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Variational approach to instantaneous and noninstantaneous impulsive system of differential equations

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Abstract

In this paper, the existence and multiplicity of solutions for a coupled system of differential equations with instantaneous and noninstantaneous impulses are studied. By the virtue of variational methods, some new existence theorems of solutions are obtained. In addition, two examples are given to demonstrate our main results.

Keywords: Differential equation; Instantaneous impulse; Noninstantaneous impulse; Variational methods

1 Introduction

In this paper, we consider the following impulsive differential system

$$\begin{cases} -u''(t) = D_uF_i(t,u(t),v(t)), & t \in (s_i,t_{i+1}], i = 0,1,2,\dots,N, \\ -v''(t) = D_vF_i(t,u(t),v(t)), & t \in (s_i,t_{i+1}], i = 0,1,2,\dots,N, \\ \Delta u'(t_i) = I_i(u(t_i)), & i = 1,2,\dots,N, \\ \Delta v'(t_i) = S_i(v(t_i)), & i = 1,2,\dots,N, \\ u'(t) = u'(t_i^+), & t \in (t_i,s_i], i = 1,2,\dots,N, \\ v'(t) = v'(t_i^+), & t \in (t_i,s_i], i = 1,2,\dots,N, \\ u'(s_i^+) = u'(s_i^-), & i = 1,2,\dots,N, \\ v'(s_i^+) = v'(s_i^-), & i = 1,2,\dots,N, \\ u(0) = u(T) = v(0) = v(T) = 0, \end{cases} \tag{1.1}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_N < s_N < t_{N+1} = T$, $D_uF_i, D_vF_i : (s_i,t_{i+1}] \times \mathbb{R}^2 \to \mathbb{R}$, $i = 0,1,\dots,N$, are continuous; $I_i, S_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,\dots,N$ are continuous; $\Delta u'(t_i) = u'(t_i^+) - u'(t_i^-)$, $\Delta v'(t_i) = v'(t_i^+) - v'(t_i^-)$. The noninear functions $D_uF_i(t,u,v)$, $D_vF_i(t,u,v)$ are the derivatives of $F_i(t,u,v)$ at u and v, respectively. The instantaneous impulses occur at the points t_i and the noninstantaneous impulses continue on the intervals $(t_i,s_i]$.

In recent years, the study of existence and multiplicity of solutions for the differential equations with impulsive effects via variational methods has attracted much attention.

Impulsive differential equations describe the dynamics of processes whose states change abruptly at certain moments of time. There are a number of classical tools to investigate impulsive differential equations [1–6], such as fixed point theory, topological degree theory, comparison method, and variational approach.

The pioneering work of boundary value problems for impulsive differential equations via variational approach was initiated by Tian–Ge [7] and Nieto–O'Regan [8]. Since then, a number of research results have emerged in this field, see, for instance, [9–14]. However, these studies merely focus on the differential equations with instantaneous impulsive effects. Due to the limitations of instantaneous impulses, which cannot describe all the phenomena in real life, such as earthquakes and tsunamis, Hernández–O'Regan [15] introduced the noninstantaneous impulsive differential equations. The existence of solutions for noninstantaneous impulsive differential equations have been investigated via some methods [15–21], such as the theory of analytic semigroups, fixed point theory, and variational methods. Since the variational structure of general noninstantaneous impulsive differential equations is relatively difficult to establish, Bai–Nieto [21] first studied the linear equation with noninstantaneous impulses via variational methods and continued until 2017. On the basis of [21], Tian–Zhang [22] took the instantaneous impulses into the noninstantaneous impulsive differential equations and extended the linear terms to the nonlinear terms. The second-order differential equation with instantaneous and noninstantaneous impulses is considered below:

$$\begin{cases} -u''(t) = f_i(t,u(t)), & t \in (s_i, t_{i+1}], i = 0,1,2,\dots,N, \\ \Delta u'(t_i) = I_i(u(t_i)), & i = 1,2,\dots,N, \\ u'(t) = u'(t_i^+), & t \in (t_i, s_i], i = 1,2,\dots N, \\ u'(s_i^+) = u'(s_i^-), & i = 1,2,\dots,N, \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < s_N < t_{N+1} = T$, $f_i \in C((s_i, t_{i+1}] \times \mathbb{R}, \mathbb{R})$, $I_i \in C(\mathbb{R}, \mathbb{R})$, $\Delta u'(t_i) = u'(t_i^+) - u'(t_i^-)$, the instantaneous impulses occur at the points t_i, and the noninstantaneous impulses continue on the intervals $(t_i, s_i]$. The authors showed that the problem (1.2) has at least one classical solution by using Ekeland's variational principle. Based on [22], Zhang–Liu [23], Zhou–Deng–Wang [24], and Chen–Gu–Ma [25] also studied the fractional differential equations with instantaneous and noninstantaneous impulses. They extended the results of [22] and obtained the existence of solutions by using variational methods.

On the other hand, the coupled systems of differential equations play a very important role in various fields such as biology, chemistry, and physics. They present some new phenomena, which are not appeared in the study of a single equation. From [26–29], we know that many authors have done lots of works in this field. In recent years, the coupled systems involving differential equations with impulsive effects are also widely studied by the variational approach. More precisely, in [30], Wu–Liu considered the following coupled system of instantaneous impulsive differential equations:

$$\begin{cases} -u''(t) + g(t)u(t) = f_u(u(t), v(t)), & \text{a.e. } t \in [0, T], \\ -v''(t) + h(t)v(t) = f_v(u(t), v(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = v(0) = v(T) = 0, \\ \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = I_k(u(t_k)), \\ \Delta v'(t_k) = v'(t_k^+) - v'(t_k^-) = J_k(v(t_k)), & k = 1, 2, \dots, n, \end{cases} \tag{1.3}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, $g, h \in L^\infty[0, T]$, $f_u, f_v : \mathbb{R}^2 \to \mathbb{R}$ are continuous, and $I_j, J_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \dots, n$ are continuous. They obtained that the problem (1.3) has at least one nontrivial solution via variational methods. For the recent works about instantaneous impulsive differential systems, the interested readers may refer to [31–34].

