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Nonnegative solution of a class of double phase problems with logarithmic nonlinearity

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Abstract
This manuscript proves the existence of a nonnegative, nontrivial solution to a class of double-phase problems involving potential functions and logarithmic nonlinearity in the setting of Sobolev space on complete manifolds. Some applications are also being investigated. The arguments are based on the Nehari manifold and some variational techniques.

Keywords: Double-phase problem; Existence of solutions; Sobolev space on Riemannian manifold; Variational method

1 Introduction

The goal of the present work is to prove the existence of nonnegative, nontrivial solutions to the following double-phase problems:

$$\begin{cases} -\operatorname{div}(|\nabla u(z)|^{p-2}\nabla u(z) + \mu(z)|\nabla u(z)|^{q-2}\nabla u(z)) + V(z)|u(z)|^{p-2}u(z) & \\ \quad = \lambda a(z)|u(z)|^{r-2}u(z)\log(|u(z)|) & \text{in } \mathcal{E}, \\ u = 0 & \text{on } \partial\mathcal{E}, \end{cases} \tag{1}$$

where $\mathcal{E} \subset \mathcal{M}$ is an open bounded set with a smooth boundary $\partial\mathcal{E}$, $(\mathcal{M}, \mathfrak{g})$ is a smooth complete compact Riemannian N-manifold, $\lambda > 0$ is a parameter specified later, and the functions $a(\cdot)$ and $V(\cdot)$ satisfy the following assumptions:

(H_1) The function $a \in C_b^+(\mathcal{M}) \cap L^1(\mathcal{M})$.

(H_2) The positive continuous function $V : \mathcal{M} \to \mathbb{R}$ and $V \in L^{p'}(\mathcal{M})$, with $p' = \frac{p}{p-1}$ and $1 < r < p < q < p^* =: \frac{Np}{N-p}$.

(H_3) $\mu \in L^1_+(\mathcal{M})$ and $\min_{z\in\mathcal{M}} \mu(z) = \mu_0 > 0$.

Double-phase differential operators have been attracting the attention of several researchers in the last years due to their applicability in several areas of science, especially in physical processes. For example, in the elasticity theory, Zhikov [36] has shown that the modulation coefficient $\mu(\cdot)$ determines the geometry of composites formed from two different materials with distinct curing exponents q and p. See also the work of Benkhira et al. [10]. For quantum physics, we refer to Benci et al. [9], and for reaction–diffusion systems, we refer to the pertinent work of Cherfils–Il'yasov [14].

We start our motivation by briefly going over the previous work. On the one hand, we were inspired by the work of Zhikov [35], who has introduced and investigated functionals with integrands that change ellipticity as a function of a point to give models of strongly anisotropic materials. As a kind of prototype, he took the function

$$u \mapsto \int_\Omega \left(|\nabla u|^p + \mu(x)|\nabla u|^q\right) dx,$$

where $1 < p < q$, and the weight $\mu \in L^\infty(\Omega)$. After that, several studies were done in this direction, we mention the famous works of Baroni et al. [7, 8], Colombo et al. [15, 16], Marcellini [28], Bahrouni et al. [6], Liu et al. [25], Farkas et al. [19], and Papageorgiou et al. [29, 30]. For more results, readers may refer to [1, 13, 17, 22, 23, 26, 27, 31, 33] for ideas and techniques developed to prove the existence of two nontrivial positive solutions to double-phase problems. On the other hand, the second motivation is the work of Aubin [5] who studied the qualitative properties of the Lebesgue space on Riemannian manifolds. This work was followed by Hebey [24] who developed this space and proved some new embedding results.

A nice overview of the recent work on such equations with variable exponents can be found in [32] by Ragusa et al., [34] by Shi et al., [20, 21] by Gaczkowski, et al., and [2–4] by Aberqi et al. in the context of Sobolev spaces on complete manifolds. We refer to Benslimane et al. [11, 12] for more results.

Concerning regularity results for this kind of problem, we recommend the relevant work of De Filippis and Mingione [18], which gives optimal regularity criteria for different types of nonuniform ellipticity.

The contributions to the paper are as follows. We prove a new embedding result for Sobolev space on complete manifolds. We also show the existence of nonnegative, nontrivial solution to the problem (1), which is a combined potential with vanishing behavior at infinity and a logarithmic nonlinearity, as an application using the Nehari manifold and some variational techniques. The idea behind this approach is as follows: Let $J \in C(\mathcal{X}, \mathbb{R})$ be an energy functional, with $\mathcal{X}$ being a real Banach space, so, if $u \neq 0$ is a critical point of J, then u is included in the following set:

$$\mathcal{N} = \left\{u \in \mathcal{X}\backslash\{0\} : \left\langle J'(u), u\right\rangle = 0\right\},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathcal{X}$ and its dual space $\mathcal{X}^*$. Hence, $\mathcal{N}$ is a suitable constraint for funding nontrivial critical points of J. While $\mathcal{N}$ is not a manifold in general, it is called the Nehari manifold. Thus, we search for nontrivial minimizers of the functional J in a subset of the entire space that contains the nontrivial critical points of J, namely $\mathcal{N}$. Here, we have treated the minimum of the energy functional of the type

$$w \mapsto \int_\Omega \left[G\left(x, |\nabla w|\right) - f \cdot w\right] dx,$$

with

$$G(x, t) = t^p + a(x)t^q.$$

When looking forward to treating it for a general functional of type,

$$w \mapsto \int_\Omega \bigl[G_1\bigl(x,|\nabla w|\bigr) - G_2\bigl(x,|\nabla w|\bigr) - f \cdot w\bigr]\,dx,$$

where G_i, $i = 1,2$, are two functions satisfying

$$p < \frac{G_i'(x,t)\cdot t}{G_i(x,t)} < q \quad \text{for every fixed } x.$$

To the best of our knowledge, this is the first paper that treats this kind of problem in the context of Sobolev spaces on Riemannian manifolds. This work will be of great interest to researchers working in this area.

