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Almost sure exponential stability of nonlinear stochastic delay hybrid systems driven by G-Brownian motion

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Abstract

G-Brownian motion has potential applications in uncertainty problems and risk measures, which has attracted the attention of many scholars. This study investigates the almost sure exponential stability of nonlinear stochastic delay hybrid systems driven by G-Brownian motion. Due to the non-linearity of G-expectation and distribution uncertainty of G-Brownian motion, it is difficult to study this issue. Firstly, the existence of the global unique solution is derived under the linear growth condition and local Lipschitz condition. Secondly, the almost sure exponential stability of the system is analyzed by applying the G-Lyapunov function and G-Itô formula. Finally, an example is introduced to illustrate the stability. The conclusions of this paper can be applied to the stability and risk management of uncertain financial markets.

Keywords: Nonlinear stochastic system; Almost sure exponential stability; G-Brownian motion; Markovian switching

1 Introduction

Most systems do not satisfy the principle of linear superposition. Thence, except for a small part that can be approximately regarded as linear systems, most of them are nonlinear systems, such as a turbulent system of fluids [19], simple pendulum system [1], gravitational three-body system [2], and finite channel [20]. In the physical world, the nonlinear system is the essence, and the linear system is the approximation or part of the nonlinear system. Therefore, it is necessary to discuss the properties of nonlinear systems. The research of nonlinear systems has always been a hot issue in the field of control. For example, Ding et al. [8] developed a new approach to the design of nonlinear disturbance observers for a class of nonlinear systems described by input-output differential equations. Liu et al. [18] presented an adaptive control design for a nonlinear system with time-varying full state constraints. Wang et al. [32] applied a fuzzy-logic system to approximate the unknown nonlinearities and proposed a novel adaptive finite-time control strategy.

Switching systems are an important class of hybrid dynamic systems. The dynamics of a system can be described by a finite number of subsystems or dynamic models, and there is a switching law that enables switching between subsystems. There is no jump phenomenon in the continuous state of a system at the moment of switching. In recent years, switching systems have been studied by some authors. For instance, Cheng et al. [6] concentrated on the nonstationary control for a class of nonlinear Markovian switching systems with the Tagaki–Sugeno fuzzy model. Cheng et al. [5] discussed the finite-time static output feedback control of Markovian switching systems. Qi et al. [31] studied the sliding mode control design for a class of stochastic switching systems subject to the semi-Markov process via an adaptive event-triggered mechanism. Qi et al. [30] analyzed the sliding-mode control design methodology for a nonlinear stochastic switching system subject to semi-Markovian switching parameters, T-S fuzzy strategy, uncertainty, signal quantization, and nonlinearity. Wu and Liu [34] investigated the Lyapunov and the external stability of Caputo fractional order switching systems.

Generally, because of the uncertain communication environment, time delay is always unavoidable. Therefore, it is required to be taken into consideration for stochastic systems. Liu et al. [17] developed a novel approach to identify the parameters of the linear time-delay differential system by analyzing the complex system response in the frequency domain. Othman et al. [23] used the three-phase-lag model, Green–Naghdi theory without energy dissipation, and Green–Naghdi theory with energy dissipation to study the influence of the gravity field on a two-temperature fiber-reinforced thermoelastic medium. Plonis et al. [28] presented the procedure of synthesis of the meander delay system using the Pareto-optimal multilayer perceptron network and multiple linear regression model with the M5 descriptor. Qi et al. [29] dealt with the problem of controller design for the time-delay system with stochastic disturbance and actuator saturation. Zhang et al. [37] developed a parameter-adjustable-based lemma and established a stability criterion for a linear time-delay system. Zhu et al. [39] studied an adaptive synchronization for a class of uncertain chaotic systems represented by the Takagi–Sugeno fuzzy model with time delay.

In recent years, Peng pioneered the concept of G-expectation and established a corresponding theoretical system [24–27]. Because of the wide applications in the fields of risk measurements, G-Brownian motion has attracted the attention of many scholars [7, 13]. Furthermore, stochastic systems driven by G-Brownian motion have attracted wide attention [10, 38]. For example, Chen and Yang [4] analyzed time-varying delay Hopfield neural networks. Fei et al. [11] proved the existence and uniqueness of solutions to stochastic differential delay equations driven by G-Brownian motion under local Lipschitz and linear growth conditions. By applying aperiodically intermittent adaptive control, Li et al. [14] concerned the stabilization of a stochastic complex system. Yin et al. [36] studied quasi-sure exponential stabilization of a stochastic system induced by G-Brownian motion with discrete-time feedback control.

