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Existence of positive solutions for p-Laplacian boundary value problems of fractional differential equations

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Abstract

In this paper, we study the existence and multiplicity of ρ-concave positive solutions for a p-Laplacian boundary value problem of two-sided fractional differential equations involving generalized-Caputo fractional derivatives. Using Guo–Krasnoselskii fixed point theorem and under some additional assumptions, we prove some important results and obtain the existence of at least three solutions. To establish the results, Green functions are used to transform the considered two-sided generalized Katugampola and Caputo fractional derivatives. Finally, applications with illustrative examples are presented to show the validity and correctness of the obtained results.

MSC: 34A08; 34B15

Keywords: Fractional differential equations; p-Laplacian; Generalized Caputo fractional derivative; Concave; Positive solutions

1 Introduction

Last decades witnessed an increased number of theoretical studies and practical applications of fractional differential equations in science, engineering, biology, etc. [1–10]. In particular, fractional p-Laplacian has been used in modeling different problems [11–17].

In 2007, Su *et al.* studied the existence of positive solution for a nonlinear four-point singular boundary value problem

$$\begin{cases} (\phi_p(\mathrm{q}'))'(\tau) + \hbar(\tau)\wp(\mathrm{q}(\tau)) = 0, \quad 0 < \tau < 1, \\ \eta_1\phi_p(\mathrm{q}(0)) - \eta_2\phi_p(\mathrm{q}'(\xi)) = 0, \\ \eta_3\phi_p(\mathrm{q}(1)) + \eta_4\phi_p(\mathrm{q}'(\lambda)) = 0, \end{cases} \tag{1}$$

by using the fixed point index theory, where $\eta_1, \eta_3 > 0$, $\eta_2, \eta_4 \geq 0$, $0 < \xi < \lambda < 1$, and $\hbar : (0,1) \to [0,\infty)$ [15]. Also, they applied the theory to study the existence of positive

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solutions for the nonlinear third-order two-point singular boundary value problem

$$\begin{cases} (\phi_p(\mathrm{q}^{(n-1)}))'(\tau) + \hbar(\tau)\wp(\mathrm{q}(\tau)) = 0, \quad 0 < \tau < 1, \\ \mathrm{q}(0) = \mathrm{q}'(0) = \cdots = \mathrm{q}^{(n-3)}(0) = \mathrm{q}^{(n-1)}(0) = 0, \\ \mathrm{q}(1) = \sum_{i=1}^{m-2} \eta_i \mathrm{q}(\lambda_i), \end{cases} \tag{2}$$

where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-2} < 1, \qquad \eta_i > 0,$$

with $\sum_{i=1}^{m-2} \eta_i \lambda_i^{n-2} < 1$ [18]. Chai in [19], considered the nonlinear fractional boundary value problem

$$\begin{cases} {}^{0^+}\mathbb{G}_{\mathrm{RL}}^{\sigma_1}(\phi_p({}^{0^+}\mathbb{G}_{\mathrm{RL}}^{\sigma_2}\mathrm{q}))(\tau) + \wp(\tau, \mathrm{q}(\tau)) = 0, \quad 0 < \tau < 1, \\ \mathrm{q}(0) = 0 = {}^{0^+}\mathbb{G}_{\mathrm{RL}}^{\sigma_2}\mathrm{q}(0) = \mathrm{q}(1) + \eta({}^{0^+}\mathbb{G}_{\mathrm{RL}}^{\sigma_3}\mathrm{q}(1)) = 0, \end{cases} \tag{3}$$

on a cone and obtained some results and positive solutions, where $1 < \sigma_2 \leq 2$, $0 < \sigma_1, \sigma_3 \leq 1$, $0 \leq \sigma_2 - \sigma_3 - 1$, $\eta > 0$, and p-Laplacian operator is defined as $\phi_p(\xi) = |\xi|^{p-2}\xi$, $p > 1$. Based on the coincidence degree theory, Chen *et al.* gave new results about the problem

$$\begin{cases} {}^{0^+}\mathbb{G}_{C}^{\sigma_1}\phi_p({}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(\tau)) = \wp(\tau, \mathrm{q}(\tau), {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(\tau)), \quad \tau \in [0,1], \\ {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(0) = {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(1) = 0, \end{cases} \tag{4}$$

where $0 < \sigma_1, \sigma_2 \leq 1$ $(1 < \sigma_1 + \sigma_2 \leq 2)$ [20]. In 2018, Bai used the Guo–Krasnoselskii fixed point theorem and the Banach contraction mapping principle to prove the existence and uniqueness of positive solutions for the following fractional boundary value problem:

$$\begin{cases} (\phi_p({}^{0^+}\mathbb{G}_{\mathrm{RL}}^{\sigma_1}\mathrm{q}))'(\tau) + \wp(\tau, \mathrm{q}(\tau)) = 0, \quad 0 < \tau < 1, \\ \mathrm{q}(0) = {}^{0^+}\mathbb{G}_{\mathrm{RL}}^{\sigma_1}\mathrm{q}(0) = {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(0) = {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(1) = 0, \end{cases} \tag{5}$$

where $0 < \sigma_2 \leq 1$, $2 < \sigma_1 < 2 + \sigma_2$, ${}^{0^+}\mathbb{G}_{\mathrm{RL}}^{\sigma_1}$ and ${}^{0^+}\mathbb{G}_{C}^{\sigma_2}$ are the Riemann–Liouville and Caputo fractional derivatives of orders σ_1, σ_2, respectively, $p > 1$, and $\wp : [\tau_1, \tau_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function [21]. Using the coincidence degree theory, Tang *et al.* gave a new result on the existence of positive solutions to the fractional boundary value problem

$$\begin{cases} {}^{0^+}\mathbb{G}_{C}^{\sigma_1}(\phi_p {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q})(\tau) = \wp(\tau, \mathrm{q}(\tau), {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(\tau)), \quad 0 < \sigma_1, \sigma_2 \leq 1, \\ \mathrm{q}(0) = 0, \qquad {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(0) = {}^{0^+}\mathbb{G}_{C}^{\sigma_2}\mathrm{q}(1), \end{cases} \tag{6}$$

where $1 < \sigma_1 + \sigma_2 \leq 2$ and ${}^{0^+}\mathbb{G}_{C}^{\sigma_i}$ $(i = 1, 2)$ denotes the Caputo fractional derivatives [13]. Torres studied the existence and multiplicity for a mixed-order three-point boundary value problem of fractional differential equation involving Caputo's differential operator and the boundary conditions with integer order derivatives

$$\begin{cases} (\phi_p({}^{0^+}\mathbb{G}_{C}^{\sigma}\mathrm{q}))'(\tau) + \hbar(\tau)\wp(\tau, \mathrm{q}(\tau)) = 0, \quad 0 < \tau < 1, \\ {}^{0^+}\mathbb{G}_{C}^{\sigma}\mathrm{q}(0) = \mathrm{q}(0) = \mathrm{q}''(0) = 0, \qquad \mathrm{q}'(1) = \eta \mathrm{q}'(\lambda), \end{cases} \tag{7}$$

where $\eta, \lambda \in (0,1)$, $\sigma \in (2,3]$ [12]. In 2022, Alkhazzan *et al.* proved the existence and uniqueness as well as the Hyers–Ulam stability for the following general system of nonlinear hybrid fractional differential equations under p-Laplacian operator:

$$\begin{cases} {}^{0^+}\mathbb{G}_C^{\sigma_{12}}(\phi_p({}^{0^+}\mathbb{G}_C^{\sigma_{11}}(\mathrm{q}_1(\tau) - \top_{12}(\tau, \mathrm{q}_2(\tau))))) = -\top_{11}(\tau, \mathrm{q}_2(\tau)), \\ {}^{0^+}\mathbb{G}_C^{\sigma_{22}}(\phi_p({}^{0^+}\mathbb{G}_C^{\sigma_{21}}(\mathrm{q}_2(\tau) - \top_{22}(\tau, \mathrm{q}_1(\tau))))) = -\top_{21}(\tau, \mathrm{q}_1(\tau)), \\ [\phi_p({}^{0^+}\mathbb{G}_C^{\sigma_{11}}(\mathrm{q}_1(\tau) - \top_{12}(\tau, \mathrm{q}_2(\tau))))]_{\tau=0}^{(i)} \\ \quad = [\phi_p({}^{0^+}\mathbb{G}_C^{\sigma_{11}}(\mathrm{q}_1(\tau) - \top_{12}(\tau, \mathrm{q}_2(\tau))))]'_{\tau=\lambda} = 0, \\ [\phi_p({}^{0^+}\mathbb{G}_C^{\sigma_{21}}(\mathrm{q}_2(\tau) - \top_{22}(\tau, \mathrm{q}_1(\tau))))]_{\tau=0}^{(i)} \\ \quad = [\phi_p({}^{0^+}\mathbb{G}_C^{\sigma_{21}}(\mathrm{q}_2(\tau) - \top_{22}(\tau, \mathrm{q}_1(\tau))))]'_{\tau=\lambda} = 0 \end{cases} \tag{8}$$

for $i \in \mathbb{R}_0^{m-1} \setminus \{1\}$, under the conditions

$$\left[\top_{12}\left(\tau, \mathrm{q}_2(\tau)\right)\right]_{\tau=0}^{(i)} = \left[\top_{22}\left(\tau, \mathrm{q}_1(\tau)\right)\right]_{\tau=0}^{(i)} = 0$$

for $i \in \mathbb{R}_0^{m-1}$,

$$\mathrm{q}_1^{(i)}(\tau)\|_{\tau=0} = \mathrm{q}_1^{(m-1)}(\tau)\|_{\tau=1} = 0, \qquad \mathrm{q}_2^{(i)}(\tau)\|_{\tau=0} = \mathrm{q}_2^{(m-1)}(\tau)\|_{\tau=1} = 0$$

for $i \in \mathbb{R}_1^{m-2}$, and

$$\mathrm{q}_1(1) - \frac{1}{(m-1)!}\mathrm{q}_1^{(m-1)}(0) = 0, \qquad \mathrm{q}_2(1) - \frac{1}{(m-1)!}\mathrm{q}_2^{(m-1)}(0) = 0,$$

where ${}^{0^+}\mathbb{G}_C^{\sigma_{ij}}$, $i,j = 1,2$, are the Caputo fractional derivatives with $m-1 < \sigma_{ij} \leq m$ and m is a nonnegative integer number, $\top_{ij}$ is a continuous function and belongs to $L[0,1]$, $\phi_p(\tau) = |\tau|^{p-2}\tau$ is a p-Laplacian operator, where $\phi_q = \phi_p^{-1}$ and $\frac{1}{p} + \frac{1}{q} = 1$ [14]. For more recent works of the models, we refer to [22–34].

In this work, we study the following p-Laplacian fractional boundary value problem:

$$\begin{cases} {}^{\rho_2;\grave{\imath}^-}\mathbb{G}_{\mathrm{CK}}^{\sigma_2}(\phi_p({}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\mathrm{q}))(\tau) + \hbar(\tau)\wp(\mathrm{q}(\tau)) = 0, \quad \grave{a} < \tau < \grave{\imath}, \\ \mathrm{q}(\grave{a}) - F_\circ({}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\mathrm{q}(\grave{a})) = 0, \\ \delta_{\rho_1}^2\mathrm{q}(\grave{a}) = 0, \qquad \delta_{\rho_1}^1\mathrm{q}(\grave{\imath}) = \mu\delta_{\rho_1}^1\mathrm{q}(\eta) + \lambda, \\ {}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\mathrm{q}(\grave{\imath}) = -\delta_{\rho_2}^1[\phi_p({}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\mathrm{q})](\grave{a}) \\ \qquad\qquad = \delta_{\rho_2}^2[\phi_p({}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\mathrm{q})](\grave{\imath}) = 0, \end{cases} \tag{9}$$

where ${}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}$ and ${}^{\rho_2;\grave{\imath}^-}\mathbb{G}_{\mathrm{CK}}^{\sigma_2}$, $(\rho_1, \rho_2 \in \mathbb{R} \setminus \{1\})$ are the right- and left-sided Caputo–Katugampola fractional derivatives, $2 < \sigma_1, \sigma_2 \leq 3$, ϕ_p is the p-Laplacian operator, i.e., $\phi_p(\xi) = |\xi|^{p-2}\xi$, $p > 1$,

$$\delta_\rho^k = \left(\tau^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^k,$$

$F_\circ$ is a continuous even function, $\wp$, $\hbar$ are continuous and positive functions. $\eta \in (\grave{a}, \grave{\imath})$, $0 \leq \mu < 1$, and $\lambda \geq 0$. In this paper, we obtain some sufficient conditions ensuring the existence of at least one, two, and three positive solutions for fractional boundary value problem (9). These results can be extended in some works such as [35–37].

The rest of the paper is organized as follows. Section 2 presents some basic definitions, lemmas, and preliminary results. In Sect. 3, we derive some conditions on the parameter λ to obtain the existence of at least one positive solution. We derive an interval for λ, which ensures the existence of ρ-concave positive solutions of the fractional boundary value problem in Sect. 4. In Sect. 5, we discuss the existence of multiple positive solutions. Finally, we give some illustrative examples in Sect. 6.

2 Preliminaries and background material

In addition to the notations introduced with problem (9), let $J = [\grave{a}, \grave{\imath}] \subset (0, \infty)$, and $\rho > 0$,

1: $C(J)$ denotes the Banach space of continuous functions q on J endowed with the norm $\|\mathrm{q}\|_C = \max_{\tau \in J} |\mathrm{q}(\tau)|$, and

$$C^+(J) = \{\mathrm{q} \in C(J) : \mathrm{q}(\tau) \geq 0 \ \forall \tau \in J\}.$$

2: $AC(J)$ and $C^n(J)$ denote the spaces of absolutely continuous and n times continuously differentiable functions on J respectively.

3: $L^p(\grave{a}, \grave{\imath})$ denotes the space of Lebesgue integrable functions on $(\grave{a}, \grave{\imath})$.

4: $C^n_\rho(J)$ is the Banach space of n continuously differentiable functions on J with respect to δ_ρ:

$$C^n_\rho(J) = \{\mathrm{q} \in C(J) : \delta^k_\rho \mathrm{q} \in C(J), k = 0, 1, \dots, n\},$$

endowed with the norm

$$\|\mathrm{q}\|_{C^n_\rho} = \sum_{k=0}^{n} \|\delta^k_\rho \mathrm{q}\|_C.$$

5: $[\sigma]$ is the largest integer less than or equal to σ. Throughout the paper, we use $n = [\sigma]$ if σ is an integer and $n = [\sigma] + 1$ otherwise.

2.1 Fractional calculus

We present basic definitions and lemmas from fractional calculus theory [1, 2, 5–7].

Definition 2.1 (Function space) For $r \in \mathbb{R}$, consider the Banach space

$$\mathcal{M}^p_r(\grave{a}, \grave{\imath}) = \left\{\mathrm{q} : J \to \mathbb{R} : \|\mathrm{q}\|_{\mathcal{M}^p_r} := \left(\int_{\grave{a}}^{\grave{\imath}} |\tau^r \mathrm{q}(\tau)|^p \frac{\mathrm{d}\tau}{\tau}\right)^{1/p} < +\infty\right\}.$$

Remark 2.1 If $r \in \mathbb{R}^\star_+$ and $\grave{\imath} \leq (pr)^{1/pr}$, then $C(J) \hookrightarrow \mathcal{M}^p_r(J)$ and $\|\mathrm{q}\|_{\mathcal{M}^p_r} \leq \|\mathrm{q}\|_C$ for each $\mathrm{q} \in C(J)$.

Now, we recall the Katugampola and Caputo–Katugampola fractional integrals and derivatives [38].

