Sun et al. *Boundary Value Problems* (2022) 2022:75
https://doi.org/10.1186/s13661-022-01657-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On the noncooperative Schrödinger–Kirchhoff system involving the critical nonlinearities on the Heisenberg group

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Abstract

This paper deals with the existence of solutions for the noncooperative Schrödinger–Kirchhoff system involving the p-Laplacian operator and critical nonlinearities on the Heisenberg group. Under some suitable conditions, together with the limit index theory and the concentration–compactness principle, we obtain the existence and multiplicity of solutions for this system. To our best knowledge, the existence results for the noncooperative system with p-Laplacian and critical nonlinearities are new on the Heisenberg group.

Keywords: Heisenberg group; Schrödinger–Kirchhoff system; p-Laplacian operator; Limit index theory; Critical Sobolev exponent

1 Introduction and main result

In this paper, we consider the existence of solutions for the noncooperative Schrödinger–Kirchhoff system involving the p-Laplacian operator and critical nonlinearity on the Heisenberg group:

$$\begin{cases} K(\|D_H u\|_p^p)\Delta_{H,p}u - V(\xi)|u|^{p-2}u = |u|^{p^*-2}u + F_u(\xi,u,v) & \text{in } \mathbb{H}^n, \\ -K(\|D_H v\|_p^p)\Delta_{H,p}v + V(\xi)|v|^{p-2}v = |v|^{p^*-2}v + F_v(\xi,u,v) & \text{in } \mathbb{H}^n, \end{cases} \tag{1.1}$$

where $\Delta_{H,p}$ is the p-Laplacian with $1 < p < Q$ and $p^* = Qp/(Q-p)$ is the critical Sobolev exponent on the Heisenberg group, $F = F(\xi,u,v)$, $F_u = \frac{\partial F}{\partial u}$, $F_v = \frac{\partial F}{\partial v}$, K and V are the Kirchhoff function and potential function, which satisfy the conditions given later. The operator $\Delta_{H,p}\varphi$ is called p Kohn–Spencer Laplacian, which is defined as follows:

$$\Delta_{H,p}\varphi = \operatorname{div}\left(|D_H\varphi|_H^{p-2}D_H\varphi\right)$$

for all $\varphi \in C^2(\mathbb{H}^n)$.

Kirchhoff's problem has a physical background, which is an extension of the model proposed by Kirchhoff [11] in 1883. Moreover, by considering the effect of string length changes during vibrations, Kirchhoff proposed Kirchhoff model with the following equation, which is an extension of D'Alembert wave equation:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{p_0}{\lambda}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

where ρ, p_0, λ, E, L are constants that represent some physical meanings, respectively. Later, more and more scholars carried out research on a Kirchhoff-type model. We also refer to [5, 8, 16, 18, 20, 24] for a wide list of contributions along with [1].

For our problems, first of all, we assume that the Kirchhoff function $K\colon \mathbb{R}_0^+\to\mathbb{R}_0^+$, the potential function $V(\xi)$, and the function $F(\xi,u,v)$ satisfy the following conditions:

(K_1) $K\in C(\mathbb{R}_0^+\to\mathbb{R}_0^+)$ satisfies $\inf_{t\in\mathbb{R}_0^+}K(t)\geq k_0>0$, where k_0 is a positive constant.

(K_2) There exists $\theta\in(1,p^*/p)$ satisfying

$$\theta\mathcal{K}(t)\geq K(t)t\quad\text{for all } t\geq 0,$$

where $\mathcal{K}(t)=\int_0^t K(\varsigma)\,d\varsigma$.

(V_1) $V:\mathbb{H}^n\to\mathbb{R}^+$ is a continuous function, and there exists $V_0>0$ such that $\inf_{\xi\in\mathbb{H}^n}V(\xi)\geq V_0$.

(V_2) For every $d>0$ such that

$$\operatorname{meas}\bigl(\bigl\{\xi\in\mathbb{H}^n:V(\xi)\leq d\bigr\}\bigr)<\infty,$$

where $\operatorname{meas}(\cdot)$ denotes Lebesgue measure in $\mathbb{H}^n$.

(F_1) $F\in C(\mathbb{H}^n\times\mathbb{R}^2,\mathbb{R}^+)$, where $\mathbb{R}^+=\{\xi\in\mathbb{R}|\xi\geq 0\}$; and there exist $c_0,c_1>0$, $p<q<p^*$ such that

$$\bigl|F_s(r,s,t)\bigr|+\bigl|F_t(r,s,t)\bigr|\leq c_0|s|^{\frac{p-1}{2}}+c_1|t|^{\frac{p-1}{2}}.$$

(F_2) $sF_s(\xi,s,t)\geq 0$, $tF_t(\xi,s,t)\geq 0$, and there exists $\tau\in(\theta p,p^*)$ such that

$$0<\tau F(\xi,s,t)\leq sF_s(\xi,s,t)+tF_t(\xi,s,t)$$

for any $(\xi,s,t)\in(\mathbb{H}^n\times\mathbb{R}^2,\mathbb{R}^+)$.

(F_3) F is even: $F(\xi,s,t)=F(\xi,-s,-t)$ for all $(\xi,s,t)\in\mathbb{H}^n\times\mathbb{R}^2$.

Note that condition (V_2) is weaker than the coercivity assumption $V(x)\to\infty$ as $|\xi|\to\infty$, which was first proposed by Bartsch and Wang in [2] to overcome the lack of compactness.

Now, we give some notations on the Heisenberg group. Let $\mathbb{H}^n$ be the Heisenberg Lie group of topological dimension $2n+1$, that is, the Lie group which has $\mathbb{R}^{2n+2}$ as a background manifold, endowed with the non-Abelian group law

$$\tau:\mathbb{H}^n\to\mathbb{H}^n,\qquad \tau_\xi\bigl(\xi'\bigr)=\xi\circ\xi'$$

with

$$\xi \circ \xi' = \big(x + x', y + y', t + t' + 2(x'y - y'x)\big), \quad \forall \xi, \xi' \in \mathbb{H}^n.$$

The inverse is given by $\xi^{-1} = -\xi$ so that $(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ \xi^{-1}$.

The anisotropic dilation structure on the Heisenberg group leads to the Korányi norm, which is defined by

$$r(\xi) = r(z,t) = \big(|z|^4 + t^2\big)^{\frac{1}{4}}, \quad \forall \xi = (z,t) \in \mathbb{H}^n.$$

Therefore, the Koranyi norm is homogeneous of degree 1 with respect to the dilations $\delta_R, R > 0$, that is,

$$r\big(\delta_R(\xi)\big) = r\big(Rz, R^2 t\big) = \big(|Rz|^4 + R^4 t^2\big)^{1/4} = Rr(\xi) \quad \text{for all } \xi = (z,t) \in \mathbb{H}^n. \tag{1.2}$$

For $s > 0$, a natural group of dilation on $\mathbb{H}^n$ is defined by $\delta_s(\xi) = (sx, sy, s^2 t)$. Hence, $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$. For all $\xi = (x,y,t) \in \mathbb{H}^n$, it is easy to verify that the Jacobian determinant of dilatations $\delta_s : \mathbb{H}^n \to \mathbb{H}^n$ is constant and equal to $\mathbb{R}^{2n+2}$. That is the reason why the natural number $Q = 2n + 2$ is called homogeneous dimension of $\mathbb{H}^n$ and critical exponents $Q^* := \frac{NQ}{N-Q}$.