The primary outcome of our paper can be presented as follows.

Theorem 1 *Let $(\mathcal{M},\mathfrak{g})$ satisfy the $B_{\mathrm{vol}}(\alpha,v)$ property. If assumptions (H_1)–(H_3) hold, then there exists a positive constant λ_* such that, if $0 < \lambda < \lambda_*$, problem (1) has at least one nontrivial solution.*

The paper consists of two sections. Section 2 contains some background on Sobolev spaces on Riemannian manifolds, as well as the proof of a new embedding result. Section 3 shows the existence of a nonnegative, nontrivial solution to a class of double-phase problems involving a potential that is allowed to have vanishing behavior at infinity and logarithmic nonlinearity.

2 Preliminaries

In what follows, we give some definitions and properties of Sobolev spaces on complete manifolds, which we will use to prove our main results. For more details, see [5, 24] and the references given therein.

2.1 Sobolev spaces on complete manifolds

Definition 1 ([5]) Let $(\mathcal{M},\mathfrak{g})$ be a smooth, compact Riemannian N-manifold. For an integer K, and a smooth $u : \mathcal{M} \to \mathbb{R}$, we denote by $\nabla^K u$ the Kth covariant derivative of u and by $|\nabla^K u|$ the norm of $\nabla^K u$ defined in a local chart by

$$\bigl|\nabla^K u\bigr| = \mathfrak{g}^{i_1 j_1}\cdots\mathfrak{g}^{i_K j_K}\bigl(\nabla^K u\bigr)_{i_1\cdots i_K}\bigl(\nabla^K u\bigr)_{j_1\cdots j_K}.$$

Since $(\nabla u)_i = \partial_i u$,

$$\bigl(\nabla^2 u\bigr)_{ij} = \partial_{ij} u - \Gamma_{ij}^K \partial_K u,$$

where Γ_{ij}^K is the Christoffel symbol defined as follows:

$$\Gamma_{ij}^K(z) = \frac{1}{2}\left(\left(\frac{\partial \mathfrak{g}_{mj}}{\partial z_i}\right)_z + \left(\frac{\partial \mathfrak{g}_{mi}}{\partial z_j}\right)_z - \left(\frac{\partial \mathfrak{g}_{ij}}{\partial z_m}\right)_z\right)\cdot \mathfrak{g}(z)^{mk},$$

with the g^{ij}'s representing the inverse matrix of g_{ij} such that $g_{im}g^{mj} = \delta_i^j$.

And for a real number $p \geq 1$ and a positive integer K, we define the Sobolev space as follows:

$$L^p(\mathcal{M}) = \left\{ u : \mathcal{M} \to \mathbb{R} \text{ measurable} \;\Big|\; \int_{\mathcal{M}} |u|^p \, dv_{\mathfrak{g}}(z) < \infty \right\}.$$

And the function space is

$$\mathcal{C}_K^p(\mathcal{M}) = \left\{ u \in C^\infty \;\Big|\; \forall j = 0, \dots, K, \int_{\mathcal{M}} \left|\nabla^j u\right|^p dv_{\mathfrak{g}}(z) < +\infty \right\},$$

where $\mathcal{M}$ is compact. By default, $\mathcal{C}_K^p(\mathcal{M}) = C^\infty(\mathcal{M})$ for every k and every $p \geq 1$.

Definition 2 $(\mathcal{M}, \mathfrak{g})$ has property $B_{vol}(\alpha, \nu)$ if the Ricci tensor of $\mathfrak{g}$, denoted by $Rc(\mathfrak{g})$, satisfies $Rc(\mathfrak{g}) \geq \alpha(N-1)\mathfrak{g}$, for some α and, for every $z \in \mathcal{M}$, there exists $\nu > 0$ such as $|B_1(z)|_{\mathfrak{g}} \geq \nu$, where $B_1(z)$ are balls of radius 1 centered at some point z in terms of the volume of smaller concentric balls.

Definition 3 ([5]) The Sobolev space $W^{K,p}(\mathcal{M})$ is the completion of $\mathcal{C}_K^p(\mathcal{M})$ with respect to $\| \cdot \|_{W^{K,p}}$, where

$$\|u\|_{W^{K,p}} = \|\nabla u\|_p + \|u\|_p.$$

Proposition 1 ([5]) *Let $\| \cdot \|_p$ be the norm of $L^p(\mathcal{M})$ defined by*

$$\|u\|_p = \left(\int_{\mathcal{M}} |u|^p \, dv_{\mathfrak{g}}(z) \right)^{\frac{1}{p}}.$$

Then:

1. *Any Cauchy sequence in $(\mathcal{C}_K^p(\mathcal{M}), \| \cdot \|_{W^{K,p}})$ is a Cauchy sequence in the Lebesgue space $(L^p(\mathcal{M}), \| \cdot \|_p)$.*
2. *Any Cauchy sequence in $(\mathcal{C}_K^p(\mathcal{M}), \| \cdot \|_{W^{K,p}})$ that converges to 0 in the Lebesgue space $(L^p(\mathcal{M}), \| \cdot \|_p)$ also converges to 0 in $(\mathcal{C}_K^p(\mathcal{M}), \| \cdot \|_{W^{K,p}})$.*

We note that $W^{K,p}(\mathcal{M})$ is a reflexive Banach space, and the set $\mathcal{D}(\mathcal{M})$ of smooth functions with compact support in $\mathcal{M}$ is dense in $W^{1,p}(\mathcal{M})$ for $p \geq 1$; see [5].

