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A class of biharmonic nonlocal quasilinear systems consisting of Leray–Lions type operators with Hardy potentials

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Abstract
We study the existence of multiple solutions to a nonlocal system involving fourth order Leray–Lions type operators along with singular terms under Navier boundary conditions. The method is based on the variational methods.

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Keywords: Fourth order Leray–Lions type operator; Variable exponent Sobolev space; Singular term

1 Introduction

In this paper, we consider the following biharmonic system:

$$\begin{cases} \Delta(a_i(x,\Delta u_i)) + b_i(x)\frac{|u_i|^{s_i-2}u_i}{|x|^{2s_i}} = \lambda H_{u_i}(x,u_1,\dots,u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary, $i = 1,\dots,n$, and the potentials

$$a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$$

for $i = 1,\dots,n$ are Carathéodory functions satisfying the following conditions:

(A1) $a_i(x,0) = 0$, for a.e. $x \in \Omega$.

(A2) There exists $C_i > 0$ such that

$$|a_i(x,t)| \leq C_i\bigl(1 + |t|^{p_i(x)-1}\bigr)$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, where $p_i \in C(\overline{\Omega})$ with

$$\max\left\{2, \frac{N}{2}\right\} < \inf_{x\in\Omega} p_i(x) \leq p_i(x) \leq \sup_{x\in\Omega} p_i(x) < \infty.$$

(A3) For all $s, t \in \mathbb{R}$,

$$\big(a_i(x,t) - a_i(x,s)\big)(t-s) \geq 0$$

for a.e. $x \in \Omega$.

(A4) There exists $c_i \geq 1$ such that

$$c_i|t|^{p_i(x)} \leq \min\big\{a_i(x,t)t, p_i(x)A_i(x,t)\big\}$$

for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$, where

$$A_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$$

is the antiderivative of a_i, that is,

$$A_i(x,t) := \int_0^t a_i(x,s)\, ds.$$

We assume that $1 < s_i < \frac{N}{2}$, the nonnegative functions b_i belong to $L^\infty(\Omega)$ for $i = 1, \dots, n$, λ is a positive parameter, and

$$H : \Omega \times \mathbb{R}^n \to \mathbb{R}$$

is a measurable function with respect to $x \in \Omega$ for each $(t_1, \dots, t_n) \in \mathbb{R}^n$ and is C^1 with respect to $(t_1, \dots, t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$. By H_{u_i} we denote the partial derivative of H with respect to u_i.

Biharmonic-type problems are used to describe a large class of physical phenomena such as micro-electro-mechanical systems, phase field models of multiphase systems, thin film theory, thin plate theory, surface diffusion on solids, interface dynamics, and also flow in Hele–Shaw cells. That is why many authors have looked for solutions of elliptic equations involving such operators.

The fourth order Leray–Lions problem with Navier boundary conditions

$$\begin{cases} \Delta(a(x,\Delta u)) = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

is studied in [9], where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$, $\Delta(a(x,\Delta u))$ is the fourth-order Leray–Lions operator, a satisfies a growth condition depending on p and some completion conditions, $\lambda > 0$ is a parameter, and V is a function in a generalized Lebesgue space $L^{s(x)}(\Omega)$. The functions $p, q, s \in C(\overline{\Omega})$ satisfy the inequalities

$$1 < \min_{x\in\overline{\Omega}} q(x) \leq \max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) \leq \max_{x\in\overline{\Omega}} p(x) \leq \frac{N}{2} < s(x)$$

for all $x \in \Omega$. In a particular case where $a(x,t) = |t|^{p(x)-2}t$, Boureanu et al. [2] studied the problem

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)|u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u \equiv \text{constant} & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial}{\partial\nu}(|\Delta u|^{p(x)-2}\Delta u)\, dS = 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, λ is a positive parameter, and $a \in L^\infty(\Omega)$.

