1}^m V_{\bar{y}_j}^2 + \sum_{j=1}^m V_{y_j}^2 - V_{r,h}^2 \right) \psi \right| \\
&\leq C \left[\int_{\mathbb{R}^N} \left(\sum_{i \neq j} V_{\bar{y}_i} V_{\bar{y}_j} + \sum_{i \neq j} V_{y_i} V_{y_j} + \sum_{1 \leq i, j \leq m} V_{\bar{y}_i} V_{y_j} \right)^2 \right]^{\frac{1}{2}} \|\psi\| \\
&\leq C \left(\sum_{i \neq j} e^{-(2-\tau)|\bar{y}_i - \bar{y}_j|} \right)^{\frac{1}{2}} \|\psi\| + C \left(\sum_{i \neq j} e^{-(2-\tau)|y_i - y_j|} \right)^{\frac{1}{2}} \|\psi\| \\
&\quad + C \left(\sum_{i \neq j} e^{-(2-\tau)|\bar{y}_i - y_j|} \right)^{\frac{1}{2}} \|\psi\| + C \left(\sum_{i=1}^m e^{-(2-\tau)|y_i - \bar{y}_i|} \right)^{\frac{1}{2}} \|\psi\| \\
&= C \left(m \sum_{i=2}^m e^{-(2-\tau)|\bar{y}_1 - \bar{y}_i|} \right)^{\frac{1}{2}} \|\psi\| + C \left(m \sum_{i=2}^m e^{-(2-\tau)|y_1 - y_i|} \right)^{\frac{1}{2}} \|\psi\| \\
&\quad + C \left(m \sum_{i=2}^m e^{-(2-\tau)|\bar{y}_1 - y_i|} \right)^{\frac{1}{2}} \|\psi\| + C \left(m e^{-(2-\tau)|y_1 - \bar{y}_1|} \right)^{\frac{1}{2}} \|\psi\| \\
&\leq Cm^{\frac{1}{2}} e^{-(2-\tau)\frac{\pi\sqrt{1-h^2}}{m}r} \|\psi\| + Cm^{\frac{1}{2}} e^{-(2-\tau)(1+\delta)\frac{\pi\sqrt{1-h^2}}{m}r} \|\psi\| \\
&\quad + Cm^{\frac{1}{2}} e^{-(2-\tau)rh} \|\psi\| + Cm^{\frac{1}{2}} e^{-(2-\tau)(1+\delta)rh} \|\psi\| \\
&\leq C \left(m^{\frac{1}{2}} e^{-(2-\tau)\frac{\pi\sqrt{1-h^2}}{m}r} + m^{\frac{1}{2}} e^{-(2-\tau)rh} \right) \|\psi\|,
\end{aligned} \tag{2.9}$$

where $\tau > 0$ is sufficiently small.

Similarly, we have

$$\left| \int_{\mathbb{R}^N} \left(\sum_{j=1}^m U_{\bar{y}_j}^2 + \sum_{j=1}^m U_{y_j}^2 - U_{r,h}^2 \right) \psi \right| \leq C \left(m^{\frac{1}{2}} e^{-(2-\tau)\frac{\pi\sqrt{1-h^2}}{m}r} + m^{\frac{1}{2}} e^{-(2-\tau)rh} \right) \|\psi\|,$$

where $\tau > 0$ is sufficiently small.

Since $U = \frac{\mu}{\gamma} V$, we can estimate

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \left(\sum_{j=1}^m V_{y_j} U_{y_j} + \sum_{j=1}^m V_{\bar{y}_j} U_{\bar{y}_j} - U_{r,h} V_{r,h} \right) \varphi \right| \\ &= \frac{\mu}{\gamma} \left| \int_{\mathbb{R}^N} \left[\sum_{j=1}^m (V_{\bar{y}_j}^2 + V_{y_j}^2) - V_{r,h}^2 \right] \varphi \right| \\ &\leq C \left(m^{\frac{1}{2}} e^{-(2-\tau)\frac{\pi\sqrt{1-h^2}}{m}r} + m^{\frac{1}{2}} e^{-(2-\tau)rh} \right) \|\varphi\|. \end{aligned} \quad (2.10)$$