The horizontal gradient of a C^1 function u: $\mathbb{H}^n \to R$ is defined by

$$D_H u = \sum_{j=1}^{n} \big[(X_j u)X_j + (Y_j u)Y_j\big].$$

It is obvious that $D_H u \in \operatorname{span}\{X_j, Y_j\}_{j=1}^n$. At the same time, for any horizontal vector field function $X = X(\xi)$, $X = \{x^j X_j + \widetilde{x}^j Y_j\}_{j=1}^n$ of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, we define the horizontal divergence of X by

$$\operatorname{div}_H X = \sum_{j=1}^{n} \big[X_j\big(x^j\big) + Y_j\big(\widetilde{x}^j\big)\big].$$

Also, if $u \in C^2(\mathbb{H}^n)$, then the Kohn–Spencer Laplacian in $\mathbb{H}^n$ is either equivalent to the horizontal Laplacian or the sub-Laplacian of u is

$$\Delta_{\mathbb{H}^n} u = \sum_{j=1}^{n} \big(X_j^2 + Y_j^2\big) u = \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t}\right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2},$$

where

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, X_3, \dots, Y_1, Y_2, \dots, Y_n).$$

The Lie algebra of left-invariant vector fields is generated by the following vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n.$$

The left-invariant distance d_H on $\mathbb{H}^n$ is accordingly defined by

$$d_H(\xi, \xi') = r(\xi^{-1} \circ \xi'), \quad \forall (\xi, \xi') \in \mathbb{H}^n \times \mathbb{H}^n.$$

For a complete treatment on the Heisenberg group functional setting, we refer to [7, 10, 14, 23].

The inspiration for our study to problem (1.1) is the application of the Heisenberg group. Recently, the reason why many scholars are interested in the Heisenberg group is its important applications in quantum mechanics, partial differential equations, and other fields. We try to put the same theorem applicable to a Euclidean space on the Heisenberg group, and many scholars are studying this kind of problem.

Next, we will focus on some latest progress of study related to our problem. In [4], the authors obtained the existence of solutions for the sub-elliptic equation involving small, nontrivial perturbations in the whole space $\mathbb{H}^n$

$$-\Delta_{H,Q} u + V(\xi)|u|^{Q-2}u = \frac{f(\xi, u)}{r(\xi)^\beta} + \varepsilon h(\xi), \tag{1.3}$$

where V is a continuous potential and f has exponential growth. After that, using a minimization argument and the Ekeland variational principle, Lam and Lu [13] considered that in the absence of perturbation ($\varepsilon = 0$), and they obtained the existence and multiplicity of solutions of problem (1.3). Subsequently, by variational methods, Pucci et al. [25, 26] dealt with the existence of nontrivial solutions for (p, Q) equations and (p, Q) system in the Heisenberg group $\mathbb{H}^n$ by considering the case without the potential function.

On the other hand, in the Euclidean space, Li [15] considered the strongly indefinite functional involving p-Laplacian elliptic system with the help of limit index theory:

$$\begin{cases} \Delta_p u = F_s(x, u, v) & \text{in } \Omega, \\ -\Delta_p v = F_t(x, u, v) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0. \end{cases} \tag{1.4}$$

More precisely, they obtained an unbounded sequence of solutions with appropriate conditions for F. There are many works in the literature on this subject that are very connected to this problem. Let us refer to some of them for further reference [9, 15, 17, 19, 21, 22, 27].

Inspired by the above references, our main aim in this paper is to consider the existence and multiplicity of solutions for problem (1.1) involving p-Laplacian operator and critical nonlinearity on the Heisenberg group. Undoubtedly, the lack of compactness makes us encounter serious difficulties. It is worth noting the argument developed in [9], where one of the emphases is to verify the proof the $(PS)_c$ condition. In this paper, we use the concentration–compactness principle and the concentration–compactness principle at infinity on the classical Sobolev spaces in the Heisenberg group to prove that the $(PS)_c$ condition holds. To the best of our knowledge, this paper is the first to deal with non-cooperative Schrödinger–Kirchhoff systems with p-Laplacian and critical nonlinearity on the Heisenberg group. Furthermore, although some properties are similar between Kohn Laplacian Δ_H and the classical Laplacian Δ, the similarities may be deceitful; see, e.g., [7].

Now, we are ready to state our main results.

Theorem 1.1 *Assume that V satisfies (V_1) and (V_2), K satisfies (K_1)–(K_2), and F satisfies (F_1)–(F_3), then problem (1.1) possesses infinitely solutions.*

This paper is organized as follows. In Sect. 2, we present some necessary preliminary knowledge on the Heisenberg group and collect some properties about the space $HW^{1,p}$. In Sect. 3, we recall the limit index theory due to Li [15]. In Sect. 4, we use the concentration–compactness principle to prove $(PS)_c$ conditions. Section 5 is devoted to the proofs of Theorem 1.1.

2 Preliminaries

In this section, we give some useful facts about classical Sobolev spaces on the Heisenberg group and provide some technical lemmas.

Let $1 \le p < Q$ be a real number. Denote by $HW^{1,p}(\mathbb{H}^n)$ the horizontal Sobolev space consisting of the functions $u \in L^p(\mathbb{H}^n)$ such that $D_H u \in L^p$ exists in the sense of distributions and $|D_H u|_H \in L^p(\mathbb{H}^n)$, equipped with the natural norm

$$\|u\|_{HW^{1,p}(\mathbb{H}^n)} := \left(\|u\|_{L^p}^p + \|D_H u\|_{L^p}^p\right), \qquad \|D_H u\|_{L^p(\mathbb{H}^n)} = \left(\int_{\mathbb{H}^n} |D_H u|_H^p \, d\xi\right)^{\frac{1}{p}}. \tag{2.1}$$

For brevity, we use the notation $\|\cdot\|_p = \|\cdot\|_{L^p}$ to denote the usual L^p-norm.

Let $HW_V^{1,p}(\mathbb{H}^n)$ represent the completion of $C_c^\infty(\mathbb{H}^n)$ with the norm

$$\|u\|_{HW_V^{1,p}(\mathbb{H}^n)} := \left(\|D_H u\|_{L^p}^p + \|u\|_{p,V}^p\right)^{\frac{1}{p}}, \qquad \|u\|_{p,V}^p = \int_{\mathbb{H}^n} V(\xi)|u|^p \, d\xi. \tag{2.2}$$

Due to (V_1), we can obtain $L^p(\mathbb{H}^n, V) = (L^p(\mathbb{H}^n, V), \|\cdot\|_{p,V})$ is a uniformly convex Banach space.

According to Folland and Stein [6], the elliptical Sobolev embedding theorem and C-R Yamabe problem are often closely linked together. It is valid in the more general case of the Carnot group, but we state it in the establishment of the Heisenberg group.

The continuous embedding of $HW^{1,Q}(\mathbb{H}^n) \hookrightarrow L^s(\mathbb{H}^n)$ for all $s \ge Q$ by [4]. That is, there exists a positive constant C such that

$$\|u\|_{L^s(\mathbb{H}^n)} \le C\|u\|_{HW_V^{1,p}(\mathbb{H}^n)} \quad \text{for all } u \in HW_V^{1,p}(\mathbb{H}^n). \tag{2.3}$$

From [28] that C_{p^*} is achieved in the Folland–Stein space $S^{1,p}(H^n)$. We define the optimal Sobolev constant C_{p^*} of the Folland–Stein inequality as follows:

$$C_{p^*} := \inf_{u \in S^{1,p}(\mathbb{H}^n), u \ne 0} \frac{\|D_H u\|_p^p}{\|u\|_{p^*}^p}. \tag{2.4}$$

Let $Y = HW_V^{1,p}(\mathbb{H}^n) \times HW_V^{1,p}(\mathbb{H}^n)$, equipped with the norm

$$\|(u,v)\| = \left(\|u\|_{HW_V^{1,p}(\mathbb{H}^n)} + \|v\|_{HW_V^{1,p}(\mathbb{H}^n)}\right)^{\frac{1}{p}}$$

for all $(u,v) \in Y$.

We will use the embedding theorem below to prove the existence of weak solutions to problem (1.1).