Definition 2.2 The Katugampola left-sided ${}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma}$ and right-sided ${}^{\rho;\grave{\imath}^-}\mathbb{I}_K^{\sigma}$ fractional integrals of noninteger order $\alpha > 0$ of a function $\mathrm{q} \in \mathcal{M}_c^p(a,T)$ are defined by

$$
\begin{aligned}
{}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma}\mathrm{q}(\tau) &= \frac{\rho^{1-\tau}}{\Gamma(\tau)}\int_{\grave{a}}^{\tau}\left(\tau^{\rho}-\xi^{\rho}\right)^{\sigma-1}\xi^{\rho-1}\mathrm{q}(\xi)\,\mathrm{d}\xi, \quad \tau > \grave{a}, \\
{}^{\rho;\grave{\imath}^-}\mathbb{I}_K^{\sigma}\mathrm{q}(\tau) &= \frac{\rho^{1-\sigma}}{\Gamma(\sigma)}\int_{\tau}^{\grave{\imath}}\left(\xi^{\rho}-\tau^{\rho}\right)^{\sigma-1}\xi^{\rho-1}\mathrm{q}(\xi)\,\mathrm{d}\xi, \quad \tau < \grave{\imath}.
\end{aligned}
$$

The Katugampola fractional derivatives of q are defined by

$$
\begin{aligned}
{}^{\rho;\grave{a}^+}\mathbb{G}_K^{\sigma}\mathrm{q}(\tau) &= \delta_{\rho}^{n}\left({}^{\rho;\grave{a}^+}\mathbb{I}_K^{n-\sigma}\mathrm{q}\right)(\tau) \\
&= \frac{\rho^{1-n+\sigma}}{\Gamma(n-\sigma)}\left(\tau^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{n}\int_{\grave{a}}^{\tau}\left(\tau^{\rho}-\xi^{\rho}\right)^{n-\sigma-1}\xi^{\rho-1}\mathrm{q}(\xi)\,\mathrm{d}\xi, \\
{}^{\rho;\grave{\imath}^-}\mathbb{D}_K^{\sigma}\mathrm{q}(\tau) &= (-1)^{n}\delta_{\rho}^{n}\left({}^{\rho;\grave{\imath}^-}\mathbb{I}_K^{n-\sigma}\right)\mathrm{q}(\tau) \\
&= \frac{(-1)^{n}\rho^{1-n+\sigma}}{\Gamma(n-\sigma)}\left(\tau^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{n}\int_{\tau}^{\grave{\imath}}\left(\xi^{\rho}-\tau^{\rho}\right)^{n-\sigma-1}\xi^{\rho-1}\mathrm{q}(\xi)\,\mathrm{d}\xi.
\end{aligned}
$$

When σ is integer, we consider the ordinary definition.

In the following, we present some properties for left-sided integrals and derivatives. But the same properties are also true for the right-sided ones.

Lemma 2.3 ([38]) *Let $r \in \mathbb{R}$, $\sigma_1, \sigma_2, \rho > 0$, and $1 \le p \le \infty$. Then, on $\mathcal{M}_r^p(\grave{a},\grave{\imath})$, we have the following*:

(i) ${}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_1} : \mathcal{M}_r^p(\grave{a},\grave{\imath}) \to \mathcal{M}_r^p(\grave{a},\grave{\imath})$;
(ii) ${}^{\rho;\grave{a}^+}\mathbb{D}_K^{\sigma_1}$ *and* ${}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_1}$ *are linear*;
(iii) ${}^{\rho;\grave{a}^+}\mathbb{D}_K^{\sigma_1} \circ {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_1} = \mathrm{I}_d$, ${}^{\rho;\grave{a}^+}\mathbb{D}_K^{\sigma_1}({}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_2}\mathrm{q})(\tau) = {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_2-\sigma_1}\mathrm{q}(\tau)$ *when* $\sigma_2 \ge \sigma_1$;
(iv) ${}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_1} \circ {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_2} = {}^{\rho;\grave{a}^+}\mathbb{D}_K^{\sigma_1+\sigma_2}$.

Definition 2.4 ([38]) The Caputo–Katugampola fractional derivatives of a function $\mathrm{q} \in C_{\delta}^{n}([\grave{a},\grave{\imath}])$ (or $\in AC_{\delta}^{n}([\grave{a},\grave{\imath}])$) are defined by

$$
{}^{\rho;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma}\mathrm{q}(\tau) = {}^{\rho;\grave{a}^+}\mathbb{I}_K^{n-\sigma}\delta_{\rho_1}^{n}\mathrm{q}(\tau)
$$

and

$$
{}^{\rho;\grave{\imath}^-}\mathbb{G}_{\mathrm{CK}}^{\sigma}\mathrm{q}(\tau) = (-1)^{n}\left({}^{\rho;\grave{\imath}^-}\mathbb{I}_K^{n-\sigma}\delta_{\rho_1}^{n}\mathrm{q}(\tau)\right).
$$

Lemma 2.5 ([38]) *The Caputo–Katugampola fractional derivatives of a function* $\mathrm{q} \in C_{\delta}^{n}(J)$ (*or* $\in AC_{\delta}^{n}(J)$) *can also be written as*

$$
{}^{\rho;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma}\mathrm{q}(\tau) = \left({}^{\rho;\grave{a}^+}\mathbb{G}_K^{\sigma}\right)\left[\mathrm{q}(\tau) - \sum_{k=0}^{n-1}\frac{\delta_{\rho}^{k}\mathrm{q}(\tau)}{k!}\bigg|_{\grave{a}}\left(\frac{\tau^{\rho}-\grave{a}^{\rho}}{\rho}\right)^{k}\right], \tag{10}
$$

$$
{}^{\rho;\grave{\imath}^-}\mathbb{G}_{\mathrm{CK}}^{\sigma}\mathrm{q}(\tau) = \left({}^{\rho;\grave{\imath}^-}\mathbb{G}_K^{\sigma}\right)\left[\mathrm{q}(\tau) - \sum_{k=0}^{n-1}(-1)^{k}\frac{\delta_{\rho}^{k}\mathrm{q}(\tau)}{k!}\bigg|_{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho}-\tau^{\rho}}{\rho}\right)^{k}\right]. \tag{11}
$$

Lemma 2.6 ([38]) *Let $\sigma_2 > \sigma_1 > 0$, $\mathrm{q} \in \mathcal{M}_r^p(\grave{a}, \grave{\imath})$, $\mathrm{q} \in AC_\delta^n(J)$, or $C_\delta^n(J)$. Then we have*

$$^{\rho;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\big({}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_2}\mathrm{q}(\tau)\big) = {}^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_2-\sigma_1}\mathrm{q}(\tau),$$

and for some real constants N_k and M_k,

$$^{\rho;\grave{a}^+}\mathbb{I}_K^{\sigma_1}\big({}^{\rho;\grave{a}^+}\mathbb{D}_{\mathrm{CK}}^{\sigma_1}\mathrm{q}\big)(\tau) = \mathrm{q}(\tau) - \sum_{k=0}^{n-1} N_k\left(\frac{\tau^\rho - \grave{a}^\rho}{\rho}\right)^k, \tag{12}$$

$$^{\rho;\grave{\imath}^-}\mathbb{I}_K^{\sigma_1}\big({}^{\rho;\grave{\imath}^-}\mathbb{D}_{\mathrm{CK}}^{\sigma_1}\mathrm{q}\big)(\tau) = \mathrm{q}(\tau) - \sum_{k=0}^{n-1} M_k\left(\frac{\grave{\imath}^\rho - \tau^\rho}{\rho}\right)^k. \tag{13}$$

Lemma 2.7 ([2]) *If $^{\rho;\grave{a}^+}\mathbb{D}_{\mathrm{CK}}^{\sigma_1}\mathrm{q} \in C(J)$, then $\mathrm{q} \in C_\rho^{n-1}(J)$.*

2.2 Fixed point theorems

Let $\mathfrak{E}$ be a real Banach function space, endowed with the infinity norm. A nonempty closed convex set $K \subset \mathfrak{E}$ is called cone

(i) if for each $\mathrm{q} \in K$ and for all $\lambda > 0$: $\lambda\mathrm{q} \in K$;

(ii) for all $\mathrm{q} \in K$, if $-\mathrm{q} \in K$, then $\mathrm{q} = 0$.

A continuous operator is called completely continuous operator if it maps bounded sets into precompact sets. Let K be a cone, $\ell > 0$,

$$\Omega_\ell = \{\mathrm{q} \in K : \|\mathrm{q}\| < \ell\},$$

and $\mathbf{i}$ is the fixed point index function.

Theorem 2.8 ([39, 40]) *Let $\mathcal{L} : K \cap \overline{\Omega}_\ell \to K$ be a completely continuous operator such that $\mathcal{L}\mathrm{q} \neq \mathrm{q}$, $\forall \mathrm{q} \in \partial\Omega_\ell$. Then*

(i) *if $\|\mathcal{L}\mathrm{q}\| \leq \|\mathrm{q}\|$ for all $\mathrm{q} \in \partial\Omega_\ell$, then $\mathbf{i}(\mathcal{L}, \Omega_\ell, K) = 1$;*

(ii) *if $\|\mathcal{L}\mathrm{q}\| \geq \|\mathrm{q}\|$ for all $\mathrm{q} \in \partial\Omega_\ell$, then $\mathbf{i}(\mathcal{L}, \Omega_\ell, K) = 0$.*

Theorem 2.9 (Guo–Krasnoselskii [1]) *Assume that Ω_1 and Ω_2 are open subsets of $\mathfrak{E}$ with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $\mathcal{L} : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator. Consider*

(D1) *$\|\mathcal{L}\mathrm{q}\| \leq \|\mathrm{q}\|$ for all $\mathrm{q} \in K \cap \partial\Omega_1$ and $\|\mathcal{L}\mathrm{q}\| \geq \|\mathrm{q}\|$ for all $\mathrm{q} \in K \cap \partial\Omega_2$;*

(D2) *$\|\mathcal{L}\mathrm{q}\| \leq \|\mathrm{q}\|$, $\forall \mathrm{q} \in K \cap \partial\Omega_2$ and $\|\mathcal{L}\mathrm{q}\| \geq \|\mathrm{q}\|$, $\forall \mathrm{q} \in K \cap \partial\Omega_1$.*

If (D1) or (D2) holds, then $\mathcal{L}$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

2.3 Convexity

Let $\mathrm{q} : J \to (0, \infty)$ be continuous.

Definition 2.10 ([41, 42]) We say that q is ρ-convex if

$$\mathrm{q}\Big(\big[(1-\eta)\tau^\rho + \eta\acute{\tau}^\rho\big]^{\frac{1}{\rho}}\Big) \leq (1-\eta)\mathrm{q}(\tau) + \eta\mathrm{q}(\acute{\tau})$$

for each $\tau, \acute{\tau} \in J$, and $\eta \in [0, 1]$. q is called ρ-concave if $(-\mathrm{q})$ is ρ-convex.

Remark 2.2 ([41, 42])

1. q is ρ-convex (concave) if and only if $\wp(\varphi^{-1})$ is convex (concave), where $\varphi(\tau) = \frac{\tau^{\rho}}{\rho}$.
2. $\wp$ is ρ-convex (concave) if and only if $\delta_{\rho}\wp(\mathrm{q})$ is increasing (decreasing).

The following technical hypotheses will be used later.

(H1) $\hbar$ does not vanish identically on any closed subinterval of $(\grave{a}, \grave{\imath})$.

(H2) $F_{\circ}$ is even and continuous on $\mathbb{R}^{+}$, and there exist $A, B > 0$:

$$Bv^{p-1} \le F_{\circ}(v) \le Av^{p-1} \quad \big(v \in \mathbb{R}^{+}\big).$$

3 Main results

We present some important lemmas which assist in proving our main results. Consider the linear generalized fractional boundary value problem associated with (9)

$$\begin{cases} {}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau) + \mathrm{w}(\tau) = 0, \quad \grave{a} < \tau < \grave{\imath}, \\ \mathrm{q}(\grave{a}) - F_{\circ}({}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\grave{a})) = 0, \\ \delta^2_{\rho_1}\mathrm{q}(\grave{a}) = 0, \qquad \delta^1_{\rho_1}\mathrm{q}(\grave{\imath}) - \mu\delta^1_{\rho_1}\mathrm{q}(\eta) = \lambda. \end{cases} \tag{14}$$

Lemma 3.1 *For* $\mathrm{w} \in C(J)$, *the integral solution of* (14) *is given by*

$$\begin{aligned} \mathrm{q}(\tau) &= \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\tau,\xi)\mathrm{w}(\xi)\,d\xi + \mu\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\mathrm{w}(\xi)\,d\xi \\ &\quad + \lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right) + F_{\circ}\big(\mathrm{w}(\grave{a})\big) \end{aligned} \tag{15}$$

for $\tau, \xi \in J$, *where*

$$\mathcal{G}_1(\tau,\xi) = \begin{cases} \frac{1}{\Gamma(\sigma_1-1)}(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1})(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-2}\xi^{\rho_1-1} \\ \quad - \frac{1}{\Gamma(\sigma_1)}(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-1}\xi^{\rho_1-1}, & \xi \le \tau, \\ \frac{1}{\Gamma(\sigma_1-1)}(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1})(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-2}\xi^{\rho_1-1}, & \tau \le \xi, \end{cases} \tag{16}$$

and

$$\mathcal{G}_2(\tau,\xi) = \begin{cases} \frac{1}{\Gamma(\sigma_1-1)}(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1})(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-2}\xi^{\rho_1-1} \\ \quad - \frac{1}{\Gamma(\sigma_1-1)}(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-2}\xi^{\rho_1-1}, & \xi \le \tau, \\ \frac{1}{\Gamma(\sigma_1-1)}(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1})(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-2}\xi^{\rho_1-1}, & \tau \le \xi. \end{cases} \tag{17}$$

Proof By applying (12), equation (14) becomes

$$\begin{aligned} \mathrm{q}(\tau) &= -l_0 - l_1\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right) - l_2\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^2 \\ &\quad - \frac{\rho_1^{1-\sigma_1}}{\Gamma(\sigma_1)}\int_{\grave{a}}^{\tau}\big(\grave{a}^{\rho_1} - \xi^{\rho_1}\big)^{\sigma_1-1}\xi^{\rho_1-1}\mathrm{w}(\xi)\,d\xi \end{aligned}$$

for some arbitrary constants $l_0, l_1, l_2 \in \mathbb{R}$. From the boundary conditions of (14) we get

$$\begin{aligned}
\mathrm{q}(\tau) &= F_\circ\big(\mathrm{w}(\grave{a})\big) + \lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right) \\
&\quad - \frac{1}{\Gamma(\sigma_1)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 1}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi \\
&\quad + \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\frac{1}{(1-\mu)\Gamma(\sigma_1 - 1)}\left[\int_{\grave{a}}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi\right. \\
&\quad \left. - \mu\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1}\mathrm{w}(\xi)\,\mathrm{d}\xi\right] \\
&= F_\circ\big(\mathrm{w}(\grave{a})\big) + \lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right) \\
&\quad - \frac{1}{\Gamma(\sigma_1)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 1}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi \\
&\quad + \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\left[\frac{1}{\Gamma(\sigma_1 - 1)}\right. \\
&\quad \left. + \frac{\mu}{\Gamma(\sigma_1 - 1)(1-\mu)}\right]\int_{\grave{a}}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi \\
&\quad - \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\frac{\mu}{(1-\mu)\Gamma(\sigma_1 - 1)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Splitting the second integral in two parts permits us to write

$$\begin{aligned}
\mathrm{q}(\tau) &= \frac{1}{\Gamma(\sigma_1)}\left[(\sigma_1 - 1)\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\int_{\grave{a}}^{\tau}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\sigma_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi\right. \\
&\quad - \int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 1}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi \\
&\quad \left. + (\sigma_1 - 1)\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)\int_{\tau}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi\right] \\
&\quad + \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right)\frac{1}{\Gamma(\sigma_1 - 1)}\left[\mu\left(\int_{\grave{a}}^{\tau}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi\right.\right. \\
&\quad \left. + \int_{\tau}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi\right) \\
&\quad \left. - \mu\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1}\right)^{\rho - 2}\xi^{\rho_1 - 1}\mathrm{w}(\xi)\,\mathrm{d}\xi\right] \\
&\quad + F_\circ\big(\mathrm{w}(\grave{a})\big) + \lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1(1-\mu)}\right) \\
&= \int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\mathrm{w}(\xi) + \mu\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{(1-\mu)\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi \\
&\quad + \lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{(1-\mu)\rho_1}\right) + F_\circ\big(\mathrm{w}(\grave{a})\big).
\end{aligned}$$

The converse follows by direct computation. The proof is completed. □

Now, consider the generalized p-Laplacian fractional boundary value problem associated with (9)

$$\begin{cases} {}^{\rho_2;\grave{\imath}^-}\mathbb{G}^{\sigma_2}_{\mathrm{CK}}(\phi_p({}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau))) = \mathrm{w}(\tau), \quad \grave{a} < \tau < \grave{\imath}, \\ \mathrm{q}(\grave{a}) - F_\circ({}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\grave{a})) = 0, \\ \delta^2_{\rho_1}\mathrm{q}(\grave{a}) = 0, \qquad \delta^1_{\rho_1}\mathrm{q}(\grave{\imath}) - \mu\delta^1_{\rho_1}\mathrm{q}(\eta) = \lambda, \\ {}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\grave{\imath}) = \delta^1_{\rho_2}[\phi_p({}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q})](\grave{a}) \\ \qquad = \delta^2_{\rho_2}[\phi_p({}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q})](\grave{\imath}) = 0. \end{cases} \tag{18}$$

Lemma 3.2 *For* $\mathrm{w}(\tau) \in C^+(J)$, *fractional boundary value problem* (18) *has a unique solution*

$$\begin{aligned} \mathrm{q}(\tau) &= \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi \\ &\quad + \mu\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi \\ &\quad + \lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) + F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right)\right), \end{aligned} \tag{19}$$

where

$$\mathcal{H}(\tau,\xi) = \begin{cases} \frac{1}{\Gamma(\sigma_2-1)}(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\rho_2})(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2})^{\sigma_2-2}\xi^{\rho_2-1} \\ \quad - \frac{1}{\Gamma(\sigma_2)}(\frac{\xi^{\rho_2}-\tau^{\rho_2}}{\rho_2})^{\sigma_2-1}\xi^{\rho_2-1}, & \tau \le \xi, \\ \frac{1}{\Gamma(\sigma_2-1)}(\frac{\grave{\imath}^{\rho_2}-\tau^{\rho_2}}{\rho_2})(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2})^{\sigma_2-2}\xi^{\rho_2-1}, & \xi \le \tau, \end{cases} \tag{20}$$

$\mathcal{G}_1(\tau,\xi)$, $\mathcal{G}_2(\tau,\xi)$ *are defined in Lemma* 3.1 *and* $\bar{p} = \frac{p}{p-1}$.

