Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global well-posedness and large time behavior of epitaxy thin film growth model

Ning Duan[1*] and Shuang Yuan[1]

*Correspondence:
duanning@mail.neu.edu.cn
[1]College of Sciences, Northeastern University, Wenhua Road, 110819, Shenyang, China

Abstract

We consider the global well-posedness and large time behavior of solutions for epitaxy thin film growth model in $\mathbb{R}^d$ with the dimensional $d \geq 3$. First, using the pure energy method and a standard continuity argument, we prove that there exists a unique global strong solution under the condition that the initial data is sufficiently small. Moreover, we also establish the suitable negative Sobolev norm estimates and obtain the optimal decay rates of the higher-order spatial derivatives of the strong solution.

Keywords: Global well-posedness; Decay rate; Epitaxy thin film growth model

1 Introduction

In this paper, we consider the following equation modeling epitaxy thin film growth

$$\partial_t h + \nu_1 \Delta h + \nu_2 \Delta^2 h - \nu_3 \nabla \cdot \left(|\nabla h|^2 \nabla h\right) + \nu_4 \Delta |\nabla h|^2 = \nu_5 |\nabla h|^2, \tag{1}$$

with the initial condition

$$h(x,0) = h_0(x), \tag{2}$$

on $\mathbb{R}^d$ with $d \geq 3$. Equation (1) arises in epitaxial growth of nanoscale thin films, where $h(x,t)$ denotes the height from the surface of the film in epitaxial growth [23, 26]. The term $\Delta^2 u$ denotes the capillarity-driven surface diffusion, $\operatorname{div}(|\nabla h|^2 \nabla h)$ correspond to the upward hopping of atoms, Δh can be used to describe the diffusion due to evaporation-condensation, $\Delta |\nabla h|^2$ is related to the equilibration of the inhomogeneous concentration of the diffusing particles on the surface, and the term $|\nabla h|^2$ is related to the density variations, respectively [15–17]. Similar to [1], in this paper, we assume that the coefficients satisfy $\nu_1, \nu_3, \nu_4, \nu_5 \geq 0$ and $\nu_2 > 0$.

Remark 1.1 There are many papers studied equation (1) with $\nu_4 = \nu_5 = 0$, see, for instance, [5, 6, 10, 11, 13, 18]. For the case $\nu_3 = \nu_5 = 0$, we refer the reader to [22, 25] and the reference therein.

In 2015, Agélas [1] studied the global regularity of solutions for the Cauchy problem (1)–(2) in 1D and 2D cases. The author assumed that the condition $\nu_2\nu_3 > \nu_4^2$ proved the existence and uniqueness of global strong solutions for problem (1)–(2). In this paper, we continue this research and study the global well-posedness of solutions for problem (1)–(2) in $\mathbb{R}^d$ with $d \geq 3$.

To study the global well-posedness of problem (1)–(2) in $\mathbb{R}^d$ with $d \geq 3$, the main challenge is caused by the strong nonlinear term $\Delta|\nabla h|^2$. Because of this term, it is difficult to obtain the higher order estimate of h, and we cannot obtain an idea of the global well-posedness result without any additional condition. Hence, in this paper, assuming that the initial data is sufficiently small and using the pure energy method, we show the following result:

Theorem 1.2 (Small initial data global well-posedness) *Let $h_0 \in H^N(\mathbb{R}^d)$ with $N \geq 2 + \frac{d}{2}$ and $d \geq 3$. Assume that there exists a constant $\delta_0 > 0$ such that if*

$$\|h_0\|_{H^{[\frac{d+1}{2}]}} \leq \delta_0, \tag{3}$$

then system (1) has a unique solution satisfying that for all $t \geq 0$,

$$\|h\|_{H^N}^2 + \int_0^t \left(\|\Delta h\|_{H^N}^2 + \|\nabla h\|_{H^N}^2\right) ds \leq C\|h_0\|_{H^N}^2. \tag{4}$$