The nonlinear characteristic makes the performance of a system more complicated, which brings difficulty to analyzing the stability of the system. Stability has always been the most fundamental and core issue in system analysis. In recent years, lots of results about stability have been reported in the literature [9, 33]. For example, by applying Lyapunov techniques, Caraballo et al. [3] analyzed the stability of a stochastic perturbed singular system under the assumption that the initial conditions are consistent. Marin et al. [21] derived some results on stability and continuous dependence in Green–Naghdi thermoelasticity of Cosserat bodies. Ngoc [22] implemented a new method to investigate the mean square exponential stability of a stochastic delay system. Wu et al. [35] considered the mean square exponential input-to-state stability for a stochastic delay reaction-diffusion neural network. Zhu and Huang [38] studied the pth moment exponential stability problem for a class of stochastic delay nonlinear systems driven by G-Brownian motion. Zong et al. [40] investigated the asymptotic properties of systems represented by stochastic functional differential equations. In [3, 9, 22, 33, 40], G-Brownian motion has not been considered. In [38], the Markovian switching and the existence of the global unique solution have not been discussed. Since G-Brownian motion has potential applications in uncertainty problems, risk measures, and superhedging in finance and switching systems are an important class of hybrid dynamic systems, it is necessary to consider these factors. Inspired by the works above, the existence of the global unique solution to a nonlinear stochastic delay hybrid system driven by the G-Brownian motion is derived in this paper. The almost sure exponential stability of the system is investigated using the G-Itô formula, the Borel–Cantelli lemma, the Gronwall inequality, the Hölder inequality, and the Chebyshev inequality.

The rest of this paper is organized as follows. In Sect. 2, some lemmas, definitions, and assumptions are introduced. In Sect. 3, the existence and uniqueness of the global solution are derived, and the almost sure exponential stability of the system is also investigated. In Sect. 4, an example is provided. In Sect. 5, the conclusion and future work are given.

2 Problem formulation and preliminaries

Let $(\{\mathscr{F}_t\}_{t\geq 0})$ be a filtration generated by the G-Brownian motion $\{B(t), t\geq 0\}$. Denote by $\mathcal{C}^{1,2}(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$ the family of positive real-valued functions $V(x,t,i)$ defined on $\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}$ that are continuously twice differentiable in $x\in\mathbb{R}^n$ and once differentiable in $t\in\mathbb{R}_+$. Let $r(t), t\geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S}=\{1,2,\ldots,N\}$ with generator $\Gamma=(\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\bigl\{r(t+\Delta)=j|r(t)=i\bigr\}=\begin{cases}\gamma_{ij}\Delta+o(\Delta), & i\neq j,\\ 1+\gamma_{ii}\Delta+o(\Delta), & i=j,\end{cases}$$

where $\Delta>0$, $\gamma_{ij}\geq 0$ denotes the transition rate from i to j if $i\neq j$ while $\gamma_{ii}=-\sum_{i\neq j}\gamma_{ij}$. Define $M_G^{p,0}([0,t],\mathbb{R}^n,\mathbb{S})=\{\alpha_t(\omega)=\sum_{j=1}^{N-1}\gamma_{ij}\beta_{t_j}(\omega)I_{[t_j,t_{j+1})};\beta_{t_j}\in L^p_{\mathscr{F}_t}(\Omega;\mathbb{R}^n), t>0\}$, $M_G^p([0,t],\mathbb{R}^n,\mathbb{S}):=$ the completion of $M_G^{p,0}([0,t],\mathbb{R}^n,\mathbb{S})$ under the norm $\|\alpha\|_{M_G^p([0,t],\mathbb{R}^n,\mathbb{S})}=(\int_0^t\widehat{\mathbb{E}}|\alpha_s|^p\,ds)^{\frac{1}{p}}$, where $L^p_{\mathscr{F}_t}(\Omega;\mathbb{R}^n):=$ the family of all $\mathscr{F}_t$ measurable $\mathbb{R}^n$-valued stochastic variables β satisfies $\widehat{\mathbb{E}}|\beta|^p<\infty$.