Lemma 1 ([5, 24]) *Let $(\mathcal{M}, \mathfrak{g})$ be a smooth, compact Riemannian N-manifold. For every real $q \in [1, N)$ with $\frac{1}{p} \geq \frac{1}{q} - \frac{1}{N}$, we have that $W^{1,q}(\mathcal{M}) \subset L^p(\mathcal{M})$. So there exists a positive constant c such that, for any $u \in \mathcal{D}(\mathbb{R}^N)$,*

$$\|u\|_p \leq c\|u\|_{W^{1,q}}.$$

Remark 1 (See [5, Proposition 2.11]) Suppose that $W^{1,1}(\mathcal{M}) \subset L^{\frac{N}{N-1}}(\mathcal{M})$. Then there exists $c > 0$ such that, for all $u \in W^{1,1}(\mathcal{M})$,

$$\left(\int_{\mathcal{M}} |u|^{\frac{N}{N-1}} \, dv_{\mathfrak{g}}(z) \right)^{\frac{N-1}{N}} \leq c \int_{\mathcal{M}} \big(|\nabla u| + |u| \big) \, dv_{\mathfrak{g}}(z).$$

Proposition 2 ([5, 24])

- *Since $\mathcal{M}$ is compact, $\mathcal{M}$ can be covered by finite numbers of charts $(\mathcal{E}_m, \varphi_m)_{m=1,\dots,N}$ such that, for all m, the components g_{ij}^m of g in $(\mathcal{E}_m, \varphi_m)$ satisfy*

$$\frac{1}{2}\delta_{ij} \leq g_{ij}^m \leq 2\delta_{ij}$$

as bilinear forms.

- *Since $\mathcal{M}$ is assumed to be compact, $(\mathcal{M}, \mathfrak{g})$ has finite volume. Hence for $1 \leq q \leq q'$, we have $L^{q'}(\mathcal{M}) \subset L^q(\mathcal{M})$.*

Lemma 2 ([5]) *Let $(\mathcal{M}, \mathfrak{g})$ be a smooth, compact, N-dimensional Riemannian manifold. Given p, q, two real numbers with $1 \leq q < N$ and $p \geq 1$ such that $\frac{1}{p} > \frac{1}{q} - \frac{1}{N}$, $W^{1,q}(\mathcal{M}) \subset L^p(\mathcal{M})$ is compact.*

The weighted Lebesgue space $L^q_\mu(\mathcal{M})$ is defined as follows:

$$L^q_\mu(\mathcal{M}) = \left\{u : \mathcal{M} \to \mathbb{R} \text{ is measurable such that } \int_{\mathcal{M}} \mu(z)\left|u(z)\right|^q dv_{\mathfrak{g}}(z) < \infty\right\},$$

endowed with the following seminorm:

$$\|u\|_{q,\mu} = \left(\int_{\mathcal{M}} \mu(z)\left|u(z)\right|^q dv_{\mathfrak{g}}(z)\right)^{\frac{1}{q}} < \infty.$$

We define

$$\rho(u) = \int_{\mathcal{M}} |\nabla u|^q \, dv_{\mathfrak{g}}(z)$$

and

$$\varrho(u) = \int_{\mathcal{M}} |\nabla u|^p \, dv_{\mathfrak{g}}(z) + \int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_{\mathfrak{g}}(z).$$

Lemma 3 *Let $u \in W_0^{1,q}(\mathcal{M})$. Then:*

(i) $\|u\| = a \Longleftrightarrow \rho(\frac{u}{a}) = 1$.

(ii) $\|u\| < 1$ (*resp.*, > 1, $= 1$) $\Longleftrightarrow \rho(u) < 1$ (*resp.*, > 1, $= 1$).

(iii) $\|u\| < 1 \Rightarrow \|u\|^q \leq \rho(u) \leq \|u\|^p$ *and* $\|u\| > 1 \Longrightarrow \|u\|^p \leq \rho(u) \leq \|u\|^q$.

(vi) $\|u\| \to 0 \Longleftrightarrow \rho(u) \to 0$ *and* $\|u\| \to \infty \Longleftrightarrow \rho(u) \to \infty$.

Proof We pursue the same steps as in the proof of Proposition 2.1 in [27]. □

Theorem 2 *Let $(\mathcal{M}, \mathfrak{g})$ be a smooth, compact, N-dimensional Riemannian manifold. Given r, q, two real numbers with $1 \leq q < N$ and $r \geq 1$ such that $\frac{1}{r} > \frac{1}{q} - \frac{1}{N}$, $W_0^{1,q}(\mathcal{M}) \hookrightarrow L^r(\mathcal{M})$ is compact.*

Proof We adopt the same technique as when proving [24, Theorem 2.6]. □

Theorem 3 *Let* $(\mathcal{M}, \mathfrak{g})$ *be a smooth, compact, N-dimensional Riemannian manifold. Given p, q, two real numbers with* $1 \le q < N$ *and* $p \ge 1$ *such that* $\frac{1}{p} > \frac{1}{q} - \frac{1}{N}$, $W_0^{1,q}(\mathcal{M}) \hookrightarrow L^p_\mu(\mathcal{M})$.