For dissipative equations, there are many different kinds of styles for large time behavior, e.g., global attractor, exponential attractors, and so on. However, since we only obtain the small initial data global well-posedness for problem (1)–(2) in $\mathbb{R}^d$, and there exists a strong nonlinear term $\Delta|\nabla h|^2$, it is difficult to consider the global or exponential attractors. In this paper, we consider another style of large-time behavior and study the temporary algebraic decay rate of strong solutions of problem (1)–(2) in $\mathbb{R}^d$ with $d \geq 3$ provided that Theorem 1.2 holds. More precisely, we prove the following theorem:

Theorem 1.3 (Large-time behavior) *Under the assumptions of Theorem 1.2, if $h_0 \in L^p(\mathbb{R}^d)$ $(\frac{d}{d-1} \leq p \leq 2)$, then for all $t \geq 0$ and $l = 0, 1, \ldots, N-1$,*

$$\left\|\nabla^l h(t)\right\|_{H^{N-l}} \leq C(1+t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})-\frac{l}{2}}. \tag{5}$$

Remark 1.4 One of the main tools to study the decay rate is the Fourier splitting method introduced by Schonbek [19, 20] in the 1980s. Since then, it has become a standard way (also a powerful tool) to establish the decay rate of solutions (see, for example, [2–4, 12, 27] and the reference therein). Here, as the structure of the equation is so complex, it is not suitable to deal with the decay rate of solutions for problem (1)–(2) using the Fourier splitting method and the Zhou method. Motivated by [8, 24], we establish suitable a priori estimates in the negative Sobolev space $\dot{H}^{-s}$ $(0 \leq s \leq \frac{d}{2})$, use the Hardy–Littlewood–Sobolev theorem $(L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = d(\frac{1}{p} - \frac{1}{2}) \in [0, \frac{d}{2}])$, and obtain the decay estimate.

In the following, ∇^l with an integral $l \geq 0$ stands for the usual spatial derivatives of order l. If $l < 0$ or l is not a positive integer, ∇^l stands for Λ^l defined by

$$(-\Delta)^\delta f(x) = \Lambda^{2\delta} f(x) = \int_{\mathbb{R}^3} |x|^{2\delta} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi. \tag{6}$$

The rest of this paper is organized as follows. We prove Theorem 1.2 in Sect. 2. The proof of Theorem 1.3 we give in Sect. 3.

2 Preliminaries

We first give the Gagliardo–Nirenberg inequality proved in [14]:

Lemma 2.1 ([14]) *Suppose that* $0 \le m, \alpha \le l$, *then*

$$\big\|\nabla^\alpha f\big\|_{L^p(\mathbb{R}^d)} \lesssim \big\|\nabla^m f\big\|_{L^q(\mathbb{R}^d)}^{1-\theta} \big\|\nabla^l f\big\|_{L^r(\mathbb{R}^d)}^{\theta}, \tag{7}$$

where $\theta \in [0,1]$ *and*

$$\frac{\alpha}{d} - \frac{1}{p} = \left(\frac{m}{d} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{d} - \frac{1}{r}\right)\theta. \tag{8}$$

Here, when $p = \infty$, *we require that* $0 < \theta < 1$.

The Kato–Ponce inequality is so important in the proofs of our main theorems.

Lemma 2.2 ([9]) *Let* $1 < p < \infty$, $s > 0$. *Then, there exists a constant* $C > 0$ *such that*

$$\big\|\Lambda^s(fg) - f\Lambda^s g\big\|_{L^p} \le C\big(\|\nabla f\|_{L^{p_1}} \big\|\Lambda^{s-1}g\big\|_{L^{p_2}} + \big\|\Lambda^s f\big\|_{L^{q_1}} \|g\|_{L^{q_2}}\big), \tag{9}$$

and

$$\big\|\Lambda^s fg\big\|_{L^p} \le C(\|f\|_{L^{p_1}} \big\|\Lambda^s g\big\|_{L^{p_2}} + \big\|\Lambda^s f\big\|_{L^{q_1}} \|g\|_{L^{q_2}}, \tag{10}$$

where $p_2, q_2 \in (1,\infty)$ *satisfying* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

We also introduce the Hardy–Littlewood–Sobolev theorem, which implies the following L^p type inequality.