In [35], Nesmoui–Abdelkade–Nieto–Ouahab considered the following noninstantaneous impulsive system of differential equations:

$$\begin{cases} -u''(t) = D_u f_i(t, u(t) - u(t_{i+1}), v(t) - v(t_{i+1})), & t \in (s_i, t_{i+1}], i = 0, 1, \dots, m, \\ -v''(t) = D_v f_i(t, u(t) - u(t_{i+1}), v(t) - v(t_{i+1})), & t \in (s_i, t_{i+1}], i = 0, 1, \dots, m, \\ u'(t) = \alpha_i, \quad t \in (t_i, s_i], i = 1, 2, \dots, m, \\ v'(t) = \beta_i, \quad t \in (t_i, s_i], i = 1, 2, \dots, m, \\ u'(s_i^+) = u'(s_i^-), \quad i = 1, 2, \dots, m, \\ v'(s_i^+) = v'(s_i^-), \quad i = 1, 2, \dots, m, \\ u'(0^+) = \alpha_0, \qquad v'(0^+) = \beta_0, \\ u(0) = u(T) = v(0) = v(T), \end{cases} \tag{1.4}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_m < s_m < t_{m+1} = T$, the impulses start abruptly at points t_i, $i = 0, 1, 2, \dots, m$, and keep the derivative constant on a finite time interval $(t_i, s_i]$. Here $u'(s_i^{\pm}) = \lim_{s \to s_i^{\pm}} u'(s)$, and α_i, β_i, $i = 0, 1, 2, \dots, m$, are given constants. For each $i = 0, 1, 2, \dots, m$, the nonlinear functions $D_u f_i$, $D_v f_i$ (the derivatives of $f_i(t, u, v)$ at u and v respectively) are Carathéodory functions on $(s_i, t_{i+1}] \times \mathbb{R}^2$. They obtained that the problem (1.4) has at least one solution.

Inspired by the above facts, in this paper, our aim is to study the variational structure of problem (1.1) in an appropriate space of functions and the existence and multiplicity of solutions for the problem (1.1) by using variational methods. Under the assumption that the nonlinearities and the impulsive functions satisfy different growth conditions, we obtain the existence of at least one classical solution and infinitely many classical solutions. Our main results generalize the existing result in [22].

Throughout this paper, we need the following conditions:

(H1) There exist $a_i, b_i > 0$, and $\gamma_1, \gamma_2 \in [0, 1)$, $i = 0, 1, \dots, N$, such that

$$\left|D_x F_i(t, x, y)\right| \le a_i + b_i |x|^{\gamma_1}, \quad \text{for every } (t, x, y) \in (s_i, t_{i+1}] \times \mathbb{R}^2,$$

and

$$\left|D_y F_i(t, x, y)\right| \le a_i + b_i |y|^{\gamma_2}, \quad \text{for every } (t, x, y) \in (s_i, t_{i+1}] \times \mathbb{R}^2.$$

(H2) There exist $c_i, d_i > 0$, and $\beta_i \in [0,1)$, $i = 1,2,\dots,N$, such that

$$\left|I_i(x)\right|, \left|S_i(x)\right| \le c_i + d_i|x|^{\beta_i}, \quad \text{for every } x \in \mathbb{R},$$

(H3) (i) I_i, S_i, $i = 1,2,\dots,N$ satisfy $I_i(x)x, S_i(x)x \ge 0$, for all $x \in \mathbb{R}$;
(ii) There exist $\theta > 2$, $\delta_i > 0$, $i = 1,2,\dots,N$, such that $\int_0^x I_i(s)\,ds$, $\int_0^x S_i(s)\,ds \le \delta_i|x|^\theta$, for $x \in \mathbb{R}\backslash\{0\}$;
(iii) $I_i(x)x \le \theta \int_0^x I_i(s)\,ds$, $S_i(x)x \le \theta \int_0^x S_i(s)\,ds$ for $x \in \mathbb{R}\backslash\{0\}$;

(H4) $|D_xF_i(t,x,y)|, |D_yF_i(t,x,y)| = o(|x| + |y|)$, as $|x| + |y| \to 0$.

(H5) There exist $C > 0$, $M > 0$, and $2 < \theta < \beta$ such that

$$F_i(t,x,y) \ge C\big(|x|^\beta + |y|^\beta\big), \quad |x| + |y| \ge M,$$

and

$$\theta F_i(t,x,y) \le xD_xF_i(t,x,y) + yD_yF_i(t,x,y).$$

The main results of this paper are as follows.

Theorem 1.1 *Assume that* (H1) *and* (H2) *hold. Then, problem* (1.1) *has at least one classical solution.*

Theorem 1.2 *Assume that* (H3)–(H5) *hold, Then, problem* (1.1) *has at least one classical solution.*

Theorem 1.3 *Assume that* (H3)–(H5) *and the following conditions hold*:
(H6) $D_xF_i(t,x,y)$, $D_yF_i(t,x,y)$, $i = 0,1,\dots,N$, *are odd as functions of* x, y, *respectively*;
(H7) $I_i(x)$, $S_i(x)$, $i = 1,2,\dots,N$ *are odd functions of* x.
Then, problem (1.1) *has infinitely many classical solutions.*

Theorem 1.4 *Assume that* (H2), (H4)–(H7) *hold. Moreover, suppose that* $I_i(x)$, $S_i(x)$, $i = 1,2,\dots,N$, *are nondecreasing. Then, problem* (1.1) *has infinitely many classical solutions.*

The rest of this paper is organized as follows. In Sect. 2, we present some preliminaries. In Sect. 3, we prove the Theorems 1.1–1.4 via the variational approach.

2 Preliminaries

In this section, we first introduce some definitions and theorems which will be needed in our argument.