Proof Let $u \in W_0^{1,q}(\mathcal{M})$. Then using the Poincaré inequality, we obtain

$$\begin{aligned}\frac{1}{c}\int_{\mathcal{M}} \left|u(z)\right|^q \, dv_{\mathfrak{g}}(z) &\le \mu_0 \int_{\mathcal{M}} \left|\nabla u(z)\right|^q \, dv_{\mathfrak{g}}(z) \\ &\le \int_{\mathcal{M}} \left(\left|\nabla u(z)\right|^p + \mu(z)\left|\nabla u(z)\right|^q\right) dv_{\mathfrak{g}}(z) = \varrho(u).\end{aligned}$$

Hence, if $u \neq 0$, we have

$$\frac{1}{c}\int_{\mathcal{M}} \left(\frac{u}{\|u\|_q}\right)^q dv_{\mathfrak{g}}(z) < 1.$$

Then, we get

$$\|u\|_{q,\mu} \le c\|u\|_E. \qquad \square$$

Remark 2 The result of Theorem 3 can be considered as a particular case of Theorem 2.22 in [2].

3 Nehari manifold and fibering maps

In the following, we suppose that $E = W_0^{1,q}(\mathcal{M}) \setminus \{0\}$ is endowed with $\|u\|_E = (\int_{\mathcal{M}} |\nabla u|^q \, dv_{\mathfrak{g}}(z))^{\frac{1}{q}}$.

Definition 4 We say that a function $u \in E$ is a weak solution to problem (1), if

$$\begin{aligned}&\int_{\mathcal{M}} \left(|\nabla u|^{p-2}\nabla u + \mu(z)|\nabla u|^{q-2}\nabla u\right)\nabla\varphi \, dv_{\mathfrak{g}}(z) + \int_{\mathcal{M}} V(z)|u|^{p-2}u\varphi \, dv_{\mathfrak{g}}(z) \\ &\quad = \lambda \int_{\mathcal{M}} a(z)|u|^{r-2}u\log\left(|u|\right)\varphi \, dv_{\mathfrak{g}}(z),\end{aligned}$$

for all $\varphi \in D(\mathcal{M})$.

Consider the functional $J_\lambda : E \to \mathbb{R}$ defined by

$$\begin{aligned}J_\lambda(u) = {}& \frac{1}{p}\int_{\mathcal{M}} |\nabla u|^p \, dv_{\mathfrak{g}}(z) + \frac{1}{q}\int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_{\mathfrak{g}}(z) + \frac{1}{p}\int_{\mathcal{M}} V(z)|u|^p \, dv_{\mathfrak{g}}(z) \\ &- \frac{\lambda}{r}\int_{\mathcal{M}} a(z)|u|^r\log\left(|u|\right) dv_{\mathfrak{g}}(z) + \frac{\lambda}{r^2}\int_{\mathcal{M}} a(z)|u|^r \, dv_{\mathfrak{g}}(z)\end{aligned}$$

for all $u \in E$.

Then J_λ is well defined and belongs to $C^1(E)$. Furthermore, we have

$$\begin{aligned}\left\langle J_\lambda'(u), \varphi\right\rangle = {}& \int_{\mathcal{M}} \left(|\nabla u|^{p-2}\nabla u + \mu(z)|\nabla u|^{q-2}\nabla u\right)\nabla\varphi \, dv_{\mathfrak{g}}(z) \\ &+ \int_{\mathcal{M}} V(z)|u|^{p-2}u\varphi \, dv_{\mathfrak{g}}(z) - \lambda\int_{\mathcal{M}} a(z)|u|^{r-2}u\log\left(|u|\right)\varphi \, dv_{\mathfrak{g}}(z)\end{aligned}$$

for all $u, \varphi \in E$.

Consider the Nehari set defined by

$$\mathcal{N} = \{u \in E : \langle J'_\lambda(u), u\rangle = 0\}.$$

We can deduce that the critical points of J_λ lie on $\mathcal{N}$ and further that $u \in \mathcal{N}$ if and only if u is a weak solution to problem (1). Let us define the maps $\psi_u : \mathbb{R}^+ \to \mathbb{R}$ by $\psi_u(t) = J_\lambda(tu)$ and analyze $\mathcal{N}$ in terms of the stationary points of fibering maps ψ_u.