Recently, the study of the biharmonic problems in various spaces is an interesting problem. For example, the existence of at least one positive radial solution of the weighted p-biharmonic problem

$$\begin{aligned} &\Delta_{\mathbb{H}^n}\big(w(\xi)|\Delta_{\mathbb{H}^n}u|^{p-2}\Delta_{\mathbb{H}^n}u\big) + R(\xi)w(\xi)|u|^{p-2}u \\ &\quad = \sum_{i=1}^{m} a_i\big(|\xi|_{\mathbb{H}^n}\big)|u|^{q_i-2}u - \sum_{j=1}^{k} b_j\big(|\xi|_{\mathbb{H}^n}\big)|u|^{r_j-2}u \end{aligned}$$

with Navier boundary conditions on a Korányi ball has been proved [21], where $w \in A_s$ is a Muckenhoupt weight function, and $\Delta^2_{\mathbb{H}^n,p}$ is the Heisenberg p-biharmonic operator.

Motivated by the works mentioned, we study the existence of multiple weak solutions for problem (1.1) consisting of fourth order Leray–Lions type operators and singular terms.

Before ending this section, we state the definition of a weak solution for problem (1.1) and recall the critical point theorem of [1].

Definition 1.1 We say that

$$u = (u_1, \dots, u_n) \in \prod_{i=1}^{n}\big(W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega)\big) \setminus \{0\}$$

is a weak solution of problem (1.1) if $u_i = 0$ on $\partial\Omega$ for each $1 \le i \le n$ and the following integral equality is true:

$$\begin{aligned} &\sum_{i=1}^{n}\int_\Omega a_i(x,\Delta u_i)\Delta v_i\, dx + \sum_{i=1}^{n}\int_\Omega b_i(x)\frac{|u_i|^{s_i-2}u_i v_i}{|x|^{2s_i}}\, dx \\ &\quad - \lambda \sum_{i=1}^{n}\int_\Omega H_{u_i}(x,u_1,\dots,u_n)v_i\, dx = 0 \end{aligned}$$

for every

$$v = (v_1, \dots, v_n) \in \prod_{i=1}^{n}\big(W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega)\big).$$

Theorem 1.1 *Let X be a reflexive real Banach space, and let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux-differentiable, and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*. Let $\Psi : X \to \mathbb{R}$ be a* *continuously Gâteaux-differentiable functional whose Gâteaux derivative is compact and such that*

$$\inf_{x\in X} \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ *and* $\bar{x} \in X$, *with* $r < \Phi(\bar{x})$ *such that*

(i) $\frac{\sup_{\Phi(x)<r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(ii) *For each*

$$\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)<r} \Psi(x)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ *is coercive*.

Then for each $\lambda \in \Lambda_r$, *the functional* $I_\lambda = \Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

The rest of the paper is organized as follows. in Sect. 2, we present a brief survey of notions and results related to our problem. In Sect. 3, we state the main result of the paper and prove it by variational techniques and applying Theorem 1.1 on three critical points.

2 Variational framework

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary. We suppose that $1 < s_i < \frac{N}{2}$ and $p_i \in C(\overline{\Omega})$, $i = 1, \dots, n$, satisfy the following condition:

$$\max\left\{2, \frac{N}{2}\right\} < p_i^- := \inf_{x\in\Omega} p_i(x) \leq p(x) \leq p_i^+ := \sup_{x\in\Omega} p_i(x) < +\infty. \tag{2.1}$$

The variable exponent Lebesgue space $L^{p_i(x)}(\Omega)$, $i = 1, \dots, n$, is defined as

$$L^{p_i(x)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} : u \text{ is measurable and } \int_\Omega \left|u(x)\right|^{p_i(x)} dx < \infty \right\},$$

with the Luxemburg norm

$$|u|_{p_i(x)} := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p_i(x)} dx \leq 1 \right\}.$$

Notice that if $q(\cdot) \equiv q$, $q \in \{s_i : i = 1, \dots, n\} \cup \{1\}$, then this norm is equal to the standard norm on $L^q(\Omega)$,

$$|u|_q = \left(\int_\Omega |u|^q \, dx \right)^{\frac{1}{q}}.$$

It is well known that for any $u \in L^{p_i(x)}(\Omega)$ and $v \in L^{p_i'(x)}(\Omega)$, where $L^{p_i'(x)}(\Omega)$ is the conjugate space of $L^{p_i(x)}(\Omega)$, we have the Hölder-type inequality

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{p_i^-} + \frac{1}{p_i'^-} \right) |u|_{p_i(x)} |v|_{p_i'(x)}.$$

The following theorem is [11, Theorem 2.8].

