Qiu and Yan *Boundary Value Problems* (2022) 2022:62
https://doi.org/10.1186/s13661-022-01641-x

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A new existence result for some nonlocal problems involving Orlicz spaces and its applications

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Abstract

This paper studies some quasilinear elliptic nonlocal equations involving Orlicz–Sobolev spaces. On the one hand, a new sub-supersolution theorem is proved via the pseudomonotone operator theory; on the other hand, using the obtained theorem, we present an existence result on the positive solutions of a singular elliptic nonlocal equation. Our work improves the results of some previous researches.

Keywords: Orlicz–Sobolev space; Sub-supersolution; Pseudomonotone Operator Theorem; Luxemburg norm; Φ-Laplace operator

1 Introduction

This paper is concerned with the problem

$$\begin{cases} -\Delta_\Phi u = h_1(u)\|u\|^{\alpha}_{L^\Psi} + h_2(u)\|u\|^{\gamma}_{L^\Lambda}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where α, γ are positive constants, $\|\cdot\|_{L^\Psi}$ (resp. $\|\cdot\|_{L^\Lambda}$) is a norm in $L^\Psi(\Omega)$ (resp. $L^\Lambda(\Omega)$) and the nonlinearities h_1, h_2: $[0,+\infty) \to [0,+\infty)$ are continuous functions, $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is bounded with $\partial\Omega \in C^2$, $\Delta_\Phi u = \operatorname{div}(\rho(|\nabla u|)\nabla u)$, where

$$\Phi(t) := \int_0^{|t|} \rho(s)s\,ds. \tag{1.2}$$

Here $\rho \in C^1 : [0,+\infty) \to [0,+\infty)$ and it satisfies (see [10])

(ρ_1) $t\rho(t)$ is differentiable for $\forall t > 0$,

(ρ_2) $\lim_{t\to 0^+} t\rho(t) = 0$, $\lim_{t\to +\infty} t\rho(t) = +\infty$,

and that there exist $\kappa, s \in (1, N)$ such that

(ρ_3) $\kappa - 1 \leq \frac{(\rho(t)t)'}{\rho(t)} \leq s - 1$, $\forall t > 0$.

Note that (ρ_3) implies that

(ρ_3)′ $\kappa \leq \frac{\rho(t)t^2}{\Phi(t)} \leq s$, $\forall t > 0$.

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Problem (1.1) was proposed in [10] and generalizes some problems in [3, 5, 6, 8, 17–20]. As the authors of [10] pointed out, there are some difficulties to study problem (1.1): (1) variational methods cannot be used directly because of the nonlocal terms; (2) the presence of the concave–convex nonlinearities leads to invalidness of the Galerkin method; (3) there is no ready-made sub-supersolutions method as in [2] and [7] because of the Φ-Laplacian operator. In [10], for the first time, using monotone iterative technique, Figueiredo et al. obtained the sub-supersolution theorem for problem (1.1) in which they needed an important condition that $h_1, h_2\colon [0,+\infty) \to \mathbb{R}$ are nondecreasing. As its application, the authors discussed the following problem:

$$\begin{cases} -\Delta_\Phi u = u^\beta \|u\|^\alpha_{L^\Psi}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

with the assumption that $\alpha, \beta \geq 0$ with $0 < \alpha + \beta < \kappa - 1$, and got the existence of a positive solution.

Another interesting work appeared in [9], in which Dos Santos et al. studied the problem as follows:

$$\begin{cases} -A(x, \|u\|_{L^{r(x)}})\Delta_{p_1(x)} u = h_1(u,x)\|u\|^{\alpha_1(x)}_{L^{q(x)}} + h_2(u,x)\|u\|^{\gamma_1(x)}_{L^{s(x)}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Note that h_1 and h_2 are not nondecreasing in this paper.

Motivated by [10] and [9], we try to present the sub-supersolution approach for problem (1.1) without the assumptions that h_1 and h_2 are nondecreasing.

Our paper is divided into four sections. In Sect. 2, some needed properties of Orlicz spaces and the main results are listed. In Sect. 3, we prove a new sub-supersolution theorem for problem (1.1) via the pseudomonotone operator theory and, using obtained theorem, we present a new existence result on positive solutions of problem (1.3) when $\alpha \geq 0$, $-1 < \beta < 0$, with $0 < \alpha < \kappa - 1$. Our work complements the conclusions in [10] and [9]: (1) we obtain the existence of a nontrivial solution of problem (1.1) when h_1 and h_2 have no monotonicity; (2) problem (1.3) is studied when $\beta \in (-1, 0)$.

2 Preliminaries and main results

Now we shall list some main definitions, properties, and conclusions in the setting of Orlicz–Sobolev spaces. For more information, please refer to the literature [1, 4, 13, 15, 16, 22].

In (1.1), because of the existence of assumption $(\rho_3)'$, it is easily to see that the Δ_2 condition is true for $\Phi(t)$ (see [10]).

Lemma 2.1 *The function* Φ *is nondecreasing on* $[0, +\infty)$.

Proof Obviously, it is enough to prove that for any $0 < \omega_1 < \omega_2$, we always have the result $\Phi(\omega_1) \leq \Phi(\omega_2)$. Since Φ is convex from the definition of an $\boldsymbol{N}$-function, we have

$$\frac{\Phi(\omega_1) - \Phi(0)}{\omega_1 - 0} \leq \frac{\Phi(\omega_2) - \Phi(\omega_1)}{\omega_2 - \omega_1},$$

that is,

$$\frac{\Phi(\omega_1) - 0}{\omega_1 - 0} \leq \frac{\Phi(\omega_2) - \Phi(\omega_1)}{\omega_2 - \omega_1}.$$

Then we have $\Phi(\omega_2) - \Phi(\omega_1) \geq 0$, that is, $\Phi(\omega_2) \geq \Phi(\omega_1)$. Therefore, the function Φ is nondecreasing on $[0, +\infty)$. □

Definition 2.2 If a positive function $\overline{w}^*$ with $\overline{w}^* \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ satisfies

$$\begin{cases} -\Delta_\Phi \overline{w}^* \geq h_1(\overline{w}^*) J_1(\overline{w}^*) + h_2(\overline{w}^*) J_2(\overline{w}^*), & x \in \Omega, \\ \overline{w}^* \geq 0, & x \in \partial\Omega, \end{cases}$$

then $\overline{w}^*(x)$ is called a supersolution of problem (1.1).