Hence,

$$|l(\varphi, \psi)| \leq Cm \left(\frac{1}{r^s} + \frac{1}{r^t} + m^{\frac{1}{2}} e^{-(2-\tau)\frac{\pi\sqrt{1-h^2}}{m}r} + m^{\frac{1}{2}} e^{-(2-\tau)rh} \right) \|(\varphi, \psi)\|. \quad \square$$

Lemma 2.5 *There holds,*

$$\begin{aligned} & I(U_{r,h}, V_{r,h}) \\ &= m \left(A + \frac{aB}{r^s} + \frac{bC}{r^t} \right) - 2m\gamma \left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2 \right) B_1 e^{-2\pi\sqrt{1-h^2}\frac{r}{m}} \\ &\quad - m\gamma \left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2 \right) B_1 e^{-2rh} + mO(e^{-2(1+\delta)rh}) + mO(e^{-2\pi(1+\delta)\sqrt{1-h^2}\frac{r}{m}}) \\ &\quad + mO\left(\frac{1}{r^{s+\theta}} + \frac{1}{r^{t+\varepsilon}}\right), \end{aligned}$$

where $A = \int_{\mathbb{R}^N} (|\nabla U|^2 + |\nabla V|^2 + U^2 + V^2 - \frac{2\beta}{3}V^3 - \alpha U^2 V)$, $B = \int_{\mathbb{R}^N} U^2$, $C = \int_{\mathbb{R}^N} V^2$ and $B_1 > 0$ is defined in [8], $\delta > 0$ is sufficiently small.

Proof Here, we calculate $I(U_{r,h}, V_{r,h})$:

$$\begin{aligned} & I(U_{r,h}, V_{r,h}) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla U_{r,h}|^2 + P(x)U_{r,h}^2 + |\nabla V_{r,h}|^2 + Q(x)V_{r,h}^2) - \frac{\alpha}{2} \int_{\mathbb{R}^N} U_{r,h}^2 V_{r,h} - \frac{\beta}{3} \int_{\mathbb{R}^N} V_{r,h}^3 \\ &= \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} |\nabla U_{\bar{y}_j}|^2 + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} \nabla U_{\bar{y}_i} \cdot \nabla U_{\bar{y}_j} + \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} |\nabla U_{y_j}|^2 \\ &\quad + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} \nabla U_{y_i} \cdot \nabla U_{y_j} + \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} \nabla U_{\bar{y}_i} \cdot \nabla U_{y_j} + \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} |\nabla V_{\bar{y}_j}|^2 \\ &\quad + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} \nabla V_{\bar{y}_i} \cdot \nabla V_{\bar{y}_j} + \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} |\nabla V_{y_j}|^2 + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} \nabla V_{y_i} \cdot \nabla V_{y_j} \\ &\quad + \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} \nabla V_{\bar{y}_i} \cdot \nabla V_{y_j} - \frac{\alpha}{2} \int_{\mathbb{R}^N} U_{r,h}^2 V_{r,h} - \frac{\beta}{3} \int_{\mathbb{R}^N} V_{r,h}^3 + \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} P(x)U_{\bar{y}_j}^2 \\ &\quad + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} P(x)U_{\bar{y}_i} U_{\bar{y}_j} + \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} P(x)U_{y_j}^2 + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} P(x)U_{y_i} U_{y_j} \end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} P(x) U_{\bar{y}_i} U_{y_j} + \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} Q(x) V_{\bar{y}_j}^2 + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} Q(x) V_{\bar{y}_i} V_{\bar{y}_j} \\
& + \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} Q(x) V_{y_j}^2 + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} Q(x) V_{y_i} V_{y_j} + \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} Q(x) V_{\bar{y}_i} V_{y_j} \\
& = m \int_{\mathbb{R}^N} |\nabla U|^2 + m \int_{\mathbb{R}^N} |\nabla V|^2 + m \int_{\mathbb{R}^N} U^2 + m \int_{\mathbb{R}^N} V^2 - \frac{2\beta}{3} m \int_{\mathbb{R}^N} V^3 \\
& - \alpha m \int_{\mathbb{R}^N} U^2 V + \frac{1}{2} \int_{\mathbb{R}^N} (P(x) - 1) U_{r,h}^2 + \frac{1}{2} \int_{\mathbb{R}^N} (Q(x) - 1) V_{r,h}^2 \\
& - \frac{\alpha}{2} \int_{\mathbb{R}^N} U_{r,h}^2 V_{r,h} - \frac{\beta}{3} \int_{\mathbb{R}^N} V_{r,h}^3 + \frac{\beta}{3} \sum_{j=1}^m \int_{\mathbb{R}^N} V_{\bar{y}_j}^3 + \frac{\beta}{3} \sum_{j=1}^m \int_{\mathbb{R}^N} V_{y_j}^3 \\
& + \frac{\alpha}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} U_{\bar{y}_i} V_{\bar{y}_i} U_{\bar{y}_j} + \frac{\alpha}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} U_{y_i} V_{y_i} U_{y_j} + \frac{\alpha}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} U_{\bar{y}_j}^2 V_{\bar{y}_j} \\
& + \frac{\alpha}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} U_{y_j}^2 V_{y_j} + \frac{\alpha}{4} \sum_{i \neq j} \int_{\mathbb{R}^N} U_{\bar{y}_i}^2 V_{\bar{y}_j} + \frac{\beta}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} V_{\bar{y}_i}^2 V_{\bar{y}_j} \\
& + \frac{\alpha}{4} \sum_{i \neq j} \int_{\mathbb{R}^N} U_{y_i}^2 V_{y_j} + \frac{\beta}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} V_{y_i}^2 V_{y_j} + \frac{\alpha}{2} \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} U_{\bar{y}_i} V_{\bar{y}_i} U_{y_j} \\
& + \frac{\alpha}{4} \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} U_{y_i}^2 V_{\bar{y}_j} + \frac{\alpha}{2} \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} U_{y_i} V_{y_i} U_{\bar{y}_j} + \frac{\alpha}{4} \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} U_{\bar{y}_i}^2 V_{y_j} \\
& + \frac{\beta}{2} \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} V_{\bar{y}_i}^2 V_{y_j} + \frac{\beta}{2} \sum_{1 \leq i, j \leq m} \int_{\mathbb{R}^N} V_{y_i}^2 V_{\bar{y}_j}. \tag{2.11}
\end{aligned}$$