Theorem 2.1 *Assume that* Ω *is a bounded and smooth set in* $\mathbb{R}^N$ *and that* $p, q \in C_+(\overline{\Omega})$. *Then*

$$L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

if and only if $q(x) \leq p(x)$ *a.e.* $x \in \Omega$; *moreover, there exists a constant* M_q *such that*

$$|u|_{q(x)} \leq M_q |u|_{p(x)}. \tag{2.2}$$

Following [13], for any $\kappa > 0$, we set

$$\kappa^{\check{r}} := \begin{cases} \kappa^{r^+}, & \kappa < 1, \\ \kappa^{r^-}, & \kappa \geq 1, \end{cases}$$

and

$$\kappa^{\hat{r}} := \begin{cases} \kappa^{r^-}, & \kappa < 1, \\ \kappa^{r^+}, & \kappa \geq 1, \end{cases}$$

for $r \in \{p_i : i = 1, \dots, n\}$. We rewrite the well-known [8, Proposition 2.7] as follows.

Proposition 2.1 *For each* $u \in L^{p(x)}(\Omega)$, *we have*

$$|u|_{p(x)}^{\check{p}} \leq \int_\Omega \left|u(x)\right|^{p(x)} dx \leq |u|_{p(x)}^{\hat{p}}.$$

For $m = 1, 2$ and $p \in \{p_i : i = 1, \dots, n\}$, by $W^{m,p(x)}(\Omega)$ we denote the variable exponent Sobolev space, that is,

$$W^{m,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq m\right\}$$

endowed with the norm

$$\|u\|_{m,p(x)} := \sum_{|\alpha| \leq m} \left|D^\alpha u\right|_{p(x)}.$$

Let us point out that the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ are separable, reflexive, and uniform convex Banach spaces [4]. Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. We set

$$Y := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$$

for $p \in \{p_i : i = 1, \dots, n\}$. It is a reflexive Banach space respect to the norm

$$\begin{aligned} \|u\|_Y &:= \|u\|_{W^{2,p(x)}(\Omega)} + \|u\|_{W_0^{1,p(x)}(\Omega)} \\ &= \left|u\right|_{p(x)} + \left||\nabla u|\right|_{p(x)} + |\Delta u|_{p(x)}, \end{aligned}$$

where

$$\nabla u = \left(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_N}(x)\right)$$

is the gradient of u at $x = (x_1, \dots, x_n)$, $\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$ is the Laplace operator, and $|\nabla u| = (\sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^2)^{\frac{1}{2}}$,.

Using the Poincaré inequality and [22], the norms $\|\cdot\|_Y$ and $|\Delta(\cdot)|_{p(x)}$ are equivalent on Y, where

$$|\Delta u|_{p(x)} := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{\Delta u}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

We have the following lemma by Theorem 2.1.

Lemma 2.1 *If $p(x) \le q(x)$ a.e. $x \in \Omega$, then*

$$W^{m,q(x)}(\Omega) \hookrightarrow W^{m,p(x)}(\Omega). \tag{2.3}$$

In a particular case, for p_i, $i = 1, \dots, n$, with condition (2.1),

$$W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega) \hookrightarrow W^{2,p_i^-}(\Omega) \cap W_0^{1,p_i^-}(\Omega)$$

is embedded continuously, and since $p_i^- > \frac{N}{2}$, we have the following compact embedding

$$W^{2,p_i^-}(\Omega) \cap W_0^{1,p_i^-}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega}).$$

Then

$$W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega}).$$

So, in particular, there exist positive constants $k_i > 0$, $i = 1, \dots, n$, such that

$$|u|_\infty \le k_i |\Delta u|_{p_i(x)} \tag{2.4}$$

for each $u \in W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega)$, where $|u|_\infty := \sup_{x\in\Omega} |u(x)|$.