If a positive function $\underline{w}_*$ with $\underline{w}_* \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ satisfies

$$\begin{cases} -\Delta_\Phi \underline{w}_* \leq h_1(\underline{w}_*) J_1(\underline{w}_*) + h_2(\underline{w}_*) J_2(\underline{w}_*), & x \in \Omega, \\ \underline{w}_* \leq 0, & x \in \partial\Omega, \end{cases}$$

then $\underline{w}_*(x)$ is called a subsolution of problem (1.1).

For more information on $L^\Phi(\Omega)$ and its norm, please refer to the literature [10]. Let

$$\begin{aligned} &\zeta(u,x) := \max\bigl\{\underline{w}_*(x), \min\bigl\{u, \overline{w}^*(x)\bigr\}\bigr\}, \\ &\nu \in (0,1), \quad \gamma(t,x) := -(\underline{w}_* - t)_+^\nu + \bigl(t - \overline{w}^*\bigr)_+^\nu, \\ &J_1(u) := \bigl\|\zeta(u,x)\bigr\|_{L^\Psi}^\alpha = \inf{}^\alpha \biggl\{\varsigma > 0 : \int_\Omega \Psi\biggl(\frac{|\zeta(u,x)|}{\varsigma}\biggr) \leq 1\biggr\}, \\ &J_2(u) := \bigl\|\zeta(u,x)\bigr\|_{L^\Lambda}^\gamma = \inf{}^\gamma \biggl\{\varsigma > 0 : \int_\Omega \Lambda\biggl(\frac{|\zeta(u,x)|}{\varsigma}\biggr) \leq 1\biggr\}. \end{aligned}$$

In addition, Ψ and Λ are ***N***-functions satisfying the Δ_2 condition, and they are also nondecreasing on $[0, +\infty)$.

For an ***N***-function Φ, the corresponding Orlicz–Sobolev space is defined as the Banach space

$$W^{1,\Phi}(\Omega) := \biggl\{ v \in L^\Phi(\Omega) \biggm| \frac{\partial v}{\partial x_i} \in L^\Phi(\Omega) \text{ for } i = 1, \ldots, N \biggr\}$$

endowed with the norm

$$\|v\|_{1,\Phi} = \|\nabla v\|_{L^\Phi} + \|v\|_{L^\Phi}.$$

Specially,

$$W_0^{1,\Phi}(\Omega) := \biggl\{ v \in L^\Phi(\Omega) \biggm| \frac{\partial v}{\partial x_i} \in L^\Phi(\Omega) \text{ for } i = 1, \ldots, N \text{ and } v = 0, x \in \partial\Omega \biggr\}.$$

For their properties, one can refer to the literature [10].

Lemma 2.3 ([10]) *Let Φ be an **N**-function defined in* (1.2) *and satisfying* (ρ_1), (ρ_2), *and* (ρ_3). *Denote*

$$\xi_0(t) = \min\{t^\kappa, t^s\}$$

and

$$\xi_1(t) = \max\{t^\kappa, t^s\}, \quad t \geq 0,$$

then

$$\xi_0(t)\Phi(\varrho) \leq \Phi(\varrho t) \leq \xi_1(t)\Phi(\varrho), \quad \varrho, t > 0,$$
$$\xi_0(\|u\|_{L^\Phi}) \leq \int_\Omega \Phi(u) \leq \xi_1(\|u\|_{L^\Phi}), \quad u \in L^\Phi(\Omega).$$

Lemma 2.4 ([10]) *Let $\lambda > 0$, let Φ be given by* (1.2), *and suppose $\Omega \subset \mathbb{R}^N$ is an admissible domain. Consider the problem*

$$\begin{cases} -\Delta_\Phi z_\lambda = \lambda, & x \in \Omega, \\ z_\lambda = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

where z_λ is the unique solution. Define

$$\rho_0 = \frac{1}{2|\Omega|^{\frac{1}{N}} C_0}.$$

If $\lambda \geq \rho_0$, then

$$|z_\lambda|_{L^\infty} \leq C^* \lambda^{\frac{1}{\kappa-1}},$$

and

$$|z_\lambda|_{L^\infty} \leq C_* \lambda^{\frac{1}{s-1}}$$

if $\lambda < \rho_0$. Here $C^* > 0$ and $C_* > 0$ depend on n, s, N, and Ω.

For z_λ which is defined in Lemma 2.4, it follows that $z_\lambda \in C^1(\overline{\Omega})$ with $z_\lambda > 0$ in Ω.

Lemma 2.5 ([11]) *There is a $k_0 > 0$ satisfying*

$$\big(\rho(|\zeta|)\zeta - \rho(|\epsilon|)\epsilon\big) \cdot (\zeta - \epsilon) \geq k_0 \frac{\Phi(|\zeta - \epsilon|)^{\frac{\kappa+1}{\kappa}}}{(\Phi(|\zeta|) + \Phi(|\epsilon|))^{\frac{1}{\kappa}}}$$

for $\zeta, \epsilon \in \mathbb{R}^N$, $\zeta \neq 0$.

Theorem 2.6 *If the functions $h_1, h_2 : [0, +\infty) \to \mathbb{R}$ are continuous and nonnegative, $\alpha, \gamma \geq 0$, $\overline{w}^*$ is a supersolution and $\underline{w}_*$ is a subsolution with $0 < \underline{w}_* \leq \overline{w}^*$, problem* (1.1) *possesses a nontrivial solution u with $\underline{w}_* \leq u \leq \overline{w}^*$.*

Theorem 2.7 *Suppose that* $0 < \alpha < \kappa - 1$ *and* $-1 < \beta < 0$, *where* κ *is given in* (ρ_3). *Then equation* (1.3) *has a positive solution.*

3 Proofs of the main results

Proof of Theorem 2.6 We consider

$$\begin{cases} -\Delta_\Phi u = H(u,x,h_1(\zeta(u,x)),h_2(\zeta(u,x))), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{3.1}$$

where

$$H(u,x,s,t) = J_1(u)s + J_2(u)t - \gamma(u,x).$$

We have the following claims:

Claim 1. Problem (3.1) has a solution in $W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$.

Define $B : W_0^{1,\Phi}(\Omega) :\to W^{-1,\Phi}(\Omega)$ as

$$\begin{aligned} (B(u),w) &= \int_\Omega -\Delta_\Phi u w - \int_\Omega \big[h_1\big(\zeta(u,x)\big)J_1(u) + h_2\big(\zeta(u,x)\big)J_2(u) - \gamma(u,x)\big]w \\ &= \int_\Omega \rho\big(|\nabla u|\big)(\nabla u \cdot \nabla w) - \int_\Omega \big[h_1\big(\zeta(u,x)\big)J_1(u) + h_2\big(\zeta(u,x)\big)J_2(u) \\ &\quad - \gamma(u,x)\big]w, \quad \forall u,w \in W_0^{1,\Phi}(\Omega), \end{aligned}$$

where ρ satisfies (ρ_1), (ρ_2), and (ρ_3).