Proof From Lemma 2.6, equation (18) is equivalent to the equation

$$\begin{aligned} \phi_{\bar{p}}\left({}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau)\right) &= -l_0 - l_1\left(\frac{\grave{\imath}^{\rho_2} - \tau^{\rho_2}}{\rho_2}\right) - l_2\left(\frac{\grave{\imath}^{\rho_2} - \tau^{\rho_2}}{\rho_2}\right)^2 \\ &\quad + {}^{\rho_2;\grave{\imath}^-}\mathbb{G}^{\sigma_2}_{\mathrm{CK}}\mathrm{w}(\tau) \end{aligned}$$

for some constants $l_0, l_1, l_2 \in \mathbb{R}$. Using the second boundary condition, we get

$$\begin{aligned} \phi_{\bar{p}}\left({}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau)\right) &= {}^{\rho_2;\grave{\imath}^-}\mathbb{I}^{\sigma_2}_{K}\mathrm{w}(\tau) \\ &\quad - \left(\frac{\grave{\imath}^{\rho_2} - \tau^{\rho_2}}{\rho_2}\right)\frac{1}{\Gamma(\sigma_2 - 1)}\int_{\grave{a}}^{\grave{\imath}}\left(\frac{\xi^{\rho_2} - \grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1}\mathrm{w}(\xi)\,\mathrm{d}\xi \\ &= -\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau,\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Consequently,

$${}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau) = -\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau,\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right).$$

Thus, problem (18) can be written as

$$\begin{cases} {}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\mathrm{q}(\tau) + \phi_{\bar{p}}(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau,\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi) = 0, \quad \tau \in (\grave{a},\grave{\imath}), \\ \mathrm{q}(\grave{a}) - F_\circ({}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\mathrm{q}(\grave{a})) = 0, \\ \delta_{\rho_1}^2\mathrm{q}(\grave{a}) = 0, \qquad \delta_{\rho_1}^1\mathrm{q}(\grave{\imath}) = \mu\delta_{\rho_1}^1\mathrm{q}(\eta) + \lambda, \end{cases} \tag{21}$$

which, according to Lemma 3.1, has a unique solution of the form (19). □

Lemma 3.3 *The functions $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{H}$, equations* (16), (17), *and* (20) *satisfy the following*:

(i) $\mathcal{G}_1(\tau,\xi)$, $\mathcal{G}_2(\tau,\xi)$, *and* $\mathcal{H}(\tau,\xi)$ *are continuous on* $[\grave{a},\grave{\imath}] \times [\grave{a},\grave{\imath}]$.

(ii) *For all* $(\tau,\xi) \in [\grave{a},\grave{\imath}] \times [\grave{a},\grave{\imath}]$,

$$\begin{aligned} \mathcal{G}_1(\tau,\xi) &\le \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-1} \frac{\grave{\imath}^{\rho_1-1}}{\Gamma(\sigma_1-1)} \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\tau,\xi)\,\mathrm{d}\xi \\ &= \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\Gamma(\sigma_1)\rho_1}\right)\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-1} \\ &\quad - \frac{1}{\Gamma(\sigma_1+1)}\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1}, \\ \mathcal{G}_2(\tau,\xi) &\le \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2} \frac{\grave{\imath}^{\rho_1-1}}{\Gamma(\sigma_1-1)} \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi \\ &= \frac{1}{\Gamma(\sigma_1)}\left(\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-1} - \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-1}\right), \\ \mathcal{H}(\tau,\xi) &\le \left(\frac{\grave{\imath}^{\rho_2} - \grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-1} \frac{\grave{\imath}^{\rho_2-1}}{\Gamma(\sigma_2-1)} \int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau,\xi)\,\mathrm{d}\xi \\ &= \frac{\grave{\imath}^{\rho_2} - \xi^{\rho_2}}{\rho_2\Gamma(\sigma_2)}\left(\left(\frac{\grave{\imath}^{\rho_2} - \grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-1} - \frac{1}{\sigma_2}\left(\frac{\grave{\imath}^{\rho_2} - \xi^{\rho_2}}{\rho_2}\right)^{\sigma_2-1}\right). \end{aligned}$$

(iii) *For all* $(\tau,\xi) \in [\grave{a},\grave{\imath}]^2 : \mathcal{G}_1(\tau,\xi) \ge 0$, $\mathcal{G}_2(\tau,\xi) \ge 0$, $\mathcal{H}(\tau,\xi) \ge 0$.

(iv) *For all* $\xi \in J$, *the function* $\tau \to \mathcal{G}_1(\tau,\xi)$ *is increasing and* $\tau \to \mathcal{H}(\tau,\xi)$ *is decreasing. In addition,* $\forall(\tau,\xi) \in (\grave{a},\grave{\imath})^2$ *we have*

$$\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1} \mathcal{G}_1(\grave{\imath},\xi) \le \mathcal{G}_1(\tau,\xi)$$

and

$$\left(\frac{\grave{\imath}^{\rho_2} - \tau^{\rho_2}}{\grave{\imath}^{\rho_2} - \grave{a}^{\rho_2}}\right)^{\sigma_2-1} \mathcal{H}(\grave{a},\xi) \le \mathcal{H}(\tau,\xi).$$

(v) *For all* $(\tau,\xi) \in (\grave{a},\grave{\imath})^2$, *we have*

$$\begin{aligned} &\frac{\tau^{\rho_1-1}\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\left[1 - \left(\frac{\tau}{\grave{\imath}}\right)^{\rho_1(\sigma_1-2)}\right]\mathcal{G}_1(\grave{\imath},\xi) \\ &\quad \le \mathcal{G}'_{1\tau}(\tau,\xi) \le \frac{\sigma_1 - 1}{\sigma_1 - 2}\frac{\tau^{\rho_1-1}\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\mathcal{G}_1(\grave{\imath},\xi). \end{aligned}$$

Proof Using the definitions of $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{H}$, (i) and (ii) are obtained straightforwardly. For property (iii), we only consider the case $\xi \leq \tau$ as the other case is straightforward. When $\xi \leq \tau$, we have

$$\begin{aligned}
\mathcal{G}_1(\tau,\xi) &\geq \frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\grave{a}^{\rho_1-1} \\
&\quad - \frac{1}{\Gamma(\sigma_1)}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-1}\grave{a}^{\rho_1-1} \\
&\geq \left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-1}\grave{a}^{\rho_1-1}\left[\frac{1}{\Gamma(\sigma_1-1)}-\frac{1}{\Gamma(\sigma_1)}\right] \geq 0,
\end{aligned}$$

because $\Gamma(\sigma_1-1) \leq \Gamma(\sigma_1)$ for $2 < \sigma_1 \leq 3$. Similarly, we can easily prove that $\mathcal{G}_2(\tau,\xi) \geq 0$ and $\mathcal{H}(\tau,\xi) \geq 0$, $\forall(\tau,\xi) \in J^2$. Now, for property (iv), we first check that $\mathcal{G}_1(\tau,\xi)$ is nondecreasing w.r.t. $\tau \in J$.

$$\frac{\partial \mathcal{G}_1}{\partial \tau}(\tau,\xi) = \begin{cases} \frac{\tau^{\rho_1-1}}{\Gamma(\sigma_1-1)}(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-2}\xi^{\rho_1-1} \\ \quad -\frac{\tau^{\rho_1-1}}{\Gamma(\sigma_1-1)}(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-2}\xi^{\rho_1-1}, & \xi \leq \tau, \\ \frac{\tau^{\rho_1-1}}{\Gamma(\sigma_1-1)}(\frac{\grave{\imath}^{\rho_1}-\xi^{\rho_1}}{\rho_1})^{\sigma_1-2}\xi^{\rho_1-1}, & \tau \leq \xi. \end{cases} \tag{22}$$

Thus, $\mathcal{G}_1(\tau,\xi)$ is increasing with respect to $\tau \in J$, and therefore $\mathcal{G}_1(\tau,\xi) \leq \mathcal{G}_1(\grave{\imath},\xi)$ for $\grave{a} \leq \tau$, $\xi \leq \grave{\imath}$. Furthermore, for $\tau \leq \xi$, we have

$$\begin{aligned}
\frac{\partial \mathcal{H}(\tau,\xi)}{\partial \tau} &= -\frac{\tau^{\rho_2-1}}{\Gamma(\sigma_2-1)}\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1} \\
&\quad + \frac{(\sigma_2-1)\tau^{\rho_2-1}}{\Gamma(\sigma_2)}\left(\frac{\xi^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1} \\
&= \frac{\tau^{\rho_2-1}}{\Gamma(\sigma_2-1)}\xi^{\rho_2-1}\left[\left(\frac{\xi^{\rho_2}-\tau^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}-\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\right] \\
&\leq \frac{\tau^{\rho_2-1}}{\Gamma(\sigma_2-1)}\xi^{\rho_2-1}\left[\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}-\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\right] = 0,
\end{aligned}$$

and for $\xi \leq \tau$, we have

$$\frac{\mathcal{H}(\tau,\xi)}{\partial \tau} = \frac{-\tau^{\rho_2-1}}{\Gamma(\sigma_2-1)}\left(\frac{\xi^{\rho_2}-\grave{a}^{\rho_2}}{\rho_2}\right)^{\sigma_2-2}\xi^{\rho_2-1} \leq 0.$$

Thus, $\mathcal{H}(\tau,\xi)$ is nonincreasing with respect to τ. Consequently, $\mathcal{H}(\tau,\xi) \leq \mathcal{H}(\grave{a},\xi)$, $\forall\tau$, $\xi \in J$. On the other hand, when $\tau \geq \xi$,

$$\begin{aligned}
\frac{\mathcal{G}_1(\tau,\xi)}{\mathcal{G}_1(\grave{\imath},\xi)} &= \frac{(\sigma_1-1)(\tau^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\sigma_1-1)(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}} \\
&= \frac{1}{(\sigma_1-1)(\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\imath}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}} \\
&\quad \times \left[(\sigma_1-1)\left(\tau^{\rho_1}-\grave{a}^{\rho_1}\right)\left(\grave{\imath}^{\rho_1}-\xi^{\rho_1}\right)^{\alpha-2}\right. \\
&\quad \left. -\left(\tau^{\rho_1}-\xi^{\rho_1}\right)^{\sigma_1-1}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\tau^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\right].
\end{aligned}$$

As

$$\left(\frac{\tau^{\rho}-\xi^{\rho}}{\tau^{\rho}-\grave{a}^{\rho}}\right)^{\sigma}\leq\left(\frac{\grave{\iota}^{\rho}-\xi^{\rho}}{\grave{\iota}^{\rho}-\grave{a}^{\rho}}\right)^{\sigma},$$

for $\sigma>0$, we obtain

$$\begin{aligned}
\frac{\mathcal{G}_1(\tau,\xi)}{\mathcal{G}_1(\grave{\iota},\xi)} &\geq \frac{1}{(\sigma_1-1)(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}} \\
&\quad\times\Bigg[(\sigma_1-1)\big(\tau^{\rho_1}-\grave{a}^{\rho_1}\big)\big(\grave{\iota}^{\rho_1}-\xi^{\rho_1}\big)^{\sigma_1-2} \\
&\quad-\big(\tau^{\rho_1}-\xi^{\rho_1}\big)^{\sigma_1-1}\left(\frac{\grave{\iota}^{\rho_1}-\xi^{\rho_1}}{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\Bigg] \\
&\geq \frac{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}} \\
&\quad\times\frac{1}{(\sigma_1-1)(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}} \\
&\quad\times\big[(\sigma_1-1)\big(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}\big)^{\sigma_1-1}\big(\tau^{\rho_1}-\grave{a}^{\rho_1}\big)^{2-\sigma_1}\big(\grave{\iota}^{\rho_1}-\xi^{\rho_1}\big)^{\sigma_1-2} \\
&\quad-\big(\grave{\iota}^{\rho_1}-\xi^{\rho_1}\big)^{\sigma_1-1}\big] \\
&\geq \frac{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}} \\
&\quad\times\frac{1}{(\sigma_1-1)(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}} \\
&\quad\times\Bigg[\left(\frac{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}{\tau^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-2}(\sigma_1-1)\big(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}\big)\big(\grave{\iota}^{\rho_1}-\xi^{\rho_1}\big)^{\sigma_1-2} \\
&\quad-\big(\grave{\iota}^{\rho_1}-\xi^{\rho_1}\big)^{\sigma_1-1}\Bigg] \\
&\geq \frac{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}.
\end{aligned}$$

For $\tau\leq\xi$, we have

$$\frac{\mathcal{G}_1(\tau,\xi)}{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}=\frac{\rho_1^{\sigma_1-1}\xi^{\rho_1-1}}{\Gamma(\sigma_1-1)}\big(\grave{\iota}^{\rho_1}-\xi^{\rho_1}\big)^{\sigma_1-2}\frac{1}{(t^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-2}},$$

which is a nonincreasing function as $\sigma_1\geq 0$. Consequently,

$$\frac{\mathcal{G}_1(\tau,\xi)}{(\tau^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}}\geq\frac{\mathcal{G}(\grave{\iota},\xi)}{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})^{\sigma_1-1}},$$

which implies

$$\mathcal{G}_1(\tau,\xi)\geq\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\mathcal{G}_1(\grave{\iota},\xi).$$

Using similar techniques, one can prove that

$$\mathcal{H}(\tau,\xi) \geq \left(\frac{\grave{\iota}^{\rho_2}-\tau^{\rho_2}}{\grave{\iota}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma-1}\mathcal{H}(\grave{a},\xi)$$

for $\grave{a} \leq \xi, \tau < \grave{\iota}$. Therefore (iv) of Lemma 3.3 holds. Finally, for property (v), we can consider two cases. Nevertheless, we prove the results for the case $\xi \leq \tau$ only. The simpler case $\grave{a} \leq \tau \leq \xi < \grave{\iota}$ can be treated with similar arguments. When $\xi \leq \tau$, we have

$$\frac{{\mathcal{G}_1}'_{\tau}(\tau,\xi)}{\mathcal{G}_1(\grave{\iota},\xi)}\frac{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})}{\tau^{\rho_1-1}\rho_1(\sigma_1-1)} = \frac{(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}}{(\sigma_1-1)(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-\frac{(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})}}.$$

Consequently,

$$\begin{aligned}
\frac{{\mathcal{G}_1}'_{\tau}(\tau,\xi)}{\mathcal{G}_1(\grave{\iota},\xi)}\frac{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})}{\tau^{\rho_1-1}\rho_1(\sigma_1-1)} &\leq \frac{(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}}{(\sigma_1-1)(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-\frac{(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})}} \\
&\leq \frac{1}{(\sigma_1-1)-\frac{(\grave{\iota}^{\rho_1}-\xi^{\rho_1})}{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})}} \\
&\leq \frac{1}{(\sigma_1-2)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{{\mathcal{G}_1}'_{\tau}(\tau,\xi)}{\mathcal{G}_1(\grave{\iota},\xi)}\frac{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})}{\tau^{\rho_1-1}\rho_1} &= \frac{(\sigma_1-1)[(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}]}{(\sigma_1-1)(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}-\frac{(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-1}}{(\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1})}} \\
&\geq 1-\frac{(\tau^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}}{(\grave{\iota}^{\rho_1}-\xi^{\rho_1})^{\sigma_1-2}} \\
&\geq 1-\left(\frac{\tau}{\grave{\iota}}\right)^{\rho_1(\sigma_1-2)}\left(\frac{1-(\frac{\xi}{\tau})^{\rho_1}}{1-(\frac{\xi}{\grave{\iota}})^{\rho_1}}\right)^{\sigma_1-2} \\
&\geq 1-\left(\frac{\tau}{\grave{\iota}}\right)^{\rho_1(\sigma_1-2)}.
\end{aligned}$$