We have

$$\begin{aligned}\psi'_u(t) &= t^{p-1}\int_{\mathcal{M}} |\nabla u|^p \, dv_g(z) + t^{q-1}\int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_g(z) + t^{p-1}\int_{\mathcal{M}} V(z)|u|^p \, dv_g(z) \\ &\quad - \lambda t^{r-1}\int_{\mathcal{M}} a(z)|u|^r \log(|u|)\, dv_g(z) - \lambda t^{r-1}\log(t)\int_{\mathcal{M}} a(z)|u|^r \, dv_g(z)\end{aligned}$$

and

$$\begin{aligned}\psi''_u(t) &= (p-1)t^{p-2}\int_{\mathcal{M}} |\nabla u|^p \, dv_g(z) + (q-1)t^{q-2}\int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_g(z) \\ &\quad + (p-1)t^{p-2}\int_{\mathcal{M}} V(z)|u|^p \, dv_g(z) - \lambda(r-1)t^{r-2}\int_{\mathcal{M}} a(z)|u|^r \log(|u|)\, dv_g(z) \\ &\quad - \lambda(r-1)t^{r-2}\log(t)\int_{\mathcal{M}} a(z)|u|^r \, dv_g(z) - \lambda t^{r-2}\int_{\mathcal{M}} a(z)|u|^r \, dv_g(z).\end{aligned}$$

It is easy to verify that $tu \in \mathcal{N} \Longleftrightarrow \psi'_u(t) = 0$ for any $u \in E$ and $t > 0$.

We shall split $\mathcal{N}$ into three subsets which correspond to local minima, local maxima, and points of inflection of fibering maps, that is,

$$\begin{aligned}\mathcal{N}^+ &= \{u \in \mathcal{N} : \psi''_u(1) > 0\} \\ &= \left\{u \in E : (q-p)\int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_g(z) + \lambda(p-r)\int_{\mathcal{M}} a(z)|u|^r \log(|u|)\, dv_g(z) \right. \\ &\quad \left. > \lambda\int_{\mathcal{M}} a(z)|u|^r \, dv_g(z)\right\}, \\ \mathcal{N}^0 &= \{u \in \mathcal{N} : \psi''_u(1) = 0\} \\ &= \left\{u \in E : (q-p)\int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_g(z) + \lambda(p-r)\int_{\mathcal{M}} a(z)|u|^r \log(|u|)\, dv_g(z) \right. \\ &\quad \left. = \lambda\int_{\mathcal{M}} a(z)|u|^r \, dv_g(z)\right\}, \\ \mathcal{N}^- &= \{u \in \mathcal{N} : \psi''_u(1) < 0\} \\ &= \left\{u \in E : (q-p)\int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_g(z) + \lambda(p-r)\int_{\mathcal{M}} a(z)|u|^r \log(|u|)\, dv_g(z) \right. \\ &\quad \left. < \lambda\int_{\mathcal{M}} a(z)|u|^r \, dv_g(z)\right\}.\end{aligned}$$

Lemma 4 *Let $u_0 \notin \mathcal{N}^0$. Then u_0 is a critical point of J_λ if u_0 is a local minimizer of J_λ on $\mathcal{N}$.*

Proof We remark that u_0 is a solution to the optimization problem to minimize J_λ subject to $I(u) = 0$, where

$$I(u) = \int_{\mathcal{M}} \big(|\nabla u|^p + \mu(z)|\nabla u|^q\big)\,dv_{\mathrm{g}}(z) + \int_{\mathcal{M}} V(z)|u|^p\,dv_{\mathrm{g}}(z) \\ - \lambda \int_{\mathcal{M}} a(z)|u|^r \log\big(|u|\big)\,dv_{\mathrm{g}}(z),$$

and, since u_0 is a local minimizer of J_λ on $\mathcal{N}$, we have

$$I(u_0) = \big\langle J'_\lambda(u_0), u_0\big\rangle. \tag{2}$$

Then, there exists a Lagrange multiplier $\alpha \in \mathbb{R}$ such that $J'_\lambda(u_0) = \alpha I'(u_0)$, namely $0 = \langle J'_\lambda(u_0), u_0\rangle = \alpha\langle I'(u_0), u_0\rangle$.

Furthermore, $\langle I'(u_0), u_0\rangle \neq 0$ since $u_0 \notin \mathcal{N}^0$ which implies $\alpha = 0$ and, actually, that u_0 is a critical point of J_λ. □

Lemma 5 *There exists a positive constant λ_0 such that, for any $0 < \lambda < \lambda_0$, the functional J_λ is bounded and coercive on $\mathcal{N}$.*

Proof Letting $u \in E$ with $\|u\|_E > 1$, we obtain

$$J_\lambda(u) \geq \frac{1}{q}\bigg(\int_{\mathcal{M}} |\nabla u|^p\,dv_{\mathrm{g}}(z) + \int_{\mathcal{M}} \mu(z)|\nabla u|^q\,dv_{\mathrm{g}}(z) + V_0 \int_{\mathcal{M}} |u|^p\,dv_{\mathrm{g}}(z)\bigg) \\ - \frac{\lambda}{r}\int_{\mathcal{M}} a(z)|u|^r \log\big(|u|\big)\,dv_{\mathrm{g}}(z) + \frac{\lambda}{r^2}\int_{\mathcal{M}} a(z)|u|^r\,dv_{\mathrm{g}}(z),$$

and we know that

$$\log(s) \leq \frac{s^\alpha}{\alpha e} \quad \text{for all } \alpha > 0 \text{ and } s > 0, \tag{3}$$

thus

$$J_\lambda(u) \geq \frac{\mu_0}{q}\|u\|_E^q - \frac{\lambda\|a\|_\infty}{r(q-r)e}\int_{\mathcal{M}} |u|^q\,dv_{\mathrm{g}}(z)$$

with $\alpha = q - r$.