Lemma 2.3 ([7, 21]) *Let* $0 \le s < \frac{d}{2}$, $1 < p \le 2$ *and* $\frac{1}{2} + \frac{s}{d} = \frac{1}{p}$, *then*

$$\|f\|_{\dot{H}^{-s}} \lesssim \|f\|_{L^p}. \tag{11}$$

The following special Sobolev interpolation lemma will be used in this paper.

Lemma 2.4 ([21]) *Let* $s, k \ge 0$ *and* $l \ge 0$, *then*

$$\big\|\nabla^l f\big\|_{L^2} \le \big\|\nabla^{l+1} f\big\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^{-s}}^{\theta}, \quad \text{with } \theta = \frac{1}{l+1+s}. \tag{12}$$

3 Proof of Theorem 1.2

Rewrite problem (1)–(2) as

$$\begin{cases} \partial_t h - \nu_2 \Delta h + \nu_2 \Delta^2 h + \nu_4 \Delta|\nabla h|^2 \\ \quad = \nu_3 \nabla \cdot [(\nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0) \cdot (\nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0)\nabla h] + \nu_5|\nabla h|^2, \\ h(x,0) = h_0(x), \end{cases} \tag{13}$$

where ω_0 is a unit vector. Assume that for sufficiently small $\delta > 0$,

$$\sqrt{\mathcal{E}_0^{[\frac{d+1}{2}]}(t)} = \|u(t)\|_{H^{[\frac{d+1}{2}]}} \le \delta. \tag{14}$$

Applying ∇^k to (13)$_1$, multiplying by $\nabla^k h$, and integrating over $\mathbb{R}^d$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^k h\|_{L^2}^2 + \nu_2\|\nabla^{k+2}h\|_{L^2}^2 + \nu_1\|\nabla^{k+1}h\|_{L^2}^2 \\
&\quad = \nu_3\int_{\mathbb{R}^d}\nabla^k\left\{\nabla\cdot\left[\left(\nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\cdot\left(\nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\nabla h\right]\right\}\cdot\nabla^k h\,dx \\
&\qquad - \nu_4\int_{\mathbb{R}^d}\nabla^k|\nabla h|^2\cdot\nabla^k\Delta h dx + \nu_5\int_{\mathbb{R}^d}\nabla^k|\nabla h|^2\cdot\nabla^k h\,dx.
\end{aligned} \tag{15}$$

For the first term of the right-hand side of (15), applying the Kato–Ponce inequality, we estimate as

$$\begin{aligned}
&\nu_3\int_{\mathbb{R}^d}\nabla^k\left\{\nabla\cdot\left[\left(\nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\cdot\left(\nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\nabla h\right]\right\}\cdot\nabla^h\,dx \\
&\quad\le C\|\nabla^{k+1}h\|_{L^{\frac{2d}{d-2}}}\left\|\nabla^k\left[\left(\nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\cdot\left(\nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\nabla h\right]\right\|_{L^{\frac{2d}{d+2}}} \\
&\quad\le C\|\nabla^{k+1}h\|_{L^{\frac{2d}{d-2}}}\left(\left\|\nabla^k\left(\nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\right\|_{L^{\frac{2d}{d-2}}}\left\|\nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right\|_{L^d}\|\nabla h\|_{L^d}\right. \\
&\qquad + \left\|\nabla^k\left(\nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\right\|_{L^{\frac{2d}{d-2}}}\left\|\nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right\|_{L^d}\|\nabla h\|_{L^d} \\
&\qquad \left. + \left\|\nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right\|_{L^d}\left\|\nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right\|_{L^d}\|\nabla^{k+1}h\|_{L^{\frac{2d}{d-2}}}\right) \\
&\quad\le C\|\nabla h\|_{H^{\frac{[d+1]}{2}-1}}^2\|\nabla^{k+2}h\|_{L^2}^2 \le C\delta^2\|\nabla^{k+2}h\|_{L^2}^2.
\end{aligned} \tag{16}$$