Thus, the proof is completed. □

Now, consider the Banach space $\mathbb{E} = C^3_{\rho_1}(J)$. Suppose that ${}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau)$ is continuous on J for all $\mathrm{q} \in \mathbb{E}$, then from Definition 2.6 and Lemma 2.4 we can define the norm on $\mathbb{E}$ as follows:

$$\|\mathrm{q}\| = \begin{cases} \max\{\breve{M}_1, \max_{\tau\in J} |{}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau)|\}, & 2 < \sigma_1 < 3, \\ \max\{\breve{M}_1, \max_{\tau\in J} |\delta^3_{\rho_1}\mathrm{q}(\tau)|\}, & \sigma_1 = 3, \end{cases}$$

in which

$$\breve{M}_1 = \max\left\{\max_{\tau\in J}\left|\mathrm{q}(\tau)\right|, \max_{\tau\in J}\left|\delta^1_{\rho_1}\mathrm{q}(\tau)\right|, \max_{\tau\in J}\left|\delta^2_{\rho_1}\mathrm{q}(\tau)\right|\right\},$$

and the cone

$$K = \{\mathrm{q} \in \mathbb{E} : \mathrm{q} \text{ is nonnegative, increasing, and } \rho_1\text{-concave}\}.$$

Lemma 3.4 *Assume* (H2) *and let* q *be the unique solution of fractional boundary value problem* (18) *associated with given* $\mathrm{w}(\tau) \in C^+(J)$. *Then* $\mathrm{q} \in K$ *and the following inequalities hold for* $\tau \in [\grave{a}_\circ, \grave{\imath}_\circ] \subset (\grave{a}, \grave{\imath})$:

$$\max_{\tau\in J}\left|\mathrm{q}(\tau)\right| \le \left(\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1}\left(\frac{\grave{\imath}^{\rho_2} - \grave{\imath}_\circ^{\rho_2}}{\grave{\imath}^{\rho_2} - \grave{a}^{\rho_2}}\right)^{\sigma_1-1}\right)^{-1}\mathrm{q}(\tau), \tag{23}$$

$$\max_{\tau\in J}\left|\delta^1_{\rho_1}\mathrm{q}(\tau)\right| \le \frac{\sigma_1 - 1}{\sigma_1 - 2}\frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\max_{\tau\in J}\left|\mathrm{q}(\tau)\right|, \tag{24}$$

$$\begin{aligned}\max_{\tau\in J}\left|\delta^2_{\rho_1}\mathrm{q}(\tau)\right| \le{}& \left(\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\imath}_\circ)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi\right)^{-1}\\ &\times\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\max_{\tau\in J}\left|\mathrm{q}(\tau)\right|,\end{aligned} \tag{25}$$

$$\begin{aligned}\max_{\tau\in J}\left|{}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau)\right| \le{}& \left(\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\imath}_\circ)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi\right)^{-1}\\ &\times\max_{\tau\in J}\left|\mathrm{q}(\tau)\right|, \quad \forall\sigma_1\in(2,3],\end{aligned} \tag{26}$$

$$\min_{\tau\in[\grave{a}_\circ,\grave{\imath}_\circ]}\mathrm{q}(\tau) \ge \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1}\breve{M}_2\|\mathrm{q}\|, \tag{27}$$

where

$$Z(\tau) = \phi_{\bar{p}}\left(\left(\frac{\grave{\imath}^{\rho_2} - \tau^{\rho_2}}{\grave{\imath}^{\rho_2} - \grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)$$

and

$$\begin{aligned}\breve{M}_2 = \min&\left\{1, \frac{\sigma_1-2}{\sigma_1-1}\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right),\right.\\ &\min\left\{\Gamma(\sigma_1-1)\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{2-\sigma_1}, 1\right\}\\ &\left.\times\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1}\times Z(\grave{\imath}_\circ)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi\right\}.\end{aligned} \tag{28}$$

Proof From Lemma 3.2, we have

$$\begin{aligned}\mathrm{q}(\tau) ={}& \int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &+\mu\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &+\lambda\left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) + F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right)\right).\end{aligned}$$

(1) The functions $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{H}$ are nonnegative (Lemma 3.3(iii)). In addition, $F_\circ(\nu)$ is nonnegative for $\nu \geq 0$ (thanks to (H2)). Thus, q is also nonnegative. Furthermore, as $\mathcal{G}_1$ is increasing w.r.t. τ (Lemma 3.3(iv)), so it is the function q. To prove that q is ρ_1-concave, we need to show that $\delta^1_{\rho_1}\mathrm{q}(\tau)$ is decreasing on J (Remark 2.2), which can be obtained from the negativity of the derivative

$$\begin{aligned}\left(\delta^1_{\rho_1}\mathrm{q}(\tau)\right)' &= -\frac{\tau^{\rho_1-1}}{\Gamma(\sigma_1-2)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-3}\xi^{\rho_1-1}\\ &\quad\times \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi \leq 0.\end{aligned}$$

(2) As q is nonnegative and increasing, we have

$$\begin{aligned}\max_{\tau\in J}\left|\mathrm{q}(\tau)\right| &= \mathrm{q}(\grave{\imath})\\ &= \int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &\quad+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &\quad+\lambda\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right)\right).\end{aligned}$$

For $\tau\in[\grave{a}_\circ,\grave{\imath}_\circ]$, using (iv) of Lemma 3.3 and the fact that

$$\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)<1,$$

we get

$$\begin{aligned}\mathrm{q}(\tau) &\geq \int_{\grave{a}}^{\grave{\imath}}\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &\quad+\mu\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-2}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\\ &\quad\times\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &\quad+\lambda\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-2}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\\ &\quad+\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right)\right).\end{aligned}$$

Consequently,

$$\mathrm{q}(\tau)\geq\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\max_{t\in J}\left|\mathrm{q}(\tau)\right|,$$

and thus (23) holds.

(3) We have

$$\delta_{\rho_1}^1 q(\tau) = \tau^{1-\rho_1}\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1{}'_{\tau}(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi + \frac{\mu}{(1-\mu)}\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi + \frac{\lambda}{(1-\mu)}.$$

From Lemma 3.3 ((iii) and (v)), we can deduce that $\delta_{\rho_1}^1 q(\tau) \geq 0$ and

$$\begin{aligned}
\delta_{\rho_1}^1 q(\tau) &\leq \int_{\grave{a}}^{\grave{\imath}} \frac{\sigma_1-1}{\sigma_1-2}\frac{\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi \\
&\quad + \frac{\mu}{(1-\mu)}\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi + \frac{\lambda}{(1-\mu)} \\
&\leq \frac{\sigma_1-1}{\sigma_1-2}\frac{\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\Bigg[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi \\
&\quad + \mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)ss\right)d\xi \\
&\quad + \lambda\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\Bigg] \\
&\leq \frac{\sigma_1-1}{\sigma_1-2}\frac{\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\Bigg[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi \\
&\quad + \lambda\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right) \\
&\quad + \mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi \\
&\quad + F_{\circ}\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,d\xi\right)\right)\Bigg] \\
&\leq \frac{\sigma_1-1}{\sigma_1-2}\frac{\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}q(\grave{\imath}).
\end{aligned}$$

Thus, we obtain (24).

(4) A straightforward calculus gives

$$\delta_{\rho_1}^2 q(\tau) = -\frac{1}{\Gamma(\sigma_1-2)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-3}\xi^{\rho_1-1} \times \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\mathrm{w}(s)\,ds\right)d\xi.$$

Then we get

$$\begin{aligned}
\left|\delta_{\rho_1}^2 q(\tau)\right| &\leq \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,d\xi\right) \\
&\quad \times \frac{1}{\Gamma(\sigma_1-2)}\int_{\grave{a}}^{\tau}\left(\frac{\tau^{\rho_1}-\xi^{\rho_1}}{\rho_1}\right)^{\sigma_1-3}\xi^{\rho_1-1}\,d\xi \\
&\leq \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,d\xi\right)\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}.
\end{aligned}$$

Thus,

$$\max_{\tau\in J}\left|\delta^2_{\rho_1}\mathrm{q}(\tau)\right| \le \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right)\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}.$$

By multiplying both sides of the previous inequality by

$$\phi_{\bar{p}}\left(\left(\frac{\grave{\imath}^{\rho_2}-\xi^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right),$$

we get

$$\begin{aligned}
&\phi_{\bar{p}}\left(\left(\frac{\grave{\imath}^{\rho_2}-\xi^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)\max_{\tau\in J}\left|\delta^2_{\rho_1}\mathrm{q}(\tau)\right| \\
&\quad\le \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\left(\frac{\grave{\imath}^{\rho_2}-\xi^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right) \\
&\qquad\times\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2},
\end{aligned}$$

using Lemma 3.3(iv), we get

$$\begin{aligned}
&\phi_{\bar{p}}\left(\left(\frac{\grave{\imath}^{\rho_2}-\xi^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)\max_{\tau\in J}\left|\delta^2_{\rho_1}\mathrm{q}(\tau)\right| \\
&\quad\le \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\tau,\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right)\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}.
\end{aligned}\tag{29}$$

Multiplying both sides by $\mathcal{G}_1(\tau,\xi)$ and integrating over J w.r.t. ξ, we get

$$\begin{aligned}
&\max_{\tau\in J}\left|\delta^2_{\rho_1}\mathrm{q}(\tau)\right|\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\left(\frac{\grave{\imath}^{\rho_2}-\xi^{\rho_2}}{\grave{\imath}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)\mathrm{d}\xi \\
&\quad\le\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi) \\
&\qquad\times\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi \\
&\quad\le\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\left[\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\right. \\
&\qquad\times\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi+\lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right) \\
&\qquad+\mu\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\mathrm{w}(s)\,\mathrm{d}s\right)\mathrm{d}\xi \\
&\qquad\left.+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right)\right)\right] \\
&\quad=\frac{1}{\Gamma(\alpha-1)}\left(\frac{\grave{\imath}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\mathrm{q}(\tau) \\
&\quad\le\frac{1}{\Gamma(\sigma_1-1)}\left(\frac{\grave{\imath}^{\rho_1}-\grave{\grave{a}}^{\rho_1}}{\rho_1}\right)^{\sigma_1-2}\max_{\tau\in J}\left|\mathrm{q}(\tau)\right|.
\end{aligned}$$

Furthermore, for $\tau \in [\grave{a}_\circ, \grave{\iota}_\circ]$,

$$\begin{aligned}&\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\left(\frac{\grave{\iota}^{\rho_2}-\xi^{\rho_2}}{\grave{\iota}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)\mathrm{d}\xi \\ &\quad \geq \left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\alpha-1} Z(\grave{\iota}_\circ)\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\grave{\iota},\xi)\,\mathrm{d}\xi\end{aligned}$$

and

$$\begin{aligned}&\max_{\tau\in J}\left|\delta^2_{\rho_1}\mathrm{q}(\tau)\right|\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\left(\frac{\grave{\iota}^{\rho_2}-\xi^{\rho_2}}{\grave{\iota}^{\rho_2}-\grave{a}^{\rho_2}}\right)^{\sigma_2-1}\right)\mathrm{d}\xi \\ &\quad \geq \left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\iota}_\circ)\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\grave{\iota},\xi)\,\mathrm{d}\xi \max_{\tau\in J}\left|\delta^2_{\rho_1}\mathrm{q}(\tau)\right|.\end{aligned}$$

Thus, we obtain (25).

(5) From the first equation in (21), one can see that

$$ {}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau) = -\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\tau,\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right) \quad (2<\sigma_1\leq 3). \tag{30}$$

Thus,

$$\max_{\tau\in J}\left|{}^{\rho_1;\grave{a}^+}\mathbb{G}^{\sigma_1}_{\mathrm{CK}}\mathrm{q}(\tau)\right| \leq \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\grave{a},\xi)\mathrm{w}(\xi)\,\mathrm{d}\xi\right) \quad (2<\sigma_1\leq 3).$$

As in (2), we can deduce (26).

(6) Equation (27) is a direct consequence of the previous results. □

Then, for given $[\grave{a}_\circ,\grave{\iota}_\circ]\subset(\grave{a},\grave{\iota})$, we define the cone

$$\Upsilon = \left\{\mathrm{q}\in K: \min_{\tau\in[\grave{a}_\circ,\grave{\iota}_\circ]}\mathrm{q}(\tau)\geq\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\breve{M}_2\|\mathrm{q}\|\right\},$$

and the integral operator $\mathcal{N}_\lambda:\Upsilon\to\mathbb{E}$ is defined for $\tau\in[\grave{a}_\circ,\grave{\iota}_\circ]$ by

$$\begin{aligned}\mathcal{N}_\lambda(\mathrm{q})(\tau) &= \int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}}\mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\right)\mathrm{d}\xi \\ &\quad + \mu\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\iota}}\mathcal{G}_2(\tau,\xi) \\ &\quad \times\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}}\mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\right)\mathrm{d}\xi + \lambda\left(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right) \\ &\quad + F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi\right)\right).\end{aligned} \tag{31}$$

When (H2) holds, we have $\mathcal{N}_\lambda(\Upsilon)\subset\Upsilon$, and the fixed points of $\mathcal{N}_\lambda$ are the solutions of (9). To use some fixed point theorems, we need to show that $\mathcal{N}_\lambda$ is completely continuous.

Lemma 3.5 ([19]) *Let* $c,s>0$. *For any* $x,y\in[0,c]$, *the following propositions hold*:

(1) *If* $s > 1$, *then* $|x^s - y^s| \le sc^{s-1}|x - y|$;
(2) *If* $0 < s \le 1$, *then* $|x^s - y^s| \le |x - y|^s$.

Lemma 3.6 *Assume* (H2) *is true. Then* $\mathcal{N}_\lambda : \Upsilon \to \Upsilon$ *is continuous and compact.*

Proof The continuity of $\mathcal{N}_\lambda$ is a consequence of the continuity and positiveness of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{H}$, $\hbar$, and $\wp$. To prove that $\mathcal{N}_\lambda$ is compact, let us consider a bounded subset $\Omega \subset \Upsilon$. Then there exists $L > 0$ such that for any $\mathrm{q} \in \Omega$ we have $|\wp(\mathrm{q}(\tau))| \le L$. For any $\mathrm{q} \in \Omega$, as $\mathcal{N}_{\mathrm{q}}$ is positive and $\mathcal{G}_1$ is increasing w.r.t. τ, we have

$$\max_{\tau \in J} \left|\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\right| = \mathcal{N}_\lambda\big(\mathrm{q}(\grave{\imath})\big).$$

Consequently, using the previous inequality and hypothesis (H2), we get

$$\begin{aligned} \max_{\tau \in J} \left|\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\right| \le{}& \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, s)\hbar(s)L\,\mathrm{d}s\right)\mathrm{d}\xi \\ &+ \mu\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \\ &\times \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, s)\hbar(s)L\,\mathrm{d}s\right)\mathrm{d}\xi \\ &+ \lambda\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) + A\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)L\,\mathrm{d}\xi =: \bar{L}. \end{aligned} \tag{32}$$

Then, as in Lemma 3.4, we obtain $\|\mathcal{N}_\lambda \mathrm{q}\| \le \breve{M}_3\bar{L}$, where

$$\begin{aligned} \breve{M}_3 = \max\Bigg\{&1, \frac{\sigma_1 - 1}{\sigma_1 - 2}\left(\frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right), \\ &\max\left\{\frac{1}{\Gamma(\sigma_1 - 1)}\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}, 1\right\} \\ &\times\left[\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} Z(\grave{\imath}_\circ)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi)\,\mathrm{d}\xi\right]^{-1}\Bigg\}. \end{aligned}$$

Hence, $\mathcal{N}_\lambda(\Omega)$ is uniformly bounded. Furthermore, by using Lemmas (3.2), (3.5), (3.3), and the Lebesgue dominated convergence theorem, we deduce the equicontinuity of $\mathcal{N}_\lambda(\Omega)$. Therefore, $\mathcal{N}_\lambda$ is completely continuous by the Arzelà–Ascoli theorem. □

4 Existence of solutions in a cone

In this section, we derive an interval for λ, which ensures the existence of ρ_1-concave positive solutions of the fractional boundary value problem.