According to Theorem 2 and Poincaré inequality, there exists a positive constant C_q such that

$$J_\lambda(u) \geq \frac{\mu_0}{q}\|u\|_E^q - \frac{\lambda\|a\|_\infty}{r(q-r)e}C_q\|u\|_E^q \geq \bigg(\frac{\mu_0}{q} - \frac{\lambda\|a\|_\infty}{r(q-r)e}C_q\bigg)\|u\|_E^q.$$

Choosing $0 < \lambda < \lambda_0 = \frac{r(q-r)e}{qC_q\|a\|_\infty}$ implies that J_λ is coercive.

Moreover, we have

$$\begin{aligned} J_\lambda(u) &\leq \frac{1}{p}\left(\varrho(u) + \int_{\mathcal{M}} V(z)|u|^p \, dv_{\mathrm{g}}(z) - \lambda \int_{\mathcal{M}} a(z)|u|^r \log\bigl(|u|\bigr) \, dv_{\mathrm{g}}(z)\right) \\ &\quad + \frac{\lambda}{r^2} \int_{\mathcal{M}} a(z)|u|^r \, dv_{\mathrm{g}}(z) \\ &= \frac{\lambda}{r^2} \int_{\mathcal{M}} a(z)|u|^r \, dv_{\mathrm{g}}(z) \leq \frac{\lambda}{r^2} \|a\|_\infty \int_{\mathcal{M}} |u|^r \, dv_{\mathrm{g}}(z). \end{aligned}$$

Thanks to Theorem 2, there exists $C_r > 0$ such that

$$J_\lambda(u) \leq C_r \frac{\lambda}{r^2} \|a\|_\infty \|u\|_E^r. \qquad \square$$

Lemma 6 *Let* $\lambda_1 = \frac{(q-p)\mu_0}{\|a\|_{L^\infty} C_1}$ *where* $C_1(r,q,\mathcal{M})$ *is a constant to be specified later. Then, for any* λ *such that* $0 < \lambda < \lambda_1$, *we have* $\mathcal{N}^0 \cup \mathcal{N}^- = \emptyset$ *and* $\mathcal{N}^+ \neq \emptyset$.

Proof We proceed by contradiction to prove that $\mathcal{N}^0 \cup \mathcal{N}^- \neq \emptyset$.

Indeed, let $u \in \mathcal{N}^0 \cup \mathcal{N}^-$, then we get

$$\begin{aligned} &(q-p) \int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_{\mathrm{g}}(z) + \lambda(p-r) \int_{\mathcal{M}} a(z)|u|^r \log\bigl(|u|\bigr) \, dv_{\mathrm{g}}(z) \\ &\quad \leq \lambda \int_{\mathcal{M}} a(z)|u|^r \, dv_{\mathrm{g}}(z), \end{aligned}$$

thus

$$(q-p) \int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_{\mathrm{g}}(z) \leq \lambda \int_{\mathcal{M}} a(z)|u|^r \, dv_{\mathrm{g}}(z) \leq \lambda \|a\|_{L^\infty} \int_{\mathcal{M}} |u|^r \, dv_{\mathrm{g}}(z).$$

Using the fact that $L^q(\mathcal{M}) \subset L^r(\mathcal{M})$ and Poincaré inequality, there exists a positive constant $C_1(r,q,\mathcal{M})$ such that

$$\int_{\mathcal{M}} |\nabla u|^q \, dv_{\mathrm{g}}(z) \geq C_1 \left(\int_{\mathcal{M}} |u|^r \, dv_{\mathrm{g}}(z)\right)^{\frac{q}{r}},$$

hence

$$\begin{aligned} (q-p)\mu_0 C_1 \left(\int_{\mathcal{M}} |u|^r \, dv_{\mathrm{g}}(z)\right)^{\frac{q}{r}} &\leq (q-p) \int_{\mathcal{M}} \mu(z)|\nabla u|^q \, dv_{\mathrm{g}}(z) \\ &\leq \lambda \|a\|_{L^\infty} \int_{\mathcal{M}} |u|^r \, dv_{\mathrm{g}}(z), \end{aligned}$$

thus

$$\left(\int_{\mathcal{M}} |u|^r \, dv_{\mathrm{g}}(z)\right)^{\frac{q}{r}-1} \leq \lambda \frac{\|a\|_{L^\infty}}{(q-p)\mu_0 C_1},$$

and, when $\lambda \to 0$, we have $u = 0$, which is a contradiction.

Now, according to Lemma 5, the set $\mathcal{N}^+ \neq \emptyset$. $\square$

4 Existence of weak solutions

Lemma 7 *If the sequence $\{u_n\}$ is bounded, hence weakly converges to u in the space $W_0^{1,q}(\mathcal{M})$, then we have*

$$\lim_{n\to+\infty}\int_{\mathcal{M}} a(z)|u_n|^r u_n \log\bigl(|u_n|\bigr)\,dv_g(z) = \int_{\mathcal{M}} a(z)|u|^r u \log\bigl(|u|\bigr)\,dv_g(z). \tag{4}$$

Proof We know that for $\alpha, \beta > 0$, there exists a constant $C(\alpha,\beta)$ such that

$$\log(t) \le C(\alpha,\beta)\bigl(t^{\alpha} + t^{-\beta}\bigr), \quad \text{for every } t > 0.$$