First, we want to show that B is continuous, bounded, and coercive.

It is easy to see that the conditions on ρ and the continuity of h_1 and h_2 guarantees that B is bounded and continuous.

According to $(\rho_3)'$, there exist $\kappa, s \in (1,N)$ such that

$$\kappa \le \frac{\rho(t)t^2}{\Phi(t)} \le s, \quad \forall t > 0,$$

which implies that

$$\begin{aligned} \frac{(B(u),u)}{\|u\|_{1,\Phi}} &= \frac{\int_\Omega \rho(|\nabla u|)|\nabla u|^2 - \int_\Omega [h_1(\zeta(u,x))J_1(u) + h_2(\zeta(u,x))J_2(u) - \gamma(u,x)]u}{\|u\|_{1,\Phi}} \\ &\ge \frac{\kappa \int_\Omega \Phi(|\nabla u|) - \int_\Omega [h_1(\zeta(u,x))J_1(u) + h_2(\zeta(u,x))J_2(u) - \gamma(u,x)]u}{\|u\|_{1,\Phi}}. \end{aligned}$$

From the Lemma 2.3 and Lemma 2.1 in [12], we have

$$\min\big\{\|\nabla u\|_{L^\Phi}^\kappa, \|\nabla u\|_{L^\Phi}^s\big\} = \xi_0\big(\|\nabla u\|_{L^\Phi}\big) \le \int_\Omega \Phi\big(|\nabla u|\big)$$

and

$$\int_\Omega \Phi\big(|\nabla u|\big) \ge \int_\Omega \Phi\left(\frac{|u|}{d}\right),$$

then we deduce

$$\frac{(B(u),u)}{\|u\|_{1,\Phi}} \geq \frac{\frac{\kappa}{2}\min\{\|\nabla u\|^{\kappa}_{L^{\Phi}}, \|\nabla u\|^{s}_{L^{\Phi}}\} + \frac{\kappa}{2}\min\{\|\frac{u}{d}\|^{\kappa}_{L^{\Phi}}, \|\frac{u}{d}\|^{s}_{L^{\Phi}}\}}{\|u\|_{1,\Phi}} - \frac{\int_{\Omega}[h_1(\zeta(u,x))J_1(u) + h_2(\zeta(u,x))J_2(u) - \gamma(u,x)]u}{\|u\|_{1,\Phi}}.$$

It follows that

$$\begin{aligned}\frac{\kappa\int_{\Omega}\Phi(|\nabla u|)}{\|u\|_{1,\Phi}} &= \frac{\kappa\int_{\Omega}\Phi(|\nabla u|)}{|\nabla u|_{L^{\Phi}} + |u|_{L^{\Phi}}} \\ &\geq \frac{\frac{\kappa}{2}\min\{\|\nabla u\|^{\kappa}_{L^{\Phi}}, \|\nabla u\|^{s}_{L^{\Phi}}\} + \frac{\kappa}{2}\min\{\|\frac{u}{d}\|^{\kappa}_{L^{\Phi}}, \|\frac{u}{d}\|^{s}_{L^{\Phi}}\}}{|\nabla u|_{L^{\Phi}} + |u|_{L^{\Phi}}} \\ &= \frac{\kappa}{2}\frac{\min\{\|\nabla u\|^{\kappa}_{L^{\Phi}}, \|\nabla u\|^{s}_{L^{\Phi}}\} + \min\{\|\frac{u}{d}\|^{\kappa}_{L^{\Phi}}, \|\frac{u}{d}\|^{s}_{L^{\Phi}}\}}{|\nabla u|_{L^{\Phi}} + |u|_{L^{\Phi}}} \to \infty\end{aligned}$$

if $\|u\|_{1,\Phi} \to \infty$. Then we have

$$\frac{(B(u),u)}{\|u\|_{1,\Phi}} \to \infty \quad \bigl(\|u\|_{1,\Phi} \to \infty\bigr).$$

Hence we can conclude that the operator B is coercive.

In the end, we will prove that operator B is pseudomonotone, i.e., if

$$u_n \rightharpoonup u \quad \text{in } W^{1,\Phi}_0(\Omega) \cap L^{\infty}(\Omega)$$

and

$$\lim_{n\to\infty}\sup\bigl(B(u_n),(u_n - u)\bigr) \leq 0,$$

then

$$\lim_{n\to\infty}\inf\bigl(B(u_n),(u_n - w)\bigr) \geq \bigl(B(u),(u - w)\bigr), \quad \forall w \text{ in } W^{1,\Phi}_0(\Omega) \cap L^{\infty}(\Omega). \tag{3.2}$$

From

$$\int_{\Omega}\bigl[h_1\bigl(\zeta(u_n,x)\bigr)J_1(u_n) + g\bigl(\zeta(u_n,x)\bigr)J_2(u_n) - \gamma(u_n,x)\bigr](u_n - u) \to 0$$

and

$$\limsup_{n\to\infty}\bigl(B(u_n),(u_n - u)\bigr) \leq 0,$$

we obtain

$$\limsup_{n\to\infty}\int_{\Omega}\rho\bigl(|\nabla u_n|\bigr)\bigl(\nabla u_n \cdot \nabla(u_n - u)\bigr) \leq 0. \tag{3.3}$$

From Lemma 3.1 in [12], we infer

$$\|\nabla(u_n-u)\|_{L^\Phi} \le \int_\Omega \Phi\bigl(|\nabla(u_n-u)|\bigr). \tag{3.4}$$