Proposition 2.1 implies the following lemma.

Lemma 2.2 *For each $u \in Y$, we have*

$$|\Delta u|_{p(x)}^{\check{p}} \le \rho(u) := \int_\Omega \left|\Delta u(x)\right|^{p(x)} dx \le |\Delta u|_{p(x)}^{\hat{p}}.$$

Now we recall the classical Hardy–Rellich inequality mentioned in [3].

Lemma 2.3 *Let $1 < s < \frac{N}{2}$. Then for $u \in W_0^{1,s}(\Omega) \cap W^{2,s}(\Omega)$, we have*

$$\int_\Omega \frac{|u(x)|^s}{|x|^{2s}}\,dx \le \frac{1}{\mathcal{H}} \int_\Omega |\Delta u(x)|^s\,dx,$$

where $\mathcal{H} := (\frac{N(s-1)(N-2p)}{s^2})^s$.

Lemma 2.4 *Let $1 < s_i < \frac{N}{2}$, and let $p_i \in C(\Omega)$ be as in relation (2.1) for $i = 1,\dots,n$. Then there exists κ such that*

$$\int_\Omega \frac{|u(x)|^{s_i}}{|x|^{2s_i}}\,dx \le \frac{\kappa}{\mathcal{H}} |\Delta u|_{p_i(x)}^{s_i}$$

for $u \in W_0^{1,p_i(x)}(\Omega) \cap W^{2,p_i(x)}(\Omega)$, where $\mathcal{H}$ is as in Lemma 2.3.

Proof Since $s_i < p_i(x)$ a.e. in Ω for each $i = 1,\dots,n$, according to relation (2.3), we have

$$W_0^{1,p_i(x)}(\Omega) \cap W^{2,p_i(x)}(\Omega) \hookrightarrow W_0^{1,s_i}(\Omega) \cap W^{2,s_i}(\Omega).$$

Moreover, there exist constants κ_{s_i} such that

$$|\Delta u|_{s_i} \le \kappa_{s_i} |\Delta u|_{p_i(x)}.$$

From Lemma 2.3 we get

$$\int_\Omega \frac{|u(x)|^{s_i}}{|x|^{2s_i}}\,dx \le \frac{1}{\mathcal{H}} \int_\Omega \bigl|\Delta u(x)\bigr|^{s_i}\,dx$$

for $u \in W_0^{1,s_i}(\Omega) \cap W^{2,s_i}(\Omega)$. Then we deduce that

$$\int_\Omega \frac{|u(x)|^{s_i}}{|x|^{2s_i}}\,dx \le \frac{\kappa_{s_i}^{s_i}}{\mathcal{H}} |\Delta u|_{p_i(x)}^{s_i}.$$

It suffices to set $\kappa = \max_{1\le i\le n} \kappa_{s_i}^{s_i}$. □

Lemma 2.5 *Assume that conditions $(A1)$–$(A4)$ hold. Then for $i = 1,\dots,n$, we have*

(I) *$A_i(x,t)$ is a C^1-Carathéodory function, i.e., for every $t \in \mathbb{R}$,*

$$A_i(\cdot,t) : \Omega \to \mathbb{R}$$

is measurable, and for a.e. $x \in \Omega$, $A_i(x,\cdot)$ is of class C^1.

(II) *There exist constants C_i', $i = 1,\dots,n$, such that*

$$\frac{c_i}{p_i(x)} |t|^{p_i(x)} \le \bigl|A_i(x,t)\bigr| \le C_i'\bigl(|t| + |t|^{p_i(x)}\bigr)$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, where the constants c_i, $i = 1,\dots,n$, are as in condition $(A4)$.