Theorem 4.1 *Assume that all conditions* (H1) *and* (H2) *hold, and that there exist* $0 < \ell_1 < \ell_2$ *and*

$$m_1 \in (0, \breve{M}_4), \qquad m_2 \in (\Lambda_6, \infty), \tag{33}$$

here $\breve{M}_4 = \min\{\frac{\Lambda_1}{4}, \frac{\Lambda_2}{4}, \frac{\Lambda_3}{2}, \Lambda_4, \Lambda_5\}$ *such that*

(H3) *For all* $\mathrm{q} \in [0, \ell_1]$, *we have* $\wp(\mathrm{q}) \leq \min\{\phi_p(m_1\ell_1), m_1\ell_1\}$;
(H4) *For all* $\mathrm{q} \in [\gamma\ell_2, \ell_2]$, *we have* $\wp(\mathrm{q}) \geq \phi_p(m_2\ell_2)$.

Then fractional boundary value problem (9) *has at least one* ρ_1*-concave positive solution for* $\lambda > 0$ *small enough, where*

$$\gamma := \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} \breve{M}_2 \tag{34}$$

and

$$\begin{aligned}
\Lambda_1 &:= \left[A \int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\,\mathrm{d}\xi\right]^{-1}, \\
\Lambda_2 &:= \left[\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi)\,\mathrm{d}\xi + \mu\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \rho_1\mu}\right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\grave{\imath}, \xi)\,\mathrm{d}\xi\right)\right. \\
&\qquad \left. \times \phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1}, \\
\Lambda_3 &:= \left[\frac{\sigma_1 - 1}{\sigma_1 - 2}\frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi)\,\mathrm{d}\xi\right.\right. \\
&\qquad \left.\left. + \frac{\mu}{1 - \mu}\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\grave{\imath}, \xi)\,\mathrm{d}\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1}, \\
\Lambda_4 &:= \left[\frac{1}{\Gamma(\sigma_1 - 1)}\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1}\right)^{\sigma_1 - 2}\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1}, \\
\Lambda_5 &:= \left[\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1}, \\
\Lambda_6 &:= \left[\gamma\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} Z(\grave{\imath}_\circ)\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi)\right.\right. \\
&\qquad \left.\left. + \mu\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\mathcal{G}_2(\grave{\imath}, \xi)\,\mathrm{d}\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]^{-1}.
\end{aligned} \tag{35}$$

Proof Let $\Omega_{\ell_1} = \{\mathrm{q} \in K : \|\mathrm{q}\| \leq \ell_1\}$ and λ satisfy

$$0 < \lambda \leq \frac{1}{2}(1 - \mu)\ell_1 \min\left\{1, \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right\}, \tag{36}$$

so that

$$2\lambda\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) \leq \ell_1,$$

and $2\lambda \leq \ell_1(1 - \mu)$. Let $\mathrm{q} \in K \cap \partial\Omega_{\ell_1}$, i.e., $\|\mathrm{q}\| = \ell_1$. From (H2) and (H3), we get

$$\begin{aligned}
F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi\right)\right) &\leq A\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi \\
&\leq m_1\ell_1 A\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi)\hbar(\xi)\,\mathrm{d}\xi, \\
\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau, \xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi\right) &\leq m_1\ell_1\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\tau, \xi)\hbar(\xi)\,\mathrm{d}\xi\,\tau\right).
\end{aligned}$$

However,

$$\begin{aligned}\max_{\tau\in J}\left|\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\right| &= \mathcal{N}_\lambda\big(\mathrm{q}(\grave{\imath})\big)\\ &= \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &\quad+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &\quad+\lambda\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi\right)\right).\end{aligned}$$

Then

$$\begin{aligned}\max_{\tau\in J}\left|\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\right| &\le \frac{\Lambda_2\ell_1}{4}\left[\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\right)\right.\\ &\quad\left.\times\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\right]+\frac{\ell_1}{2}+\frac{\Lambda_1\ell_1 A}{4}\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi.\end{aligned}$$

Consequently,

$$\max_{\tau\in J}\left|\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\right| \le \frac{\ell_1}{4}+\frac{\ell_1}{2}+\frac{\ell_1}{4}=\|\mathrm{q}\|.$$

Similarly, we obtain

$$\max\left\{\max_{k\in\{1,2\}}\max_{\tau\in J}\left|\delta_{\rho_1}^k\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\right|,\max_{\tau\in J}\left|{}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\right|\right\}\le\|\mathrm{q}\|.$$

Therefore, we conclude that $\|\mathcal{N}_\lambda \mathrm{q}\|\le\|\mathrm{q}\|$ for all $\mathrm{q}\in K\cap\partial\Omega_{\ell_1}$. Then Theorem 2.8 implies that

$$\mathbf{i}(\mathcal{N}_\lambda,\Omega_{\ell_1},K)=1. \tag{37}$$

On the other hand, let us consider $\Omega_{\ell_2}=\{\mathrm{q}\in K:\|\mathrm{q}\|\le\ell_2\}$. Then, for any $\mathrm{q}\in K\cap\partial\Omega_{\ell_2}$, by Lemma 3.4 one has $\ell_2\ge\min_{\tau\in[\grave{a}_\circ,\grave{\imath}_\circ]}\mathrm{q}(\tau)\ge\gamma\ell_2$. Using hypothesis (H4), we get

$$\begin{aligned}\mathcal{N}_\lambda\big(\mathrm{q}(\grave{\imath})\big) &\ge \left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\left[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\right)\mathrm{d}\xi\right.\\ &\quad+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\right)\mathrm{d}\xi\\ &\quad\left.+\lambda\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)+F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi\right)\right)\right]\\ &\ge\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1} m_2\ell_2\gamma Z(\grave{\imath}_\circ)\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\right.\\ &\quad\left.+\mu\left(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\right)\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\right)\end{aligned}$$

$$\geq \left(\frac{\grave{a}_{\circ}^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} \Lambda_6 \ell_2 \gamma Z(\grave{\imath}_{\circ}) \biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) + \mu \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) \mathcal{G}_2(\tau, \xi)\, \mathrm{d}\xi \biggr) \phi_{\bar{p}} \biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) \hbar(\xi)\, \mathrm{d}\xi \biggr) := \ell_2 = \|\mathrm{q}\|,$$

which implies that $\|\mathcal{N}_\lambda \mathrm{q}\| \geq \|\mathrm{q}\|$ for any $\mathrm{q} \in K \cap \partial\Omega_{\ell_2}$. Hence Theorem 2.8 implies that

$$\mathbf{i}(\mathcal{N}_\lambda, \Omega_{\ell_2}, K) = 0. \tag{38}$$

Therefore, by equations (37), (38) and $\ell_1 < \ell_2$, we have

$$\mathbf{i}(\mathcal{N}_\lambda, \overline{\Omega_{\ell_2}} \setminus \Omega_{\ell_1}, K) = 1.$$

By employing Theorem 2.9, one can see that the operator $\mathcal{N}_\lambda$ has at least one fixed point $\mathrm{q} \in K \cap \overline{\Omega_{\ell_2}} \setminus \Omega_{\ell_1}$, which is a ρ_1-concave positive solution of fractional boundary value problem (9). □

Theorem 4.2 *Assume that all conditions* (H1), (H2), *and* (H4) *hold. Then* $\mathbb{FBVP}$ (9) *has no* ρ_1*-concave positive solution for* λ *large enough.*

Proof Suppose that $\exists \check{\mathrm{N}} \in \mathbb{N}$ and $(\lambda_j)_j$ such that $\lim_{j\to\infty} \lambda_j = +\infty$ and fractional boundary value problem (9) has ρ_1-concave positive solution q_j ($j \geq \check{\mathrm{N}}$), i.e.,

$$\begin{aligned}
\mathrm{q}_j(\tau) = {} & \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\tau, \xi) \phi_{\bar{p}} \biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) \hbar(s) \wp\bigl(\mathrm{q}(s)\bigr)\, \mathrm{d}s \biggr) \mathrm{d}\xi \\
& + \mu \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) \hbar(s) \wp\bigl(\mathrm{q}(s)\bigr)\, \mathrm{d}s \biggr) \mathrm{d}\xi \\
& + \lambda_j \left(\frac{\tau^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) + F_{\circ} \biggl(\phi_{\bar{p}} \biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) \hbar(\xi) \wp\bigl(u(\xi)\bigr)\, \mathrm{d}\xi \biggr) \biggr).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{q}_j(\grave{\imath}) \geq {} & \left(\frac{\grave{a}_{\circ}^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1 - 1} \biggl[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath}, \xi) \phi_{\bar{p}} \biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) \hbar(s) \wp\bigl(\mathrm{q}(s)\bigr)\, \mathrm{d}s \biggr) \mathrm{d}\xi \\
& + \mu \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau, \xi) \phi_{\bar{p}} \biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi, s) \hbar(s) \wp\bigl(\mathrm{q}(s)\bigr)\, \mathrm{d}s \biggr) \mathrm{d}\xi \\
& + \lambda_j \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) + F_{\circ} \biggl(\phi_{\bar{p}} \biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a}, \xi) \hbar(\xi) \wp\bigl(\mathrm{q}(\xi)\bigr)\, \mathrm{d}\xi \biggr) \biggr) \biggr].
\end{aligned}$$

Consequently,

$$\mathrm{q}_j(\grave{\imath}) \geq \left(\frac{\grave{a}_{\circ}^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\alpha - 1} \lambda_j \left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right).$$

Without loss of generality, we can suppose that $\check{\mathrm{N}}$ is large enough to get, for $j \geq \check{\mathrm{N}}$,

$$\lambda_j > j \left(\frac{\rho_1 - \mu\rho_1}{\grave{\imath}^{\rho} - \grave{a}^{\rho}}\right) \left(\frac{\grave{a}_{\circ}^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{1 - \sigma_1}. \tag{39}$$

Then we have $q_j(\grave{\iota}) > j$. Consequently, $\lim_{j\to+\infty}\|q_j\| = +\infty$. Using (H4), we deduce that there exist $m_2 > \Lambda_6$ and $\ell_2 > 0$ such that $\wp(q) \geq \phi_p(m_2\ell_2)$ for all $q \in [\gamma\ell_2, \ell_2]$. Again, we can choose $\breve{N}$ large enough to get $\|q_j\| \geq \ell_2$, $\forall j \geq \breve{N}$. By writing $m_2 = \Lambda_6 + \varpi$, where $\varpi > 0$, we get

$$\begin{aligned}
\|q_j\| &\geq q_j(\grave{\iota})\\
&\geq \left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\iota}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1}\left[\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\grave{\iota},\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\xi,s)\hbar(s)\wp\big(q(s)\big)\,ds\right)d\xi\right.\\
&\quad + \mu\left(\frac{\grave{\iota}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\xi,s)\hbar(s)\wp\big(q(s)\big)\,ds\right)d\xi\\
&\quad \left. + \lambda\left(\frac{\grave{\iota}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) + F_\circ\left(\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\big(q(\xi)\big)\,d\xi\right)\right)\right]\\
&\geq (\Lambda_6 + \varpi)\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\iota}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\iota}_\circ)\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\grave{\iota},\xi)\right.\\
&\quad \left. + \mu\left(\frac{\grave{\iota}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\mathcal{G}_2(\tau,\xi)\,d\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\phi_p\big(q(\xi)\big)\,d\xi\right)\\
&\geq \|q_j\|(\Lambda_6 + \varpi)\gamma\left(\frac{\grave{a}_\circ^{\rho_1} - \grave{a}^{\rho_1}}{\grave{\iota}^{\rho_1} - \grave{a}^{\rho_1}}\right)^{\sigma_1-1} Z(\grave{\iota}_\circ)\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\grave{\iota},\xi)\right.\\
&\quad \left. + \mu\left(\frac{\grave{\iota}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\mathcal{G}_2(\tau,\xi)\,d\xi\right)\phi_{\bar{p}}\left(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi\right)\\
&= \|q_j\|\big(1 + \varpi\Lambda_6^{-1}\big),
\end{aligned}$$

which leads to a contradiction $\|q_j\|\varpi\Lambda_6^{-1} \leq 0$. The proof is completed. □

Remark 4.1 Let

$$\wp_0 := \lim_{q\to0^+}\frac{\wp(q)}{\min\{\phi_p(q), q\}}, \qquad \wp_\infty = \lim_{q\to\infty}\frac{\wp(q)}{\phi_p(q)}. \tag{40}$$

If $\wp_0 = 0$ and $\wp_\infty = \infty$ hold, then conditions (H3) and (H4) hold respectively. Moreover, if the functions $\wp$ and $F_\circ$ are nondecreasing, the following theorem holds.

Theorem 4.3 *Assume that the hypotheses of Theorem 4.1 hold and that $\wp$ and $F_\circ$ are nondecreasing. Then there exists $\lambda^* > 0$ such that fractional boundary value problem (9) has at least one ρ-concave positive solution for $\lambda \in (0,\lambda^*)$ and has no ρ_1-concave positive solution for $\lambda \in (\lambda^*,\infty)$.*

Proof Let $\acute{\Upsilon} \subset \mathbb{R}_+^*$ be the set of all λ such that fractional boundary value problem (9) has at least one ρ_1-concave positive solution and $\lambda^* = \sup\acute{\Upsilon}$. It follows from Theorem 4.1 that $\acute{\Upsilon} \neq \emptyset$, and thus λ^* exists. We denote by q_0 the solution of fractional boundary value problem (9) associated with λ_0 and

$$\mathcal{K}(q_0) = \big\{q \in K : q(\tau) < q_0(\tau), \forall\tau \in J\big\}.$$

Let $\lambda \in (0, \lambda_0)$ and $\mathrm{q} \in \mathcal{K}(\mathrm{q}_0)$. It follows from the definition of $\mathcal{N}_\lambda$ (31) and the monotonicity of f that, for any $\tau \in J$,

$$\mathcal{N}_\lambda\bigl(\mathrm{q}(\tau)\bigr) \le \mathcal{N}_\lambda\bigl(\mathrm{q}_0(\tau)\bigr) = \mathrm{q}_0(\tau).$$

Thus $\mathcal{N}_\lambda(\mathcal{K}(\mathrm{q}_0)) \subseteq \mathcal{K}(\mathrm{q}_0)$. Now, Schauder's fixed point theorem implies that there exists a fixed point $\mathrm{q} \in \mathcal{K}(\mathrm{q}_0)$ such that it is a positive solution of (9). The proof is completed. □

Theorem 4.4 *Suppose that conditions* (H1) *and* (H2) *hold. Assume that* $\wp$ *also satisfies*:

(H5) $\wp_0 = \varpi_1 \in [0, \min\{k^{p-1}, k\})$, $k = \frac{1}{4}\breve{M}_4$;

(H6) $\wp_\infty = \varpi_2 \in ((\frac{2\Lambda_6}{\gamma})^{p-1}, \infty)$.

Then fractional boundary value problem (9) *has at least one* ρ_1*-concave positive solution for* λ *small enough.*

Proof Firstly, from the definition of $\wp_0$, for all $\epsilon > 0$, there exists an adequate small positive number $\bar{\delta}(\epsilon)$ such that

$$\wp(\mathrm{q}) \le (\epsilon + \varpi_1)\min\bigl\{\mathrm{q}^{p-1}, \mathrm{q}\bigr\} \le (\epsilon + \varpi_1)\min\bigl\{\bar{\delta}^{p-1}, \bar{\delta}\bigr\},$$

$\forall \mathrm{q} \in [0, \bar{\delta}(\epsilon)]$. Then, for $\epsilon = \min\{k^{p-1}, k\} - \varpi_1$, we have

$$\begin{aligned}\wp(\mathrm{q}) &\le \min\bigl\{k^{p-1}, k\bigr\}\min\bigl\{\bar{\delta}(\epsilon)^{p-1}, \bar{\delta}(\epsilon)\bigr\}\\ &\le \min\bigl\{k^{p-1}\bar{\delta}(\epsilon)^{p-1}, k\bar{\delta}(\epsilon)\bigr\}\\ &\le \min\bigl\{\bigl(2k\bar{\delta}(\epsilon)\bigr)^{p-1}, 2k\bar{\delta}(\epsilon)\bigr\}.\end{aligned}$$

It is enough to take $\ell_1 = \bar{\delta}(\epsilon)$ and $m_1 = 2k \in (0, \breve{M}_4)$, i.e., condition (H3) holds. Next, since (H6) holds, then for every $\epsilon > 0$ there exists an adequate big positive number $\ell_2 \neq \ell_1$ such that

$$\wp(\mathrm{q}) \ge (\varpi_2 - \epsilon)\mathrm{q}^{p-1} \ge (\varpi_2 - \epsilon)(\gamma\ell_2)^{p-1} \quad (\mathrm{q} \ge \gamma\ell_2).$$

Hence, for $\epsilon = \varpi_2 - (\frac{2\Lambda_6}{\gamma})^{p-1}$, we get

$$\wp(\mathrm{q}) \ge \left(\frac{2\Lambda_6}{\gamma}\right)^{p-1}(\gamma\ell_2)^{p-1} = (2\Lambda_6\ell_2)^{p-1}. \tag{41}$$

By considering $m_2 = 2\Lambda_6 > \Lambda_6$, condition (H4) holds by Theorem 4.1, we complete the proof. □

5 Several solutions in a cone

In order to show the existence of multiple solutions, we will use the Leggett–Williams fixed point theorem [43]. For this, we define the following subsets of a cone K:

$$\begin{aligned}\Omega_c &= \bigl\{\mathrm{q} \in K : \|\mathrm{q}\| < c\bigr\},\\ \Omega_\varphi(b, d) &= \bigl\{\mathrm{q} \in K : b \le \varphi(\mathrm{q}), \|\mathrm{q}\| \le d\bigr\}.\end{aligned}$$

A map $\Pi : K \to [0,\infty)$ is said to be a nonnegative continuous concave functional on a cone K of a real Banach space $\mathfrak{E}$, if it is continuous and

$$\Pi\big(\bar{\lambda}\mathrm{q} + (1-\bar{\lambda})\acute{\mathrm{q}}\big) \geq \bar{\lambda}\Pi(\mathrm{q}) + (1-\bar{\lambda})\Pi(\acute{\mathrm{q}})$$

for all $\mathrm{q}, \acute{\mathrm{q}} \in K$ and $\bar{\lambda} \in [0,1]$.