The second term of the right-hand of (15) satisfies

$$\begin{aligned}
-\nu_4\int_{\mathbb{R}^d}\nabla^k|\nabla h|^2\cdot\nabla^k\Delta h\,dx &\le C\|\nabla^{k+2}h\|_{L^2}\|\nabla h\|_{L^d}\|\nabla^{k+1}h\|_{L^{\frac{2d}{d-2}}} \\
&\le C\|\nabla h\|_{H^{\frac{[d+1]}{2}-1}}\|\nabla^{k+2}h\|_{L^2}^2 \le C\delta\|\nabla^{k+2}h\|_{L^2}^2.
\end{aligned} \tag{17}$$

Moreover, the third therm of the right-hand of (15) can be estimated as

$$\begin{aligned}
\nu_5\int_{\mathbb{R}^d}\nabla^k|\nabla h|^2\cdot\nabla^k h\,dx &\le C\|\nabla^k h\|_{L^{\frac{2d}{d-2}}}\|\nabla h\|_{L^{\frac{d}{2}}}\|\nabla^{k+1}h\|_{L^{\frac{2d}{d-2}}} \\
&\le C\|\nabla h\|_{H^{\frac{[d+1]}{2}-2}}\|\nabla^{k+1}h\|_{L^2}^2\|\nabla^{k+2}h\|_{L^2}^2 \\
&\le C\delta\left(\|\nabla^{k+2}h\|_{L^2}^2 + \|\nabla^{k+1}h\|_{L^2}^2\right).
\end{aligned} \tag{18}$$

Combining (15)–(18) together gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^k h\|_{L^2}^2+\nu_2\|\nabla^{k+2}h\|_{L^2}^2+\nu_1\|\nabla^{k+1}h\|_{L^2}^2\\&\quad\le(\delta^2+\delta)\left(\|\nabla^{k+2}h\|_{L^2}^2+\|\nabla^{k+1}h\|_{L^2}^2\right).\end{aligned}\tag{19}$$

We close the energy estimates at each lth level in the weak sense. Suppose that $N\ge 1$ and $0\le l\le m-1$ with $1\le m\le N$. Summing up the estimates (19) from $k=l$ to m, we arrive at

$$\begin{aligned}&\frac{d}{dt}\sum_{l\le k\le m}\|\nabla^k h\|_{L^2}^2+C_1\sum_{l\le k\le m}\left(\|\nabla^{k+1}h\|_{L^2}^2+\|\nabla^{k+2}h\|_{L^2}^2\right)\\&\quad\le C_2(\delta+\delta^2)\sum_{l\le k\le m}\left(\|\nabla^{k+1}h\|_{L^2}^2+\|\nabla^{k+2}h\|_{L^2}^2\right).\end{aligned}\tag{20}$$

Since $\delta>0$ is sufficiently small, there exists a positive constant C_3 such that for $0\le l\le m-1$,

$$\frac{d}{dt}\sum_{l\le k\le m}\|\nabla^k h\|_{L^2}^2+C_3\sum_{l\le k\le m}\left(\|\nabla^{k+1}h\|_{L^2}^2+\|\nabla^{k+2}h\|_{L^2}^2\right)\le 0.\tag{21}$$

Define $\mathcal{E}_l^m(t)$ to be $\frac{1}{C_3}$ times the expression under the time derivative in (21). Hence, we may rewrite (21) as

$$\frac{d}{dt}\mathcal{E}_l^m(t)+\left(\|\nabla^{k+1}h\|_{L^2}^2+\|\nabla^{k+2}h\|_{L^2}^2\right)\le 0,\quad\text{for } 0\le l\le m-1.\tag{22}$$