In what follows, we set

$$X := \prod_{i=1}^{n} \left(W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega) \right)$$

endowed with the norm

$$\|u\| = \left\|(u_1, \ldots, u_n)\right\| = \sum_{i=1}^{n} |\Delta u_i|_{p_i(x)}$$

for $u = (u_1, \ldots, u_n) \in X$. From Remark 2.1 we conclude that the embedding

$$X \hookrightarrow C^0(\overline{\Omega}) \times \cdots \times C^0(\overline{\Omega})$$

is compact, and if we put

$$K := \max_{1 \le i \le n} k_i,$$

where $k_i, 1 \le i \le n$, are as in relation (2.4), then it is clear that $K > 0$ and

$$|u_i|_\infty \le K |\Delta u_i|_{p_i(x)}, \quad i = 1, \ldots, n. \tag{2.5}$$

We define the functional $\Phi : X \longrightarrow \mathbb{R}$ by

$$\Phi(u_1, \ldots, u_n) := \sum_{i=1}^{n} \int_\Omega A_i(x, \Delta u_i)\, dx + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{|u_i(x)|^{s_i}}{s_i |x|^{2s_i}}\, dx.$$

Lemma 2.6 *There exists a positive constant $\hat{C}$ such that*

$$\frac{c_i}{p_i^+} |\Delta u_i|_{p_i(x)}^{\check{p}_i} \le \Phi(u_1, \ldots, u_n) \le \hat{C} \sum_{i=1}^{n} \left(|\Delta u_i|_{p_i(x)}^{\hat{p}_i} + |\Delta u_i|_{p_i(x)}^{s_i} \right)$$

for all $1 \le i \le n$ and $u = (u_1, \ldots, u_n) \in X$.

Proof By (2.2) and Lemma 2.5, for every $1 \le i \le n$, we have the estimate

$$\begin{aligned}
\frac{c_i}{p_i^+} |\Delta u_i|_{p_i(x)}^{\check{p}_i}\, dx &\le \frac{c_i}{p_i^+} \int_\Omega |\Delta u_i|^{p_i(x)}\, dx \\
&\le \sum_{i=1}^{n} \frac{c_i}{p_i^+} \int_\Omega |\Delta u_i|^{p_i(x)}\, dx \\
&\le \Phi(u_1, \ldots, u_n) \\
&= \sum_{i=1}^{n} \int_\Omega A_i(x, \Delta u_i)\, dx + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{|u_i|^{s_i}}{s_i |x|^{2s_i}}\, dx \\
&\le \sum_{i=1}^{n} C_i' \int_\Omega \left(|\Delta u_i| + |\Delta u_i|^{p_i(x)} \right) dx + \frac{\kappa}{\mathcal{H}} \sum_{i=1}^{n} \frac{|b_i|_\infty}{s_i} |\Delta u_i|_{p_i(x)}^{s_i}
\end{aligned}$$

$$\leq \sum_{i=1}^{n} C_i'(M_1+1)|\Delta u_i|_{p_i(x)}^{\hat{p}_i} + \frac{\kappa}{\mathcal{H}} \sum_{i=1}^{n} |b_i|_\infty |\Delta u_i|_{p_i(x)}^{s_i}.$$

It suffices to set $\hat{C} = (M_1+1)\max_{1\leq i\leq n} C_i' + \frac{\kappa}{\mathcal{H}} \max_{1\leq i\leq n} |b_i|_\infty$. □

Remark 2.1 Lemma 2.6 ensures that Φ is coercive.

Proof Let $u = (u_1,\dots,u_n) \in X$ and $\|u\| \to \infty$. By the definition of $\|\cdot\|$ there exists $1 \leq i_0 \leq n$ such that $|\Delta u_{i_0}|_{p_{i_0}(x)} \to \infty$. Then Lemma 2.6 implies that $\Phi(u) \to \infty$. □

Furthermore, Φ is sequentially weakly lower semicontinuous, and it is known that Φ is continuously Gâteaux-differentiable functional. Moreover,

$$\Phi'(u_1,\dots,u_n)(v_1,\dots,v_n) = \sum_{i=1}^{n} \int_\Omega \left(a_i(x,\Delta u_i)\Delta v_i + b_i(x)\frac{|u_i|^{s_i-2}u_i v_i}{|x|^{2s_i}} \right) dx$$

for each $(v_1,\dots,v_n) \in X$.