The energy functional corresponding to problem (1.1) is defined as follows:

$$\begin{aligned}\mathcal{J}(u,v) &= -\frac{1}{p}\mathcal{K}\left(\|D_H u\|_p^p\right) - \frac{1}{p}\int_{\mathbb{H}^n} V(\xi)|u|^p\,d\xi + \frac{1}{p}\mathcal{K}\left(\|D_H v\|_p^p\right)\\ &\quad + \frac{1}{p}\int_{\mathbb{H}^n} V(\xi)|v|^p\,d\xi - \frac{1}{p^*}\int_{\mathbb{H}^n}|u|^{p^*}\,d\xi\\ &\quad - \frac{1}{p^*}\int_{\mathbb{H}^n}|v|^{p^*}\,d\xi - \int_{\mathbb{H}^n}F(\xi,u,v)\,d\xi.\end{aligned} \tag{2.5}$$

Under the assumption, it is obvious that the energy $\mathcal{J}: Y \to \mathbb{R}$ connected with problem (1.1) is well defined and $\mathcal{J} \in C^1(Y \to \mathbb{R})$,

$$\begin{aligned}\left\langle \mathcal{J}'(u,v),(\varphi,\phi)\right\rangle = -K\left(\|D_H u\|_p^p\right)\langle u,\varphi\rangle_p - \int_{\mathbb{H}^n} V(\xi)|u|^{p-2}u\varphi\,d\xi\\ + K\left(\|D_H v\|_p^p\right)\langle v,\varphi\rangle_p + \int_{\mathbb{H}^n} V(\xi)|v|^{p-2}v\phi\,d\xi\\ - \int_{\mathbb{H}^n}|u|^{p^*-2}u\varphi\,d\xi - \int_{\mathbb{H}^n}|v|^{p^*-2}v\phi\,d\xi\\ - \int_{\mathbb{H}^n}F_u(\xi,u,v)\varphi\,d\xi - \int_{\mathbb{H}^n}F_v(\xi,u,v)\phi\,d\xi\end{aligned}$$

for all $(u,v) \in Y$ and $(\varphi,\phi) \in Y$. Therefore, the critical points of function $\mathcal{J}$ are weak solutions of problem (1.1).

Next, we recall the concentration–compactness principle of the p Laplacian on the Heisenberg group.

Definition 2.1 Let $\mathcal{M}(\mathbb{H})$ show the finite nonnegative Radon measures space in $\mathbb{H}^n$. For any $\mu \in \mathcal{M}(\mathbb{H}^n)$, $\mu(\mathbb{H}^n) = \|\mu\|$ holds. We say that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{H}^n)$ if $(\mu_n,\eta) \to (\mu,\eta)$ holds for all $\eta \in C_0(\mathbb{H}^n)$ as $n \to \infty$.

Proposition 2.1 *Let $\sigma \in (-\infty,\mathcal{H}_p)$ and let $(u_k)_k$ be a sequence in $S^{1,p}(\mathbb{H}^n)$ such that $u_k \rightharpoonup u$ in $S^{1,p}(\mathbb{H}^n)$, and $|u_k|^{p^*}\,d\xi \overset{*}{\rightharpoonup} \nu$, $|D_H u_k|_H^p\,d\xi \overset{*}{\rightharpoonup} \mu$, $|u_k|^p\psi^p\frac{d\xi}{r(\xi)^p} \overset{*}{\rightharpoonup} \omega$ in $\mathcal{M}(\mathbb{H}^n)$ for some appropriate $u \in S^{1,p}(\mathbb{H}^n)$ and finite nonnegative Radon measure μ, ν, ω on $\mathbb{H}^n$. Then there exists an at most countable set J, a family of points $\{\xi_k\}_{j\in J} \subset \mathbb{H}^n$, two families of nonnegative numbers $\{\mu_k\}_{j\in J}$, and $\{\nu_k\}_{j\in J}$, and there are nonnegative numbers ν_0, μ_0, ω_0 such that*

$$\nu = |u|^{p^*}\,d\xi + \nu_0\delta_0 + \sum_{j\in J}\nu_j\delta_{\xi_j}, \tag{2.6}$$

$$\mu \geq |D_H u|_H^p\,d\xi + \mu_0\delta_0 + \sum_{j\in J}\mu_j\delta_{\xi_j}, \tag{2.7}$$

$$\omega = |u|^p\psi^p\frac{d\xi}{r(\xi)^p} + \omega_0\delta_0, \tag{2.8}$$

$$\nu_j^{p/p^*} \leq \frac{\mu_j}{C_{p^*}} \quad \textit{for all } j \in J, \qquad \nu_0^{p/p^*} \leq \frac{\mu_0 - \sigma\omega_0}{I_\sigma}, \tag{2.9}$$

where $C_{p^*} = I_0$ *and* $I_\sigma = \inf_{u\in S^{1,p}(\mathbb{H}^n),u\neq 0} \frac{\|D_H u\|_p^p - \sigma\|u\|_{Hp}^p}{\|u\|_{p^*}^p}$, *while* δ_0, δ_{ξ_j} *are the Dirac functions at the points* O *and* ξ_j *of* $\mathbb{H}^n$, *respectively.*

Proposition 2.2 *Let* $(u_k)_k$ *be a sequence in* $S^{1,p}(\mathbb{H}^n)$ *as in Proposition* 2.1 *and define*

$$\nu_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{B_R^c}|u_k|^{p^*}\,d\xi, \qquad \mu_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{B_R^c}|D_H u_k|_H^p\,d\xi \tag{2.10}$$

and

$$\omega_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{B_R^c}|u_k|^p\psi^p\frac{d\xi}{r(\xi)^p}. \tag{2.11}$$

Then

$$\begin{aligned} &\limsup_{k\to\infty}\int_{\mathbb{H}^n}|u_k|^{p^*}\,d\xi = \nu\big(\mathbb{H}^n\big) + \nu_\infty, \\ &\limsup_{k\to\infty}\int_{\mathbb{H}^n}|D_H u_k|_H^p\,d\xi = \mu\big(\mathbb{H}^n\big) + \mu_\infty, \end{aligned} \tag{2.12}$$

$$\limsup_{k\to\infty}\int_{\mathbb{H}^n}|u_k|^p\psi^p\frac{d\xi}{r(\xi)^p} = \omega\big(\mathbb{H}^n\big) + \omega_\infty, \qquad \nu_\infty^{p/p^*} \le \frac{\mu_\infty - \sigma\omega_\infty}{I_\sigma}, \tag{2.13}$$

where μ, ν, ω *are the measures introduced in Proposition* 2.1. *Moreover, the following inequality holds if* $\sigma = 0$:

$$C_{p^*}\nu_\infty^{p/p^*} \le \mu_\infty.$$

Remark 2.1 Weak (resp. strong) convergence is denoted by $\rightharpoonup$ (resp., $\to$). $G_1 = O(Q)$ is the group of orthogonal linear transformations in $\mathbb{H}^n$. $E = HW^{1,p}(\mathbb{H}^n)$, $E_{G_1} = HW^{1,p}_{O(Q)} := \{u \in HW^{1,p}(\mathbb{H}^n) : gu(\xi) = u(g^{-1}\xi) = u(\xi), g \in O(Q)\}$. $G_2 = Z_2$, $Y = E\times E$, $X = Y_{G_1} = E_{G_1}\times E_{G_1}$, c denotes a positive constant.

3 Limit index theory

In this section, we review the limit index theory by Li [15] and give the following definitions.

Definition 3.1 ([15, 29]) The action of a topological group G on a normed space Z is a continuous map

$$G\times Z\to Z : [g,z]\mapsto gz$$

such that

$$1\cdot z = z, \qquad (gh)z = g(hz), \qquad z\mapsto gz \text{ is linear}, \quad \forall g,h\in G.$$

The action is isometric if

$$\|gz\| = \|z\| \quad \text{for all } g\in G, z\in Z,$$

and in this case Z is called the G-space.