Theorem 5.1 ([43]) *Let $\mathcal{T} : \overline{\Omega_c} \to \overline{\Omega_c}$ be a completely continuous operator and φ be a nonnegative continuous concave functional on K such that $\varphi(\mathrm{q}) \leq \|\mathrm{q}\|$ for all $\mathrm{q} \in \overline{\Omega_c}$. Suppose that there exist constants $0 < \mathring{a} < b < d \leq c$ such that*

(D3) $\{\mathrm{q} \in \Omega_\varphi(b,d) : \varphi(\mathrm{q}) > b\} \neq \emptyset$ *and* $\varphi(\mathcal{T}\mathrm{q}) > b$ *if* $\mathrm{q} \in K_\varphi(b,d)$;

(D4) $\|\mathcal{T}\mathrm{q}\| < \mathring{a}$ *if* $\mathrm{q} \in \Omega_{\mathring{a}}$;

(D5) $\varphi(\mathcal{T}\mathrm{q}) > b$ *for* $\mathrm{q} \in \Omega_\varphi(b,c)$ *with* $\|\mathcal{T}\mathrm{q}\| > d$.

Then $\mathcal{T}$ has at least three fixed points q_1, q_2, and q_3 such that $\|\mathrm{q}_1\| < \mathring{a}$, $b < \varphi(\mathrm{q}_2)$, and $\|\mathrm{q}_3\| > \mathring{a}$ with $\varphi(\mathrm{q}_3) < b$.

Theorem 5.2 *Suppose that conditions* (H1) *and* (H2) *hold, if there exist $\mathring{a}$, b, c with $0 < \mathring{a} < \gamma b < b \leq c$ such that*

(H7) $\wp(\mathrm{q}(\tau)) < \min\{\phi_p(m_1\mathring{a}), m_1\mathring{a}\}$ *for* $(\tau,\mathrm{q}) \in J \times [0,\mathring{a}]$;

(H8) $\wp(\mathrm{q}(\tau)) \geq \phi_p(m_2\gamma b)$ *for* $(\tau,\mathrm{q}) \in [\grave{a}_\circ, \grave{\iota}_\circ] \times [\gamma b, b]$;

(H9) $\wp(\mathrm{q}(\tau)) \leq \min\{\phi_p(m_1 c), m_1 c\}$ *for* $(\tau,\mathrm{q}) \in J \times [0,c]$;

(H10) $0 < \lambda < \frac{(1-\mu)\mathring{a}}{2} \min\{1, \frac{\rho_1}{\grave{\iota}^{\rho_1} - \grave{a}^{\rho_1}}\}$;

where the constants m_2 and m_1 are defined in (33). *Then fractional boundary value problem* (9) *has at least three positive ρ_1-concave solutions q_1, q_2, and q_3 satisfying $\|\mathrm{q}_1\| < \mathring{a}$, $\gamma b < \varphi(\mathrm{q}_2)$, and $\|\mathrm{q}_3\| > \mathring{a}$ with $\varphi(\mathrm{q}_3) < b\gamma$ for λ small enough.*

Proof We prove that fractional boundary value problem (9) has at least three positive ρ_1-concave solutions for $\lambda > 0$ small enough. By Lemma 3.6, $\mathcal{N}_\lambda : \Upsilon \to \Upsilon$ is completely continuous. Let $\varphi(\mathrm{q}) = \min_{\tau \in [\grave{a}_\circ, \grave{\iota}_\circ]} \mathrm{q}(\tau)$. Obviously, $\varphi(\mathrm{q})$ is a nonnegative, continuous, and concave functional on K with $\varphi(\mathrm{q}) \leq \|\mathrm{q}\|$ for $\mathrm{q} \in \overline{\Omega_c}$. Now we will show that all conditions of Theorem 5.1 are satisfied. Suppose that $\mathrm{q} \in \overline{\Omega_c}$, that is, $\|\mathrm{q}\| \leq c$. For $\tau \in J$, by equation (31), Lemmas 3.4, 3.5, we acquire

$$\begin{aligned}\max_{\tau\in J}\big|\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\big| ={}& \int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\grave{\iota},\xi)\phi_{\bar{p}}\mathrm{q}(\tau)\bigg(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi \\ &+ \mu\bigg(\frac{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi \\ &+ \lambda\bigg(\frac{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg) + F_\circ\bigg(\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi\bigg)\bigg).\end{aligned}$$

From (H2), (H9), and (H10), we get

$$\begin{aligned}\max_{\tau\in J}\big|\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\big| \leq{}& \int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_1(\grave{\iota},\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\grave{a},s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi \\ &+ \mu\bigg(\frac{\grave{\iota}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\int_{\grave{a}}^{\grave{\iota}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\iota}} \mathcal{H}(\grave{a},s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi\end{aligned}$$

$$
\begin{aligned}
&+\frac{c}{2}+A\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi\\
&\le m_1c\bigg[\bigg(\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi\\
&\quad+\mu\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\rho_1\mu}\bigg)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\bigg)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\bigg)\\
&\quad+A\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\bigg]+\frac{c}{2}\\
&\le\frac{\Lambda_2c}{4}\bigg[\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi\\
&\quad+\mu\bigg(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\rho_1\mu}\bigg)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi\bigg)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi\bigg)\bigg]\\
&\quad+\frac{A\Lambda_1c}{4}\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi+\frac{c}{2}\\
&=\frac{c}{4}+\frac{c}{4}+\frac{c}{2}=c
\end{aligned}
$$

and

$$
\max\Big\{\max_{k\in\{1,2\}}\max_{\tau\in J}\big|\delta_{\rho_1}^{k}\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\big|,\max_{\tau\in J}\big|{}^{\rho_1;\grave{a}^+}\mathbb{G}_{\mathrm{CK}}^{\sigma_1}\mathcal{N}_\lambda\big(\mathrm{q}(\tau)\big)\big|\Big\}\le\|\mathrm{q}\|.
$$

Therefore, we have

$$
\big\|\mathcal{N}_\lambda\mathrm{q}(\tau)\big\|\le c\quad(\forall\mathrm{q}\in\Omega_c).
$$

This implies that $\mathcal{N}_\lambda:\overline{\Omega_c}\to\overline{\Omega_c}$. By the same method, if $\mathrm{q}\in\overline{\Omega_{\mathring{a}}}$, then we can get $\|\mathcal{N}_\lambda\mathrm{q}(\tau)\|<$ å, therefore (D4) has been checked. Next, we assert that

$$
\big\{\mathrm{q}\in\Omega_\varphi(\gamma b,b):\varphi(\mathrm{q})>\gamma b\big\}\neq\emptyset
$$

and $\varphi(\mathcal{N}_\lambda(\mathrm{q}))>\gamma b$ for all $\mathrm{q}\in\Omega_\varphi(\gamma b,b)$. In fact, the constant function $\frac{\gamma b+b}{2}\in\Omega_\varphi(\gamma b,b)$ and $\varphi(\frac{\gamma b+b}{2})>\gamma b$. On the other hand, for $\mathrm{q}\in\Omega_\varphi(\gamma b,b)$, we have

$$
\gamma b\le\varphi(\mathrm{q})=\min\mathrm{q}(\tau)\le\|\mathrm{q}\|=b\quad\big(\forall t\in[\grave{a}_\circ,\grave{\imath}_\circ]\big).
$$

Thus, in view of (31), Lemmas 3.3, 3.4, 3.5, and (H8), we have

$$
\begin{aligned}
\varphi(\mathcal{N}_\lambda\mathrm{q})=\min_{t\in[\grave{a}_\circ,\grave{\imath}_\circ]}\bigg[&\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi\\
&+\mu\bigg(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\xi,s)\hbar(s)\wp\big(\mathrm{q}(s)\big)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi\\
&+\lambda\bigg(\frac{\tau^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1}\bigg)+F_\circ\bigg(\phi_{\bar{p}}\bigg(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\big(\mathrm{q}(\xi)\big)\,\mathrm{d}\xi\bigg)\bigg)\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \gamma \biggl[\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\phi_{\bar{p}}\biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\bigl(\mathrm{q}(s)\bigr)\,\mathrm{d}s \biggr)\,\mathrm{d}\xi \\
&\quad + \mu\biggl(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1} \biggr) \int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\tau,\xi)\phi_{\bar{p}}\biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\xi,s)\hbar(s)\wp\bigl(\mathrm{q}(s)\bigr)\,\mathrm{d}s \biggr)\,\mathrm{d}\xi \\
&\quad + \lambda\biggl(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1} \biggr) + F_{\circ}\biggl(\phi_{\bar{p}}\biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\wp\bigl(\mathrm{q}(\xi)\bigr)\,\mathrm{d}\xi \biggr)\biggr)\biggr] \\
&> \gamma m_2 b\biggl[\gamma\biggl(\frac{\grave{a}_{\circ}^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}} \biggr)^{\sigma_1-1} Z(\grave{\imath}_{\circ})\biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi) \\
&\quad + \mu\biggl(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1} \biggr)\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi \biggr)\phi_{\bar{p}}\biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi \biggr)\biggr] \\
&> \gamma \Lambda_6 b\biggl[\gamma\biggl(\frac{\grave{a}_{\circ}^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}} \biggr)^{\sigma_1-1} Z(\grave{\imath}_{\circ})\biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi) \\
&\quad + \mu\biggl(\frac{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}{\rho_1-\mu\rho_1} \biggr)\mathcal{G}_2(\tau,\xi)\,\mathrm{d}\xi \biggr)\phi_{\bar{p}}\biggl(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi \biggr)\biggr] \\
&= \gamma b.
\end{aligned}
$$

Thus, (D3) has been verified. Finally, we need to show that if $\mathrm{q} \in \Omega_{\varphi}(\gamma b, b)$ with $\|\mathcal{N}\lambda\mathrm{q}\| > b$, then $\|\mathcal{N}_{\lambda}\mathrm{q}\| > \gamma b$. In fact, to see this, suppose that $\mathrm{q} \in \Omega_{\varphi}(\gamma b, b)$ with $\|\mathcal{N}_{\lambda}\mathrm{q}\| > b$, then through Lemma 3.4 we have

$$\varphi(\mathcal{N}_{\lambda}\mathrm{q}) = \min_{\grave{a}_{\circ}\leq t\leq \grave{\imath}_{\circ}} (\mathcal{N}_{\lambda}\mathrm{q})(\tau) \geq \gamma\|\mathcal{N}_{\lambda}\mathrm{q}\| > \gamma b.$$

Thus (D5) is satisfied. Hence, an application of Theorem 5.1 completes the proof. □

Corollary 5.1 *Suppose that conditions* (H1) *and* (H2) *hold. If there exist constants*

$$0 < r_1 < b_1 < \gamma b_1 \leq r_2 < b_2 < \gamma b_2 \leq \cdots \leq r_n$$

for $1 \leq j \leq n-1$ *and the following conditions are satisfied*:

(H11) $\wp(\mathrm{q}(\tau)) < \min\{\phi_p(m_1 r_j), m_1 r_j\}$ *for* $(\tau,\mathrm{q}) \in J \times [0, r_j]$;

(H12) $\wp(\mathrm{q}(\tau)) > \phi_p(m_2 b_j)$ *for* $(\tau,\mathrm{q}) \in [\grave{a}_{\circ}, \grave{\imath}_{\circ}] \times [\gamma b_j, b_j]$;

(H13) $0 < \lambda < \frac{(1-\mu)r_1}{2}\max\{1, \frac{\rho_1}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\}$.

Then fractional boundary value problem (9) *has at least* $2n-1$ *positive* ρ_1*-concave solutions.*

Proof By the induction method, we get the proof. □

6 Applications

In this section, we give some examples to illustrate the usefulness of our main results.

Example 6.1 Let us consider the following p-Laplacian fractional boundary value problem:

$$\begin{cases} {}^{\rho_2;e^{2-}}\mathbb{G}^{5/2}_{\mathrm{CK}}(\phi_p({}^{\rho_1;e^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q}))(\tau) \\ \quad + \frac{\mathrm{q}^{3/2}(\tau)}{\sqrt{(2.2-\ln(\tau))(\ln(\tau)-0.9)}} = 0, \quad e < \tau < e^2, \\ \mathrm{q}(e) - \sqrt{|{}^{\rho_1;e^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q}(e)|} = 0, \\ \delta^2_{0^+}\mathrm{q}(e) = 0, \qquad \delta^1_{0^+}\mathrm{q}(e^2) = \frac{1}{2}\delta^1_{0^+}\mathrm{q}(e^{\frac{8}{5}}) + \lambda, \\ {}^{\rho_1;e^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q}(e^2) = -\delta^1_{0^+}(\phi_p({}^{\rho_1;e^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q}))(e) \\ \qquad = \delta^2_{0^+}(\phi_p({}^{\rho_1;e^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q}))(e^2) = 0. \end{cases} \tag{42}$$

Here, $J = [e, e^2]$, $\sigma_1 = \sigma_2 = \frac{5}{2} \in (2, 3]$,

$$\mu = \frac{1}{2} \in (0,1), \qquad \eta = e^{\frac{8}{5}} \in J, \qquad [\grave{a}_\circ, \grave{\imath}_\circ] = [e^{3/2}, e^{7/4}] \subset J.$$

We put

$$\rho_1 = 0.5 \in \mathbb{R} \setminus \{1\}, \qquad \rho_2 = 1.3 \in \mathbb{R} \setminus \{1\}, \qquad p = \frac{3}{2},$$

and so $\bar{p} = 3$, $A = \frac{3}{2}$, $B = \frac{1}{2}$. ${}^{\rho_1;e^+}\mathbb{G}^{5/2}_{\mathrm{CK}}$ and ${}^{\rho_2;e^{2-}}\mathbb{G}^{5/2}_{\mathrm{CK}}$ are the left- and right-sided Caputo–Katugampola fractional derivatives, $F_\circ(\nu) = \sqrt{|\nu|}$ and

$$\hbar(\tau) = \frac{1}{\sqrt{(2.2 - \ln(\tau))(\ln(t) - 0.9)}}.$$

We can easily show that (H1), (H2) hold, and from (40) we get $\wp(\mathrm{q}(\tau)) = (\mathrm{q}(\tau))^{3/2}$ satisfies

$$\wp_0 = \lim_{\mathrm{q}\to 0^+} \frac{\wp(\mathrm{q})}{\min\{\phi_{\frac{3}{2}}(\mathrm{q}), \mathrm{q}\}} = \lim_{\mathrm{q}\to 0^+} \frac{\mathrm{q}^{\frac{3}{2}}}{\min\{\frac{\mathrm{q}}{\sqrt{|\mathrm{q}|}}, \mathrm{q}\}} = 0,$$

$$\wp_\infty = \lim_{\mathrm{q}\to\infty} \frac{\wp(\mathrm{q})}{\phi_{\frac{3}{2}}(\mathrm{q})} = \lim_{\mathrm{q}\to\infty} \frac{\mathrm{q}^{\frac{3}{2}}}{\mathrm{q}|\mathrm{q}|^{\frac{3}{2}-2}} = \lim_{\mathrm{q}\to\infty} \frac{\mathrm{q}^{\frac{1}{2}}}{|\mathrm{q}|^{\frac{-1}{2}}} = \infty.$$