Now suppose that the function

$$H : \Omega \times \mathbb{R}^n \to \mathbb{R}$$

is a measurable function with respect to $x \in \Omega$ for each $(t_1,\dots,t_n) \in \mathbb{R}^n$ and is C^1 with respect to $(t_1,\dots,t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$. By H_{u_i} we denote the partial derivative of H with respect to u_i. We define $\Psi : \mathbb{R}^n \to \mathbb{R}$ by

$$\Psi(u_1,\dots,u_n) := \int_\Omega H(x,u_1,\dots,u_n)\,dx.$$

The functional Ψ is well defined, continuously Gâteaux-differentiable with compact derivative, whose Gâteaux derivative at a point $u = (u_1,\dots,u_n) \in X$ is

$$\Psi'(u_1,\dots,u_n)(v_1,\dots,v_n) = \sum_{i=1}^{n} \int_\Omega H_{u_i}\big(x,u_1(x),\dots,u_n(x)\big)v_i(x)\,dx$$

for every $(v_1,\dots,v_n) \in X$. Notice that the energy functional corresponding to the problem is

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for each $u = (u_1,\dots,u_n)$, or, equivalently, weak solutions of (1.1) are exactly the critical points of I_λ. We set

$$\delta(x) := \sup\big\{\delta > 0 : B(x,\delta) \subseteq \Omega\big\}$$

and define

$$R := \sup_{x\in\Omega} \delta(x). \tag{2.6}$$

Obviously, there exists $x^0 = (x_1^0, \dots, x_N^0) \in \Omega$ such that

$$B(x^0, R) \subseteq \Omega.$$

In the next section, we prove the main result of the paper.

3 Three distinct weak solutions

Here we prove the existence of at least three distinct weak solutions to problem (1.1) by Theorem 3.1. The main result of the paper is the following:.

Theorem 3.1 *Assume that conditions* $(A1)$–$(A4)$ *hold and* $H : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ *satisfies the following conditions*:

$(H1)$ $H(x, 0, \dots, 0) = 0$ *for a.e.* $x \in \Omega$;

$(H2)$ *There exist* $\eta \in L^1(\Omega)$ *and* n *positive continuous functions* γ_i, $1 \le i \le n$, *with* $\gamma_i(x) < p_i(x)$ *a.e. in* Ω *such that*

$$0 \le H(x, u_1, \dots, u_n) \le \eta(x)\left(1 + \sum_{i=1}^{n} |u_i|^{\gamma_i(x)}\right);$$

$(H3)$ *There exist* $r > 0$, $\delta > 0$, *and* $1 \le i_* \le n$ *such that*

$$\frac{c_{i_*}}{p_{i_*}^+}\left(\frac{2\delta N}{R^2 - (\frac{R}{2})^2}\right)^{\check{p_{i_*}}} m\left(R^N - \left(\frac{R}{2}\right)^N\right) > r,$$

where $m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ *is the measure of unit ball of* $\mathbb{R}^N$, *and* Γ *is the gamma function. Suppose that*

$$A_r < B_\delta, \tag{3.1}$$

where

$$A_r := \frac{|\eta|_1}{r}\left(1 + \sum_{i=1}^{n} K^{\hat{\gamma}_i}\left(\frac{p_i^+}{c_i} r\right)^{\frac{\hat{\gamma}_i}{\check{p}_i}}\right),$$

and

$$B_\delta := \frac{\sum_{i=1}^{n} \inf_{x \in \Omega} F(x, \delta, \dots, \delta)}{\hat{C} \sum_{i=1}^{n} ((\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^{\hat{p}_i} + (\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^{s_i})(2^N - 1)}.$$

Then for each

$$\lambda \in \Lambda_{r,\delta} := \left(\frac{1}{B_\delta}, \frac{1}{A_r}\right),$$

problem (1.1) *possesses at least three distinct weak solutions in* X.