From Lemma 2.5, we can obtain a $k_0>0$ such that

$$\begin{aligned}
&\Phi\bigl(|\nabla(u_n-u)|\bigr)\\
&\quad\le \frac{[\Phi(|\nabla u_n|)+\Phi(|\nabla u|)]^{\frac{1}{\kappa+1}}}{k_0^{\frac{\kappa}{\kappa+1}}}\\
&\qquad\times\bigl[\rho\bigl(|\nabla u_n|\bigr)\bigl(\nabla u_n\cdot\nabla(u_n-u)\bigr)-\rho\bigl(|\nabla u|\bigr)\bigl(\nabla u\cdot\nabla(u_n-u)\bigr)\bigr]^{\frac{\kappa}{\kappa+1}},
\end{aligned}\tag{3.5}$$

that is,

$$\begin{aligned}
&\int_\Omega \Phi\bigl(|\nabla(u_n-u)|\bigr)\\
&\quad\le \int_\Omega\biggl\{\frac{[\Phi(|\nabla u_n|)+\Phi(|\nabla u|)]^{\frac{1}{\kappa+1}}}{k_0^{\frac{\kappa}{\kappa+1}}}\\
&\qquad\times\bigl[\rho\bigl(|\nabla u_n|\bigr)\bigl(\nabla u_n\cdot\nabla(u_n-u)\bigr)-\rho\bigl(|\nabla u|\bigr)\bigl(\nabla u\cdot\nabla(u_n-u)\bigr)\bigr]^{\frac{\kappa}{\kappa+1}}\biggr\}\\
&\quad\le\biggl\{\int_\Omega\biggl[\frac{[\Phi(|\nabla u_n|)+\Phi(|\nabla u|)]^{\frac{1}{\kappa+1}}}{k_0^{\frac{\kappa}{\kappa+1}}}\biggr]^{\kappa+1}\biggr\}^{\frac{1}{\kappa+1}}\\
&\qquad\times\biggl\{\int_\Omega\bigl[\rho\bigl(|\nabla u_n|\bigr)\bigl(\nabla u_n\cdot\nabla(u_n-u)\bigr)-\rho\bigl(|\nabla u|\bigr)\bigl(\nabla u\cdot\nabla(u_n-u)\bigr)\bigr]\biggr\}^{\frac{\kappa}{\kappa+1}}.
\end{aligned}\tag{3.6}$$

Since $u_n\rightharpoonup u$, we have

$$\int_\Omega \rho\bigl(|\nabla u|\bigr)\bigl(\nabla u\cdot\nabla(u_n-u)\bigr)\to 0,$$

which, together with (3.3), guarantees that

$$\int_\Omega \rho\bigl(|\nabla u_n|\bigr)\bigl(\nabla u_n\cdot\nabla(u_n-u)\bigr)-\rho\bigl(|\nabla u|\bigr)\bigl(\nabla u\cdot\nabla(u_n-u)\bigr)\to 0\quad\text{as } n\to+\infty. \tag{3.7}$$

From (3.5), (3.6), and (3.7), we have

$$\int_\Omega \Phi\bigl(|\nabla(u_n-u)|\bigr)\to 0,$$

that is,

$$\|\nabla(u_n-u)\|_{L^\Phi}\to 0.$$

Therefore,

$$\|u_n-u\|_{1,\Phi}=\|u_n-u\|_{L^\Phi}+\|\nabla(u_n-u)\|_{L^\Phi}\to 0,$$

which implies that (3.2) is true.

According to Lemma 2.2.2 in [21], there is a $u \in W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ such that for $\forall w \in W_0^{1,\Phi}(\Omega)$,

$$\big(B(u), w\big) = 0.$$

Therefore, we know that u is a (weak) solution of problem (3.1).

Claim 2. We show that the solution u of problem (3.1) obtained above is a solution of (1.1).

We shall prove that

$$\underline{w}_* \le u \le \overline{w}^* \quad \text{in } \Omega. \tag{3.8}$$

Choosing $w = (u - \overline{w}^*)_+$ as a test function, we have

$$\begin{aligned} \int_\Omega -\Delta_\Phi u \big(u - \overline{w}^*\big)_+ &= \int_\Omega \big[H\big(x, u, h_1\big(\zeta(u,x)\big), h_2\big(\zeta(u,x)\big)\big) - \gamma(u,x)\big]\big(u - \overline{w}^*\big)_+ \\ &= \int_\Omega \big[h_1\big(\zeta(u,x)\big)J_1(u) + h_2\big(\zeta(u,x)\big)J_2(u) - \gamma(u,x)\big]\big(u - \overline{w}^*\big)_+. \end{aligned} \tag{3.9}$$

Define

$$\Omega_1 := \big\{x \in \Omega \mid u > \overline{w}^*\big\}.$$

Then

$$\begin{aligned} &\int_\Omega \big[h_1\big(\zeta(u,x)\big)J_1(u) + h_2\big(\zeta(u,x)\big)J_2(u) - \gamma(u,x)\big]\big(u - \overline{w}^*\big)_+ \\ &\quad = \int_{\Omega_1} + \int_{\Omega - \Omega_1} \big[h_1\big(\zeta(u,x)\big)J_1(u) + h_2\big(\zeta(u,x)\big)J_2(u) - \gamma(u,x)\big]\big(u - \overline{w}^*\big)_+ \\ &\quad = \int_{\Omega_1} \big[h_1\big(\zeta(u,x)\big)J_1(u) + h_2\big(\zeta(u,x)\big)J_2(u) - \gamma(u,x)\big]\big(u - \overline{w}^*\big)_+ + 0 \\ &\quad = \int_{\Omega_1} \big[h_1\big(\overline{w}^*\big)J_1(u) + h_2\big(\overline{w}^*\big)J_2(u) - \big(u - \overline{w}^*\big)_+^\nu\big]\big(u - \overline{w}^*\big)_+. \end{aligned} \tag{3.10}$$

Since Ψ and Λ are increasing, from Lemma 2.1 and $|\zeta(u,x)| \le \overline{w}^*$, we have

$$\left\{\varsigma > 0 \,\middle|\, \int_\Omega \Psi\left(\frac{|\zeta(u,x)|}{\varsigma}\right) \le 1\right\} \supseteq \left\{\varsigma > 0 \,\middle|\, \int_\Omega \Psi\left(\frac{\overline{w}^*}{\varsigma}\right) \le 1\right\}$$

and

$$\left\{\varsigma > 0 \,\middle|\, \int_\Omega \Lambda\left(\frac{|\zeta(u,x)|}{\varsigma}\right) \le 1\right\} \supseteq \left\{\varsigma > 0 \,\middle|\, \int_\Omega \Lambda\left(\frac{\overline{w}^*}{\varsigma}\right) \le 1\right\},$$

which implies that

$$J_1\big(\zeta(u,x)\big) \le J_1\big(\overline{w}^*\big), \qquad J_2\big(\zeta(u,x)\big) \le J_2\big(\overline{w}^*\big). \tag{3.11}$$

From (3.9), (3.10), and (3.11), we have

$$\int_{\Omega} -\Delta_{\Phi} u\left(u-\overline{w}^{*}\right)_{+} \leq \int_{\Omega}\left[h_{1}\left(\overline{w}^{*}\right) J_{1}\left(\overline{w}^{*}\right)+h_{2}\left(\overline{w}^{*}\right) J_{2}\left(\overline{w}^{*}\right)-\left(u-\overline{w}^{*}\right)_{+}^{\nu}\right]\left(u-\overline{w}^{*}\right)_{+}.$$