Here, we consider the following nonlinear stochastic system driven by the G-Brownian motion:

$$\begin{aligned} dx(t)={}&f\bigl(x(t),x\bigl(t-\tau(t)\bigr),t,r(t)\bigr)\,dt\\ &+g\bigl(x(t),x\bigl(t-\tau(t)\bigr),t,r(t)\bigr)\,d\langle B\rangle(t)\\ &+h\bigl(x(t),x\bigl(t-\tau(t)\bigr),t,r(t)\bigr)\,dB(t),\quad t\geq 0,\end{aligned}\tag{1}$$

where $0\leq\tau(t)\leq\tau$, the nonrandom initial data $\{x(t)=\xi(t):-\tau\leq t\leq 0\}=\xi\in\mathcal{C}([-\tau,0];\mathbb{R}^n)$, $r(0)=r_0\in\mathbb{S}$, $B(t)$ represents a one-dimensional G-Brownian motion with $G(a):=\frac{1}{2}\widehat{\mathbb{E}}[aB^2(1)]=\frac{1}{2}(\overline{\sigma}^2a^++\underline{\sigma}a^-)$, for $a\in\mathbb{R}$, where $a^+=\max\{a,0\}$, $a^-=\max\{-a,0\}$, $\overline{\sigma}^2=$

$\widehat{\mathbb{E}}[B^2(1)]$, $\underline{\sigma}^2 = -\widehat{\mathbb{E}}[-B^2(1)]$, $\langle B\rangle(t)$ denotes the quadratic variation process of the G-Brownian motion $B(t)$, $\widehat{\mathbb{E}}$ stands for the G-expectation. $f : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n$. We assume that the Markov chain is ergodic.

Firstly, we introduce some assumptions, definitions, and lemmas that are very important for the proof of the main results.

Assumption 1 There exists a positive constant L_K, for $\forall t \geq 0$, $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq K$ and $i \in S$, $|f(x_1,y_1,t,i) - f(x_2,y_2,t,i)| \vee |g(x_1,y_1,t,i) - g(x_2,y_2,t,i)| \vee |h(x_1,y_1,t,i) - h(x_2,y_2,t,i)| \leq L_K(|x_1 - x_2| + |y_1 - y_2|)$.

Assumption 2 There exists nonnegative function $V(x,y,t,i) \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S};\mathbb{R}_+)$ and positive constant b_1 such that

$$\lim_{|x|\to\infty} \inf_{t\geq 0, i\in S} V(x,y,t,i) = \infty \quad \text{and} \quad \mathcal{L}V(x,y,t,i) \leq b_1 V(x,y,t,i).$$

Assumption 3 $f(0,t,i) \equiv 0$, $g(0,t,i) \equiv 0$, $h(0,t,i) \equiv 0$, $\forall i \in \mathbb{S}$.

Definition 1 *If there exists a constant* $\lambda > 0$ *satisfying*

$$\lim_{t\to\infty} \sup \frac{1}{t} \log\bigl(\bigl|x(t;x_0,r_0)\bigr|\bigr) < -\lambda,$$

for any $x_0 \in \mathscr{F}_0$ *and* $r_0 \in \mathbb{S}$. *The system* (1) *is almost sure exponential stability.*

Given $V \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S};\mathbb{R}_+)$, *we define the operator* $\mathcal{L}V$ *by*

$$\begin{aligned}
&\mathcal{L}V(x,y,t,i)\\
&\quad = V_t(x,y,t,i) + V_x(x,y,t,i)f(x,y,t,i)\\
&\qquad + G\bigl(2V_x(x,y,t,i)g(x,y,t,i) + h^T(x,y,t,i)V_{xx}(x,y,t,i)h(x,y,t,i)\bigr).
\end{aligned}$$

Lemma 1 ([27]) *For any* $0 \leq s \leq t < \infty$,

$$\begin{aligned}
&\widehat{\mathbb{E}}\biggl[\biggl|\int_0^t \nu_s\, d\langle B\rangle(s)\biggr|\biggr] \leq \overline{\sigma}^2 \widehat{\mathbb{E}}\biggl[\int_0^t |\nu_s|\, d(s)\biggr], \quad \forall \nu_s \in M_G^1\bigl([0,t],\mathbb{R}^n,\mathbb{S}\bigr),\\
&\widehat{\mathbb{E}}\biggl[\biggl(\int_0^t \nu_s\, dB(s)\biggr)^2\biggr] = \widehat{\mathbb{E}}\biggl[\int_0^t \bigl|\nu_s^2\bigr|\, d\langle B\rangle(s)\biggr], \quad \forall \nu_s \in M_G^2\bigl([0,t],\mathbb{R}^n,\mathbb{S}\bigr),\\
&\widehat{\mathbb{E}}\biggl[\biggl(\int_0^t |\nu_s|^2\, ds\biggr)^2\biggr] \leq \int_0^t \widehat{\mathbb{E}}\bigl[|\nu_s|^2\bigr]\, ds, \quad \forall \nu_s \in M_G^2\bigl([0,t],\mathbb{R}^n,\mathbb{S}\bigr),
\end{aligned}$$