Proof We apply Theorem 1.1. According to the previous section, the space

$$X=\prod_{i=1}^{n}\bigl(W^{2,p_i(x)}(\Omega)\cap W_0^{1,p_i(x)}(\Omega)\bigr)$$

and the functionals $\Phi,\Psi:X\to\mathbb{R}$ defined as above satisfy the regularity assumptions of Theorem 1.1. From the definition of Φ and Ψ and condition $(H1)$ it is clear that

$$\inf_{x\in X}\Phi=\Phi(0)=\Psi(0)=0.$$

Fix $\delta>0$ and R defined as in (2.6). We denote by w the function on the space $W^{2,p_i(x)}(\Omega)\cap W_0^{1,p_i(x)}(\Omega)$, $1\le i\le n$, defined by

$$w(x):=\begin{cases}0, & x\in\Omega\setminus B(x^0,R),\\ \delta, & x\in B(x^0,\frac{R}{2}),\\ \frac{\delta}{R^2-(\frac{R}{2})^2}(R^2-\sum_{i=1}^{N}(x_i-x_i^0)^2), & x\in B(x^0,R)\setminus B(x^0,\frac{R}{2}),\end{cases}$$

where $x=(x_1,\dots,x_N)\in\Omega$. Then

$$\sum_{i=1}^{N}\frac{\partial^2 w}{\partial x_i^2}(x)=\begin{cases}0 & x\in(\Omega\setminus B(x^0,R))\cup B(x^0,\frac{R}{2}),\\ -\frac{2\delta N}{R^2-(\frac{R}{2})^2} & x\in B(x_0,R)\setminus B(x^0,\frac{R}{2}).\end{cases}$$

By Lemma 2.6, for $1\le i_*\le n$, we have

$$\begin{aligned}&\frac{c_{i_*}}{p_{i_*}^+}\biggl(\frac{2\delta N}{R^2-(\frac{R}{2})^2}\biggr)^{\check{p_{i_*}}}m\biggl(R^N-\biggl(\frac{R}{2}\biggr)^N\biggr)\\ &\quad<\Phi(w,\dots,w)\\ &\quad\le\hat{C}\sum_{i=1}^{n}\biggl(\biggl(\frac{2\delta N}{R^2-(\frac{R}{2})^2}\biggr)^{\hat{p}_i}+\biggl(\frac{2\delta N}{R^2-(\frac{R}{2})^2}\biggr)^{s_i}\biggr)m\biggl(R^N-\biggl(\frac{R}{2}\biggr)^N\biggr).\end{aligned}$$

Then by assumption $(H3)$ we have $\Phi(w,\dots,w)>r$. On the other hand, we have

$$\begin{aligned}\Psi(w,\dots,w)&\ge\sum_{i=1}^{n}\int_{B(x_0,\frac{R}{2})}H(x,w,\dots,w)\,dx\\ &\ge\sum_{i=1}^{n}\inf_{x\in\Omega}H(x,\delta,\dots,\delta)m\biggl(\frac{R}{2}\biggr)^N,\end{aligned}$$

where m is the measure of the unit ball of $\mathbb{R}^N$, and so

$$\begin{aligned}\frac{\Psi(w,\dots,w)}{\Phi(w,\dots,w)}&\ge\frac{\sum_{i=1}^{n}\inf_{x\in\Omega}H(x,\delta,\dots,\delta)m(\frac{R}{2})^N}{\hat{C}\sum_{i=1}^{n}((\frac{2\delta N}{R^2-(\frac{R}{2})^2})^{\hat{p}_i}+(\frac{2\delta N}{R^2-(\frac{R}{2})^2})^{s_i})m(R^N-(\frac{R}{2})^N)}\\ &=\frac{\sum_{i=1}^{n}\inf_{x\in\Omega}H(x,\delta,\dots,\delta)}{\hat{C}\sum_{i=1}^{n}((\frac{2\delta N}{R^2-(\frac{R}{2})^2})^{\hat{p}_i}+(\frac{2\delta N}{R^2-(\frac{R}{2})^2})^{s_i})(2^N-1)}=B_\delta.\end{aligned}\tag{3.2}$$