By Definition 2.2, we have

$$\int_{\Omega} -\Delta_{\Phi} u\left(u-\overline{w}^{*}\right)_{+} \leq \int_{\Omega}\left[-\Delta_{\Phi} \overline{w}^{*}-\left(u-\overline{w}^{*}\right)_{+}^{\nu}\right]\left(u-\overline{w}^{*}\right)_{+}.$$

Hence

$$\int_{\Omega} -\Delta_{\Phi} u\left(u-\overline{w}^{*}\right)_{+}+\int_{\Omega} \Delta_{\Phi} \overline{w}^{*}\left(u-\overline{w}^{*}\right)_{+} \leq \int_{\Omega}\left[-\left(u-\overline{w}^{*}\right)_{+}^{\nu+1}\right] \leq 0,$$

i.e.,

$$\int_{\Omega}\left(\rho(|\nabla u|) \nabla u-\rho\left(\left|\nabla \overline{w}^{*}\right|\right) \nabla \overline{w}^{*}\right) \cdot \nabla\left(u-\overline{w}^{*}\right)_{+} \leq \int_{\Omega}\left[-\left(u-\overline{w}^{*}\right)_{+}^{\nu+1}\right] \leq 0. \tag{3.12}$$

From Lemma 2.5, there exists a $k_0 > 0$ such that

$$\begin{aligned} &\int_{\Omega}\left(\rho(|\nabla u|) \nabla u-\rho\left(\left|\nabla \overline{w}^{*}\right|\right) \nabla \overline{w}^{*}\right) \cdot \nabla\left(u-\overline{w}^{*}\right)_{+} \\ &\quad \geq \int_{\Omega} k_{0} \frac{\Phi(|\nabla u-\nabla \overline{w}^{*}|)^{\frac{\kappa+1}{\kappa}}}{(\Phi(|\nabla u|)+\Phi(|\nabla \overline{w}^{*}|))^{\frac{1}{\kappa}}} \frac{\nabla(u-\overline{w}^{*})_{+}}{\nabla(u-\overline{w}^{*})}. \end{aligned} \tag{3.13}$$

Since

$$\int_{\Omega} k_{0} \frac{\Phi(|\nabla u-\nabla \overline{w}^{*}|)^{\frac{\kappa+1}{\kappa}}}{(\Phi(|\nabla u|)+\Phi(|\nabla \overline{w}^{*}|))^{\frac{1}{\kappa}}} \frac{\nabla(u-\overline{w}^{*})_{+}}{\nabla(u-\overline{w}^{*})}=\int_{\Omega_{1}} k_{0} \frac{\Phi(|\nabla u-\nabla \overline{w}^{*}|)^{\frac{\kappa+1}{\kappa}}}{(\Phi(|\nabla u|)+\Phi(|\nabla \overline{w}^{*}|))^{\frac{1}{\kappa}}}$$

and Φ is continuous, we obtain that there is an $M_1 > 0$ such that

$$\int_{\Omega_{1}} k_{0} \frac{\Phi(|\nabla u-\nabla \overline{w}^{*}|)^{\frac{\kappa+1}{\kappa}}}{(\Phi(|\nabla u|)+\Phi(|\nabla \overline{w}^{*}|))^{\frac{1}{\kappa}}}=\frac{k_{0}}{M_{1}} \int_{\{u>\overline{w}^{*}\}} \Phi\left(\left|\nabla u-\nabla \overline{w}^{*}\right|\right)^{\frac{\kappa+1}{\kappa}}. \tag{3.14}$$

From (3.12), (3.13), and (3.14), we have

$$\int_{\{u>\overline{w}^{*}\}} \Phi\left(\left|\nabla u-\nabla \overline{w}^{*}\right|\right)^{\frac{\kappa+1}{\kappa}} \leq 0.$$

From Lemma 2.2 in [11] and [14], we obtain

$$\int_{\{u>\overline{w}^{*}\}} \Phi\left(\frac{|u-\overline{w}^{*}|}{d}\right)^{\frac{\kappa+1}{\kappa}} \leq \int_{\{u>\overline{w}^{*}\}} \Phi\left(\left|\nabla u-\nabla \overline{w}^{*}\right|\right)^{\frac{\kappa+1}{\kappa}} \leq 0,$$

where $d = \operatorname{diam}(\Omega)$. Therefore, we can conclude that

$$\left|\left\{u>\overline{w}^{*}\right\}\right|=0,$$

and then $u \leq \overline{w}^{*}$.

A similar argument shows that $u \geq \underline{w}_*$.

Therefore, (3.8) is true and thus u is a solution of problem (1.1).

The proof is completed. □

Proof of Theorem 2.7 In order to get positive solutions of problem (1.3), we study the following problem:

$$\begin{cases} -\Delta_\Phi u = (u + \frac{1}{n})^\beta \|u\|_{L^\Psi}^\alpha, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.15}$$

for $n \geq 1$. We will use Theorem 2.6 to discuss problem (3.15).

First, we will construct a supersolution $\overline{u}$ of problem (3.15).

From Lemma 2.4, problem (2.1) has a unique positive $z_\lambda \in W_0^{1,\Psi}(\Omega)$ which satisfies

$$0 < z_\lambda(x) \leq K\lambda^{\frac{1}{\kappa-1}}, \quad x \in \Omega \tag{3.16}$$

for $\lambda > 0$ big enough, where K is independent of λ.

Let $M = K\lambda^{\frac{1}{\kappa-1}}$. Then

$$K\lambda^{\frac{1}{\kappa-1}} < z_\lambda(x) + M \leq 2K\lambda^{\frac{1}{\kappa-1}}, \quad x \in \Omega.$$

The condition $0 < \alpha < \kappa - 1$ implies that there is a $\lambda > 1$ big enough such that

$$\lambda^{\frac{\alpha}{\kappa-1}} \|2K\|_{L^\Psi}^\alpha \leq \lambda, \qquad M = K\lambda^{\frac{1}{\kappa-1}} > 1$$

and (3.16) holds. Hence

$$\left(z_\lambda + M + \frac{1}{n}\right)^\beta \|z_\lambda + M\|_{L^\Psi}^\alpha \leq \|z_\lambda + M\|_{L^\Psi}^\alpha \leq \lambda^{\frac{\alpha}{\kappa-1}} \|2K\|_{L^\Psi}^\alpha \leq \lambda$$

and

$$-\Delta_\Phi(z_\lambda + M) = -\Delta_\Phi z_\lambda = \lambda \geq \left(z_\lambda + M + \frac{1}{n}\right)^\beta \|z_\lambda + M\|_{L^\Psi}^\alpha.$$

Therefore, $z_\lambda + M$ is a supersolution of (3.15).