By symmetry and Lemma 3.1 in [8], we have

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^N} (P(x) - 1) U_{r,h}^2 \\
& = m \int_{\Omega_1^+} (P(x) - 1) \left(U_{\bar{y}_1} + U_{y_1} + \sum_{j=2}^m U_{\bar{y}_j} + \sum_{j=2}^m U_{y_j} \right)^2 \\
& = m \int_{\Omega_1^+} (P(x) - 1) \left(U_{\bar{y}_1} + O\left(e^{-\frac{1}{2}hr} e^{-\frac{1}{2}|x-\bar{y}_1|} + e^{-\frac{1}{2}\sqrt{1-h^2}r\frac{\pi}{m}} e^{-\frac{1}{2}|x-\bar{y}_1|}\right) \right)^2 \\
& = m \int_{\Omega_1^+} (P(x) - 1) U_{\bar{y}_1}^2 + mO\left(\frac{1}{r^{s+\theta}}\right) \\
& = m \frac{a \int_{\mathbb{R}^N} U^2}{r^s} + mO\left(\frac{1}{r^{s+\theta}}\right). \tag{2.12}
\end{aligned}$$

Analogously,

$$\frac{1}{2} \int_{\mathbb{R}^N} (Q(x) - 1) V_{r,h}^2 = m \frac{b \int_{\mathbb{R}^N} V^2}{r^t} + mO\left(\frac{1}{r^{t+\varepsilon}}\right). \tag{2.13}$$