Taking $l=0$ and $m=[\frac{d+1}{2}]$ in (22) and integrating directly in time, we have

$$\begin{aligned}&\|h(t)\|_{H^{[\frac{d+1}{2}]}}^2+\int_0^t\left(\|\nabla h(t)\|_{H^{[\frac{d+1}{2}]}}^2+\|\Delta h(t)\|_{H^{[\frac{d+1}{2}]}}^2\right)dt\\&\quad\le C\mathcal{E}_0^{[\frac{d+1}{2}]}(0)\le C\|h_0\|_{H^{[\frac{d+1}{2}]}}^2.\end{aligned}\tag{23}$$

Then by a standard continuity argument, this closes the a priori estimates (14) if at the initial time $\mathcal{E}_0^{[\frac{d+1}{2}]}=\|h_0\|_{H^{[\frac{d+1}{2}]}}^2$ is sufficiently small. This in turn allows us to take $l=0$ and $m=N$ in (23) and then integrate it directly in time to obtain (4).

4 Proof of Theorem 1.3

In this section, we consider the temporary decay rate of strong solutions of problem (1)–(2) provided that Theorem 1.2 holds. First of all, one need to derive the evolution of the negative Sobolev norms of solution to problem (13) (which is equivalent to problem (1)–(2)). To estimate the nonlinear terms, we need to restrict ourselves to that $s\in(0,\frac{d}{2})$.

Applying Λ^{-s} to (13)$_1$, multiplying the resulting identity by $\Lambda^{-s}h$, and then integrating over $\mathbb{R}^d$ by parts, we derive that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{-s}h\|_{L^2}^2+\nu_2\|\Lambda^{-s}\Delta h\|_{L^2}^2+\nu_1\|\Lambda^{-s}\nabla h\|_{L^2}^2\\
&\quad=\nu_3\int_{\mathbb{R}^d}\Lambda^{-s}\Big\{\nabla\cdot\Big[\Big(\nabla h+\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\\
&\qquad\times\Big(\nabla h-\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\nabla h\Big]\Big\}\cdot\Lambda^{-s}h\,dx\\
&\qquad-\nu_4\int_{\mathbb{R}^d}\Lambda^{-s}\big[\Delta|\nabla h|^2\big]\cdot\Lambda^{-s}h dx+\nu_5\int_{\mathbb{R}^d}\Lambda^{-s}|\nabla h|^2\cdot\Lambda^{-s}h\,dx.
\end{aligned}\tag{24}$$

First, we assume that $0<s<\frac{d}{2}-1$, then $\frac{1}{2}+\frac{s}{d}<1$. Moreover, on the basis of Theorem 1.2, we have $\|\Delta h\|_{L^2}+\|\Delta\nabla h\|_{L^2}\le C$. Using estimate (11) of the Riesz potential in Lemma 2.3, we find that

$$\begin{aligned}
&\nu_5\int_{\mathbb{R}^d}\Lambda^{-s}|\nabla h|^2\cdot\Lambda^{-s}h\,dx\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\|\Lambda^{-s}|\nabla h|^2\|_{L^2}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\||\nabla h|^2\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{d}}}}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\|\nabla h\|_{L^2}\|\nabla h\|_{L^{\frac{d}{s}}}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\|\nabla h\|_{L^2}\|\Delta h\|_{L^2}^{2-\frac{d}{2}+s}\|\nabla\Delta h\|_{L^2}^{\frac{d}{2}-s-1}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\big(\|\nabla h\|_{L^2}^2+\|\Delta h\|_{L^2}^2+\|\nabla\Delta h\|_{L^2}^2\big),
\end{aligned}\tag{25}$$

$$\begin{aligned}
&-\nu_4\int_{\mathbb{R}^d}\Lambda^{-s}\big[\Delta|\nabla h|^2\big]\cdot\Lambda^{-s}h\,dx\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\|\Lambda^{-s}\Delta|\nabla h|^2\|_{L^2}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\|\Delta|\nabla h|^2\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{d}}}}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\|\Delta\nabla h\|_{L^2}\|\nabla h\|_{L^{\frac{d}{s}}}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\|\Delta\nabla h\|_{L^2}\|\Delta h\|_{L^2}^{2-\frac{d}{2}+s}\|\nabla\Delta h\|_{L^2}^{\frac{d}{2}-s-1}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\big(\|\Delta h\|_{L^2}^2+\|\nabla\Delta h\|_{L^2}^2\big),
\end{aligned}\tag{26}$$