Second, we will construct a positive subsolution $\underline{u}_*$ of problem (3.15).

Define $d(x) := \operatorname{dist}(x, \partial\Omega)$, then by a direct calculation one can deduce that $|\nabla d(x)| = 1$. Because $\partial\Omega$ is C^2, we can get a constant $\tau > 0$ such that $d \in C^2(\overline{\Omega_{3\tau}})$ with $\overline{\Omega_{3\tau}} := \{x \in \overline{\Omega} : d(x) \leq 3\tau\}$ (see [9, 10]). Let $\varpi \in (0, \tau)$. Define

$$\eta(x) := \begin{cases} e^{\vartheta d(x)} - 1, & \text{for } d(x) < \varpi, \\ e^{\vartheta\varpi} - 1 + \int_\varpi^{d(x)} \vartheta e^{\vartheta d(x)} (\frac{2\tau - t}{2\tau - \varpi})^{\frac{s}{\kappa-1}}\, dt, & \text{for } \varpi \leq d(x) \leq 2\tau, \\ e^{\vartheta\varpi} - 1 + \int_\varpi^{2\tau} k e^{\vartheta d(x)} (\frac{2\tau - t}{2\tau - \varpi})^{\frac{s}{\kappa-1}}\, dt, & \text{for } 2\tau < d(x), \end{cases}$$

where $\vartheta > 0$ is an arbitrary number. Direct computations imply that

$$-\Delta_\Phi(\mu\eta) = \begin{cases} -\vartheta\Theta(x)\frac{d}{dt}(\rho(t)t)|_{t=\Theta(x)} - \rho(\Theta(x))\Theta(x)\Delta d, & \text{for } d(x) < \varpi, \\ \frac{\Theta_0(\frac{s}{\kappa-1})\chi(x)^{\frac{s}{\kappa-1}-1}}{2\tau-\varpi}\frac{d}{dt}(\rho(t)t)|_{t=\Theta_0\chi(x)^{\frac{s}{\kappa-1}}} & \\ \quad - \rho(\Theta_0\chi(x)^{\frac{s}{\kappa-1}})\Theta_0\chi(x)^{\frac{s}{\kappa-1}}\Delta d, & \text{for } \varpi \le d(x) \le 2\tau, \\ 0, & \text{for } 2\tau < d(x), \end{cases}$$

with $\Theta(x) = \mu\vartheta e^{\vartheta d(x)}$, $\Theta_0 = \mu\vartheta e^{\vartheta\varpi}$, and $\chi(x) = \frac{2\tau-d(x)}{2\tau-\varpi}$ for all $\mu > 0$.

There are three cases: (1) $d(x) < \varpi$; (2) $\varpi < d(x) < 2\tau$; and (3) $d(x) > 2\tau$.

(1) We consider the case $d(x) < \varpi$.

Since Δd is a bounded function near $\partial\Omega$ and $\kappa > 1$, there is a ϑ large enough such that

$$\begin{aligned} -\Delta_\Phi(\mu\eta) &= -\mu\vartheta^2 e^{\vartheta d(x)}\frac{d}{dt}\big(\rho(t)t\big)\Big|_{t=\mu\vartheta e^{\vartheta d(x)}} - \rho\big(\mu\vartheta e^{\vartheta d(x)}\big)\mu\vartheta e^{\vartheta d(x)}\Delta d \\ &\le -\vartheta^2\mu e^{\vartheta d(x)}(\kappa-1)\rho\big(\mu\vartheta e^{\mu\vartheta e^{\vartheta d(x)}}\big) - \rho\big(\mu\vartheta e^{\vartheta d(x)}\big)\mu\vartheta e^{\vartheta d(x)}\Delta d \\ &= \mu\vartheta e^{\vartheta d(x)}\rho\big(\mu\vartheta e^{\vartheta d(x)}\big)\big(-\vartheta(\kappa-1) - \Delta d\big) \\ &\le 0, \end{aligned}$$

which implies that

$$-\Delta_\Phi(\mu\eta) \le 0 \le (\mu\eta)^\beta|\mu\eta|^\alpha_{L^\Psi},$$

when $d(x) < \varpi$ and ϑ is large enough.

(2) We consider the case $\varpi < d(x) < 2\delta$.

From the condition (ρ_3) and Lemma 2.3, we have

$$\begin{aligned} &\mu\vartheta e^{\vartheta\varpi}\left(\frac{s}{\kappa-1}\right)\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\frac{s}{\kappa-1}-1}\left(\frac{1}{2\tau-\varpi}\right)\frac{d}{dt}\big(\rho(t)t\big)\Big|_{t=\mu\vartheta e^{\vartheta\varpi}(\frac{2\tau-d(x)}{2\tau-\varpi})^{\frac{s}{\kappa-1}}} \\ &\quad\le \mu\vartheta e^{\vartheta\varpi}\left(\frac{s}{\kappa-1}\right)\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\frac{s}{\kappa-1}-1}\left(\frac{s-1}{2\tau-\varpi}\right)\rho\left(\mu\vartheta e^{\vartheta\varpi}\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\frac{s}{\kappa-1}}\right) \\ &\quad\le \left(\frac{s}{\kappa-1}\right)\left(\frac{s-1}{2\tau-\varpi}\right)\frac{s\Phi(\mu\vartheta e^{\vartheta\varpi}(\frac{2\tau-d(x)}{2\tau-\varpi})^{\frac{s}{\kappa-1}})}{\mu\vartheta e^{\vartheta\varpi}(\frac{2\delta-d(x)}{2\tau-\varpi})^{\frac{s}{\kappa-1}}}\frac{1}{\frac{2\tau-d(x)}{2\tau-\varpi}} \\ &\quad\le \left(\frac{s^2}{\kappa-1}\right)\left(\frac{s-1}{2\tau-\varpi}\right)\max\left\{\left(\mu\vartheta e^{\vartheta\varpi}\right)^{s-1}\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{s(\frac{s}{\kappa-1})-(\frac{s}{\kappa-1}+1)}, \right. \\ &\qquad\left.\left(\mu\vartheta e^{\vartheta\varpi}\right)^{\kappa-1}\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\kappa(\frac{s}{\kappa-1})-(\frac{s}{\kappa-1}+1)}\right\}\Phi(1). \end{aligned} \tag{3.17}$$