Using symmetry, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^N} \left(V_{r,h}^3 - \sum_{j=1}^m V_{\bar{y}_j}^3 - \sum_{j=1}^m V_{y_j}^3 - \frac{3}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} V_{\bar{y}_i}^2 V_{\bar{y}_j} - \frac{3}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} V_{y_i}^2 V_{y_j} \right. \\
& \quad \left. - \frac{3}{2} \sum_{1 \leq i, j \leq m} V_{\bar{y}_i}^2 V_{y_j} - \frac{3}{2} \sum_{1 \leq i, j \leq m} V_{y_i}^2 V_{\bar{y}_j} \right) \\
&= 2m \int_{\Omega_1^+} \left(V_{r,h}^3 - \sum_{j=1}^m V_{\bar{y}_j}^3 - \sum_{j=1}^m V_{y_j}^3 - \frac{3}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} V_{\bar{y}_i}^2 V_{\bar{y}_j} - \frac{3}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} V_{y_i}^2 V_{y_j} \right. \\
& \quad \left. - \frac{3}{2} \sum_{1 \leq i, j \leq m} V_{\bar{y}_i}^2 V_{y_j} - \frac{3}{2} \sum_{1 \leq i, j \leq m} V_{y_i}^2 V_{\bar{y}_j} \right) \\
&= 2m \int_{\Omega_1^+} \left(\left(V_{\bar{y}_1} + \sum_{j=2}^m V_{\bar{y}_j} + V_{y_1} + \sum_{j=2}^m V_{y_j} \right)^3 - \left(V_{\bar{y}_1}^3 + \sum_{j=2}^m V_{\bar{y}_j}^3 \right) - \left(V_{y_1}^3 + \sum_{j=2}^m V_{y_j}^3 \right) \right. \\
& \quad - \frac{3}{2} V_{\bar{y}_1}^2 \sum_{j=2}^m V_{\bar{y}_j} - \frac{3}{2} V_{\bar{y}_1} \sum_{j=2}^m V_{\bar{y}_j}^2 - \frac{3}{2} \sum_{i \neq j \geq 2} V_{\bar{y}_i}^2 V_{\bar{y}_j} - \frac{3}{2} V_{y_1}^2 \sum_{j=2}^m V_{y_j} - \frac{3}{2} V_{y_1} \sum_{j=2}^m V_{y_j}^2 \\
& \quad - \frac{3}{2} \sum_{i \neq j \geq 2} V_{y_i}^2 V_{y_j} - \frac{3}{2} V_{\bar{y}_1}^2 \sum_{j=1}^m V_{y_j} - \frac{3}{2} V_{y_1} \sum_{j=2}^m V_{\bar{y}_j}^2 \\
& \quad \left. - \frac{3}{2} \sum_{2 \leq i, j \leq m} V_{\bar{y}_i}^2 V_{y_j} - \frac{3}{2} V_{y_1}^2 \sum_{j=1}^m V_{\bar{y}_j} - \frac{3}{2} V_{\bar{y}_1} \left(\sum_{j=2}^m V_{y_j}^2 \right) - \frac{3}{2} \sum_{2 \leq i, j \leq m} V_{y_i}^2 V_{\bar{y}_j} \right) \\
&= 2m \int_{\Omega_1^+} \left[\frac{3}{2} V_{\bar{y}_1}^2 \sum_{j=2}^m V_{\bar{y}_j} + \frac{3}{2} V_{\bar{y}_1}^2 \left(\sum_{j=1}^m V_{y_j} \right) + \left(\sum_{j=2}^m V_{\bar{y}_j} \right)^3 - \sum_{j=2}^m V_{\bar{y}_j}^3 - \frac{3}{2} \sum_{i \neq j \geq 2} V_{\bar{y}_i}^2 V_{\bar{y}_j} \right. \\
& \quad + \left(\sum_{j=2}^m V_{y_j} \right)^3 - \sum_{j=2}^m V_{y_j}^3 - \frac{3}{2} \sum_{i \neq j \geq 2} V_{y_i}^2 V_{y_j} + 3 V_{\bar{y}_1} \left(\sum_{j=2}^m V_{\bar{y}_j} \right)^2 - \frac{3}{2} V_{\bar{y}_1} \sum_{j=2}^m V_{\bar{y}_j}^2 \\
& \quad + 3 V_{y_1} \left(\sum_{j=2}^m V_{y_j} \right)^2 - \frac{3}{2} V_{y_1} \sum_{j=2}^m V_{y_j}^2 + 3 V_{y_1} \left(\sum_{j=2}^m V_{\bar{y}_j} \right)^2 - \frac{3}{2} V_{y_1} \sum_{j=2}^m V_{\bar{y}_j}^2 \\
& \quad + 3 V_{\bar{y}_1} \left(\sum_{j=2}^m V_{y_j} \right)^2 - \frac{3}{2} V_{\bar{y}_1} \left(\sum_{j=2}^m V_{y_j}^2 \right) + 6 V_{\bar{y}_1} \left(\sum_{j=2}^m V_{\bar{y}_j} \right) V_{y_1} \\
& \quad + 6 V_{\bar{y}_1} \left(\sum_{j=2}^m V_{\bar{y}_j} \right) \left(\sum_{j=2}^m V_{y_j} \right) + \frac{3}{2} V_{\bar{y}_1} V_{y_1}^2 + 6 V_{\bar{y}_1} V_{y_1} \left(\sum_{j=2}^m V_{y_j} \right) \\
& \quad + \frac{3}{2} V_{y_1}^2 \left(\sum_{j=2}^m V_{y_j} \right) + \frac{3}{2} \left(\sum_{j=2}^m V_{\bar{y}_j} \right) V_{y_1}^2 + 6 V_{y_1} \left(\sum_{j=2}^m V_{y_j} \right) \left(\sum_{j=2}^m V_{\bar{y}_j} \right) \\
& \quad + 3 \left(\sum_{j=2}^m V_{\bar{y}_j} \right)^2 \left(\sum_{j=2}^m V_{y_j} \right) + 3 \left(\sum_{j=2}^m V_{\bar{y}_j} \right) \left(\sum_{j=2}^m V_{y_j} \right)^2 \\
& \quad \left. - \frac{3}{2} \sum_{2 \leq i, j \leq m} V_{\bar{y}_i}^2 V_{y_j} - \frac{3}{2} \sum_{2 \leq i, j \leq m} V_{y_i}^2 V_{\bar{y}_j} \right]
\end{aligned}$$