and

$$\begin{aligned}
&\nu_3\int_{\mathbb{R}^d}\Lambda^{-s}\Big\{\nabla\cdot\Big[\Big(\nabla h+\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\cdot\Big(\nabla h-\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\nabla h\Big]\Big\}\cdot\Lambda^{-s}h\,dx\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\Big\|\Lambda^{-s}\Big\{\nabla\cdot\Big[\Big(\nabla h+\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\cdot\Big(\nabla h-\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\nabla h\Big]\Big\}\Big\|_{L^2}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\Big\|\nabla\cdot\Big[\Big(\nabla h+\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\cdot\Big(\nabla h-\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\nabla h\Big]\Big\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{d}}}}\\
&\quad\le C\|\Lambda^{-s}h\|_{L^2}\Big(\Big\|\Delta\Big(\nabla h+\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big)\Big\|_{L^2}\Big\|\nabla h-\sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\Big\|_{L^\infty}\|\nabla h\|_{L^{\frac{d}{s}}}
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \Delta\left(\nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0\right)\right\|_{L^2} \left\| \nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0 \right\|_{L^\infty} \|\nabla h\|_{L^{\frac{d}{s}}} \\
&+ \left\| \nabla h + \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0 \right\|_{L^{\frac{d}{s}}} \left\| \nabla h - \sqrt{\frac{\nu_1+\nu_2}{\nu_3}}\omega_0 \right\|_{L^\infty} \|\Delta h\|_{L^2}\Bigg) \\
\le{}& C\left\|\Lambda^{-s}h\right\|_{L^2}\big(\|\Delta\nabla h\|_{L^2}\|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}\|\Delta h\|_{L^2}^{2-\frac{d}{2}+s}\|\nabla\Delta h\|_{L^2}^{\frac{d}{2}-s-1} \\
&+ \|\Delta\nabla h\|_{L^2}\|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}\|\Delta h\|_{L^2}^{2-\frac{d}{2}+s}\|\nabla\Delta h\|_{L^2}^{\frac{d}{2}-s-1} \\
&+ \|\Delta h\|_{L^2}^{2-\frac{d}{2}+s}\|\nabla\Delta h\|_{L^2}^{\frac{d}{2}-s-1}\|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}\|\Delta\nabla h\|_{L^2}\big) \\
\le{}& C\left\|\Lambda^{-s}h\right\|_{L^2}\big(\|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}\|\Delta h\|_{L^2}^{2-\frac{d}{2}+s}\|\nabla\Delta h\|_{L^2}^{\frac{d}{2}-s-1} \\
&+ \|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}\|\Delta h\|_{L^2}^{2-\frac{d}{2}+s}\|\nabla\Delta h\|_{L^2}^{\frac{d}{2}-s-1} + \|\Delta h\|_{L^2}^{2-\frac{d}{2}+s}\|\nabla\Delta h\|_{L^2}^{\frac{d}{2}-s-1}\|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}\big) \\
\le{}& C\left\|\Lambda^{-s}h\right\|_{L^2}\big(\|\Delta h\|_{L^2}^2 + \|\nabla\Delta h\|_{L^2}^2 + \|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}^2\big).
\end{aligned}
\tag{27}
$$

Combining (24)–(27) together gives

$$
\begin{aligned}
&\frac{d}{dt}\left\|\Lambda^{-s}h\right\|_{L^2}^2 + \nu_2\left\|\Lambda^{-s}\Delta h\right\|_{L^2}^2 + \nu_1\left\|\Lambda^{-s}\nabla h\right\|_{L^2}^2 \\
&\quad\le C\left\|\Lambda^{-s}h\right\|\big(\|\nabla h\|_{H^2}^2 + \|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}^2\big).
\end{aligned}
\tag{28}
$$