The set of invariant points is defined by

$$\mathrm{Fix}(G) := \{z \in Z : gz = z, \forall g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant $\varphi \circ g = \varphi$ for every $g \in G$, $z \in Z$. A map $f : Z \to Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Assume that Z is a G-Banach space, that is, there is a G isometric action on Z. Let

$$\Sigma := \{A \subset Z : A \text{ is closed and } gA = A, \forall g \in G\}$$

be a family of all G-invariant closed subsets of Z, and let

$$\Gamma := \left\{h \in C^0(Z,Z) : h(gu) = g(hu), \forall g \in G\right\}$$

be a class of all G-equivariant mappings of Z. Finally, we call the set

$$O(u) := \{gu : g \in G\}$$

the G-orbit of u.

Definition 3.2 ([15]) An index for $(G, \sum, \Gamma)$ is a mapping $i := \sum \to \mathcal{Z}_+ \cup \{+\infty\}$ (where $\mathcal{Z}_+$ is the set of all nonnegative integers) such that, for all $A, B \in \Sigma$, $h \in \Gamma$, the following conditions are satisfied:

(1) $i(A) = 0 \Leftrightarrow A = \emptyset$;
(2) (Monotonicity) $A \subset B \Rightarrow i(A) \le i(B)$;
(3) (Subadditivity) $i(A \cup B) \le i(A) + i(B)$;
(4) (Supervariance) $i(A) \le i(\overline{h(A)})$, $\forall h \in \Gamma$;
(5) (Continuity) If A is compact and $A \cap \mathrm{Fix}(G) = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighborhood N of A such that $i(\overline{N}) = i(A)$;
(6) (Normalization) If $\xi \notin \mathrm{Fix}(G)$, then $i(O(\xi)) = 1$.

Definition 3.3 ([3]) An index theory is said to satisfy the d-dimensional property if there is a positive integer d such that

$$i\big(V^{dk} \cap S_1\big) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \mathrm{Fix}(G) = \{0\}$, where S_1 is the unit sphere in Z.

Suppose that U and V are G-invariant closed subspaces of Z such that

$$Z = U \oplus V,$$

where V is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is a dn_j-dimensional G-invariant subspace of V, $j = 1, 2, \dots$, and $V_1 \subset V_1 \subset \cdots \subset V_1 \subset \cdots$. Let

$$Z_j = U \oplus V_j$$

and $\forall A \in \sum$, let

$$A_j = A \oplus Z_j.$$

Definition 3.4 ([15]) Let i be an index theory satisfying the d-dimensional property. A limit index with respect to (Z_j) introduced by i is a mapping

$$i^\infty : \sum \to \mathcal{Z} \cup \{-\infty, +\infty\}$$

given by

$$i^\infty(A) = \limsup_{j\to\infty} \bigl(i(A_j) - n_j\bigr).$$

Proposition 3.1 ([15]) *Let $A, B \in \sum$, then i^∞ satisfies*:

(1) $A = \emptyset \Rightarrow i^\infty = -\infty$;
(2) (*Monotonicity*) $A \subset B \Rightarrow i^\infty(A) \le i^\infty(B)$;
(3) (*Subadditivity*) $i^\infty(A \cup B) \le i^\infty(A) + i^\infty(B)$;
(4) *If* $V \cap \operatorname{Fix}\{G\} = 0$, *then* $i^\infty(S_\varrho \cap V) = 0$, *where* $S_\varrho = \{z \in Z : \|z\| = \varrho\}$;
(5) *If* Y_0 *and* $\widetilde{Y}_0$ *are* G-*invariant closed subspaces of* V *such that* $V = Y_0 \oplus \widetilde{Y}_0$, $\widetilde{Y}_0 \subset V_{j_0}$ *for some* j_0 *and* $\dim(Y_0) = dm$, *then* $i^\infty(S_\varrho \cap Y_0) \ge -m$.

Definition 3.5 ([29]) A functional $\mathcal{J} \in C^1(Z, R)$ is said to satisfy the condition $(PS)_c^*$ if any sequence $\{u_{n_k}\}, u_{n_k} \in Z_{u_{n_k}}$ such that

$$\mathcal{J}(u_{n_k}) \to c, \qquad d\mathcal{J}(u_{n_k}) \to 0, \quad \text{as } k \to \infty,$$

possesses a convergent subsequence, where $Z_{u_{n_k}}$ is the n_k-dimensional subspace of Z, $\mathcal{J}_{n_k} = \mathcal{J}|_{Z_{u_{n_k}}}$.

Theorem 3.1 ([15]) *Assume that*

(B_1) $\mathcal{J} \in C^1(Z, R)$ *is* G-*invariant*;
(B_2) *There are* G-*invariant closed subspaces* U *and* V *such that* V *is infinite dimensional and* $Z = U \oplus V$;
(B_3) *There is a sequence of* G-*invariant finite dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots, \qquad \dim(V_j) = dn_j,$$

such that $V = \overline{\bigcup_{j=1}^\infty V_j}$;
(B_4) *There is an index theory* i *on* Z *satisfying the* d-*dimensional property*;
(B_5) *There are* G-*invariant subspaces* Y_0, $\widetilde{Y}_0$, Y_1 *of* V *such that* $V = Y_0 \oplus \widetilde{Y}_0, Y_1$, $\widetilde{Y}_0 \subset V_{j_0}$ *for some* j_0 *and* $\dim(\widetilde{Y}_0) = dm < dk = \dim(Y_1)$;

(B_6) *There are* α, β *and* $\alpha < \beta$ *such that* f *satisfies* $(PS)_c^*$, $\forall c \in [\alpha, \beta]$;
(B_7)

$$\begin{cases} \text{(a)} \quad \textit{either} \quad \operatorname{Fix}(G) \subset U \oplus Y_1 \quad \textit{or} \quad \operatorname{Fix}(G) \cap V = \{0\}, \\ \text{(b)} \quad \textit{there is} \quad \varrho > 0 \quad \textit{such that} \quad \forall u \in Y_0 \cap S_\varrho, \quad f(z) \geq \alpha, \\ \text{(c)} \quad \forall z \in U \oplus Y_1, \quad f(z) \leq \beta, \end{cases}$$

if i^∞ *is the limit index corresponding to* i, *then the numbers*

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{z \in A} f(u), \quad -k+1 \leq j \leq -m$$

are critical values of f, *and* $\alpha \leq c_{-k+1} \leq \cdots \leq c_{-m} \leq \beta$. *Moreover, if* $c = c_l = \cdots = c_{l+r}$, $r \geq 0$, *then* $i(\mathbb{K}_c) \geq r+1$, *where* $\mathbb{K}_c = \{z \in Z : df(z) = 0, f(z) = c\}$.

4 Verification of the $(PS)_c$ condition

In this section, with the help of the concentration–compactness principle in the classical Sobolev space on the Heisenberg group stated above, we show a careful analysis of the behavior of minimizing sequences, which is able to recover compactness below some critical threshold.

It is obvious that $\mathcal{J} \in C^1$. Moreover, the weak solution of problem (1.1) is consistent with the critical point of $\mathcal{J}$. According to (H_4) and (H_6), we can obtain that $\mathcal{J}$ is $O(Q)$-invariant. Combining with the principle of symmetric criticality due to Krawcewicz and Marzantowicz [12], we get that (u,v) is a critical point of $\mathcal{J}$ if and only if (u,v) is a critical point of $\widetilde{\mathcal{J}} = \mathcal{J}|_{X=E_{G_1} \times E_{G_1}}$. So, we are going to focus on verifying the existence of a sequence of critical points of $\mathcal{J}$ on X. At the same time, let E be a real Banach space and $\widetilde{\mathcal{J}} : E \to \mathbb{R}$ be a function of class C^1.