$$\begin{aligned}
&= 2m \int_{\Omega_1^+} \left[\frac{3}{2} V_{\bar{y}_1}^2 \sum_{j=2}^m V_{\bar{y}_j} + \frac{3}{2} V_{\bar{y}_1}^2 \left(\sum_{j=1}^m V_{\bar{y}_j} \right) \right] + mO \left(\int_{\Omega_1^+} V_{\bar{y}_1} \left(\sum_{j=2}^k V_{\bar{y}_j}^2 \right) \right) \\
&\quad + mO \left(\int_{\Omega_1^+} V_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=2}^k V_{\bar{y}_j} \right) \right) + mO \left(\int_{\Omega_1^+} V_{\bar{y}_1} V_{\bar{y}_1}^2 \right). \quad (2.14)
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(u_{r,h}^2 V_{r,h} - \sum_{j=1}^m u_{\bar{y}_j}^2 V_{\bar{y}_j} - \sum_{j=1}^m u_{y_j}^2 V_{y_j} - \sum_{i \neq j} u_{\bar{y}_i} V_{\bar{y}_i} u_{\bar{y}_j} - \sum_{i \neq j} u_{y_i} V_{y_i} u_{y_j} \right. \\
&\quad - \frac{1}{2} \sum_{i \neq j} u_{\bar{y}_i}^2 V_{\bar{y}_j} - \frac{1}{2} \sum_{i \neq j} u_{y_i}^2 V_{y_j} - \sum_{1 \leq i, j \leq m} u_{\bar{y}_i} V_{\bar{y}_i} u_{y_j} - \sum_{1 \leq i, j \leq m} u_{y_i} V_{y_i} u_{\bar{y}_j} \\
&\quad \left. - \frac{1}{2} \sum_{1 \leq i, j \leq m} u_{y_i}^2 V_{\bar{y}_j} - \frac{1}{2} \sum_{1 \leq i, j \leq m} u_{\bar{y}_i}^2 V_{y_j} \right) \\
&= 2m \int_{\Omega_1^+} \left(u_{r,h}^2 V_{r,h} - \sum_{j=1}^m u_{\bar{y}_j}^2 V_{\bar{y}_j} - \sum_{j=1}^m u_{y_j}^2 V_{y_j} - \sum_{i \neq j} u_{\bar{y}_i} V_{\bar{y}_i} u_{\bar{y}_j} - \sum_{i \neq j} u_{y_i} V_{y_i} u_{y_j} \right. \\
&\quad - \frac{1}{2} \sum_{i \neq j} u_{\bar{y}_i}^2 V_{\bar{y}_j} - \frac{1}{2} \sum_{i \neq j} u_{y_i}^2 V_{y_j} - \sum_{1 \leq i, j \leq m} u_{\bar{y}_i} V_{\bar{y}_i} u_{y_j} - \sum_{1 \leq i, j \leq m} u_{y_i} V_{y_i} u_{\bar{y}_j} \\
&\quad \left. - \frac{1}{2} \sum_{1 \leq i, j \leq m} u_{y_i}^2 V_{\bar{y}_j} - \frac{1}{2} \sum_{1 \leq i, j \leq m} u_{\bar{y}_i}^2 V_{y_j} \right) \\
&= 2m \int_{\Omega_1^+} \left(\frac{1}{2} u_{\bar{y}_1}^2 \left(\sum_{j=2}^m V_{\bar{y}_j} \right) + \frac{1}{2} u_{\bar{y}_1}^2 \left(\sum_{j=1}^m V_{y_j} \right) + u_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=2}^m u_{\bar{y}_j} \right) \right. \\