Lemma 2 ([15]) *For* $0 \leq s \leq t < \infty$,

$$\widehat{\mathbb{E}}\biggl[\sup_{s\leq d\leq t}\biggl|\int_s^d \nu_s\, d\langle B\rangle(s)\biggr|^2\biggr] \leq \overline{\sigma}^4 |t-s|^2 \int_s^t \widehat{\mathbb{E}}\bigl[|\nu_s|^2\bigr]\, d_s, \quad \forall \nu_s \in M_G^1\bigl([0,t],\mathbb{R}^n,\mathbb{S}\bigr).$$

Lemma 3 ([16])

$$\widehat{\mathbb{E}}\biggl[\sup_{s\leq d\leq t}\biggl|\int_s^d \nu_s\, dB(s)\biggr|^2\biggr] \leq 4\overline{\sigma}^4 |t-s|^2 \int_s^t \widehat{\mathbb{E}}\bigl[|\nu_s|^2\bigr]\, d_s, \quad \forall \nu_s \in M_G^2\bigl([0,t],\mathbb{R}^n,\mathbb{S}\bigr).$$

Lemma 4 (G-Itô formula, [25]) *Let* $\varphi \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ *and*

$$X_t = X_0 + \int_0^t f_s\,ds + \int_0^t g_s\,d\langle B\rangle_s + \int_0^t h_s\,dB_s,$$

where $f, g, h \in M_G^2(0,T;\mathbb{R}^n)$. *Then, for* $\forall t > 0$,

$$\begin{aligned}
&\varphi(X_t,t) - \varphi(X_0,t) \\
&\quad = \int_0^t \bigl[\partial_s\varphi(X_s,s) + \partial_x\varphi(X_s,s)f_s + G\bigl(2\partial_x\varphi(X_s,s)g_s\bigr) + \partial_{xx}\varphi(X_s,s)h_s^2\bigr]\,ds \\
&\qquad + \int_0^t \partial_x\varphi(X_s,s)h_s\,dB_s + \int_0^t \biggl[\partial_x\varphi(X_s,s)g_s + \frac{1}{2}\partial_{xx}\varphi(X_s,s)h_s^2\biggr]\,d\langle B\rangle_s \\
&\qquad - \int_0^t G\bigl(2\partial_x\varphi(X_s,s)g_s + \partial_{xx}\varphi(X_s,s)h_s^2\bigr)\,ds.
\end{aligned}$$

3 Main results

In the following theorem, we proved the existence of the global unique solution under the linear growth condition and the local Lipschitz condition.

Theorem 1 *When Assumptions 1–3 hold, the system (1) has a global unique solution* $\{x(t), t \ge 0\}$.

Proof 1 Let the initial value $|x_0| \le \xi$. For $k \ge \xi$, $k \in \mathbb{N}$, we suppose that

$$\begin{aligned}
f^{(k)}(x,y,t,i) &= f\biggl(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, t, i\biggr), \\
g^{(k)}(x,y,t,i) &= g\biggl(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, t, i\biggr), \\
h^{(k)}(x,y,t,i) &= h\biggl(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, t, i\biggr),
\end{aligned} \tag{2}$$

where $(\frac{|x| \wedge k}{|x|}x) = 0$ when $x = 0$.

We obtain that $f^{(k)}$, $g^{(k)}$, and $h^{(k)}$ satisfy the linear growth condition and the local Lipschitz condition. Thus,

$$\begin{aligned}
dx_k(t) &= f^{(k)}\bigl(x_k(t), x_k\bigl(t-\tau(t)\bigr), t, r(t)\bigr)\,dt \\
&\quad + g^{(k)}\bigl(x_k(t), x_k\bigl(t-\tau(t)\bigr), t, r(t)\bigr)\,d\langle B\rangle(t) \\
&\quad + h^{(k)}\bigl(x_k(t), x_k\bigl(t-\tau(t)\bigr), t, r(t)\bigr)\,dB(t)
\end{aligned} \tag{3}$$

has the global unique solution.