Now $s,\kappa > 1$ implies $\kappa(\frac{s}{\kappa-1}) - s(\frac{s}{\kappa-1}+1), s(\frac{s}{\kappa-1}) - s(\frac{s}{\kappa-1}+1) > 0$, which, together with $0 \le \frac{2\tau-d(x)}{2\tau-\varpi} \le 1$ and (3.17), guarantees that

$$\mu\vartheta e^{\vartheta\varpi}\left(\frac{s}{\kappa-1}\right)\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\frac{s}{\kappa-1}-1}\left(\frac{1}{2\tau-\varpi}\right)\frac{d}{dt}\big(\rho(t)t\big)\Big|_{t=\mu\vartheta e^{\vartheta\varpi}(\frac{2\delta-d(x)}{2\tau-\varpi})^{\frac{s}{\kappa-1}}}$$

$$\begin{aligned}
&\le \left(\frac{s^2}{\kappa-1}\right)\left(\frac{s-1}{2\tau-\varpi}\right)\Phi(1)\max\left\{\left(\mu\vartheta e^{\vartheta\varpi}\right)^{s-1},\left(\mu\vartheta e^{\vartheta\varpi}\right)^{\kappa-1}\right\} \\
&= C_1\left(\frac{1}{2\tau-\varpi}\right)\max\left\{\left(\mu\vartheta e^{\vartheta\varpi}\right)^{s-1},\left(\mu\vartheta e^{\vartheta\varpi}\right)^{\kappa-1}\right\},
\end{aligned} \tag{3.18}$$

where $C_1 = \frac{s^2(s-1)\Phi(1)}{\kappa-1}$ is a constant independent of μ and ϑ. Similarly, one has

$$\begin{aligned}
&\left|\rho\left(\mu\vartheta e^{\vartheta\varpi}\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\frac{s}{\kappa-1}}\right)\mu\vartheta e^{\vartheta\varpi}\frac{(2\tau-d(x))^{\frac{s}{\kappa-1}}}{(2\tau-\varpi)^{\frac{s}{\kappa-1}}}\Delta d\right| \\
&\le \rho\left(\mu\vartheta e^{\vartheta\varpi}\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\frac{s}{r-1}}\right)\mu\vartheta e^{\vartheta\varpi}\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\frac{s}{\kappa-1}}\sup_{\overline{\Omega_{3\tau}}}|\Delta d| \\
&\le C\frac{\Phi(\mu\vartheta e^{\vartheta\varpi}(\frac{2\tau-d(x)}{2\tau-\varpi})^{\frac{s}{\kappa-1}})}{\mu\vartheta e^{\vartheta\varpi}(\frac{2\tau-d(x)}{2\tau-\varpi})^{\frac{s}{\kappa-1}}} \\
&\le C\max\left\{\left(\mu\vartheta e^{\vartheta\varpi}\right)^{s-1}\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{s(\frac{s}{\kappa-1})-(\frac{s}{\kappa-1}+1)},\right. \\
&\qquad\left.\left(\mu\vartheta e^{\vartheta\varpi}\right)^{\kappa-1}\left(\frac{2\tau-d(x)}{2\tau-\varpi}\right)^{\kappa(\frac{s}{\kappa-1})-(\frac{s}{\kappa-1}+1)}\right\} \\
&\le C_2\max\left\{\left(\mu\vartheta e^{\vartheta\varpi}\right)^{s-1},\left(\mu\vartheta e^{\vartheta\varpi}\right)^{\kappa-1}\right\},
\end{aligned} \tag{3.19}$$

where C_2 is a constant independent of ϖ, ϑ, and μ. Thus from (3.18) and (3.19) we have

$$-\Delta_\Phi u \le \max\left\{\frac{C_1}{2\tau-\varpi},C_2\right\}\max\left\{\left(\mu\vartheta e^{\vartheta\varpi}\right)^{s-1},\left(\mu\vartheta e^{\vartheta\varpi}\right)^{\kappa-1}\right\},$$

when $\varpi < d(x) < 2\tau$.

Let $\varpi = \frac{ln2}{\vartheta}$ and $\mu = e^{-\vartheta}$, then $e^{\vartheta\varpi} = 2$. Since

$$\begin{aligned}
\eta(x) &= e^{\vartheta\varpi} - 1 + \int_\varpi^{d(x)}\vartheta e^{\vartheta d(x)}\left(\frac{2\tau-t}{2\tau-\varpi}\right)^{\frac{s}{\kappa-1}}dt \\
&> 2-1+2\vartheta\int_\varpi^{d(x)}\left(\frac{2\tau-t}{2\tau-\varpi}\right)^{\frac{s}{\kappa-1}}dt \\
&= 1+\vartheta C_3 \\
&\ge 1,
\end{aligned}$$

where $C_3 > 0$ is a constant, we have that when μ is small enough and n is large enough,

$$\begin{aligned}
\left(\mu\eta+\frac{1}{n}\right)^\beta|\mu\eta|^\alpha_{L^\Psi} \ge |\mu\eta|^\alpha_{L^\Psi}
&= \inf{}^\alpha\left\{\varsigma>0:\int_\Omega\Psi\left(\frac{|\mu\eta|}{\varsigma}\right)<1\right\} \\
&= \inf{}^\alpha\left\{\tau\mu>0:\int_\Omega\Psi\left(\frac{|\mu\eta|}{\tau\mu}\right)<1\right\} \\
&= \mu^\alpha\inf{}^\alpha\left\{\tau>0:\int_\Omega\Psi\left(\frac{|\eta|}{\tau}\right)<1\right\}
\end{aligned}$$

$$\geq \mu^\alpha C_4,$$

where $C_4 > 0$ is a constant independent of $\vartheta > 0$.

Since $0 < \alpha < \kappa - 1$, we have the result

$$\lim_{\vartheta \to +\infty} \frac{\vartheta^{\kappa-1}}{e^{\vartheta(\kappa-1-\alpha)}} = 0.$$

In view of

$$-\Delta_\Phi(\mu\eta) \leq \max\left\{\frac{C_1}{2\tau - \varpi}, C_2\right\} \max\left\{2^{s-1}, 2^{\kappa-1}\right\}\left(\frac{\vartheta}{e^\vartheta}\right)^{\kappa-1},$$

choose a $\vartheta_0 > 0$ large enough such that

$$C_4 \geq \max\left\{\frac{C_1}{2\tau - \frac{\ln 2}{\vartheta}}, C_2\right\} \max\left\{2^{s-1}, 2^{\kappa-1}\right\}\left(\frac{\vartheta^{\kappa-1}}{e^{\vartheta(\kappa-1-\alpha)}}\right)$$

for all $\vartheta \geq \vartheta_0$.