&\quad + u_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=1}^m u_{y_j} \right) + \left(\sum_{j=2}^m u_{\bar{y}_j} + u_{y_1} + \sum_{j=2}^m u_{y_j} \right)^2 \left(\sum_{j=1}^m V_{\bar{y}_j} + \sum_{j=1}^m V_{y_j} \right) \\
&\quad + 2u_{\bar{y}_1} \left(\sum_{j=2}^m u_{\bar{y}_j} + \sum_{j=1}^m u_{y_j} \right) \left(\sum_{j=2}^m V_{\bar{y}_j} + \sum_{j=1}^m V_{y_j} \right) - u_{\bar{y}_1} \left(\sum_{j=2}^m u_{\bar{y}_j} V_{\bar{y}_j} \right) \\
&\quad - \sum_{i \neq j \geq 2} u_{\bar{y}_i} V_{\bar{y}_i} u_{\bar{y}_j} - \sum_{j=1}^m u_{y_j}^2 V_{y_j} - \sum_{j=2}^m u_{\bar{y}_j}^2 V_{\bar{y}_j} - \sum_{1 \leq i, j \leq m} u_{y_i} V_{y_i} u_{y_j} - \sum_{1 \leq i, j \leq m} u_{y_i} V_{y_i} u_{\bar{y}_j} \\
&\quad - \frac{1}{2} V_{\bar{y}_1} \left(\sum_{j=2}^m u_{\bar{y}_j}^2 \right) - \frac{1}{2} \sum_{i \neq j \geq 2} u_{\bar{y}_i}^2 V_{\bar{y}_j} - \frac{1}{2} \sum_{1 \leq i, j \leq m} u_{y_i}^2 V_{y_j} - u_{y_1} \left(\sum_{j=2}^m u_{\bar{y}_j} V_{\bar{y}_j} \right) \\
&\quad \left. - \sum_{2 \leq i, j \leq m} u_{\bar{y}_i} V_{\bar{y}_i} u_{y_j} - \frac{1}{2} V_{\bar{y}_1} \left(\sum_{j=2}^m u_{\bar{y}_j}^2 \right) - \frac{1}{2} \sum_{2 \leq i, j \leq m} u_{\bar{y}_i}^2 V_{y_j} - \frac{1}{2} \sum_{1 \leq i, j \leq m} u_{y_i}^2 V_{\bar{y}_j} \right) \\
&= 2m \int_{\Omega_1^+} \left(\frac{1}{2} u_{\bar{y}_1}^2 \left(\sum_{j=2}^m V_{\bar{y}_j} \right) + \frac{1}{2} u_{\bar{y}_1}^2 \left(\sum_{j=1}^m V_{y_j} \right) + u_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=2}^m u_{\bar{y}_j} \right) \right. \\
&\quad \left. + u_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=1}^m u_{y_j} \right) \right) + mO \left(\int_{\Omega_1^+} u_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=2}^m u_{\bar{y}_j} \right) \right)
\end{aligned}$$