Lemma 4.1 *Suppose that condition* V *satisfies* (V_1) *and* (V_2), K *satisfies* (K_1)–(K_2), F *satisfies* (F_1)–(F_3). *Let* $\{(u_{n_k}, v_{n_k})\}$ *be a sequence such that* $\{(u_{n_k}, v_{n_k})\} \in X_{n_k}$.

$$\mathcal{J}(u_{n_k}, v_{n_k}) \to c < \left(\frac{1}{\tau} - \frac{1}{p^*}\right)(k_0 C_{p^*})^{\frac{Q}{p}}, \qquad d\mathcal{J}_{n_k}(u_{n_k}, v_{n_k}) \to 0 \quad \textit{as } k \to \infty,$$

where $\mathcal{J}_{n_k} = \mathcal{J}|_{X_{n_k}}$, *then* $\{(u_{n_k}, v_{n_k})\}$ *contains a subsequence converging strongly in* X.

Proof To this aim, we need two steps as follows.

Step 1. We show that $\{(u_{n_k}, v_{n_k})\}$ is bounded in X.

By conditions (M_1) and (F_2), we obtain that

$$\begin{aligned} o(1)\|u_{n_k}\|_p &\geq \langle -d\mathcal{J}_{u_{n_k}}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle \\ &= K(\|D_H u\|_p^p)\|D_H u\|_p^p + \int_{\mathbb{H}^n} V(\xi)|u_{n_k}|^p \, d\xi \\ &\quad + \int_{\mathbb{H}^n} |u_{n_k}|^{p^*} \, d\xi + \int_{\mathbb{H}^n} F_u(\xi, u_{n_k}, v_{n_k}) u_{n_k} \, d\xi \\ &\geq k_0 \|D_H u_{n_k}\|_p^p + \int_{\mathbb{H}^n} V(\xi)|u_{n_k}|^p \, d\xi \\ &\geq \min\{k_0, 1\}\|u_{n_k}\|_p^p. \end{aligned} \tag{4.1}$$

Since $p > 1$, from (4.1), we obtain that $\|u_{n_k}\|$ is bounded.

On the other hand, using condition (F_2), we have

$$\begin{aligned}
c + o(1)\|v_{n_k}\|_p &= \mathcal{J}_{n_k}(0, v_{n_k}) - \frac{1}{\tau}\big\langle d\mathcal{J}_{n_k}(v_{n_k}, v_{n_k}), (0, v_{n_k})\big\rangle \\
&= \frac{1}{p}\mathcal{K}\big(\|D_H v_{n_k}\|_p^p\big) - \frac{1}{\tau}K\big(\|D_H v_{n_k}\|_p^p\big)\|D_H v_{n_k}\|_p^p \\
&\quad + \left(\frac{1}{p} - \frac{1}{\tau}\right)\int_{\mathbb{H}^n} V(\xi)|v_{n_k}|^p\,d\xi + \left(\frac{1}{\tau} - \frac{1}{p^*}\right)\int_{\mathbb{H}^n} |v_{n_k}|^{p^*}\,d\xi \\
&\quad - \int_{\mathbb{H}^n}\left[F(\xi, 0, v_{n_k}) - \frac{1}{\tau}F_v(\xi, 0, v_{n_k})v_{n_k}\right]d\xi \\
&\ge \left(\frac{1}{p} - \frac{1}{\tau}\right)k_0\|D_H v_{n_k}\|_p^p + \left(\frac{1}{p} - \frac{1}{\tau}\right)\int_{\mathbb{H}^n} V(\xi)|v_{n_k}|^p\,d\xi \\
&\ge \left(\frac{1}{p} - \frac{1}{\tau}\right)\min\{k_0, 1\}\|v_{n_k}\|_p^p.
\end{aligned}$$

This shows that $\{v_{n_k}\}$ is bounded in E. Therefore, $\|u_{n_k}\|_p + \|v_{n_k}\|_p$ is bounded in X.

Step 2. We prove that $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in X.

Note that $\{u_{n_k}\}$ is bounded in E_{G_1}. Therefore, up to a sequence, $u_{n_k} \rightharpoonup u_0$ weakly in E_{G_1} and $u_{n_k}(\xi) \to u_0(\xi)$, a.e. in $\mathbb{H}^n$.

(I). We claim that $u_{n_k}(\xi) \to u_0(\xi)$ strongly in E_{G_1}. It follows from conditions (K_1) and (F_2) that

$$\begin{aligned}
0 &\leftarrow \big\langle -d\mathcal{J}_{n_k}(u_{n_k} - u_0, v_{n_k}), (u_{n_k} - u_0, 0)\big\rangle \\
&= K\big(\big\|D_H(u_{n_k} - u_0)\big\|_p^p\big)\big\|D_H(u_{n_k} - u_0)\big\|_p^p + \int_{\mathbb{H}^n} V(\xi)|u_{n_k} - u_0|^p\,d\xi \\
&\quad + \int_{\mathbb{H}^n} |u_{n_k} - u_0|^{p^*}\,d\xi + \int_{\mathbb{H}^n} F_u(\xi, u_{n_k} - u_0, v_{n_k})(u_{n_k} - u_0)\,d\xi \\
&\ge k_0\big\|D_H(u_{n_k} - u_0)\big\|_p^p + \int_{\mathbb{H}^n} V(\xi)|v_{n_k} - v_0|^p\,d\xi \\
&\ge \min\{k_0, 1\}\|u_{n_k} - u_0\|_p^p.
\end{aligned}$$

This fact implies that

$$u_{n_k} \to u_0 \quad \text{strongly in } E_{G_1}. \tag{4.2}$$

(II). In the following we will prove that there exists $v \in E_{G_1}$ such that

$$v_{n_k} \to v_0 \quad \text{strongly in } E_{G_1}. \tag{4.3}$$

Since $\{v_{n_k}\}$ is also bounded in E, we suppose that there exist v_0 and a subsequence, still denoted by $\{v_{n_k}\} \subset E$, such that

$$\begin{aligned}
&v_{n_k}(\xi) \rightharpoonup v_0 \quad \text{weakly in } E, \\
&v_{n_k}(\xi) \to v_0 \quad \text{strongly in } L^t_{\mathrm{loc}}\big(\mathbb{H}^n\big) \text{ for all } t \in \big[1, p^*\big), \\
&v_{n_k}(\xi) \to v_0 \quad \text{a.e. } \xi \in \mathbb{H}^n.
\end{aligned}$$

From Proposition 2.1, we obtain that

$$\begin{aligned}
&v_{n_k}(\xi) \rightharpoonup v_0 \quad \text{in } E,\\
&|u_{n_k}|^{p^*}\,d\xi \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}(\mathbb{H}^n),\\
&|D_H u_k|_H^p \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\mathbb{H}^n).
\end{aligned}$$

Then

$$\begin{aligned}
&\nu = |u|^{p^*}\,d\xi + \nu_0\delta_0 + \sum_{j\in J}\nu_j\delta_{\xi_j},\\
&\mu \ge |D_H u|_H^p\,d\xi + \mu_0\delta_0 + \sum_{j\in J}\mu_j\delta_{\xi_j},\\
&\omega = |u|^p\psi^p\frac{d\xi}{r(\xi)^p} + \omega_0\delta_0,\\
&\nu_j^{p/p^*} \le \frac{\mu_j}{C_{p^*}} \quad \text{for all } j\in J, \qquad \nu_0^{p/p^*} \le \frac{\mu_0 - \sigma\omega_0}{I_\sigma},
\end{aligned} \tag{4.4}$$

where J is at most countable, $\{\nu_j\}_j \subset \mathbb{H}^n$, δ_O, δ_{x_j} are the Dirac functions at the points O and ξ_j of $\mathbb{H}^n$, respectively.