Definition 2.1 Let X be a Banach space and $\Phi : X \to (-\infty, +\infty]$. Functional Φ is said to be weakly lower semicontinuous if $\liminf_{m\to\infty} \Phi(x_m) \ge \Phi(x)$ as $x_m \rightharpoonup x$ in X.

Definition 2.2 ([36], (PS) condition) Let X be a real reflexive Banach space. For any sequence $\{u_m\} \subset X$, if $\{\Phi(u_m)\}$ is bounded and $\Phi'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence, then we say that Φ satisfies the Palais–Smale condition.

Theorem 2.3 ([36]) *Let X be a reflexive Banach space. If $\Phi : X \to (-\infty, +\infty]$ is coercive, then Φ has a bounded minimizing sequence.*

Theorem 2.4 ([36]) *Let X be a reflexive Banach space and let $\Phi : X \to (-\infty, +\infty]$ be weakly lower semicontinuous on X. If Φ has a bounded minimizing sequence, then Φ has a minimum on X.*

Theorem 2.5 ([37], Mountain Pass Theorem) *Let X be a real Banach space and suppose $\Phi \in C^1(X, \mathbb{R})$ satisfies the (PS) condition with $\Phi(0) = 0$. If Φ satisfies the following conditions:*

(i) *there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho} \geq \alpha$;*

(ii) *there exists an $e \in X \backslash B_\rho$ such that $\Phi(e) \leq 0$,*

then Φ possesses a critical value $c \geq \alpha$. Moreover, c is given by $c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \Phi(g(s))$, where

$$\Gamma = \{g \in C([0,1], X) \mid g(0) = 0, g(1) = e\}.$$

Theorem 2.6 ([37], Symmetric Mountain Pass Theorem) *Let X be an infinite-dimensional real Banach space. Let $\Phi \in C^1(X, \mathbb{R})$ be an even functional which satisfies the (PS) condition, and $\Phi(0) = 0$. Suppose that $X = V \oplus Y$, where V is finite dimensional, and Φ satisfies:*

(i) *there exist $\alpha > 0$ and $\rho > 0$ such that $\Phi|_{\partial B_\rho \cap Y} \geq \alpha$;*

(ii) *for each finite-dimensional subspace $W \subset X$, there is $R = R(W)$ such that $\Phi(u) \leq 0$ on $W \backslash B_{R(W)}$,*

then Φ possesses an unbounded sequence of critical values.

In the Sobolev space $H_0^1(0, T)$, we consider the inner products

$$(u, v)_1 = \int_0^T u'(t)v'(t)\,dt$$

and

$$(u, v)_2 = \int_0^T u(t)v(t)\,dt + \int_0^T u'(t)v'(t)\,dt,$$

which induce the corresponding norms

$$\|u\|_1 = \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}$$

and

$$\|u\|_2 = \left(\int_0^T \left|u(t)\right|^2 dt + \int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

We also define the norms in $C[0, T]$, $L^2(0, T)$ as $\|u\|_\infty = \max_{t \in [0,T]} |u(t)|$ and $\|u\|_{L^2} = (\int_0^T |u(t)|^2\,dt)^{\frac{1}{2}}$, respectively.

Obviously, $H_0^1(0,T)$ is a Hilbert space. By Poincaré inequality,

$$\int_0^T \left|u(t)\right|^2 dt \le \frac{1}{\lambda_1}\int_0^T \left|u'(t)\right|^2 dt,$$

where $\lambda_1 = \frac{\pi^2}{T^2}$ is the first eigenvalue of the Dirichlet problem

$$\begin{cases} -u''(t) = \lambda u(t), & t \in [0,T], \\ u(0) = u(T) = 0. \end{cases} \tag{2.1}$$

It is easy to verify that the norms $\|u\|_1$ and $\|u\|_2$ are equivalent. Set $X = H_0^1(0,T) \times H_0^1(0,T)$. In the Hilbert space X, for any $(u,v) \in X$, we consider the norm

$$\left\|(u,v)\right\|_X = \left(\|u\|_1^2 + \|v\|_1^2\right)^{\frac{1}{2}}.$$

Taking $(x,y) \in X$ and multiplying the two sides of the equalities

$$-u''(t) = D_u F_i\left(t,u(t),v(t)\right)$$

and

$$-v''(t) = D_v F_i\left(t,u(t),v(t)\right)$$

by x and y, respectively, then integrating from s_i to t_{i+1}, we have

$$-\int_{s_i}^{t_{i+1}} u''(t)x(t)\,dt = \int_{s_i}^{t_{i+1}} D_u F_i\left(t,u(t),v(t)\right)x(t)\,dt \tag{2.2}$$

and

$$-\int_{s_i}^{t_{i+1}} v''(t)y(t)\,dt = \int_{s_i}^{t_{i+1}} D_v F_i\left(t,u(t),v(t)\right)y(t)\,dt. \tag{2.3}$$

The first term of (2.2) is

$$-\int_{s_i}^{t_{i+1}} u''(t)x(t)\,dt = -u'\left(t_{i+1}^-\right)x\left(t_{i+1}^-\right) + u'\left(s_i^+\right)x\left(s_i^+\right) + \int_{s_i}^{t_{i+1}} u'(t)x'(t)\,dt.$$

Hence,

$$\begin{aligned} &\sum_{i=0}^{N} -u'\left(t_{i+1}^-\right)x\left(t_{i+1}^-\right) + \sum_{i=0}^{N} u'\left(s_i^+\right)x\left(s_i^+\right) + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} u'(t)x'(t)\,dt \\ &\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_u F_i\left(t,u(t),v(t)\right)x(t)\,dt = 0. \end{aligned} \tag{2.4}$$

Since $u'(t) = u'(t_i^+)$, $t \in (t_i, s_i]$, $i = 1,2,\ldots,N$,

$$\int_{t_i}^{s_i} u'(t)x'(t)\,dt = u'\left(t_i^+\right)x\left(s_i^-\right) - u'\left(t_i^+\right)x\left(t_i^+\right).$$