Define

$$
\mathcal{E}_{-s}(t) := \left\|\Lambda^{-s}h(t)\right\|_{L^2}^2.
$$

Consider inequality (28), integrating in time, we find that

$$
\begin{aligned}
\mathcal{E}_{-s}(t) &\le \mathcal{E}_{-s}(0) + \int_0^t \big(\|\nabla h\|_{H^2}^2 + \|\nabla h\|_{\dot{H}^{[\frac{d+1}{2}]}}^2\big)\sqrt{\mathcal{E}_{-s}(\tau)}\,d\tau \\
&\le C_0\Big(1 + \sup_{0\le\tau\le t}\sqrt{\mathcal{E}_{-s}(\tau)}\,d\tau\Big),
\end{aligned}
\tag{29}
$$

where we have used the inequality (4) in the above. It follows from (29) that

$$
\left\|\Lambda^{-s}h(t)\right\|_{L^2}^2 \le C_0, \quad \forall s \in \left[0, \frac{d}{2}-1\right].
\tag{30}
$$

Moreover, using Lemma 2.4, if $l = 1, 2, \dots, N-1$, we have

$$
\left\|\nabla^{l+1}f\right\|_{L^2} \ge C\left\|\Lambda^{-s}f\right\|_{L^2}^{-\frac{1}{l+s}}\left\|\nabla^l f\right\|_{L^2}^{1+\frac{1}{l+s}}.
$$

Then, by this facts and (30), we get

$$
\left\|\nabla^{l+1}h\right\|_{L^2}^2 \ge C_0\big(\left\|\nabla^l h\right\|_{L^2}^2\big)^{1+\frac{1}{l+s}}.
\tag{31}
$$

Hence, for $1 = 1, 2, \dots, N-1$, the following inequality holds:

$$
\left\|\nabla^{l+1}h\right\|_{H^{N-l-1}}^2 \ge C_0\big(\left\|\nabla^l h\right\|_{H^{N-l}}^2\big)^{1+\frac{1}{l+s}}.
$$

Thus, we deduce from (22) with $m = N$ the following inequality

$$\frac{d}{dt}\mathcal{E}_l^N + C_0\big(\mathcal{E}_l^N\big)^{1+\frac{1}{l+s}} \le 0, \quad \text{for } l = 1, 2, \ldots, N-1. \tag{32}$$

Solving (32), we find that

$$\mathcal{E}_l^N(t) \le C_0(1+t)^{-l-s}, \quad \text{for } l = 1, 2, \ldots, N-1. \tag{33}$$

Note that the Hardy–Littlewood–Sobolev theorem implies that for $p \in [\frac{d}{d-1}, 2]$, $L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = d(\frac{1}{p} - \frac{1}{2}) \in [0, \frac{d}{2} - 1]$. Therefore, based on Theorem 1.2 and (33), we obtain

$$\big\|\nabla^l h(t)\big\|_{H^{N-l}} \le C(1+t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})-\frac{l}{2}}, \quad \text{for } l = 0, 1, \ldots, N-1, \tag{34}$$

which complete the proof of Theorem 1.3.

Acknowledgements
The author would like to thank Dr. Xiaopeng Zhao for their helpful suggestions.

Funding
This paper was supported by the Fundamental Research Funds for the Central Universities (grant No. N2005031).

Availability of data and materials
All data generated or analyzed during this study are included in this published article.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
ND completed the proof of global well-posedness, SY completed the proof of decay estimate. All authors reviewed the manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 1 March 2022 Accepted: 26 September 2022 Published online: 18 October 2022