Concentration at infinity of the sequence v_{n_k} is described by the following quantities:

$$\begin{aligned}
&\nu_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{B_R^c}|u_k|^{p^*}\,d\xi,\\
&\mu_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{B_R^c}|D_H u_k|_H^p\,d\xi,\\
&\omega_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{B_R^c}|u_{n_k}|^p\psi^p\frac{d\xi}{r(\xi)^p}.
\end{aligned}$$

Now, we will prove that the following three claims hold.

Claim 1.

$$J \text{ is finite and for } j\in J, \qquad \text{either } \nu_j = 0 \quad \text{or} \quad \nu_j \ge (k_0 C_{p^*})^{\frac{Q}{p}}.$$

In fact, fix $\phi \in C_c^\infty(\mathbb{H}^n)$ such that $0 \le \phi \le 1$, $\phi(O) = 1$ and $\operatorname{supp}\phi = \overline{B_1}$. Take $\varepsilon > 0$ and put $\phi_\varepsilon(\xi) = \phi(\delta_{1/\varepsilon}(\xi))$, $\xi \in \mathbb{H}^n$.

Then we get that $\{v_{n_k}\phi_\varepsilon\}$ is bounded in E.

Hence, we obtain that

$$\lim_{n\to\infty}\langle d\mathcal{J}_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k}\phi_\varepsilon)\rangle = 0,$$

that is,

$$\begin{aligned}
&K(\|D_H v_{n_k}\|_p^p)\|D_H v_{n_k}\phi_\varepsilon\|_p^p + \int_{\mathbb{H}^n}V(\xi)|v_{n_k}|^p\phi_\varepsilon\,d\xi - \int_{\mathbb{H}^n}|v_{n_k}|^{p^*}\phi_\varepsilon\,d\xi\\
&\quad - \int_{\mathbb{H}^n}F_v(\xi, u_{n_k}, v_{n_k})v_{n_k}\phi_\varepsilon(\xi)\,d\xi
\end{aligned} \tag{4.5}$$

$$= K\big(\|D_H u_{n_k}\|_p^p\big)\bigg(\int_{\mathbb{H}^n} |\phi_\varepsilon|^p |D_H u_{n_k}|_H^p\, d\xi + \|D_H\phi_\varepsilon u_{n_k}\|_p^p\bigg)$$
$$+ \int_{\mathbb{H}^n} V(\xi)|v_{n_k}|^p\phi_\varepsilon\, d\xi - \int_{\mathbb{H}^n} |v_{n_k}|^{p^*}\phi_\varepsilon\, d\xi - \int_{\mathbb{H}^n} F_v(\xi, u_{n_k}, v_{n_k})v_{n_k}\phi_\varepsilon(\xi)\, d\xi$$
$$= 0.$$

In the following, we estimate each term in (4.5).

In fact, arguing as before, we have

$$\int_{\mathbb{H}^n} |\phi_\varepsilon|^p |D_H v_{n_k}|_H^p\, d\xi + \|D_H\phi_\varepsilon v_{n_k}\|_p^p = \int_{\mathbb{H}^n} |\phi_\varepsilon|^p |D_H v_{n_k}|_H^p\, d\xi + o(1). \tag{4.6}$$

Now, by the compactness result (see Proposition 2.1), we know that

$$\lim_{\varepsilon\to 0^+}\lim_{k\to\infty}\int_{\mathbb{H}^n} |\phi_\varepsilon|^p |D_H v_{n_k}|_H^p\, d\xi = \int_{\mathbb{H}^n} |\phi_\varepsilon|^p\, d\mu \ge \int_{\mathbb{H}^n} |D_H v_0|_H^p \big|\phi_\varepsilon(\xi)\big|^p\, d\xi + \mu_j. \tag{4.7}$$

Therefore, from (4.6) and (4.7), we deduce that

$$\lim_{\varepsilon\to 0^+}\lim_{k\to\infty} K\big(\|D_H v_{n_k}\|_p^p\big)\|D_H v_{n_k}\phi_\varepsilon\|_p^p \ge k_0\mu_j. \tag{4.8}$$

Also, we can get that

$$\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\int_{\mathbb{H}^n} V(\xi)|v_{n_k}|^p\phi_\varepsilon\, d\xi = \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\int_{B_\varepsilon} V(\xi)|v_{n_k}|^p\phi_\varepsilon\, d\xi = 0, \tag{4.9}$$

$$\lim_{n\to\infty}\int_{\mathbb{H}^n} |v_n|^{p^*}\phi_\varepsilon\, d\xi = \int_{\mathbb{H}^n} \phi_\varepsilon\, d\nu = \int_{\mathbb{H}^n} |v_0|^{p^*}\phi_\varepsilon\, d\xi + \nu_j \tag{4.10}$$

and

$$\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\int_{\mathbb{H}^n} F_v(\xi, u_{n_k}, v_{n_k})v_{n_k}\phi_\varepsilon(\xi)\, d\xi = 0. \tag{4.11}$$

Hence, from (4.5) and the aforementioned arguments, we obtain

$$k_0\mu_j - \nu_j \le 0. \tag{4.12}$$

Combining with (4.4), we can deduce

$$\text{either}\quad \text{(i)}\quad \nu_j = 0 \quad \text{or}\quad \text{(ii)}\quad \nu_j \ge (k_0 C_{p^*})^{\frac{Q}{p}}.$$

Claim 2. Analyzing the concentration at ∞.

Choosing a appropriate cut-off function $\Psi \in C_c^\infty(\mathbb{H}^n)$ such that $0 \le \Psi \le 1$, $\Psi = 0$ in B_1 and $\Psi = 1$ in B_2^c. We take $R > 0$ and choose $\Psi_R(\xi) = \Psi(\delta_{1/R(\xi)})$, $\xi \in \mathbb{H}^n$, then $\{v_{n_k}\Psi_R\}$ is bounded in E and

$$\lim_{n\to\infty}\big\langle d\mathcal{J}_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k}\Psi_R)\big\rangle = 0,$$

that is,

$$\begin{aligned} & K\big(\|D_H v_{n_k}\|_p^p\big)\|D_H v_{n_k}\Psi_R\|_p^p + \int_{\mathbb{H}^n} V(\xi)|v_{n_k}|^p\Psi_R\,d\xi - \int_{\mathbb{H}^n}|v_{n_k}|^{p^*}\Psi_R\,d\xi \\ &\quad - \int_{\mathbb{H}^n} F_v(\xi,u_{n_k},v_{n_k})v_{n_k}\Psi_R\,d\xi \\ &\quad = K\big(\|D_H v_{n_k}\|_p^p\big)\bigg(\int_{\mathbb{H}^n}|\Psi_R|^p|D_H v_{n_k}|_H^p\,d\xi + \|D_H\Psi_R v_{n_k}\|_p^p\bigg) \\ &\qquad + \int_{\mathbb{H}^n} V(\xi)|v_{n_k}|^p\Psi_R\,d\xi - \int_{\mathbb{H}^n}|v_{n_k}|^{p^*}\Psi_R\,d\xi - \int_{\mathbb{H}^n} F_v(\xi,v_{n_k},v_{n_k})v_{n_k}\Psi_R(\xi)\,d\xi \\ &\quad = 0. \end{aligned} \tag{4.13}$$

In the following, we estimate each term in (4.13).