Therefore,

$$\sum_{i=1}^{N}\int_{t_i}^{s_i} u'(t)x'(t)\,dt - \sum_{i=1}^{N} u'\bigl(t_i^+\bigr)x\bigl(s_i^-\bigr) + \sum_{i=1}^{N} u'\bigl(t_i^+\bigr)x\bigl(t_i^+\bigr) = 0. \tag{2.5}$$

Combining (2.4) and (2.5), one has

$$\begin{aligned}&\int_0^T u'(t)x'(t)\,dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_uF_i\bigl(t,u(t),v(t)\bigr)x(t)\,dt\\&\quad+\sum_{i=1}^{N}\bigl(u'\bigl(t_i^+\bigr)-u'\bigl(t_i^-\bigr)\bigr)x(t_i)=0,\end{aligned}$$

i.e.,

$$\int_0^T u'(t)x'(t)\,dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_uF_i\bigl(t,u(t),v(t)\bigr)x(t)\,dt + \sum_{i=1}^{N} I_i\bigl(u(t_i)\bigr)x(t_i) = 0. \tag{2.6}$$

On the other hand, similarly we obtain

$$\int_0^T v'(t)y'(t)\,dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_vF_i\bigl(t,u(t),v(t)\bigr)y(t)\,dt + \sum_{i=1}^{N} S_i\bigl(v(t_i)\bigr)y(t_i) = 0. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned}&\int_0^T u'(t)x'(t)\,dt + \int_0^T v'(t)y'(t)\,dt + \sum_{i=1}^{N} I_i\bigl(u(t_i)\bigr)x(t_i) + \sum_{i=1}^{N} S_i\bigl(v(t_i)\bigr)y(t_i)\\&\quad= \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_uF_i\bigl(t,u(t),v(t)\bigr)x(t)\,dt + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_vF_i\bigl(t,u(t),v(t)\bigr)y(t)\,dt.\end{aligned} \tag{2.8}$$

Now, we introduce the concept of a weak solution for problem (1.1).

Definition 2.7 We say that a pair of functions $(u,v)\in X$ is a weak solution for problem (1.1) if identity (2.8) holds for any $(x,y)\in X$.

For any $(u,v)\in X$, we define the following functional on X:

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2}\int_0^T \bigl|u'(t)\bigr|^2\,dt + \frac{1}{2}\int_0^T \bigl|v'(t)\bigr|^2\,dt + \sum_{i=1}^{N}\int_0^{u(t_i)} I_i(s)\,ds\\&\quad+\sum_{i=1}^{N}\int_0^{v(t_i)} S_i(s)\,ds - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i\bigl(t,u(t),v(t)\bigr)\,dt\\&=\frac{1}{2}\bigl\|(u,v)\bigr\|_X^2 + \sum_{i=1}^{N}\int_0^{u(t_i)} I_i(s)\,ds + \sum_{i=1}^{N}\int_0^{v(t_i)} S_i(s)\,ds\\&\quad-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i\bigl(t,u(t),v(t)\bigr)\,dt.\end{aligned} \tag{2.9}$$

Using the continuity of D_uF_i, D_vF_i, $i = 0, 1, \ldots, N$, and I_i, S_i, $i = 1, 2, \ldots, N$, we can show that the functional $\Phi \in C^1(X, \mathbb{R})$. For any $(x, y) \in X$, we have

$$\begin{aligned}\Phi'(u,v)(x,y) = {} & \int_0^T u'(t)x'(t)\,dt + \int_0^T v'(t)y'(t)\,dt + \sum_{i=1}^{N} I_i\bigl(u(t_i)\bigr)x(t_i) \\ & + \sum_{i=1}^{N} S_i\bigl(v(t_i)\bigr)y(t_i) - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} D_uF_i\bigl(t, u(t), v(t)\bigr)x(t)\,dt \\ & - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} D_vF_i\bigl(t, u(t), v(t)\bigr)y(t)\,dt. \end{aligned} \tag{2.10}$$

Thus, the weak solutions of problem (1.1) are the corresponding critical points of Φ.

Lemma 2.8 *If $(u, v) \in X$ is a weak solution of problem (1.1), then $(u, v) \in X$ is a classical solution of problem (1.1).*

Proof Proceeding as in the proof of Lemma 2.2 in [22], we can prove that Lemma 2.8 holds. Thus, the proof is omitted here. □

Lemma 2.9 *For any $(u, v) \in X$, there exists a constant $c > 0$ such that $\|u\|_\infty, \|v\|_\infty \leq c\|(u, v)\|_X$.*

Proof For any $(u, v) \in X$, it follows from the mean value theorem that

$$u(\tau) = \frac{1}{T}\int_0^T u(s)\,ds$$

for some $\tau \in [0, T]$. Furthermore, using Hölder and Poincaré inequality, we have

$$\begin{aligned}\bigl|u(t)\bigr| &= \left|u(\tau) + \int_\tau^T u'(s)\,ds\right| \\ &\leq \frac{1}{T}\int_0^T \bigl|u(s)\bigr|\,ds + \int_0^T \bigl|u'(s)\bigr|\,ds \\ &\leq T^{-\frac{1}{2}}\|u\|_{L^2} + T^{\frac{1}{2}}\bigl\|u'\bigr\|_{L^2} \\ &\leq \bigl((\lambda_1 T)^{-\frac{1}{2}} + T^{\frac{1}{2}}\bigr)\bigl\|u'\bigr\|_{L^2} \\ &\leq \bigl((\lambda_1 T)^{-\frac{1}{2}} + T^{\frac{1}{2}}\bigr)\bigl\|(u,v)\bigr\|_X.\end{aligned}$$

Hence, there exists a constant $c = (\lambda_1 T)^{-\frac{1}{2}} + T^{\frac{1}{2}} > 0$ such that

$$\|u\|_\infty \leq c\bigl\|(u,v)\bigr\|_X.$$

Similarly, we can obtain

$$\|v\|_\infty \leq c\bigl\|(u,v)\bigr\|_X.$$ □

3 Main results

In this section, we give the proofs of our main results.