References

1. Agelas, L.: Global regularity of solutions of equation modeling epitaxy thin film growth in $\mathbb{R}^d$, $d = 1, 2$. J. Evol. Equ. **15**, 89–106 (2015)
2. Anh, C.T., Trang, P.T.: Decay rate of solutions to 3D Navier–Stokes–Voigt equations in H^m spaces. Appl. Math. Lett. **61**, 1–7 (2016)
3. Chae, D., Schonbek, M.: On the temporal decay for the Hall-magnetohydrodynamic equations. J. Differ. Equ. **255**, 3971–3982 (2013)
4. Dai, M., Qing, J., Schonbek, M.: Asymptotic behavior of solutions to liquid crystal systems in $\mathbb{R}^3$. Commun. Partial Differ. Equ. **37**, 2138–2164 (2012)
5. Fan, J., Alsaedi, A., Hayat, T., Zhou, Y.: Global well-posedness and regularity criteria for epitaxial growth modles. Comput. Math. Appl. **74**, 459–465 (2017)
6. Fan, J., Zhou, Y.: Global well-posedness for the 4D epitaxial growth models. Appl. Math. Lett. **49**, 28–32 (2015)
7. Grafakos, L.: Classical and Modern Fourier Analysis. Pearson, Prentice-Hall (2004)
8. Guo, Y., Wang, Y.: Decay of dissipative equations and negative Sobolev spaces. Commun. Partial Differ. Equ. **37**, 2165–2208 (2012)
9. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier–Stokes equations. Commun. Pure Appl. Math. **41**, 891–907 (1988)
10. King, B., Stein, O., Winkler, M.: A fourth-order parabolic equation modeling epitaxial thin film growth. J. Math. Anal. Appl. **203**, 459–490 (2003)
11. Li, B., Liu, J.G.: Thin film epitaxy with or without slope selection. Eur. J. Appl. Math. **14**, 713–743 (2003)
12. Li, B., Zhu, H., Zhao, C.: Time decay rate of solutions to the hyperbolic MHD equations in $\mathbb{R}^3$. Acta Math. Sci. **36**, 1369–1382 (2016)

13. Li, D., Qiao, Z., Tang, T.: Gradient bounds for a thin film epitaxy equation. J. Differ. Equ. **262**, 1720–1746 (2017)
14. Nirenberg, L.: On elliptic partial differential equations. Ann. Sc. Norm. Super. Pisa **13**, 115–162 (1959)
15. Raible, M., Linz, S., Hänggi, P.: Amorphous thin film growth: minimal deposition equation. Phys. Rev. E **62**, 1691–1694 (2000)
16. Raible, M., Linz, S., Hänggi, P.: Amorphous thin film growth: modeling and pattern formation. Adv. Solid State Phys. **41**, 391–403 (2001)
17. Raible, M., Linz, S., Hänggi, P.: Amorphous thin film growth: effects of density inhomogeneities. Phys. Rev. E **64**, 031506 (2001)
18. Robert, V.K., Yan, X.: Upper bound on the coarsening rate for an epitaxial growth model. Commun. Pure Appl. Math. **56**, 1549–1564 (2003)
19. Schonbek, M.E.: L^2 decay for weak solutions of the Navier–Stokes equations. Arch. Ration. Mech. Anal. **88**(2), 209–222 (1985)
20. Schonbek, M.E.: Large time behaviour of solutions to the Navier–Stokes equations. Commun. Partial Differ. Equ. **11**(7), 733–763 (1986)
21. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Unversity Press, Princeton (1970)
22. Stein, O., Winkler, M.: Amorphous molecular beam epitaxy: global solutions and absorbing sets. Eur. J. Appl. Math. **16**, 767–798 (2005)
23. Villain, J.: Continuum models of crystal growth from atomic beams with and without desorption. J. Phys. I **1**, 19–42 (1991)
24. Wang, Y.: Decay of the Navier–Stokes–Poisson equations. J. Differ. Equ. **253**, 273–297 (2012)
25. Winkler, M.: Global solutions in higher dimensions to a fourth order parabolic equation modeling epitaxial thin film growth. Z. Angew. Math. Phys. **62**, 575–608 (2011)
26. Zangwill, A.: Some causes and a consequence of epitaxial roughening. J. Cryst. Growth **163**, 8–21 (1996)
27. Zhao, C., Liang, Y., Zhao, M.: Upper and lower bounds of time decay rate of solutions to a class of incompressible third grade fluid equations. Nonlinear Anal., Real World Appl. **15**, 229–238 (2014)