Thus

$$\int_{\mathcal{M}} a(z)|u_n|^r \log\bigl(|u_n|\bigr)\,dv_g(z) \le C(p-r,\delta)\int_{\mathcal{M}} \bigl|a(z)\bigr|\bigl(|u_n|^p + |u_n|^{r-\delta}\bigr)\,dv_g(z),$$

for some $\delta \in (1, r-1)$. Furthermore, since $\{u_n\}$ is bounded, we get $u_n \to u$ a.e. in $\mathcal{M}$, and then

$$a(z)|u_n|^r \log\bigl(|u_n|\bigr) \to a(z)|u|^r \log\bigl(|u|\bigr) \quad \text{a.e. in } \mathcal{M} \text{ as } n \to +\infty.$$

Then, thanks to Lebesgue's theorem, we get the required result. □

Lemma 8 *There exist two positive constants λ_2, λ_3 such that, for any $\lambda \in (0, \min(\lambda_2, \lambda_3))$, we have*

1. $m_\lambda^+ = \inf_{u\in\mathcal{N}^+} J_{\lambda(u)} < 0$,
2. *There exists $u^+ \in \mathcal{N}^+$ such that $J_\lambda(u^+) = m_\lambda^+$.*

Proof 1. Let $u \in \mathcal{N}^+$, then $\psi_u''(1) > 0$, thus

$$\begin{aligned}&\int_{\mathcal{M}} |\nabla u|^p\,dv_g(z) + \int_{\mathcal{M}} \mu(z)|\nabla u|^q\,dv_g(z) + \int_{\mathcal{M}} V(z)|\nabla u|^p\,dv_g(z)\\ &\quad - \lambda\int_{\mathcal{M}} a(z)|u|^r \log\bigl(|u|\bigr)\,dv_g(z) = 0,\end{aligned}$$

and

$$(q-p)\int_{\mathcal{M}} \mu(z)|\nabla u|^q\,dv_g(z) + (p-r)\int_{\mathcal{M}} a(z)|u|^r \log\bigl(|u|\bigr)\,dv_g(z) > \lambda\int_{\mathcal{M}} a(z)|u|^r\,dv_g(z).$$

Combining the definition of $J_\lambda(u)$ with the above, we get

$$\begin{aligned}J_\lambda(u) &\le \frac{1}{p}\biggl(\int_{\mathcal{M}} |\nabla u|^p\,dv_g(z) + \int_{\mathcal{M}} V(z)|\nabla u|^p\,dv_g(z)\biggr)\\ &\quad + \biggl(\frac{1}{q} + \frac{q-p}{r^2}\biggr)\int_{\mathcal{M}} \mu(z)|\nabla u|^q\,dv_g(z)\\ &\quad + \biggl(\frac{p-r}{r^2} - \frac{1}{r}\biggr)\lambda\int_{\mathcal{M}} a(z)|u|^r \log\bigl(|u|\bigr)\,dv_g(z)\\ &= \biggl(\frac{1}{p} - \frac{1}{q} - \frac{q-p}{r^2}\biggr)\biggl(\int_{\mathcal{M}} |\nabla u|^p\,dv_g(z) + \int_{\mathcal{M}} V(z)|\nabla u|^p\,dv_g(z)\biggr)\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{q}+\frac{q-p}{r^2}+\frac{p-r}{r^2}-\frac{1}{r}\right)\lambda\int_{\mathcal{M}} a(z)|u|^r\log\bigl(|u|\bigr)\,dv_{g}(z)\\
&=\frac{(q-p)(r^2-pq)}{pqr^2}\left(\int_{\mathcal{M}}|\nabla u|^p\,dv_{g}(z)+\int_{\mathcal{M}}V(z)|u|^p\,dv_{g}(z)\right)\\
&+\frac{(q-r)^2}{qr^2}\lambda\int_{\mathcal{M}} a(z)|u|^r\log\bigl(|u|\bigr)\,dv_{g}(z).
\end{aligned}
$$

Using (3) and Poincaré inequality, we obtain that there exists a positive constant C_2 such that

$$
J_\lambda(u)\le\frac{(q-p)(r^2-pq)}{pqr^2}C_2\int_{\mathcal{M}}|u|^p\,dv_{g}(z)+\lambda\frac{(q-r)^2}{(p-r)eqr^2}\|a\|_{L^\infty}\int_{\mathcal{M}}|u|^p\,dv_{g}(z).
$$

Taking $\lambda_2=\frac{(q-p)(pq-r^2)}{pqr^2}C_2\cdot\frac{(p-r)eqr^2}{(q-r)^2\|a\|_{L^\infty}}$, we conclude that $m_\lambda^+<0$, since $u\neq 0$.

2. Consider a minimizing sequence $\{u_n\}\subset\mathcal{N}^+$ such that $\lim_{n\to+\infty}J_\lambda(u_n)=\inf_{u\in\mathcal{N}^+}J_\lambda(u)$.