Now let $u = (u_1,\dots,u_n) \in \Phi^{-1}(-\infty,r)$. From Lemma 2.6 we get

$$|\Delta u|_{p_i(x)} \le \left(\frac{p_i^+}{c_i}\Phi(u_1,\dots,u_n)\right)^{\frac{1}{\check{p}_i}} \le \left(\frac{p_i^+}{c_i}r\right)^{\frac{1}{\check{p}_i}} \tag{3.3}$$

for each $i = 1,\dots,n$. Then for every $u = (u_1,\dots,u_n) \in \Phi^{-1}(-\infty,r)$, using condition $(H2)$, the Hölder inequality, and (2.2), we have

$$\begin{aligned}\int_\Omega H(x,u_1,\dots,u_n)\,dx &\le \int_\Omega \sup_{u\in\Phi^{-1}(-\infty,r)} H(x,u_1,\dots,u_n)\,dx \\ &\le \int_\Omega \eta(x)\left(1+\sum_{i=1}^n |u_i|^{\gamma_i(x)}\right)dx \\ &\le |\eta|_1\left(1+\sum_{i=1}^n |u_i|_\infty^{\hat{\gamma}_i}\right) \\ &\le |\eta|_1\left(1+\sum_{i=1}^n K^{\hat{\gamma}_i}|\Delta u_i|_{p_i(x)}^{\hat{\gamma}_i}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{r}\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u) &= \frac{1}{r}\sup_{u\in\Phi^{-1}(-\infty,r)}\int_\Omega H(x,u_1,\dots,u_n)\,dx \\ &\le \frac{|\eta|_1}{r}\left(1+\sum_{i=1}^n K^{\hat{\gamma}_i}\left(\frac{p_i^+}{c_i}r\right)^{\frac{\hat{\gamma}_i}{\check{p}_i}}\right) = A_r.\end{aligned} \tag{3.4}$$

From assumption (3.1) and relations (3.2) and (3.4) we have

$$\frac{1}{r}\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u_i) < \frac{\Psi(w,\dots,w)}{\Phi(w,\dots,w)},$$

and so condition (i) of Theorem 1.1 is verified. Now we prove that for each $\lambda > 0$, I_λ is coercive.

With the same arguments as used before, we have

$$\Psi(u) = \int_\Omega H(x,u_1,\dots,u_n)\,dx \le |\eta|_1\left(1+\sum_{i=1}^n K^{\hat{\gamma}_i}|\Delta u_i|_{p_i(x)}^{\hat{\gamma}_i}\right).$$

The last inequality and Lemma 2.6 lead to

$$I_\lambda(u) \ge \frac{c_i}{p_i^+}|\Delta u_i|_{p_i(x)}^{\check{p}_i} - \lambda|\eta|_1\left(1+\sum_{i=1}^n K^{\hat{\gamma}_i}|\Delta u_i|_{p_i(x)}^{\hat{\gamma}_i}\right)$$

for each $i = 1,\dots,n$. Now suppose that $u \in X$ and $\|u\| \to \infty$. So, there exists $1 \le i_0 \le n$ such that $|\Delta u_{i_0}|_{p_{i_0}(x)} \to \infty$. Since according to our assumptions, $\gamma_{i_0}(x) < p_{i_0}(x)$ a.e. in Ω, the coercivity of I_λ is obtained.

Taking into account that

$$\Lambda_{\delta,r} := \left(\frac{1}{B_\delta}, \frac{1}{A_r}\right) \subseteq \left(\frac{\Phi(w,\dots,w)}{\Psi(w,\dots,w)}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u_i)}\right),$$

Theorem 1.1 ensures that for each $\lambda \in \Lambda_{r,\delta}$, the functional I_λ admits at least three critical points in X, which are weak solutions of problem (1.1). □

Remark 3.1 An interesting problem is to probe the existence and multiplicity of solutions of this system under Steklov boundary conditions [10] or in the Heisenberg–Sobolev and Orlicz–Sobolev spaces. The interested reader can read the details on these spaces in [5–7, 12–21] and references therein.

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