In fact, in the first integral of the right-hand side of (4.13), using the compactness result (see Proposition 2.2), we have

$$\lim_{n\to\infty}\int_{\mathbb{H}^n}|\Psi_R|^p|D_H v_{n_k}|^p\,d\xi = \int_{\mathbb{H}^n}\Psi_R\,d\mu = \int_{\mathbb{H}^n}|\Psi_R|^p|D_H v|^p\,d\xi + \mu_\infty. \tag{4.14}$$

For the second integral of the right-hand side of (4.13), it follows that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\|D_H\Psi_R v_{n_k}\|_p^p = 0. \tag{4.15}$$

Hence, we get

$$\lim_{R\to\infty}\lim_{n\to\infty} K\big(\|D_H u_{n_k}\|_p^p\big)\|D_H v_{n_k}\Psi_R\|_p^p \ge k_0\mu_\infty. \tag{4.16}$$

We also can have

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{H}^n} V(\xi)|v_{n_k}|^{p^*}\Psi_R\,d\xi = \lim_{R\to\infty}\lim_{n\to\infty}\int_{B_{2R}^c} V(\xi)|v_{n_k}|^{p^*}\Psi_R\,d\xi = 0 \tag{4.17}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{H}^n}|v_{n_k}|^{p^*}\Psi_R\,d\xi = \int_{\mathbb{H}^n}\Psi_R\,d\nu = \int_{\mathbb{H}^n}|v_0|^{p^*}\,d\xi + \nu_\infty. \tag{4.18}$$

Note that $v_{n_k} \rightharpoonup v$ weakly in E, then $\int_{\mathbb{H}^n} F_v(\xi,u,v)(v_{n_k}-v)\Psi_R\,d\xi \to 0$. As

$$\begin{aligned} \bigg|\int_{\mathbb{H}^n}\big(F_v\big(\xi,u_{n_k},v_{n_k} - F_v(\xi,u,v)\big)v_{n_k}\Psi_R\bigg| &\le c\big|F_v\big(\xi,u_{n_k},v_{n_k} - F_v(\xi,u,v)\big)\Psi_R\big|_{(p^*)'}|v_{n_k}|_{(p^*)} \\ &\le c\big|F_v\big(\xi,u_{n_k},v_{n_k} - F_v(\xi,u,v)\big)\big|_{(p^*)',\mathbb{H}^n\setminus B_R(0)}, \end{aligned}$$

by condition (F_1), for any $\varepsilon > 0$, there exists $R_1 > 0$ such that, when $R > R_1$,

$$\big|F_v\big(\xi,u_{n_k},v_{n_k} - F_v(\xi,u,v)\big)\big|_{(p^*)',\mathbb{H}^n\setminus B_R(0)} < \varepsilon \quad \text{for any } n \in \mathbb{N}.$$

Note that $\int_{\mathbb{H}^n} F_v(\xi,u,v)v\Psi_R\,d\xi \to 0$ as $R\to\infty$. Thus, we can deduce that

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{H}^n} F_v(\xi,u_{n_k},v_{n_k})v_{n_k}\Psi_R\,d\xi \\
&\quad=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{H}^n}\big(F_v(\xi,u_{n_k},v_{n_k})-F_v(\xi,u,v)\big)v_{n_k}\Psi_R \\
&\qquad+F_v(\xi,u,v)(v_{n_k}-v)\Psi_R+F_v(\xi,u,v)v\Psi_R\,d\xi \\
&\quad=\lim_{R\to\infty}\left(\limsup_{n\to\infty}\int_{\mathbb{H}^n}\big(F_v(\xi,u_{n_k},v_{n_k})-F_v(\xi,u,v)\big)v_{n_k}\Psi_R\,d\xi+F_v(\xi,u,v)v\Psi_R\,d\xi\right) \\
&\quad=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{H}^n}\big(F_v(\xi,u_{n_k},v_{n_k})-F_v(\xi,u,v)\big)v_{n_k}\Psi_R\,d\xi \\
&\qquad+\lim_{R\to\infty}\int_{\mathbb{H}^n}F_v(\xi,u,v)v\Psi_R\,d\xi \\
&\quad=0.
\end{aligned}$$

Due to Proposition 2.2, we deduce that

$$\text{either}\quad \text{(iii)}\quad \nu_\infty=0\quad \text{or}\quad \text{(iv)}\quad \nu_\infty\ge (k_0C_{p^*})^{\frac{Q}{p}}.$$

Claim 3. (ii) and (iv) cannot occur.

By contradiction, we assume that for some $j\in J$, then by condition (F_2), we have

$$\begin{aligned}
c&=\lim_{n\to\infty}\left(\mathcal{J}_{n_k}(0,v_{n_k})-\frac{1}{\tau}\langle d\mathcal{J}_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k})\rangle\right) \\
&=\lim_{n\to\infty}\left(\frac{1}{p}\mathcal{K}\big(\|D_Hu\|_p^p\big)-\frac{1}{\tau}K\big(\|D_Hu\|_p^p\big)\|D_Hu\|_p^p\right)+\left(\frac{1}{p}-\frac{1}{\tau}\right)\int_{\mathbb{H}^n}V(\xi)|v_{n_k}|^p\,d\xi \\
&\quad+\left(\frac{1}{\tau}-\frac{1}{p^*}\right)\int_{\mathbb{H}^n}|v_{n_k}|^{p^*}\,d\xi-\int_{\mathbb{H}^n}\left[F(\xi,0,v_{n_k})-\frac{1}{\tau}F_v(\xi,0,v_{n_k})v_{n_k}\right]d\xi \\
&\ge\left(\frac{1}{p}-\frac{1}{\tau}\right)k_0\|D_Hu_{n_k}\|_p^p+\left(\frac{1}{p}-\frac{1}{\tau}\right)\int_{\mathbb{H}^n}V(\xi)|v_{n_k}|^p\,d\xi+\left(\frac{1}{\tau}-\frac{1}{p^*}\right)\int_{\mathbb{H}^n}|v_{n_k}|^{p^*}\,d\xi \\
&\ge\left(\frac{1}{\tau}-\frac{1}{p^*}\right)\nu_\infty\ge\left(\frac{1}{\tau}-\frac{1}{p^*}\right)(k_0C_{p^*})^{\frac{Q}{p}}.
\end{aligned}$$

This is impossible. Consequently, $\nu_\infty=0$. Similarly, we can prove that (ii) cannot hold for each j. Hence

$$\lim_{n\to\infty}\int_{\mathbb{H}^n}v_{n_k}^{p^*}\,d\xi=\int_{\mathbb{H}^n}v_0^{p^*}\,d\xi. \tag{4.19}$$

Moreover, the Brézis–Lieb lemma implies that

$$\lim_{n\to\infty}\int_{\mathbb{H}^n}|v_{n_k}-v_0|^{p^*}\,d\xi=0. \tag{4.20}$$

Now, we define an operator as follows:

$$\langle\mathcal{L}(v),\varpi\rangle=\int_{\mathbb{H}^n}|D_Hv|_H^{p-2}|D_Hv|_H|D_H\varpi|_H\,d\xi+\int_{\mathbb{H}^n}V(\xi)|v|^{p-2}v\varpi\,d\xi$$

for any $v, \varpi \in E$. The Hölder inequality gives

$$\left|\langle \mathcal{L}(v), \varpi \rangle\right| \leq \|D_H v\|_p^{p-1} \|D_H \varpi\|_p + \|v\|_{p,V}^{p-1} \|\varpi\|_{p,V} \leq \|v\|_{HW^{1,p}}^{p-1} \|\varpi\|_{HW^{1,p}}.$$

Hence, for each $v \in E$, the linear functional $\mathcal{L}(v)$ is continuous on E. Hence, the weak convergence of v_{n_k} in E gives that

$$\lim_{k\to\infty} \langle \mathcal{L}(v_{n_k}), v_0 \rangle = \langle \mathcal{L}(v_0), v_0 \rangle \quad \text{and} \quad \lim_{k\to\infty} \langle \mathcal{L}(v_0), v_{n_k} - v_0 \rangle = 0. \tag{4.21}$$

Clearly, $\langle \mathcal{J}(v_{n_k}), v_{n_k} - v_0 \rangle \to 0$ as $n \to \infty$. Hence, by (4.21), one has

$$\lim_{k\to\infty} \langle \mathcal{L}(v_{n_k}) - \mathcal{L}(v_0), v_{n_k} - v_0 \rangle = 0. \tag{4.22}$$

By famous Simon inequalities,

$$|s-p|^p \leq \begin{cases} C_p'(|s|^{p-2}s - |t|^{p-2}t) \cdot (s-t), & \text{for } p \geq 2, \\ C_p''[(|s|^{p-2}s - |t|^{p-2}t)(s-t)]^{\frac{p}{2}} \cdot (|s|^p + |t|^p)^{\frac{2-p}{2}}, & \text{for } 1 < p < 2, \end{cases} \tag{4.23}$$

for all $s, t \in H^n$, where C_p' and C_p'' are positive constants depending only on p.