Let

$$\eta_k = \inf\bigl\{t \ge 0 : \bigl|x_k(t)\bigr| \ge k\bigr\}, \tag{4}$$

where $k \in \mathbb{N}$, $\inf\phi = \infty$.

When $0 \leq t \leq \eta_k$, $x_k(t) = x_{k+1}$. Then, $\{\eta_k\}$ is increasing. Thus, there exists a stopping time η such that

$$\eta = \lim_{k\to\infty} \eta_k. \tag{5}$$

Let

$$x(t) = \lim_{k\to\infty} x_k(t), \quad -\tau \leq t < \eta. \tag{6}$$

It can be confirmed that when $-\tau \leq t < \eta$, $x(t)$ is the unique solution of system (1).

Using the G-Itô formula, for $t \geq 0$, we obtain

$$\begin{aligned}
&V\big(x_k(t\wedge\eta_k), x_k\big(t\wedge\eta_k - \tau(t\wedge\eta_k)\big), t\wedge\eta_k, r(t\wedge\eta_k)\big) \\
&\quad = V\big(\xi(0), x_k\big(-\tau(0)\big), 0, r_0\big) + \int_0^{t\wedge\eta_k} \mathcal{L}^{(k)} V\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big)\, ds \\
&\qquad + \int_0^{t\wedge\eta_k} V_x\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) h\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big)\, dB(s) \\
&\qquad + \int_0^{t\wedge\eta_k} \Big[V_x\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) g\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \\
&\qquad + \frac{1}{2} h^T\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) V_{xx}\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \\
&\qquad \times h\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \Big]\, d\langle B\rangle(s) \\
&\qquad - \int_0^{t\wedge\eta_k} G\big(2V_x\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) g\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \\
&\qquad + h^T\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) V_{xx}\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \\
&\qquad \times h\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big)\big)\, ds
\end{aligned}$$

where $\mathcal{L}^{(k)} V(x_k(s), x_k(s-\tau(s)), s, r(s)) = \mathcal{L} V(x_k(s), x_k(s-\tau(s)), s, r(s))$ when $0 \leq s \leq t \wedge \eta_k$.

From [27], we know that

$$\widehat{\mathbb{E}}\left[\int_0^{t\wedge\eta_k} V_x\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) h\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big)\, dB(s)\right] = 0,$$

and

$$\begin{aligned}
&\widehat{\mathbb{E}}\bigg[\int_0^{t\wedge\eta_k} \Big[V_x\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) g\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \\
&\quad + \frac{1}{2} h^T\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) V_{xx}\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \\
&\quad \times h\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \Big]\, d\langle B\rangle(s) \\
&\quad - \int_0^{t\wedge\eta_k} G\big(2V_x\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) g\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big)
\end{aligned}$$

$$+ h^T\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) V_{xx}\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big) \times h\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big)\big)\, ds\bigg] \le 0.$$

Then, it can be confirmed that

$$\begin{aligned} &\widehat{\mathbb{E}}\big[V\big(x_k(t\wedge\eta_k), x_k\big(t\wedge\eta_k-\tau(t\wedge\eta_k)\big), t\wedge\eta_k, r(t\wedge\eta_k)\big)\big] \\ &\quad\le \widehat{\mathbb{E}}\big[V\big(\xi(0), x_k\big(-\tau(0)\big), 0, r_0\big)\big] + \widehat{\mathbb{E}}\bigg[\int_0^{t\wedge\eta_k} \mathcal{L}^{(k)} V\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big)\, ds\bigg] \\ &\quad\le \widehat{\mathbb{E}}\big[V\big(\xi(0), x_k\big(-\tau(0)\big), 0, r_0\big)\big] + b_1 \int_0^{t\wedge\eta_k} \widehat{\mathbb{E}}\big[V\big(x_k(s), x_k\big(s-\tau(s)\big), s, r(s)\big)\big]\, ds. \end{aligned}$$

According to the Gronwall inequality, we obtain

$$\begin{aligned} &\widehat{\mathbb{E}}\big[V\big(x_k(t\wedge\eta_k), x_k\big(t\wedge\eta_k-\tau(t\wedge\eta_k)\big), t\wedge\eta_k, r(t\wedge\eta_k)\big)\big] \\ &\quad\le \widehat{\mathbb{E}}\big[V\big(\xi(0), x_k\big(-\tau(0)\big), 0, r_0\big)\big] e^{b_1(t\wedge\eta_k)}. \end{aligned} \tag{7}$$