Thus,

$$-\Delta_\Phi(\mu\eta) \leq \left(\mu\eta + \frac{1}{n}\right)^\beta |\mu\eta|^\alpha_{L^\Psi}$$

in the case $\varpi < d(x) < 2\tau$ for $\vartheta > 0$ large enough.

(3) We consider the case $d(x) > 2\tau$.

Obviously,

$$-\Delta_\Phi(\mu\eta) = 0 \leq \left(\mu\eta + \frac{1}{n}\right)^\beta |\mu\eta|^\alpha_{L^\Psi}.$$

It is obvious that $\underline{w}_* \leq \overline{w}^*$ if M is large enough and μ is small enough. And $(\underline{w}_*, \overline{w}^*)$ is a sub-supersolution pair of problem (3.15). Now Theorem 2.6 guarantees that problem (3.15) has a solution u_n which satisfies $0 < \mu\eta \leq u_n \leq z_\lambda + M$.

Now we consider the set $\{u_n\}$.

From Lemma 2.2 in [12], one has that $\|u\|_{1,\Phi}$ and $\||\nabla u|\|_{L^\Phi}$ defined on $W_0^{1,\Phi}$ are equivalent. And from the proof of the coercivity of the operator B, we know that if $\||\nabla u|\|_{L^\Phi} > 1$, then

$$\int_\Omega \Phi\big(|\nabla u|\big) \geq \||\nabla u|\|_{L^\Phi},$$

that is,

$$\int_\Omega \Phi\big(|\nabla u|\big) \geq \|u\|_{1,\Phi},$$

when $\|u\|_{1,\Phi} > 1$.

If $\|u_n\|_{1,\Phi} \leq 1$, then u_n is bounded in $W_0^{1,\Phi}(\Omega)$ naturally.

If $\|u_n\|_{1,\Phi} > 1$, then

$$\|u_n\|_{1,\Phi} \le \int_\Omega \Phi(|\nabla u_n|).$$

By the condition $(\rho_3)'$ and due to

$$\int_\Omega -\Delta_\Phi u_n u_n = \int_\Omega u_n \left(u_n + \frac{1}{n}\right)^\beta \|u_n\|_{L^\Psi}^\alpha,$$

we have

$$\kappa \int_\Omega \Phi(|\nabla u_n|) \le \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 = \int_\Omega u_n \left(u_n + \frac{1}{n}\right)^\beta \|u_n\|_{L^\Psi}^\alpha,$$

which, together with $\alpha \ge 0$, $-1 < \beta < 0$, gives

$$\int_\Omega \Phi(|\nabla u_n|) \le \frac{1}{\kappa} \int_\Omega \overline{w}^{*\beta+1} \|\overline{w}^*\|_{L^\Psi}^\alpha,$$

that is,

$$\|u_n\|_{1,\Phi} \le \frac{1}{\kappa} \int_\Omega \overline{w}^{*\beta+1} \|\overline{w}^*\|_{L^\Psi}^\alpha.$$

Therefore, $\{u_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$.

Since $W_0^{1,\Phi}(\Omega)$ is reflexive, $\{u_n\}$ has weakly convergent subsequences in $W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$, and we still use u_n to denote its subsequence. From the analysis in [3], we have

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$$

and

$$u_n(x) \xrightarrow{\text{a.e.}} u(x), \quad x \in \Omega.$$

Since

$$\underline{w}_* \le u_n \le \overline{w}^*, \quad x \in \Omega,$$

Lebesgue theorem implies

$$u_n \to u \quad \text{in } L^q(\Omega)\ \forall q \in [1, +\infty). \tag{3.20}$$

Since u_n is a (weak) solution of (3.15) for all $n \in \mathbb{N}^+$, we have

$$\int_\Omega -\Delta_\Phi u_n w = \int_\Omega \left(u_n + \frac{1}{n}\right)^\beta \|u_n\|_{L^\Psi}^\alpha w,$$

for all $w \in W_0^{1,\Phi}(\Omega)$.

Denoting $w = u_n - u$, we have

$$\int_\Omega -\Delta_\Phi u_n(u_n - u) = \int_\Omega \left(u_n + \frac{1}{n}\right)^\beta \|u_n\|^\alpha_{L^\Psi}(u_n - u).$$

Since

$$\left(u_n + \frac{1}{n}\right)^\beta \le \underline{w}_*^\beta, \quad x \in \Omega,$$

one has

$$\begin{aligned}\int_\Omega \left(u_n + \frac{1}{n}\right)^\beta \|u_n\|^\alpha_{L^\Psi}|u_n - u| &\le \int_\Omega \underline{w}_*^\beta |u_n - u|\|u_n\|^\alpha_{L^\Psi} \\ &\le \left[\int_\Omega \left(\underline{w}_*^\beta \|u_n\|^\alpha_{L^\Psi}\right)^p\right]^{\frac{1}{p}} \left[\int_\Omega |u_n - u|^q\right]^{\frac{1}{q}},\end{aligned}$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\beta \in (-1, 0)$. From (3.20), we have

$$\left[\int_\Omega \left(\underline{w}_*^\beta \|u_n\|^\alpha_{L^\Psi}\right)^p\right]^{\frac{1}{p}} \left[\int_\Omega |u_n - u|^q\right]^{\frac{1}{q}} \to 0,$$

and so

$$\int_\Omega \left(u_n + \frac{1}{n}\right)^\beta \|u_n\|^\alpha_{L^\Psi}|u_n - u| \to 0 \quad \text{as } n \to +\infty,$$

which implies

$$\int_\Omega -\Delta_\Phi u_n(u_n - u) \to 0.$$

Obviously,

$$\int_\Omega -\Delta_\Phi u(u_n - u) \to 0. \tag{3.21}$$

Similar to the previous proof, from (3.4), (3.6), and (3.21), we have

$$u_n \to u \quad \text{in } W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega),$$

and so

$$\|u_n\|^\alpha_{L^\Psi} \to \|u\|^\alpha_{L^\Psi}.$$

Therefore, taking the limit as $n \to \infty$ in (3.15), we have

$$-\Delta_\Phi u = u^\beta \|u\|^\alpha_{L^\Psi}.$$

The limit value u is just the solution which we are looking for, and it satisfies $\underline{w}_* \le u \le \overline{w}^*$, obviously. Therefore, the proof is finished. □

Acknowledgements
This work is supported by National Natural Science Foundation of China (62073203).

Funding
This research was funded by the NSFC of China (62073203).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
Methodology and investigation, BY; Formal analysis, XQ. All authors have read and agreed to the published version of the manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 November 2021 Accepted: 23 August 2022 Published online: 03 September 2022

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