Then, obviously, $Z(\grave{\imath}_\circ) = 0.05549$,

$$\breve{M}_4 = \min\left\{\frac{\Lambda_1}{4}, \frac{\Lambda_2}{4}, \frac{\Lambda_3}{2}, \Lambda_4, \Lambda_5\right\} \simeq 0.00007, \Lambda_6 \simeq 0.000007.$$

Tables 1 and 2 show the numerical results (for getting the technique, see Algorithm 1). So, by assuming that $\lambda = 1.5$ and $\ell_1 = 12$, all conditions of Theorem 4.1 hold, then we can choose $\ell_2 > \ell_1$ and λ satisfying

$$0 < \lambda \le \frac{1}{2}(1-\mu)\ell_1 \min\left\{1, \frac{\rho_1}{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}\right\} = 2.4542789 < \ell_2,$$

$$2\lambda\left(\frac{\grave{\imath}^{\rho_1} - \grave{a}^{\rho_1}}{\rho_1 - \mu\rho_1}\right) = 3.42259 \le 12 = \ell_1,$$

Table 1 Numerical values of $\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_1(\grave{\imath},\xi)\,d\xi$, $\breve{M}_2$, γ, $\int_{\grave{a}}^{\grave{\imath}} \mathcal{G}_2(\grave{\imath},\xi)\,d\xi$, $\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi$, and $\Delta = \phi_{\bar{p}}(\int_{\grave{a}}^{\grave{\imath}} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi)$ in Example 6.1 for $\tau \in J$

τ	$\int_{\grave{a}}^{\tau} \mathcal{G}_1(\grave{\imath},\xi)\,d\xi$	$\breve{M}_2$	γ	$\int_{\grave{a}}^{\tau} \mathcal{G}_2(\grave{\imath},\xi)\,d\xi$	$\int_{\grave{a}}^{\tau} \mathcal{H}(\grave{a},\xi)\hbar(\xi)\,d\xi$	Δ
2.7183	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.0377	0.2357	0.0039	0.0011	0.3462	3.7320	13.9275
3.3947	0.5121	0.0085	0.0025	0.6944	10.0873	101.7541
3.7937	0.8293	0.0138	0.0040	1.0416	18.4145	339.0940
4.2395	1.1850	0.0197	0.0057	1.3835	28.8166	830.3948
4.7377	1.5738	0.0261	0.0076	1.7148	41.5311	1724.8351
5.2945	1.9852	0.0329	0.0095	2.0276	56.8307	3229.7341
5.9167	2.3994	0.0398	0.0115	2.3105	74.9520	5617.7976
6.6120	2.7785	0.0461	0.0133	2.5445	95.9883	9213.7513
7.3891	3.0207	0.0501	0.0145	2.6809	119.6935	14,326.5251

Table 2 Numerical values of Λ_1, Λ_2, Λ_3, Λ_4, Λ_5, Λ_6, and $\breve{M}_4$ in Example 6.1 for $\tau \in J$

τ	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	$\breve{M}_4$
2.7183	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*
3.0377	0.178637	0.585071	0.179564	0.043506	53.867322	0.071800	0.043506
3.3947	0.066090	0.045238	0.011463	0.005955	1.916555	0.009828	0.005731
3.7937	0.036203	0.009195	0.002150	0.001787	0.240558	0.002949	0.001075
4.2395	0.023135	0.002839	0.000621	0.000730	0.051977	0.001204	0.000311
4.7377	0.016052	0.001104	0.000227	0.000351	0.015226	0.000580	0.000114
5.2945	0.011731	0.000499	0.000097	0.000188	0.005456	0.000310	0.000049
5.9167	0.008895	0.000252	0.000047	0.000108	0.002278	0.000178	0.000023
6.6120	0.006945	0.000140	0.000025	0.000066	0.001090	0.000109	0.000012
7.3891	0.005570	0.000085	0.000015	0.000042	0.000612	0.000070	0.000007

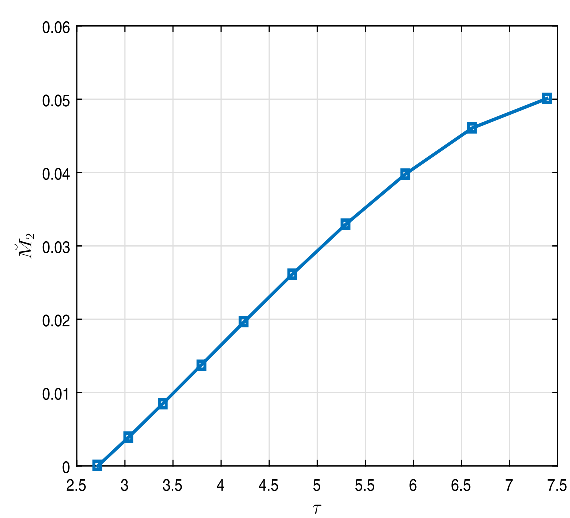
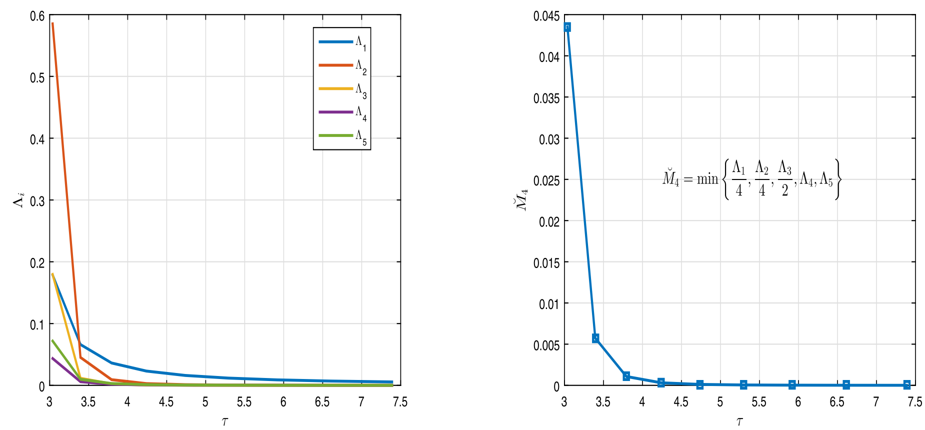


Figure 1 2D-graph of $\breve{M}_2$ for $\tau \in [e,e^2]$ in Example 6.1

and $2\lambda \leq \ell_1(1-\mu) = 10.5$ such that

$$\Omega_{\ell_1} = \{q \in K : \|q\| < \ell_1\}, \qquad \Omega_2 = \{q \in K : \|q\| < \ell_2\}.$$

Figures 1, 2, and 3 show a graphical representation of the variables. As shown in Fig. 1, $\breve{M}_2$ is directly related to $\tau \in [e,e^2]$ and increases with increasing τ. It can be seen in Fig. 2(a) that all values of Λ_i for $i = 1,2,3,4,5$ are inversely proportional to τ. Also, $\breve{M}_4$ has the



(a) Λ_i $(i = 1, 2, 3, 4, 5)$ *(b)* $\breve{M}_4$

Figure 2 Graphical representation of Λ_i ($i = 1, 2, 3, 4, 5$) and $\breve{M}_4$ for $\tau \in J$ in Example 6.1

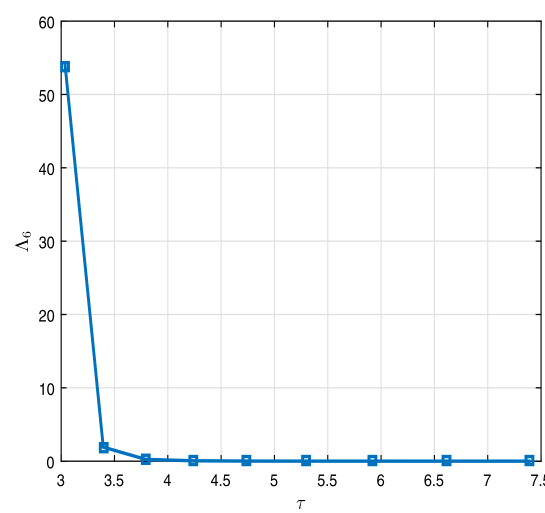


Figure 3 2D-graph of Λ_6 for $\tau \in J$ in Example 6.1

same behavior for $\tau \in J$, which can be seen in Fig. 2(b). Finally, the trend of variable Λ_6 with respect to τ is shown in Fig. 3. Then we can show that fractional boundary value problem (42) has at least a positive solution $\mathrm{q} \in K \cap (\overline{\Omega_{\ell_2}} \setminus \Omega_{\ell_1})$ for λ small enough.

Example 6.2 Let us consider the following p-Laplacian fractional boundary value problem:

$$\begin{cases} {}^{\rho_2;e^-}\mathbb{G}^{7/2}_{\mathrm{CK}}(\phi_p({}^{1/2;1^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q}))(\tau) \\ \quad + \frac{5\sqrt{\pi}}{4}\ln(\tau)\wp(\mathrm{q}(\tau)) = 0, \quad 1 < \tau < e, \\ \mathrm{q}(1) - F_\circ({}^{1/2;1^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q}(1)) = 0, \\ \delta^2_{\frac{1}{2}}\mathrm{q}(1) = 0, \qquad \delta^1_{\frac{1}{2}}\mathrm{q}(e) = \frac{1}{2}\delta^1_{\frac{1}{2}}\mathrm{q}(\sqrt{e}) + \lambda, \\ {}^{1/2;1^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q}(e) = -\delta^1_{0+}[\phi_p({}^{1/2;1^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q})](1) \\ \qquad\qquad = \delta^2_{0+}[\phi_p({}^{1/2;1^+}\mathbb{G}^{5/2}_{\mathrm{CK}}\mathrm{q})](e) = 0. \end{cases} \tag{43}$$

Here, $J=[1,e]$, $\sigma_1=\sigma_2=\frac{5}{2}\in(2,3]$,

$$\mu=\frac{1}{2}\in(0,1),\qquad \eta=\sqrt{e}\in J,\qquad [\grave{a}_\circ,\grave{\imath}_\circ]=\left[\sqrt{e},\sqrt[4]{e}\right]\subset J.$$

We put

$$\rho_1=0.5\in\mathbb{R}\setminus\{1\},\qquad \rho_2=2\in\mathbb{R}\setminus\{1\},\qquad p=\frac{3}{2},$$

and so $\bar{p}=3$, $A=\frac{3}{2}$, $B=\frac{1}{2}$. ${}^{\rho_1;1^+}\mathbb{G}^{5/2}_{\mathrm{CK}}$ and ${}^{\rho_2;e^-}\mathbb{G}^{5/2}_{\mathrm{CK}}$ are the left- and right-sided Caputo–Katugampola fractional derivatives, $F_\circ(\nu)=\sqrt{|\nu|}$ and

$$\hbar(\tau)=\frac{5\sqrt{\pi}}{4}\ln(\tau),$$

and

$$\wp(\mathrm{q})=\begin{cases}6\mathrm{q}^2, & \mathrm{q}\le 1,\\ 5+\mathrm{q}^{1/4}, & \mathrm{q}>1.\end{cases}$$

Through a simple calculation, we have $\int_1^e\mathcal{H}(e,\xi)\hbar(\xi)\,\mathrm{d}\xi=12.5716$,

$$\gamma=\left(\frac{\grave{a}_\circ^{\rho_1}-\grave{a}^{\rho_1}}{\grave{\imath}^{\rho_1}-\grave{a}^{\rho_1}}\right)^{\sigma_1-1}\breve{M}_2=\left(\frac{e^{0.25}-1}{e^{0.5}-1}\right)^{\frac{3}{2}}\times 0.4325=0.1253.$$

Tables 3 and 4 show the numerical results (for getting the technique, see Algorithm 2).

$$\breve{M}_4=\min\left\{\frac{\Lambda_1}{4},\frac{\Lambda_2}{4},\frac{\Lambda_3}{2},\Lambda_4,\Lambda_5\right\}\simeq 0.001499,$$

and $\Lambda_6\simeq 1.583636$. Figures 4, 5, and 6 show a graphical representation of the variables. As shown in Fig. 4, $\breve{M}_2$ is directly related to $\tau\in[1,e]$ and increases with increasing τ. It can be seen in Fig. 5(a) that all values of Λ_i for $i=1,2,3,4,5$ are inversely proportional to τ. Also, $\breve{M}_4$ has the same behavior for $\tau\in J$, which can be seen in Fig. 5(b). Finally, the

Table 3 Numerical values of $\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi$, $\breve{M}_2$, γ, $\int_{\grave{a}}^{\grave{\imath}}\mathcal{G}_2(\grave{\imath},\xi)\,\mathrm{d}\xi$, $\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi$, and $\Delta=\phi_{\bar{p}}(\int_{\grave{a}}^{\grave{\imath}}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi)$ in Example 6.2 for $\tau\in J$

τ	$\int_{\grave{a}}^{\tau}\mathcal{G}_1(\grave{\imath},\xi)\,\mathrm{d}\xi$	$\breve{M}_2$	γ	$\int_{\grave{a}}^{\tau}\mathcal{G}_2(\grave{\imath},\xi)\,\mathrm{d}\xi$	$\int_{\grave{a}}^{\tau}\mathcal{H}(\grave{a},\xi)\hbar(\xi)\,\mathrm{d}\xi$	Δ
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.1052	0.0602	0.1059	0.0307	0.0384	0.0119	0.0001
1.2214	0.1299	0.2284	0.0662	0.0771	0.0793	0.0063
1.3499	0.2091	0.3677	0.1065	0.1157	0.2572	0.0661
1.4918	0.2975	0.4325	0.1253	0.1539	0.6196	0.3839
1.6487	0.3942	0.4325	0.1253	0.1913	1.2649	1.5999
1.8221	0.4975	0.4325	0.1253	0.2273	2.3154	5.3612
2.0138	0.6046	0.4325	0.1253	0.2610	3.9084	15.2758
2.2255	0.7105	0.4325	0.1253	0.2913	6.1656	38.0151
2.4596	0.8056	0.4325	0.1253	0.3162	9.1216	83.2040
2.7183	0.8654	0.4325	0.1253	0.3307	12.5716	158.0444

Table 4 Numerical values of Λ_1, Λ_2, Λ_3, Λ_4, Λ_5, Λ_6, and $\breve{M}_4$ in Example 6.2 for $\tau \in J$

τ	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	$\breve{M}_4$
1.0000	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*	*Inf*
1.1052	56.016486	16.599423	46,453.302726	5493.069851	1132.699598	7060.155197	4.149856
1.2214	8.405999	7.690366	487.818288	123.697821	243.285192	158.986839	1.922592
1.3499	2.592400	4.749445	28.980434	11.764874	93.355843	15.121205	0.648100
1.4918	1.075949	3.242105	3.526105	2.026594	54.177100	2.604749	0.268987
1.6487	0.527061	2.217810	0.641322	0.486300	37.060645	0.625034	0.131765
1.8221	0.287924	1.382433	0.152208	0.145124	23.101100	0.186526	0.071981
2.0138	0.170572	0.744218	0.044104	0.050933	12.436232	0.065463	0.022052
2.2255	0.108126	0.361888	0.015127	0.020467	6.047314	0.026305	0.007563
2.4596	0.073086	0.175974	0.006109	0.009351	2.940615	0.012019	0.003055
2.7183	0.053030	0.094769	0.002998	0.004923	1.583636	0.006327	0.001499