Furthermore, as

$$\begin{aligned} &\mathbb{P}\{\eta_k \le t\} \inf_{|x|\ge n, |y|\ge n, t\ge 0, i\in S} V(x,y,t,i) \\ &\quad\le \int_{\eta_k\le t} V\big(x_k(t\wedge\eta_k), x_k\big(t\wedge\eta_k-\tau(t\wedge\eta_k)\big), t\wedge\eta_k, r(t\wedge\eta_k)\big)\, dP \\ &\quad\le \widehat{\mathbb{E}} V\big(x_k(t\wedge\eta_k), x_k\big(t\wedge\eta_k-\tau(t\wedge\eta_k)\big), t\wedge\eta_k, r(t\wedge\eta_k)\big), \end{aligned}$$

we have

$$\mathbb{P}\{\eta_k \le t\} \le \frac{\widehat{\mathbb{E}}[V(\xi(0), x_k(-\tau(0)), 0, r_0)] e^{b_1(t\wedge\eta_k)}}{\inf_{|x|\ge n, |y|\ge n, t\ge 0, i\in S} V(x,y,t,i)}. \tag{8}$$

When $t \to \infty$,

$$\mathbb{P}\{\eta \le t\} = 0. \tag{9}$$

Thus,

$$\mathbb{P}\{\eta = \infty\} = 1. \tag{10}$$

The proof is complete.

In the following theorem, the almost sure exponential stability of system (1) is discussed.

Theorem 2 *If there exists a function* $V(x,y,t,i) \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$ *and some positive constants* a_2, a_3, a_4, a_5, p, *and* $a_1 \in \mathbb{R}$ *satisfies*

$$V_t(x,y,t,i) + V_x(x,y,t,i) f(x,y,t,i) \le a_1 V(x,y,t,i), \tag{11}$$

$$h^T(x,y,t,i)V_{xx}(x,y,t,i)h(x,y,t,i) \le a_2 V(x,y,t,i), \tag{12}$$

$$V_x(x,y,t,i)g(x,y,t,i) \le a_3 V(x,y,t,i), \tag{13}$$

$$\left|V_x(x,y,t,i)h(x,y,t,i)\right|^2 \ge a_4 V^2(x,y,t,i), \tag{14}$$

$$a_5|x|^p \le V(x,y,t,i), \tag{15}$$

for $\forall(x,y,t,i) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S})$, *and* $\frac{(a_2+2a_3)\overline{\sigma}^2+2a_1-a_4\underline{\sigma}^2}{2p} < 0$, *the system* (1) *is almost sure exponential stability.*

Proof 2 Using the G-Itô formula, for $\forall i \in S$, $t > 0$, we get

$$\begin{aligned}
&\log V\big(x(t),x\big(t-\tau(t)\big),t,i\big) \\
&\quad = \log V\big(\xi(0),x\big(-\tau(0)\big),0,i\big) \\
&\qquad + \int_0^t \frac{(V_s(x(s),x(s-\tau(s)),s,i) + V_x(x(s),x(s-\tau(s)),s,i)f(x(s),x(s-\tau(s)),s,i))}{V(x(s),x(s-\tau(s)),s,i)}\,ds \\
&\qquad + \int_0^t \frac{h^T(x(s),x(s-\tau(s)),s,i)V_{xx}(x(s),x(s-\tau(s)),s,i)h(x(s),x(s-\tau(s)),s,i)}{2V(x(s),x(s-\tau(s)),s,i)}\,d\langle B\rangle(S) \\
&\qquad + \int_0^t \frac{V_x(x(s),x(s-\tau(s)),s,i)g(x(s),x(s-\tau(s)),s,i)}{V(x(s),x(s-\tau(s)),s,i)}\,d\langle B\rangle(S) \\
&\qquad - \frac{1}{2}\int_0^t \frac{[V_x(x(s),x(s-\tau(s)),s,i)h(x(s),x(s-\tau(s)),s,i)]^2}{V^2(x(s),x(s-\tau(s)),s,i)}\,d\langle B\rangle(S) \\
&\qquad + \int_0^t \frac{V_x(x(s),x(s-\tau(s)),s,i)h(x(s),x(s-\tau(s)),s,i)}{V(x(s),x(s-\tau(s)),s,i)}\,dB(s),
\end{aligned}$$

where $\int_0^t \frac{V_x(x(s),x(s-\tau(s)),s,i)h(x(s),x(s-\tau(s)),s,i)}{V(x(s),x(s-\tau(s)),s,i)}\,dB(s)$ is a continuous martingale.