Lemma 3.1 *The functional* $\Phi : X \to \mathbb{R}$ *is weakly lower semicontinuous.*

Proof Let $\{(u_m, v_m)\} \subset X$ with $(u_m, v_m) \rightharpoonup (u, v)$, then we obtain that $\{u_m\}$ and $\{v_m\}$ converge uniformly to u and v on $[0, T]$, respectively (see [36, Proposition 1.2]). In connection with the fact that $\|(u, v)\|_X \leq \liminf_{m\to\infty} \|(u_m, v_m)\|_X$, one has

$$\begin{aligned}
\liminf_{m\to\infty} \Phi(u_m, v_m) &= \liminf_{m\to\infty} \Bigg\{ \frac{1}{2} \big\|(u_m, v_m)\big\|_X^2 + \sum_{i=1}^{N} \int_0^{u_m(t_i)} I_i(s)\,ds \\
&\quad + \sum_{i=1}^{N} \int_0^{v_m(t_i)} S_i(s)\,ds - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} F_i\big(t, u_m(t), v_m(t)\big)\,dt \Bigg\} \\
&\geq \frac{1}{2} \big\|(u, v)\big\|_X^2 + \sum_{i=1}^{N} \int_0^{u(t_i)} I_i(s)\,ds \\
&\quad + \sum_{i=1}^{N} \int_0^{v(t_i)} S_i(s)\,ds - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} F_i\big(t, u(t), v(t)\big)\,dt \\
&= \Phi(u, v).
\end{aligned}$$

This implies that Φ is a weakly lower semicontinuous functional. □

Proof of Theorem 1.1 For any $(u, v) \in X$, by (H1), (H2), and Lemma 2.9, we have

$$\begin{aligned}
\Phi(u, v) &= \frac{1}{2} \big\|(u, v)\big\|_X^2 + \sum_{i=1}^{N} \int_0^{u(t_i)} I_i(s)\,ds + \sum_{i=1}^{N} \int_0^{v(t_i)} S_i(s)\,ds \\
&\quad - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} F_i\big(t, u(t), v(t)\big)\,dt \\
&\geq \frac{1}{2} \big\|(u, v)\big\|_X^2 - \sum_{i=1}^{N} \int_0^{u(t_i)} \big(c_i + d_i|s|^{\beta_i}\big)\,ds - \sum_{i=1}^{N} \int_0^{v(t_i)} \big(c_i + d_i|s|^{\beta_i}\big)\,ds \\
&\quad - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} \big(a_i|u| + b_i|u|^{\gamma_1+1} + a_i|v| + b_i|v|^{\gamma_2+1}\big)\,dt \\
&\geq \frac{1}{2} \big\|(u, v)\big\|_X^2 - NC\|u\|_\infty - D\sum_{i=1}^{N} \|u\|_\infty^{\beta_i+1} - NC\|v\|_\infty \\
&\quad - D\sum_{i=1}^{N} \|v\|_\infty^{\beta_i+1} - (N+1)AT\|u\|_\infty - (N+1)BT\|u\|_\infty^{\gamma_1+1} \\
&\quad - (N+1)AT\|v\|_\infty - (N+1)BT\|v\|_\infty^{\gamma_2+1} \\
&\geq \frac{1}{2} \big\|(u, v)\big\|_X^2 - 2NCc\big\|(u, v)\big\|_X - 2D\sum_{i=1}^{N} c^{\beta_i+1} \big\|(u, v)\big\|_X^{\beta_i+1} \\
&\quad - 2(N+1)ATc\big\|(u, v)\big\|_X - (N+1)BTc^{\gamma_1+1} \big\|(u, v)\big\|_X^{\gamma_1+1}
\end{aligned}$$

$$- (N+1)BTc^{\gamma_2+1}\big\|(u,v)\big\|_X^{\gamma_2+1},$$

where $A = \max\{a_0, a_1, \dots, a_N\}$, $B = \max\{b_0, b_1, \dots, b_N\}$, $C = \max\{c_1, c_2, \dots, c_N\}$, and $D = \max\{d_1, d_2, \dots, d_N\}$. Since $\gamma_1, \gamma_2 \in [0,1)$, $i = 0, 1, \dots, N$, $\beta_i \in [0,1)$, $i = 1, 2, \dots, N$, the above equation implies $\lim_{\|(u,v)\|_X \to \infty} \Phi(u,v) = +\infty$, i.e., Φ is coercive. By Lemma 3.1 and Theorem 2.3, we obtain that functional Φ satisfies all the conditions of Theorem 2.4. So Φ has a minimum on X, which is a critical point of Φ. Hence, problem (1.1) has at least one classical solution. □

Corollary 3.2 *Assume that D_uF_i, D_vF_i, $i = 0, 1, \dots, N$, I_i, S_i, $i = 1, 2, \dots, N$, are bounded. Then, problem* (1.1) *has at least one classical solution.*

Proof of Theorem 1.2 Obviously, $\Phi \in C^1(X, \mathbb{R})$ and $\Phi(0,0) = 0$. We divide the proof into three parts.

First, we will show that Φ satisfies the (PS) condition. Let $\{(u_m, v_m)\} \subset X$ be a sequence such that $\{\Phi(u_m, v_m)\}$ is bounded and $\Phi'(u_m, v_m) \to 0$. By (2.9), (2.10), (H3), and (H5), we have