$$+ mO\left(\int_{\Omega_1^+} U_{\bar{y}_1} V_{\underline{y}_1} \left(\sum_{j=2}^m V_{\bar{y}_j}\right)\right) + mO\left(\int_{\Omega_1^+} U_{\underline{y}_1}^2 V_{\bar{y}_1} + U_{\underline{y}_1} V_{\underline{y}_1} U_{\bar{y}_1}\right). \quad (2.15)$$

Inserting (2.12)–(2.15) into (2.11) and by Lemma 3.2 in [8], we have

$$\begin{aligned} & I(U_{r,h}, V_{r,h}) \\ &= m \int_{\mathbb{R}^N} \left(|\nabla U|^2 + |\nabla V|^2 + U^2 + V^2 - \frac{2\beta}{3} V^3 - \alpha U^2 V \right) + \frac{ma \int_{\mathbb{R}^N} U^2}{r^s} \\ & \quad + \frac{mb \int_{\mathbb{R}^N} V^2}{r^t} - \beta m \int_{\Omega_1^+} \left(V_{\bar{y}_1}^2 \left(\sum_{j=2}^m V_{\bar{y}_j} \right) + V_{\bar{y}_1}^2 \left(\sum_{j=1}^m V_{\underline{y}_j} \right) \right) \\ & \quad - \alpha m \int_{\Omega_1^+} \left(\frac{1}{2} U_{\bar{y}_1}^2 \left(\sum_{j=2}^m V_{\bar{y}_j} \right) + \frac{1}{2} U_{\bar{y}_1}^2 \left(\sum_{j=1}^m V_{\underline{y}_j} \right) \right) \\ & \quad + U_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=2}^m U_{\bar{y}_j} \right) + U_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=1}^m U_{\underline{y}_j} \right) + mO\left(\int_{\Omega_1^+} U_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=2}^m U_{\bar{y}_j}\right)\right) \\ & \quad + mO\left(\int_{\Omega_1^+} U_{\bar{y}_1} V_{\underline{y}_1} \left(\sum_{j=2}^m V_{\bar{y}_j}\right)\right) + mO\left(\int_{\Omega_1^+} U_{\underline{y}_1}^2 V_{\bar{y}_1} + U_{\underline{y}_1} V_{\underline{y}_1} U_{\bar{y}_1}\right) \\ & \quad + mO\left(\int_{\Omega_1^+} V_{\bar{y}_1} \left(\sum_{j=2}^k V_{\bar{y}_j}^2\right)\right) + mO\left(\int_{\Omega_1^+} V_{\bar{y}_1} V_{\underline{y}_1} \left(\sum_{j=2}^k V_{\bar{y}_j}\right)\right) \\ & \quad + mO\left(\int_{\Omega_1^+} V_{\bar{y}_1} V_{\underline{y}_1}^2\right) + mO\left(\frac{1}{r^{s+\theta}} + \frac{1}{r^{t+\varepsilon}}\right) \\ &= m\left(A + \frac{aB}{r^s} + \frac{bC}{r^t}\right) - m\gamma\left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2\right) \int_{\mathbb{R}^N} \left(W_{\bar{y}_1}^2 \left(\sum_{j=2}^m W_{\bar{y}_j} \right) + W_{\bar{y}_1}^2 \left(\sum_{j=1}^m W_{\underline{y}_j} \right) \right) \\ & \quad + mO\left(\int_{\Omega_1^+} U_{\bar{y}_1} V_{\bar{y}_1} \left(\sum_{j=2}^m U_{\bar{y}_j}\right)\right) + mO\left(\int_{\Omega_1^+} U_{\bar{y}_1} V_{\underline{y}_1} \left(\sum_{j=2}^m V_{\bar{y}_j}\right)\right) \\ & \quad + mO\left(\int_{\Omega_1^+} U_{\underline{y}_1}^2 V_{\bar{y}_1} + U_{\underline{y}_1} V_{\underline{y}_1} U_{\bar{y}_1}\right) + mO\left(\int_{\Omega_1^+} V_{\bar{y}_1} \left(\sum_{j=2}^k V_{\bar{y}_j}^2\right)\right) \\ & \quad + mO\left(\int_{\Omega_1^+} V_{\bar{y}_1} V_{\underline{y}_1} \left(\sum_{j=2}^k V_{\bar{y}_j}\right)\right) + mO\left(\int_{\Omega_1^+} V_{\bar{y}_1} V_{\underline{y}_1}^2\right) + mO\left(\frac{1}{r^{s+\theta}} + \frac{1}{r^{t+\varepsilon}}\right) \\ &= m\left(A + \frac{aB}{r^s} + \frac{bC}{r^t}\right) - m\gamma\left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2\right) \left(\sum_{j=2}^m B_1 e^{-|\bar{y}_1 - \bar{y}_j|} \right) \\ & \quad - m\gamma\left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2\right) \left(\sum_{j=1}^m B_1 e^{-|\bar{y}_1 - \underline{y}_j|} \right) \\ & \quad + mO\left(\sum_{j=2}^m e^{-(1+\delta)|\bar{y}_1 - \bar{y}_j|}\right) + mO\left(\sum_{j=1}^m e^{-(1+\delta)|\bar{y}_1 - \underline{y}_j|}\right) + mO\left(\frac{1}{r^{s+\theta}} + \frac{1}{r^{t+\varepsilon}}\right) \\ &= m\left(A + \frac{aB}{r^s} + \frac{bC}{r^t}\right) - 2m\gamma\left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2\right) B_1 e^{-2\pi\sqrt{1-h^2}\frac{r}{m}} \end{aligned}$$