According to Lemma 2.6 in [12], for $\forall\varepsilon \in (0,1)$ and all $\omega \in \Omega$, there exists an integer n_0, when $n \ge n_0$, we have

$$\begin{aligned}
&\int_0^t \frac{V_x(x(s),x(s-\tau(s)),s,i)h(x(s),x(s-\tau(s)),s,i)}{V(x(s),x(s-\tau(s)),s,i)}\,dB(s) \\
&\quad \le \frac{2}{\varepsilon}\log(n) + \frac{\varepsilon}{2}\int_0^t \frac{[V_x(x(s),x(s-\tau(s)),s,i)h(x(s),x(s-\tau(s)),s,i)]^2}{V^2(x(s),x(s-\tau(s)),s,i)}\,d\langle B\rangle(S).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\log V\big(x(t),x\big(t-\tau(t)\big),t,i\big) \\
&\quad \le \log V\big(\xi(0),x\big(-\tau(0)\big),0,i\big) + \frac{2}{\varepsilon}\log(n) \\
&\qquad + \int_0^t \frac{(V_s(x(s),x(s-\tau(s)),s,i) + V_x(x(s),x(s-\tau(s)),s,i)f(x(s),x(s-\tau(s)),s,i))}{V(x(s),x(s-\tau(s)),s,i)}\,ds \\
&\qquad + \int_0^t \frac{h^T(x(s),x(s-\tau(s)),s,i)V_{xx}(x(s),x(s-\tau(s)),s,i)h(x(s),x(s-\tau(s)),s,i)}{2V(x(s),x(s-\tau(s)),s,i)}\,d\langle B\rangle(S)
\end{aligned}$$

$$+\int_0^t \frac{V_x(x(s),x(s-\tau(s)),s,i)g(x(s),x(s-\tau(s)),s,i)}{V(x(s),x(s-\tau(s)),s,i)}\,d\langle B\rangle(S)$$
$$-\frac{1}{2}(1-\varepsilon)\int_0^t \frac{[V_x(x(s),x(s-\tau(s)),s,i)h(x(s),x(s-\tau(s)),s,i)]^2}{V^2(x(s),x(s-\tau(s)),s,i)}\,d\langle B\rangle(S).$$

From (11)–(14), we have

$$\begin{aligned}
&\log V\bigl(x(t),x\bigl(t-\tau(t)\bigr),t,i\bigr)\\
&\quad\le \log V\bigl(\xi(0),x\bigl(-\tau(0)\bigr),0,i\bigr)+\frac{2}{\varepsilon}\log(n)+a_1t+\frac{a_2}{2}\overline{\sigma}^2t+a_3\overline{\sigma}^2t-\frac{1}{2}(1-\varepsilon)a_4\underline{\sigma}^2t\\
&\quad=\log V\bigl(\xi(0),x\bigl(-\tau(0)\bigr),0,i\bigr)+\frac{2}{\varepsilon}\log(n)+a_1t+\left(\frac{a_2}{2}+a_3\right)\overline{\sigma}^2t-\frac{1}{2}(1-\varepsilon)a_4\underline{\sigma}^2t.
\end{aligned}$$

Thus, it can be checked that

$$\begin{aligned}
&\frac{1}{t}\log V\bigl(x(t),x\bigl(t-\tau(t)\bigr),t,i\bigr)\\
&\quad\le \frac{1}{t}\log V\bigl(\xi(0),x\bigl(-\tau(0)\bigr),0,i\bigr)+\frac{1}{t}\frac{2}{\varepsilon}\log(n)+a_1+\left(\frac{a_2}{2}+a_3\right)\overline{\sigma}^2-\frac{1}{2}(1-\varepsilon)a_4\underline{\sigma}^2.
\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\log V\bigl(x(t),x\bigl(t-\tau(t)\bigr),t,i\bigr)\le a_1+\left(\frac{a_2}{2}+a_3\right)\overline{\sigma}^2-\frac{1}{2}(1-\varepsilon)a_4\underline{\sigma}^2. \tag{16}$$

According to (15), we have

$$\limsup_{t\to\infty}\frac{1}{t}\log\bigl|x(t)\bigr|\le\frac{(a_2+2a_3)\overline{\sigma}^2-(1-\varepsilon)a_4\underline{\sigma}^2+2a_1}{2p}. \tag{17}$$

When $\varepsilon\to 0$, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\log\bigl|x(t)\bigr|\le\frac{(a_2+2a_3)\overline{\sigma}^2-a_4\underline{\sigma}^2+2a_1}{2p}<0. \tag{18}$$

Therefore, system (1) is almost sure exponential stability.
The proof is complete.