$$\begin{aligned}
&\theta\Phi(u_m,v_m) - \Phi'(u_m,v_m)(u_m,v_m) \\
&\quad= \frac{\theta}{2}\big\|(u_m,v_m)\big\|_X^2 + \theta\sum_{i=1}^N\int_0^{u_m(t_i)} I_i(s)\,ds + \theta\sum_{i=1}^N\int_0^{v_m(t_i)} S_i(s)\,ds \\
&\qquad - \theta\sum_{i=0}^N\int_{s_i}^{t_{i+1}} F_i(t,u_m,v_m)\,dt - \int_0^T \big|u_m'(t)\big|^2\,dt - \int_0^T\big|v_m'(t)\big|^2\,dt \\
&\qquad - \sum_{i=1}^N I_i\big(u_m(t_i)\big)u_m(t_i) - \sum_{i=1}^N S_i\big(v_m(t_i)\big)v_m(t_i) \\
&\qquad + \sum_{i=0}^N\int_{s_i}^{t_{i+1}} D_uF_i(t,u_m,v_m)u_m\,dt + \sum_{i=0}^N\int_{s_i}^{t_{i+1}} D_vF_i(t,u_m,v_m)v_m\,dt \\
&\quad= \left(\frac{\theta}{2}-1\right)\big\|(u_m,v_m)\big\|_X^2 + \left(\theta\sum_{i=1}^N\int_0^{u_m(t_i)} I_i(s)\,ds - \sum_{i=1}^N I_i\big(u_m(t_i)\big)u_m(t_i)\right) \\
&\qquad + \left(\theta\sum_{i=1}^N\int_0^{v_m(t_i)} S_i(s)\,ds - \sum_{i=1}^N S_i\big(v_m(t_i)\big)v_m(t_i)\right) \\
&\qquad + \left(\sum_{i=0}^N\int_{s_i}^{t_{i+1}} D_uF_i(t,u_m,v_m)u_m\,dt + \sum_{i=0}^N\int_{s_i}^{t_{i+1}} D_vF_i(t,u_m,v_m)v_m\,dt\right. \\
&\qquad \left. - \theta\sum_{i=0}^N\int_{s_i}^{t_{i+1}} F_i(t,u_m,v_m)\,dt\right) \\
&\quad\geq \left(\frac{\theta}{2}-1\right)\big\|(u_m,v_m)\big\|_X^2.
\end{aligned}$$

Since $\theta > 2$, it follows that $\{(u_m, v_m)\}$ is bounded in X. Passing, if necessary, to a subsequence, we can assume that there exist $\{(u_m, v_m)\} \in X$ such that

$$(u_m, v_m) \rightharpoonup (u,v) \quad \text{in } X,$$

$$u_m \to u, \qquad v_m \to v \quad \text{uniformly in } C\big([0,T]\big),$$

as $m \to +\infty$. Hence

$$\begin{aligned}
&\sum_{i=1}^{N}(I_i(u_m(t_i)) - I_i\big(u(t_i)\big)\big(u_m(t_i) - u(t_i)\big) \to 0,\\
&\sum_{i=1}^{N}(S_i(v_m(t_i)) - S_i\big(v(t_i)\big)\big(v_m(t_i) - v(t_i)\big) \to 0,\\
&\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\big(D_uF_i(t,u_m,v_m) - D_uF_i(t,u,v)\big)(u_m - u)\,dt \to 0,\\
&\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\big(D_vF_i(t,u_m,v_m) - D_vF_i(t,u,v)\big)(v_m - v)\,dt \to 0,
\end{aligned} \tag{3.1}$$

as $m \to +\infty$. Moreover, by (2.10), we have

$$\begin{aligned}
&\big(\Phi'(u_m,v_m) - \Phi'(u,v)\big)(u_m - u, v_m - v)\\
&\quad = \big\|(u_m - u, v_m - v)\big\|^2 + \sum_{i=1}^{N}\big(I_i\big(u_m(t_i) - I_i\big(u(t_i)\big)\big)\big(u_m(t_i) - u(t_i)\big)\\
&\qquad + \sum_{i=1}^{N}\big(S_i\big(v_m(t_i) - S_i\big(v(t_i)\big)\big)\big(v_m(t_i) - v(t_i)\big)\\
&\qquad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\big(D_uF_i(t,u_m,v_m) - D_uF_i(t,u,v)\big)(u_m - u)\,dt\\
&\qquad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\big(D_vF_i(t,u_m,v_m) - D_vF_i(t,u,v)\big)(v_m - v)\,dt.
\end{aligned} \tag{3.2}$$

Since $\Phi'(u_m,v_m) \to 0$ and $(u_m,v_m) \rightharpoonup (u,v)$, we have

$$\big(\Phi'(u_m,v_m) - \Phi'(u,v)\big)(u_m - u, v_m - v) \to 0, \quad \text{as } m \to +\infty. \tag{3.3}$$

Therefore, (3.1), (3.2), and (3.3) yield $\|(u_m - u, v_m - v)\| \to 0$ as $m \to +\infty$. That is, $(u_m,v_m) \to (u,v)$ in X, which means that the (PS) condition holds for Φ.

Second, we verify that Φ satisfies assumption (i) of Theorem 2.5. By the Sobolev embedding theorem, there exists $\gamma > 0$ such that

$$\|u\|_{L^2}^2 + \|v\|_{L^2}^2 \le \gamma\big\|(u,v)\big\|_X^2. \tag{3.4}$$

By (H4), we have

$$F_i(t,u,v) = o\big(|u|^2 + |v|^2\big), \quad \text{as } |u| + |v| \to 0.$$

Let $\varepsilon = \frac{1}{4(N+1)\gamma}$, then there exists $\delta > 0$ such that $|u| + |v| < \delta$ implies

$$F_i(t,u,v) \le \frac{1}{4(N+1)\gamma}\big(|u|^2 + |v|^2\big), \quad \forall (u,v) \in X. \tag{3.5}$$

In addition, it follows from (i) of (H3) that

$$\int_0^{u(t_i)} I_i(s)\,ds \geq 0 \quad \text{and} \quad \int_0^{v(t_i)} S_i(s)\,ds \geq 0. \tag{3.6}$$