$$\begin{aligned}
& -m\gamma\left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2\right)B_1e^{-2rh} + mO(e^{-2(1+\delta)rh}) \\
& + mO(e^{-2\pi(1+\delta)\sqrt{1-h^2}\frac{r}{m}}) + mO\left(\frac{1}{r^{s+\theta}} + \frac{1}{r^{t+\varepsilon}}\right),
\end{aligned} \quad (2.16)$$

where $A = \int_{\mathbb{R}^N} (|\nabla U|^2 + |\nabla V|^2 + U^2 + V^2 - \frac{2\beta}{3}V^3 - \alpha U^2 V)$, $B = \int_{\mathbb{R}^N} U^2$, $C = \int_{\mathbb{R}^N} V^2$ and $B_1 > 0$ is defined in [8], $\delta > 0$ is sufficiently small. $\square$

Proof of Theorem 1.3 Let $(\varphi, \psi) = (\varphi(r, h), \psi(r, h))$ be the map obtained in Proposition 2.3. Define

$$F(r, h) = I(U_{r,h} + \varphi, V_{r,h} + \psi), \quad \forall (r, h) \in \Lambda_m.$$

With a similar argument as used in [5, 11], we can prove that for m large, if (r, h) is a critical point of $F(r, h)$, then $(U_{r,h} + \varphi, V_{r,h} + \psi)$ is a critical point of I . Next, we will prove that the function $F(r, h)$ has a critical point that is an interior point of Λ_m . It follows from Proposition 2.3, and Lemmas 2.4 and 2.5 that

$$\begin{aligned}
F(r, h) &= I(U_{r,h}, V_{r,h}) + \ell(\varphi, \psi) + \frac{1}{2}L(\varphi, \psi) + R(\varphi, \psi) \\
&= I(U_{r,h}, V_{r,h}) + O(\|\ell_k\| \|(\varphi, \psi)\| + \|(\varphi, \psi)\|^2 + \|(\varphi, \psi)\|^3) \\
&= m\left(A + \frac{aB}{r^s} + \frac{bC}{r^t}\right) - 2m\gamma\left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2\right)B_1e^{-2\pi\sqrt{1-h^2}\frac{r}{m}} \\
&\quad - m\gamma\left(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2\right)B_1e^{-2rh} + mO(e^{-2(1+\delta)rh}) \\
&\quad + mO(e^{-2\pi(1+\delta)\sqrt{1-h^2}\frac{r}{m}}) + mO\left(\frac{1}{r^{s+\theta}} + \frac{1}{r^{t+\varepsilon}}\right) \\
&= m\left(A + \frac{aB}{r^s} + \frac{bC}{r^t} - De^{-2\pi\sqrt{1-h^2}\frac{r}{m}} - Ee^{-2rh}\right) \\
&\quad + mO(e^{-2(1+\delta)rh}) + mO(e^{-2\pi(1+\delta)\sqrt{1-h^2}\frac{r}{m}}) + mO\left(\frac{1}{r^{s+\theta}} + \frac{1}{r^{t+\varepsilon}}\right),
\end{aligned}$$

where $D = 2\gamma(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2)B_1$, $E = \gamma(\beta\gamma^2 + \frac{3}{2}\alpha\mu^2)B_1$.

We only prove the Theorem 1.3 for the case $s < t$, since the other case is similar.

If $s < t$, then

$$\begin{aligned}
F(r, h) &= m\left(A + \frac{aB}{r^s} - De^{-2\pi\sqrt{1-h^2}\frac{r}{m}} - Ee^{-2rh}\right) \\
&\quad + mO\left(e^{-2(1+\delta)rh} + e^{-2\pi(1+\delta)\sqrt{1-h^2}\frac{r}{m}} + \frac{1}{r^{s+\theta}}\right).
\end{aligned}$$

In [8], we know that the same function $F(r, h)$ has a maximum point (r_m, h_m) , which is an interior point of Λ_m . $\square$

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Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The idea of this research was introduced by Weiming Liu. All authors contributed to the main results. All authors read and approved the final manuscript.

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