According to Lemma 5, $\{u_n\}$ is bounded in $W_0^q(\mathcal{M})$. Then, up to a subsequence still denoted $\{u_n\}$, there exists $u^+\in W_0^q(\mathcal{M})$ such that

$$
u_n\rightharpoonup u^+\quad\text{weakly in } W_0^q(\mathcal{M}),\tag{5}
$$

and, by the compact embedding,

$$
u_n\to u^+\quad\text{in } L^r(\mathcal{M})\text{ strongly for every } 1<r<p^*,\tag{6}
$$

and

$$
u_n\to u^+\quad\text{a.e. in }\mathcal{M}.\tag{7}
$$

So, $\int_{\mathcal{M}}V(z)|u^+|^p\,dv_{g}(z)=\lim_{n\longrightarrow+\infty}\int_{\mathcal{M}}V(z)|u_n|^p\,dv_{g}(z)$, $\int_{\mathcal{M}}a(z)|u^+|^p\,dv_{g}(z)=\lim_{n\longrightarrow+\infty}\int_{\mathcal{M}}a(z)|u_n|^p\,dv_{g}(z)$, and by Lemma 7, we have

$$
\int_{\mathcal{M}}a(z)\bigl|u^+\bigr|^r\log\bigl(\bigl|u^+\bigr|\bigr)\,dv_{g}(z)=\lim_{n\longrightarrow+\infty}\int_{\mathcal{M}}a(z)|u_n|^r\log\bigl(|u_n|\bigr)\,dv_{g}(z).
$$

Thus, it remains shown that $\varrho(u^+)=\liminf_{n\to+\infty}\varrho(u_n)$.

By contradiction, let $\varrho(u^+)<\liminf_{n\to+\infty}\varrho(u_n)$. Then, since $u_n\in\mathcal{N}^+$, we obtain

$$
\begin{aligned}
J_\lambda(u_n)&=\frac{1}{p}\int_{\mathcal{M}}|\nabla u_n|^p\,dv_{g}(z)+\frac{1}{q}\int_{\mathcal{M}}\mu(z)|\nabla u_n|^q\,dv_{g}(z)+\frac{1}{p}\int_{\mathcal{M}}V(z)|u_n|^p\,dv_{g}(z)\\
&\quad-\frac{\lambda}{r}\int_{\mathcal{M}}a(z)|u_n|^r\log\bigl(|u_n|\bigr)\,dv_{g}(z)+\frac{\lambda}{r^2}\int_{\mathcal{M}}a(z)|u_n|^r\,dv_{g}(z)\\
&\ge\frac{1}{q}\varrho(u_n)-\frac{\lambda}{r}\int_{\mathcal{M}}a(z)|u_n|^r\log\bigl(|u_n|\bigr)\,dv_{g}(z).
\end{aligned}
$$

Passing to the limit as $n\to+\infty$, we get

$$
\liminf_{n\to+\infty}J_\lambda(u_n)>\frac{1}{q}\varrho\bigl(u^+\bigr)-\frac{\lambda}{r}\int_{\mathcal{M}}a(z)\bigl|u^+\bigr|^r\log\bigl(\bigl|u^+\bigr|\bigr)\,dv_{g}(z).
$$

From (3) and Poincaré inequality, there exists a positive constant C_3 such that

$$\liminf_{n\to+\infty} J_\lambda(u_n) > \frac{1}{q}\varrho\big(u^+\big) - \frac{\lambda\|a\|_{L^\infty}C_3}{r(p-r)e}\varrho\big(u^+\big) = \left(\frac{1}{q} - \frac{\lambda\|a\|_{L^\infty}C_3}{r(p-r)e}\right)\varrho\big(u^+\big),$$

and, since $\lambda < \lambda_3 = \frac{r(p-r)e}{q\|a\|_{L^\infty}C_3}$, we obtain $\liminf_{n\to+\infty} J_\lambda(u_n) = m_\lambda^+ > 0$, which is a contradiction.

Then $\varrho(u^+) = \liminf_{n\to+\infty}\varrho(u_n)$, $u^+ \in \mathcal{N}$, and $J_\lambda(u^+) = \liminf_{n\to+\infty} J_\lambda(u_n)$.

Finally, to prove that $u^+ \in \mathcal{N}^+$, it is sufficient to show that

$$(q-p)\int_{\mathcal{M}} \mu(z)\big|\nabla u^+\big|^q + (p-r)\lambda\int_{\mathcal{M}} a(z)\big|u^+\big|^r \log\big(\big|u^+\big|\big)\,dv_{\mathfrak{g}}(z) > \lambda\int_{\mathcal{M}} a(z)\big|u^+\big|^r\,dv_{\mathfrak{g}}(z).$$

Indeed, suppose that

$$(q-p)\int_{\mathcal{M}} \mu(z)\big|\nabla u^+\big|^q + (p-r)\lambda\int_{\mathcal{M}} a(z)\big|u^+\big|^r \log\big(\big|u^+\big|\big)\,dv_{\mathfrak{g}}(z) \le \lambda\int_{\mathcal{M}} a(z)\big|u^+\big|^r\,dv_{\mathfrak{g}}(z),$$

then

$$(q-p)\mu_0\int_{\mathcal{M}} \big|\nabla u^+\big|^q\,dv_{\mathfrak{g}}(z) \le \lambda\int_{\mathcal{M}} a(z)\big|u^+\big|^r\,dv_{\mathfrak{g}}(z),$$

and, in the same way as above, we obtain a contradiction. Thus $u^+ \in \mathcal{N}^+$. □

Proof of Theorem 1 For every $\lambda \in (0, \lambda_* = \min_{i=0,\dots,3}(\lambda_i))$, there exists $u^+ \in \mathcal{N}^+$ such that $J_\lambda(u^+) = \inf_{u\in\mathcal{N}^+} J_\lambda(u)$. In addition, it easy to show that $|u^+| \in \mathcal{N}^+$ and $J_\lambda(|u^+|) = J_\lambda(u^+)$. Hence, our equation (1) admits at least one nonnegative solution $u^+ \in E$. □

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