It is clear that $\|(u,v)\|_X \leq \frac{\delta}{c}$, where c is defined in Lemma 2.9, implies that $\|u\|_\infty, \|v\|_\infty < \delta$. By (2.9), (3.4), (3.5), and (3.6), we have

$$\begin{aligned}
\Phi(u,v) &= \frac{1}{2}\big\|(u,v)\big\|_X^2 + \sum_{i=1}^N \int_0^{u(t_i)} I_i(s)\,ds + \sum_{i=1}^N \int_0^{v(t_i)} S_i(s)\,ds \\
&\quad - \sum_{i=0}^N \int_{s_i}^{t_{i+1}} F_i(t,u,v)\,dt \\
&\geq \frac{1}{2}\big\|(u,v)\big\|_X^2 - \sum_{i=0}^N \int_{s_i}^{t_{i+1}} \frac{1}{4(N+1)\gamma}\big(|u|^2+|v|^2\big)\,dt \\
&\geq \frac{1}{2}\big\|(u,v)\big\|_X^2 - \frac{1}{4\gamma}\int_0^T \big(|u|^2+|v|^2\big)\,dt \\
&\geq \frac{1}{2}\big\|(u,v)\big\|_X^2 - \frac{1}{4\gamma}\big(\|u\|_{L^2}^2+\|v\|_{L^2}^2\big) \\
&= \frac{1}{4}\big\|(u,v)\big\|_X^2.
\end{aligned}$$

Choose $\alpha = \frac{\delta^2}{4c^2}$, $\rho = \frac{\delta}{c}$, then $\Phi(u,v) \geq \alpha > 0$ for any $(u,v) \in \partial B_\rho$.

Finally, we prove that assumption (ii) of Theorem 2.5 is satisfied. According to (H3), (H5), and Lemma 2.9, we have

$$\begin{aligned}
\Phi(u,v) &= \frac{1}{2}\big\|(u,v)\big\|_X^2 + \sum_{i=1}^N \int_0^{u(t_i)} I_i(s)\,ds + \sum_{i=1}^N \int_0^{v(t_i)} S_i(s)\,ds \\
&\quad - \sum_{i=0}^N \int_{s_i}^{t_{i+1}} F_i(t,u,v)\,dt \\
&\leq \frac{1}{2}\big\|(u,v)\big\|_X^2 + \sum_{i=1}^N \delta_i\big|u(t_i)\big|^\theta + \sum_{i=1}^N \delta_i\big|v(t_i)\big|^\theta - \sum_{i=0}^N \int_{s_i}^{t_{i+1}} C\big(|u|^\beta+|v|^\beta\big)\,dt \\
&\leq \frac{1}{2}\big\|(u,v)\big\|_X^2 + \sum_{i=1}^N \delta_i\|u\|_\infty^\theta + \sum_{i=1}^N \delta_i\|v\|_\infty^\theta - C\sum_{i=0}^N \int_{s_i}^{t_{i+1}} \big(|u|^\beta+|v|^\beta\big)\,dt \\
&\leq \frac{1}{2}\big\|(u,v)\big\|_X^2 + \sum_{i=1}^N \delta_i c^\theta\big\|(u,v)\big\|_X^\theta + \sum_{i=1}^N \delta_i c^\theta\big\|(u,v)\big\|_X^\theta \\
&\quad - C\sum_{i=0}^N \int_{s_i}^{t_{i+1}} \big(|u|^\beta+|v|^\beta\big)\,dt
\end{aligned} \tag{3.7}$$

Now, for any given $(u,v) \in X$ with $\|u\|_1 = \|v\|_1 = 1$, by (3.7), we have

$$\Phi(\xi u,\xi v) \leq \frac{1}{2}\big\|(\xi u,\xi v)\big\|_X^2 + \sum_{i=1}^N \delta_i c^\theta\big\|(\xi u,\xi v)\big\|_X^\theta + \sum_{i=1}^N \delta_i c^\theta\big\|(\xi u,\xi v)\big\|_X^\theta$$

$$
\begin{aligned}
&- C\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\big(|\xi u|^{\beta}+|\xi v|^{\beta}\big)\,dt \\
&= \xi^2 + 2(\sqrt{2}c)^{\theta}\xi^{\theta}\sum_{i=1}^{N}\delta_i - C\xi^{\beta}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\big(|u|^{\beta}+|v|^{\beta}\big)\,dt.
\end{aligned}
$$

Since $2<\theta<\beta$, the above inequality implies that $\Phi(\xi u,\xi v)\to-\infty$ as $\xi\to+\infty$. Therefore, there exists $\xi_0\in\mathbb{R}\backslash\{0\}$ with $\xi_0>\rho$ such that $\Phi(\xi_0 u,\xi_0 v)\le 0$. By Theorem 2.5, problem (1.1) has at least one classical solution. □

Proof of Theorem 1.3 We apply Theorem 2.6 to show this result. In view of the proof of Theorem 1.2, we obtain that $\Phi\in C^1(X,\mathbb{R})$ with $\Phi(0,0)=0$ satisfies the (PS) condition. Conditions (H6) and (H7) imply that Φ is even.

The set of all eigenvalues of (2.1) is given by the sequence of positive numbers $\lambda_n=(\frac{n\pi}{T})^2$ $(n=1,2,\dots)$. Let E_n denote the feature space corresponding to λ_n, then we obtain $H_0^1(0,T)=\overline{\bigoplus_{i\in\mathbb{N}}E_i}$ and $X=\overline{\bigoplus_{i\in\mathbb{N}}E_i\times E_i}$. Assume that $V=\bigoplus_{i=1}^{2}E_i\times E_i$ and $Y=\overline{\bigoplus_{i=3}^{+\infty}E_i\times E_i}$, then $X=V+Y$, where V is finite dimensional. As in the proof of Theorem 1.2, there exist $\rho,\alpha>0$ such that $\Phi(u,v)\ge\alpha$ for any $(u,v)\in\partial B_\rho\cap Y$. In addition, in the same way as in the proof of Theorem 1.2, we can obtain that $\Phi(\xi u,\xi v)\to-\infty$ as $\xi\to\infty$ for any $(u,v)\in W$. Hence, there exists $R=R(W)$ such that $\Phi(\xi u,\xi v)\le 0$ on $W\backslash B_{R(W)}$. By Theorem 2.6, problem (1.1) has infinitely many classical solutions. □