Remark 1 Due to the nonlinearity of G-expectation and distribution uncertainty of G-Brownian motion, it is difficult to study the existence of global unique solutions and stability of the system. Using G-Lyapunov function, G-Itô formula, Borel–Cantelli lemma, Gronwall inequality, Hölder inequality, and Chebyshev inequality, we proved the existence and uniqueness of the global solution under linear growth condition and local Lipschitz condition and provided sufficient conditions for the stability.

Remark 2 When a financial market is affected by uncertain factors and needs to carry out risk management, and the current performance of the real economy and financial markets has not fully reflected the impact of regulation, it can be described by a nonlinear stochastic delay hybrid system driven by the G-Brown motion. The conclusions of this paper can be applied to the stability and risk management of uncertain financial markets.

4 Example

Let $B(t)$ be a G-Brownian motion and $B(t) \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, $r(t) \in \mathbb{S} = \{1, 2\}$ and

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{pmatrix}$$

Consider the following nonlinear stochastic system driven by the G-Brownian motion:

$$\begin{aligned} &dx(t) \\ &\quad = f\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, r(t)\bigr)\,dt + g\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, r(t)\bigr)\,d\langle B\rangle(t) \\ &\qquad + h\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, r(t)\bigr)\,dB(t), \end{aligned}$$

where

$$\begin{aligned} &f\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, 1\bigr) = -4x(t) + x\bigl(t-\tau(t)\bigr), \\ &g\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, 1\bigr) = \frac{1}{4}x(t), \\ &h\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, 1\bigr) = \frac{1}{2}x(t), \\ &f\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, 2\bigr) = -5x(t) + x\bigl(t-\tau(t)\bigr), \\ &g\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, 2\bigr) = \frac{1}{3}x(t), \\ &h\bigl(x(t), x\bigl(t-\tau(t)\bigr), t, 2\bigr) = x(t), \\ &\tau(t) = 1 + 0.3\sin(t). \end{aligned}$$

Hence, $\tau = 1.3$. Let $V(x, y, t, i) = x^2$, $i = 1, 2$, we have

$$\begin{aligned} &V_t(x, y, t, 1) + V_x(x, y, t, 1)f(x, y, t, 1) \le -5V(x, y, t, 1), \\ &h^T(x, y, t, 1)V_{xx}(x, y, t, 1)h(x, y, t, 1) \le 3V(x, y, t, 1), \\ &V_x(x, y, t, 1)g(x, y, t, 1) \le V(x, y, t, 1), \\ &\bigl|V_x(x, y, t, 1)h(x, y, t, 1)\bigr|^2 \ge \frac{1}{2}V^2(x, y, t, 1), \\ &\frac{1}{2}|x|^2 \le V(x, y, t, 1), \\ &V_t(x, y, t, 2) + V_x(x, y, t, 2)f(x, y, t, 2) \le -5V(x, y, t, 2), \\ &h^T(x, y, t, 2)V_{xx}(x, y, t, 2)h(x, y, t, 2) \le 3V(x, y, t, 2), \\ &V_x(x, y, t, 2)g(x, y, t, 2) \le V(x, y, t, 2), \\ &\bigl|V_x(x, y, t, 2)h(x, y, t, 2)\bigr|^2 \ge \frac{1}{2}V^2(x, y, t, 2), \\ &\frac{1}{2}|x|^2 \le V(x, y, t, 2), \end{aligned}$$

where $p = 2$, $a_1 = -5$, $a_2 = 3$, $a_3 = 1$, $a_4 = \frac{1}{2}$, $a_5 = \frac{1}{2}$.

Let $\overline{\sigma}^2 = 1$ and $\underline{\sigma}^2 = -1$, we obtain

$$\frac{(a_2 + 2a_3)\overline{\sigma}^2 - a_4\underline{\sigma}^2 + 2a_1}{2p} = -\frac{9}{8} < 0.$$

Therefore, the system is almost sure exponential stability.

5 Conclusion

In this paper, we investigate the almost sure exponential stability of nonlinear stochastic delay hybrid systems driven by the G-Brownian motion. We have proved the existence and uniqueness of the global solution under linear growth condition and the local Lipschitz condition and provided sufficient conditions for stability. We will consider the stability of nonlinear stochastic hybrid systems driven by the G-Brownian motion with aperiodically intermittent control in the future.

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