If $p > 2$, then if follows from (4.23) that

$$\|v_{n_k} - v_0\|_{HW_V^{1,p}(\mathbb{H}^n)}^p \leq C_p' \lim_{k\to\infty} \langle \mathcal{L}(v_{n_k}) - \mathcal{L}(v_0), v_{n_k} - v_0 \rangle \to 0 \tag{4.24}$$

as $n \to \infty$. Hence, $v_{n_k} \to v_0$ in E.

It remains to deal with the case $1 < p < 2$. In order to reach this aim, from (4.23) we get

$$\begin{aligned} &\|v_{n_k} - v_0\|_{HW_V^{1,p}(\mathbb{H}^n)} \\ &\quad \leq C_p''\big(\langle \mathcal{L}(v_{n_k}) - \mathcal{L}(v_0), v_{n_k} - v_0 \rangle\big)^{\frac{p}{2}} \big(\|D_H v_{n_k}\|_p^p + \|D_H v_0\|_p^p\big)^{\frac{2-p}{2}} \\ &\quad \leq C\big(\langle \mathcal{L}(v_{n_k}) - \mathcal{L}(v_0), v_{n_k} - v_0 \rangle\big)^{\frac{p}{2}} \to 0, \end{aligned} \tag{4.25}$$

as $n \to \infty$. Consequently, $v_{n_k} \to v_0$ in E.

Therefore, we can deduce that $v_{n_k} \to v_0$ in E as $n \to \infty$. The proof of convergence of $\{u_{n_k}\}$ is similar, so we omit it.

Consequently, we obtain that $\{(u_{n_k}, v_{n_k})\}$ contains a sequence converging strongly in X. □

5 Proof of Theorem 1.1

Proof Now we will prove the condition of Theorem 3.1. Set

$$\begin{aligned} &X = U \oplus V, \qquad U = E_{G_1} \times 0, \qquad V = \{0\} \times E_{G_1}, \\ &Y_0 = 0 \times E_{G_1}^{m\perp}, \qquad Y_1 = \{0\} \times E_{G_1}^{(k)}, \end{aligned}$$

where m and k are to be determined. It is obvious that Y_0, Y_1 are G-invariant and $\operatorname{codim}_V Y_0 = m$, $\dim Y_1 = k$. Clearly, (B_1), (B_2), (B_4) in Theorem 3.1 are satisfied. Set

$V_j = E_{G_1}^{(j)} = \operatorname{span}\{e_1, e_2, \ldots, e_j\}$, then (B_3) is also satisfied. By $\dim \widetilde{Y}_0 = 1 < k = \dim Y_1$, (B_5) holds. In the following, we prove the condition (B_7). Since $\operatorname{Fix}(G) \cap V = 0$, (a) of (B_7) are satisfied. It remains to verify (b), (c) of (B_7).

(i) If $(0, v) \in Y_0 \cap S_{\varrho m}$ (where ϱ_m is to be determined), then by (F_1) and (F_2),

$$\begin{aligned}
\mathcal{J}(0,v) &= \frac{1}{p}\mathcal{K}\big(\|D_H v\|_p^p\big) + \frac{1}{p}\int_{\mathbb{H}^n} V(\xi)|v|^p\,d\xi - \frac{1}{p^*}\int_{\mathbb{H}^n} |v|^{p^*}\,d\xi - \int_{\mathbb{H}^n} F(\xi,0,v)\,d\xi \\
&\geq \frac{k_0}{p}\|D_H v\|_p^p + \frac{1}{p}\int_{\mathbb{H}^n} V(\xi)|v|^p\,d\xi - \frac{1}{p^*}\int_{\mathbb{H}^n} |v|^{p^*}\,d\xi - \int_{\mathbb{H}^n} F(\xi,0,v)\,d\xi \\
&\geq \min\left\{\frac{k_0}{p}, \frac{1}{p}\right\}\|v\|^p - \frac{c}{p^*}\|v\|^{p^*} - c\|v\|^q,
\end{aligned}$$

since $p < q < p^*$, there exists $\varrho > 0$ such that $\mathcal{J}(0,v) \geq \alpha$ for every $\|v\| = \varrho$, that is, (b) of (B_7) holds.

(ii) From (H_1), we have

$$\mathcal{J}(u,0) = -\frac{1}{p}\mathcal{K}\big(\|D_H u\|_p^p\big) - \frac{1}{p}\int_{\mathbb{H}^n} V(\xi)|u|^p\,d\xi - \frac{1}{p^*}\int_{\mathbb{H}^n} |u|^{p^*}\,d\xi - \int_{\mathbb{H}^n} F(\xi,u,0)\,d\xi \leq 0.$$

Therefore, we choose α such that

$$\alpha > \sup_{u \in E_{G_1}} \mathcal{J}(u,0).$$

For each $(u,v) \in U \oplus Y_1$, we get

$$\begin{aligned}
\mathcal{J}(u,v) &= -\frac{1}{p}\mathcal{K}\big(\|D_H u\|_p^p\big) - \frac{1}{p}\int_{\mathbb{H}^n} V(\xi)|u|^p\,d\xi + \frac{1}{p}\mathcal{K}\big(\|D_H v\|_p^p\big) + \frac{1}{p}\int_{\mathbb{H}^n} V(\xi)|v|^p\,d\xi \\
&\quad - \frac{1}{p^*}\int_{\mathbb{H}^n} |u|^{p^*}\,d\xi - \frac{1}{p^*}\int_{\mathbb{H}^n} |v|^{p^*}\,d\xi - \int_{\mathbb{H}^n} F(\xi,u,v)\,d\xi \\
&\leq \frac{1}{p}\|v\|^p - \frac{1}{p^*}|v|_{p^*}^{p^*} + \alpha.
\end{aligned}$$

Since all norms are equivalent on the finite dimensional space Y_1, we can choose $k > m$ and $\beta_k > \alpha_m$ such that

$$\mathcal{J}_{U \oplus Y_1} \leq \beta_k,$$

hence, we have (c) in (B_7). Using Lemma 4.1, for any $[\alpha_m, \beta_k]$, $\mathcal{J}(u,v)$ satisfies the condition of $(PS)_c^*$, then (B_6) in Theorem 3.1 is satisfied. Therefore, according to Theorem 3.1,

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{z \in A} \mathcal{J}(u,v), \qquad -k+1 \leq j \leq -m, \alpha_m \leq c_j \leq \beta_k$$

are critical values of $\mathcal{J}$. Letting $m \to \infty$, we can get an unbounded sequence of critical values c_j. And since the functional $\mathcal{J}$ is even, we have two critical points $\pm u_j$ of $\mathcal{J}$ corresponding to c_j. $\square$

Funding
X. Sun was supported by the Graduate Scientific Research Project of Changchun Normal University (SGSRPCNU [2022], Grant No. 056). Y. Song was supported by the National Natural Science Foundation of China (Grant No. 12001061), the Research Foundation of Department of Education of Jilin Province (Grant No. JJKH20220822KJ), and the Natural Science Foundation of Jilin Province (Grant No. YDZJ202201ZYTS582).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
X. Sun and Sh. Bai wrote the main manuscript text. Both authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 10 June 2022 Accepted: 4 October 2022 Published online: 11 October 2022

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