Proof of Theorem 1.4 Obviously, $\Phi\in C^1(X,\mathbb{R})$ with $\Phi(0,0)=0$ is even. First, we prove that Φ satisfies the (PS) condition. As in the proof of Theorem 1.2, by (2.9), (2.10), (H2), (H5), and Lemma 2.9, we have

$$
\begin{aligned}
&\theta\Phi(u_m,v_m)-\Phi'(u_m,v_m)(u_m,v_m)\\
&=\left(\frac{\theta}{2}-1\right)\|(u_m,v_m)\|_X^2+\theta\sum_{i=1}^{N}\int_0^{u_m(t_i)}I_i(s)\,ds-\sum_{i=1}^{N}I_i\big(u_m(t_i)\big)u_m(t_i)\\
&\quad+\theta\sum_{i=1}^{N}\int_0^{v_m(t_i)}S_i(s)\,ds-\sum_{i=1}^{N}S_i\big(v_m(t_i)\big)v_m(t_i)\\
&\quad+\Bigg(\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}D_uF_i(t,u_m,v_m)u_m\,dt+\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}D_vF_i(t,u_m,v_m)v_m\,dt\\
&\quad-\theta\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,u_m,v_m)\,dt\Bigg)\\
&\ge\left(\frac{\theta}{2}-1\right)\|(u_m,v_m)\|_X^2-\theta\sum_{i=1}^{N}\big(c_i\|u_m\|_\infty+d_i\|u_m\|_\infty^{\beta_i+1}\big)\\
&\quad-\sum_{i=1}^{N}\big(c_i\|u_m\|_\infty+d_i\|u_m\|_\infty^{\beta_i+1}\big)-\theta\sum_{i=1}^{N}\big(c_i\|v_m\|_\infty+d_i\|v_m\|_\infty^{\beta_i+1}\big)\\
&\quad-\sum_{i=1}^{N}\big(c_i\|v_m\|_\infty+d_i\|v_m\|_\infty^{\beta_i+1}\big)
\end{aligned}
$$

$$\geq \left(\frac{\theta}{2}-1\right)\|(u_m,v_m)\|_X^2 - 2(\theta+1)\left(\sum_{i=1}^{N} c_i c\|(u_m,v_m)\|_X + \sum_{i=1}^{N} d_i c^{\beta_i+1}\|(u_m,v_m)\|_X^{\beta_i+1}\right).$$

It follows that $\{(u_m, v_m)\}$ is bounded in X. The rest of the proof showing that the (PS) condition holds is similar to that in Theorem 1.2. Secondly, since I_i, S_i, $i = 1,2,\dots,N$, are odd and nondecreasing, we obtain $\int_0^{u(t_i)} I_i(s)\,ds \geq 0$ and $\int_0^{v(t_i)} S_i(s)\,ds \geq 0$. As in the proofs of Theorems 1.2 and 1.3, we can easily verify that condition (i) of Theorem 2.6 is satisfied. Finally, the proof of condition (ii) of Theorem 2.6 is also the same as that in Theorem 1.3. Hence, by Theorem 2.6, problem (1.1) has infinitely many classical solutions. □

4 Examples

In this section, we give two examples to illustrate our main results.

Example 4.1 Let $T = 1$, consider the following problem:

$$\begin{cases} -u''(t) = D_uF_i(t,u(t),v(t)), & t \in (s_i,t_{i+1}], i = 0,1, \\ -v''(t) = D_vF_i(t,u(t),v(t)), & t \in (s_i,t_{i+1}], i = 0,1, \\ \Delta u'(t_1) = I_1(u(t_1)), \\ \Delta v'(t_1) = S_1(v(t_1)), \\ u'(t) = u'(t_1^+), & t \in (t_1,s_1], \\ v'(t) = v'(t_1^+), & t \in (t_1,s_1], \\ u'(s_1^+) = u'(s_1^-), \\ v'(s_1^+) = v'(s_1^-), \\ u(0) = u(1) = v(0) = v(1) = 0. \end{cases} \tag{4.1}$$

where $D_uF_i(t,u,v) = t + u^{\frac{1}{2}}$, $D_vF_i(t,u,v) = t^2 + v^{\frac{1}{4}}$, $I_1(u) = \sin u + u^{\frac{1}{2}}$, $S_1(v) = 2\cos v + v^{\frac{1}{3}}$. Let $a_i = 1$, $b_i = 1$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{1}{4}$, $c_i = 2$, $d_i = 1$, $\beta_i = \frac{1}{2}$, then conditions (H1) and (H2) hold. Therefore, problem (4.1) has at least one classical solution by Theorem 1.1.

Example 4.2 Let $T = 1$, consider the following problem:

$$\begin{cases} -u''(t) = D_uF_i(t,u(t),v(t)), & t \in (s_i,t_{i+1}], i = 0,1, \\ -v''(t) = D_vF_i(t,u(t),v(t)), & t \in (s_i,t_{i+1}], i = 0,1, \\ \Delta u'(t_1) = I_1(u(t_1)), \\ \Delta v'(t_1) = S_1(v(t_1)), \\ u'(t) = u'(t_1^+), & t \in (t_1,s_1], \\ v'(t) = v'(t_1^+), & t \in (t_1,s_1], \\ u'(s_1^+) = u'(s_1^-), \\ v'(s_1^+) = v'(s_1^-), \\ u(0) = u(1) = v(0) = v(1) = 0. \end{cases} \tag{4.2}$$

where $D_u F_i(t,u,v) = u^7$, $D_v F_i(t,u,v) = v^9$, $I_1(u) = u^4$, $S_1(v) = v^6$. Let $\beta = 8$, $\theta = 4$, $C = \frac{1}{2}$. By simple calculations, we show that condition (H3)–(H5) are satisfied. So problem (4.2) has at least one classical solution by Theorem 1.2. Furthermore, if we take $I_1(u) = u^5$ and $S_1(v) = v^5$, conditions (H6) and (H7) are also satisfied. Applying Theorem 1.3, problem (4.2) has infinitely many classical solutions.

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