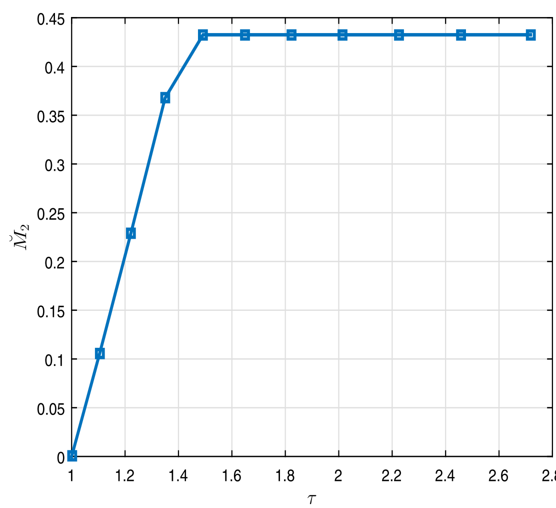


Figure 4 2D-graph of $\breve{M}_2$ for $\tau \in [1, e]$ in Example 6.2

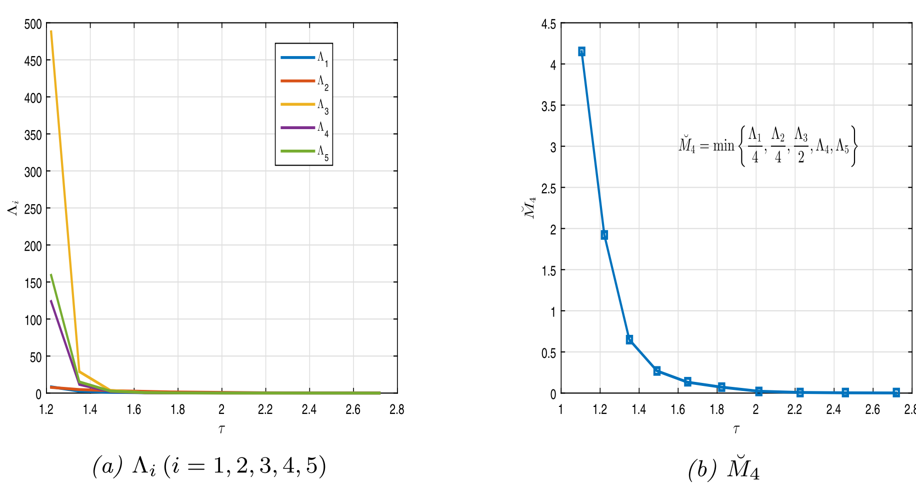


Figure 5 Graphical representation of Λ_i $(i = 1, 2, 3, 4, 5)$ and $\breve{M}_4$ for $\tau \in J$ in Example 6.2

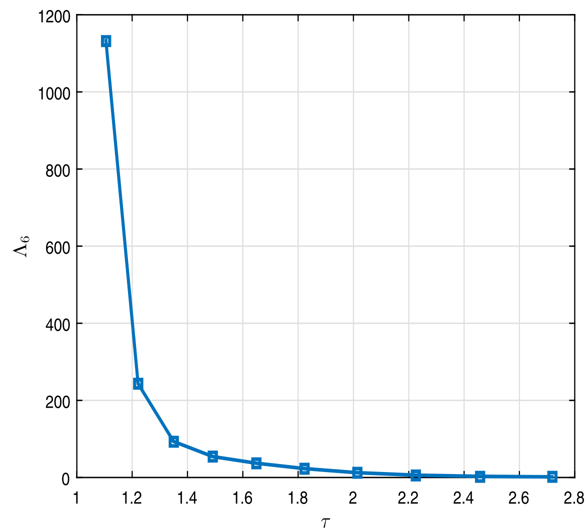


Figure 6 2D-graph of Λ_6 for $\tau \in J$ in Example 6.2

trend of variable Λ_6 with respect to τ is shown in Fig. 6. Choosing $\mathring{a} = 10^{-2}$, $b = \frac{11}{10}$, $c = 10^5$, $m_1 = 0.001 \in (0, \check{M}_4)$, $m_2 = 13 \in (\Lambda_6, \infty) = (1.583636, \infty)$, we get

$$\begin{aligned}
&\wp(\mathrm{q}) < \wp\bigl(10^{-2}\bigr) = 6 \times 10^{-4} < \min\bigl\{\phi_p(\mathring{a}m_1), \mathring{a}m_1\bigr\} \\
&\quad = \mathring{a}m_1 = 8 \times 10^{-4} \in \bigl[0, 10^{-2}\bigr], \\
&\wp\bigl(\mathrm{q}(\tau)\bigr) > 5 + (m_2\gamma b)^{1/4} = 5 + \left(0.1253 \times 13 \times \frac{11}{10}\right)^{1/4} \simeq 1.1570 > \phi_p(\gamma b m_2) \\
&\qquad \simeq 0.739467251 \in \left[\frac{11}{10}\gamma, \frac{11}{10}\right], \\
&\wp\bigl(\mathrm{q}(\tau)\bigr) < \wp\bigl(10^4\bigr) = 15 < \min\bigl\{\phi_p(cm_1), cm_1\bigr\} \\
&\qquad = \phi_p(cm_1) = \sqrt{8000}.001\mathrm{q} \in \bigl[0, 10^4\bigr], \\
&0 < \lambda \leq \frac{(1-\mu)\mathring{a}}{2} = 2.5 \times 10^{-3}.
\end{aligned}$$

Then, conditions (H7), (H8), and (H9) are satisfied. Therefore, it follows from Theorem 5.2 that fractional boundary value problem (43) has at least three $\frac{1}{2}$-concave positive solutions q_1, q_2, and q_3 such that

$$\|\mathrm{q}_1\| < 10^{-2}, \qquad \frac{11}{10}\gamma < \varphi(\mathrm{q}_2), \qquad \|\mathrm{q}_3\| > 10^{-2},$$

with $\varphi(\mathrm{q}_3) < \frac{11}{10}\gamma$.

7 Conclusion

The paper presents a new p-Laplacian boundary value problem of two-sided fractional differential equations involving generalized Caputo fractional derivatives, and we investigate the existence and multiplicity of ρ-concave positive solutions of it. We made some additional assumptions to prove some important results and obtain the existence of at least three solutions by using some fixed point theorems.

Appendix

Algorithm 1 (MATLAB lines for calculation of variable values in Example 6.1)

```
clear;
format short;
syms t r a x s b v e;
uprho_1=0.5; uprho_2=1.3;
sigma_1= 5/2; sigma_2= 5/2;
ga=exp(1); gi=exp(1)^2; gacirc=exp(1)^(3/2); gicirc=exp(1)^(7/4);
eta=exp(1)^8/5; mu=1/8;
p=3/2;
barp=p/(p-1);
A= 3/2; B = 1/2;
hslash=1/sqrt((2.2-log(x))*(log(x)-0.9));
fcirc=sqrt(abs(v));
Zedgi=upphi(barp,gi);
Zedgicirc=upphi(barp,gicirc);
column=1;
row=1;
tau=ga;
while tau<=gi
    RMatrix(row, column)=row;
    RMatrix(row, column+1)=tau;
    xi= ((gi^uprho_2-gicirc^uprho_2)...
    /(gi^uprho_2-ga^uprho_2))^(sigma_2-1);
    Zedgicirc=upphi(barp,xi);
    RMatrix(row, column+2)=Zedgicirc;
    if tau<=gi
        G1=1/gamma(s-1)*(t^r-a^r)/r*((b^r-x^r)/r)^(s-2)...
        *x^(r-1)-1/gamma(s)*((t^r-x^r)/r)^(s-1)*x^(r-1);
    else
        G1=1/gamma(s-1)*(t^r-a^r)/r*((b^r-x^r)/r)^(s-2)*x^(r-1);
    end;
    intG1=int(subs(G1, {t, s, r, a, b}, {gi, sigma_1, uprho_1, ga, gi}), x, ga,
        tau);
    RMatrix(row, column+3)=intG1;
    breveM_2 = min(min(1, (sigma_1-2)/(sigma_1-1)...
        *(gi^uprho_1-ga^uprho_1)/uprho_1), ...
        min(1, gamma(sigma_1-1)*((gi^uprho_1-ga^uprho_1)...
        /uprho_1)^(2-sigma_1)))*...
        ((gacirc^uprho_1-ga^uprho_1)/(gicirc^uprho_1-ga^uprho_1))...
        ^(sigma_1-1)*Zedgicirc*intG1);
    RMatrix(row, column+4)=breveM_2;
    vargamma=((gacirc^uprho_1-ga^uprho_1)...
        /(gi^uprho_1-ga^uprho_1))^(sigma_1-1)*breveM_2;
    RMatrix(row, column+5)=vargamma;
    if tau > ga
        GH=1/gamma(s-1)*(b^r-t^r)/r*((x^r-a^r)/r)^(s-2)*x^(r-1)...
            -1/gamma(s)*((x^r-t^r)/r)^(s-1)*x^(r-1);
    else
        GH=1/gamma(s-1)*(b^r-t^r)/r*((x^r-a^r)/r)^(s-2)*x^(r-1);
    end;
    intGHhslash=int((hslash)*subs(GH, {t, s, r, a, b}, {ga, sigma_2, uprho_2,
        ga, gi}), x, ga, tau);
    RMatrix(row, column+6)=intGHhslash;
    Lambda1=1/(A*intGHhslash);
    RMatrix(row, column+7)=Lambda1;
    if tau<=gi
        G2=1/gamma(s-1)*(t^r-a^r)/r*((b^r-x^r)/r)^(s-2)...
        *x^(r-1)-1/gamma(s-1)*((t^r-x^r)/r)^(s-2)*x^(r-1);
    else
        G2=1/gamma(s-1)*(t^r-a^r)/r*((b^r-x^r)/r)^(s-2)*x^(r-1);
    end;
    intG2=int(subs(G2, {t, s, r, a, b}, {gi, sigma_1, uprho_1, ga, gi}), x, ga,
        tau);
    RMatrix(row, column+8)=intG2;
    upphiintGH=upphi(barp, intGHhslash);
    RMatrix(row, column+9)=upphiintGH;
    temp=intG1+mu*((gi^uprho_1-ga^uprho_1)...
        /(uprho_1-uprho_1*mu))*intG2*upphiintGH;
    Lambda2=1/temp;
    RMatrix(row, column+10)=Lambda2;
    Lambda3=1/((sigma_1-1)/(sigma_1-2)*uprho_1...
        /(gi^uprho_1-ga^uprho_1)...
        *(intG1+mu/(1-mu)*intG2)*upphiintGH);
    RMatrix(row, column+11)=Lambda3;
    Lambda4=1/(1/gamma(sigma_1-1)*((gi^uprho_1-ga^uprho_1)...
```

```
            /uprho_1)^(sigma_1-2)*upphiintGH);
        RMatrix(row, column+12)=Lambda4;
        Lambda5=1/upphiintGH;
        RMatrix(row, column+13)=Lambda5;
        Zedgicirc=upphi(barp,gicirc);
        Lambda6=1/(vargamma*((gacirc^uprho_1-ga^uprho_1)...
            /(gi^uprho_1-ga^uprho_1))^(sigma_1-1)...
            *Zedgicirc*(temp));
        RMatrix(row, column+14)=Lambda6;
        RMatrix(row, column+15)=min(Lambda1/4, min(Lambda2/4, min(Lambda3/2,...
            min(Lambda4, Lambda5))));
        lambda=1.5;
        ell1=12;
        RMatrix(row, column+16)=(1-mu)/2*ell1*min(1, uprho_1/(gi^uprho_1-ga^uprho_1
            ));
        RMatrix(row, column+17)=2*lambda*((gi^uprho_1-ga^uprho_1)/(1-mu*uprho_1));
        RMatrix(row, column+18)=(1-mu)*ell1;
        tau = (exp(1))^(1+row/9);
        row=row+1;
    end
```

Algorithm 2 (MATLAB lines for calculation of variable values in Example 6.2)

```
clear;
format short;
syms t r a x s b v e;
uprho_1=0.5; uprho_2=2;
sigma_1= 5/2; sigma_2= 5/2;
ga=1; gi=exp(1); gacirc=sqrt(exp(1)); gicirc=exp(1)^(1/4);
eta=exp(1)^8/5; mu=1/8;
p=3/2;
barp=p/(p-1);
A= 3/2; B = 1/2;
hslash=5*sqrt(pi)*log(x)/4;
fcirc=sqrt(abs(v));
Zedgi=upphi(barp,gi);
Zedgicirc=upphi(barp,gicirc);
mathringa=10^(-2);
column=1;
row=1;
tau=ga;
while tau<=gi
    RMatrix(row, column)=row;
    RMatrix(row, column+1)=tau;
    xi= ((gi^uprho_2-gicirc^uprho_2)...
    /(gi^uprho_2-ga^uprho_2))^(sigma_2-1);
    Zedgicirc=upphi(barp,xi);
    RMatrix(row, column+2)=Zedgicirc;
    if tau<=gi
        G1=1/gamma(s-1)*(t^r-a^r)/r*((b^r-x^r)/r)^(s-2)...
        *x^(r-1)-1/gamma(s)*((t^r-x^r)/r)^(s-1)*x^(r-1);
    else
        G1=1/gamma(s-1)*(t^r-a^r)/r*((b^r-x^r)/r)^(s-2)*x^(r-1);
    end;
    intG1=int(subs(G1, {t, s, r, a, b}, {gi, sigma_1, uprho_1, ga, gi}), x, ga,
        tau);
    RMatrix(row, column+3)=intG1;
    breveM_2 = min(min(1, (sigma_1-2)/(sigma_1-1)...
        *(gi^uprho_1-ga^uprho_1)/uprho_1), ...
        min(1, gamma(sigma_1-1)*((gi^uprho_1-ga^uprho_1)...
        /uprho_1)^(2-sigma_1))*...
        ((gacirc^uprho_1-ga^uprho_1)/(gicirc^uprho_1-ga^uprho_1))...
        ^(sigma_1-1)*Zedgicirc*intG1);
    RMatrix(row, column+4)=breveM_2;
    vargamma=((gacirc^uprho_1-ga^uprho_1)...
        /(gi^uprho_1-ga^uprho_1))^(sigma_1-1)*breveM_2;
    RMatrix(row, column+5)=vargamma;
    if tau > ga
        GH=1/gamma(s-1)*(b^r-t^r)/r*((x^r-a^r)/r)^(s-2)*x^(r-1)...
            -1/gamma(s)*((x^r-t^r)/r)^(s-1)*x^(r-1);
    else
        GH=1/gamma(s-1)*(b^r-t^r)/r*((x^r-a^r)/r)^(s-2)*x^(r-1);
    end;
    intGHhslash=int((hslash)*subs(GH, {t, s, r, a, b}, {ga, sigma_2, uprho_2,
        ga, gi}), x, ga, tau);
    RMatrix(row, column+6)=intGHhslash;
    Lambda1=1/(A*intGHhslash);
    RMatrix(row, column+7)=Lambda1;
    if tau<=gi
        G2=1/gamma(s-1)*(t^r-a^r)/r*((b^r-x^r)/r)^(s-2)...
        *x^(r-1)-1/gamma(s-1)*((t^r-x^r)/r)^(s-2)*x^(r-1);
```

```
        else
            G2=1/gamma(s-1)*(t^r-a^r)/r*((b^r-x^r)/r)^(s-2)*x^(r-1);
        end;
        intG2=int(subs(G2, {t, s, r, a, b}, {gi, sigma_1, uprho_1, ga, gi}), x, ga,
            tau);
        RMatrix(row, column+8)=intG2;
        upphiintGH=upphi(barp, intGHhslash);
        RMatrix(row, column+9)=upphiintGH;
        temp=intG1+mu*((gi^uprho_1-ga^uprho_1)...
            /(uprho_1-uprho_1*mu))*intG2*upphiintGH;
        Lambda2=1/temp;
        RMatrix(row, column+10)=Lambda2;
        Lambda3=1/((sigma_1-1)/(sigma_1-2)*uprho_1...
            /(gi^uprho_1-ga^uprho_1)...
            *(intG1+mu/(1-mu)*intG2)*upphiintGH);
        RMatrix(row, column+11)=Lambda3;
        Lambda4=1/(1/gamma(sigma_1-1)*((gi^uprho_1-ga^uprho_1)...
            /uprho_1)^(sigma_1-2)*upphiintGH);
        RMatrix(row, column+12)=Lambda4;
        Lambda5=1/upphiintGH;
        RMatrix(row, column+13)=Lambda5;
        Zedgicirc=upphi(barp,gicirc);
        Lambda6=1/(vargamma*((gacirc^uprho_1-ga^uprho_1)...
            /(gi^uprho_1-ga^uprho_1))^(sigma_1-1)...
            *Zedgicirc*(temp));
        RMatrix(row, column+14)=Lambda6;
        RMatrix(row, column+15)=min(Lambda1/4, min(Lambda2/4, min(Lambda3/2,...
            min(Lambda4, Lambda5))));
        lambda=0.02;
        ell1=3;
        RMatrix(row, column+16)=(1-mu)/2*ell1*min(1, uprho_1/(gi^uprho_1-ga^uprho_1
            ));
        RMatrix(row, column+17)=2*lambda*((gi^uprho_1-ga^uprho_1)/(1-mu*uprho_1));
        RMatrix(row, column+18)=(1-mu)*ell1;

        tau = (exp(1))^(row/10);
        row=row+1;
    end
```

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Author contribution
FC: Actualization, methodology, formal analysis, validation, investigation, initial draft, and major contribution in writing the manuscript. MB: Methodology, formal analysis, validation, investigation, and initial draft. MH: Actualization, methodology, formal analysis, validation, investigation, initial draft, and major contribution in writing the manuscript. MES: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft, and major contribution in writing the manuscript. All authors read and approved the final manuscript.

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