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Nonstandard competing anisotropic (p, q)-Laplacians with convolution

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Abstract

A competing anisotropic (p,q)-Laplacian

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} - \mu\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\right)\frac{\partial u}{\partial x_i} = f(x, \phi \star u, \nabla(\phi \star u))$$

as a nonstandard Dirichlet problem with convolutions on a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$ is considered. Assume $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and $\phi \in L^1(\mathbb{R}^N)$. If $\mu > 0$, the existence of a generalized solution is proved. By the Galerkin basis for the space, a sequence that converges strongly to the solution is constructed. If $\mu \leq 0$, it is proved that any generalized solution is a weak solution.

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1 Introduction

The (p,q)-Laplacian comes from a general reaction-diffusion system that has a wide spectrum of applications in physics and related sciences such as biophysics, plasma physics, solid-state physics, fractional quantum mechanics in the study of particles on stochastic fields, fractional superdiffusion and fractional white-noise limit, etc. (see [1, 5–7, 23–25, 31, 32] and the references therein).

Recently, Motreanu [20] proved the existence of solutions (generalized and weak) for

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u - \mu|\nabla u|^{q-2}\nabla u) = f(x, \rho \star u, \nabla(\rho \star u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

under suitable condition of f and ρ, where he overcame the lack of ellipticity.

Here, with the inspiration of [20], the multiplicity of nontrivial solutions for the nonstandard Dirichlet problem with an anisotropic competing (p,q)-Laplacian

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i-2} - \mu|\frac{\partial u}{\partial x_i}|^{q_i-2})\frac{\partial u}{\partial x_i} = f(x, \phi \star u, \nabla(\phi \star u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

is proved, where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, with a Lipschitz boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, $\phi \in L^1(\mathbb{R}^N)$, $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, and the convolution $\phi \star u(x)$ is defined by

$$\phi \star u(x) := \int_{\mathbb{R}^N} \phi(x-y)u(y)\,dy \quad \text{for a.e. } x \in \mathbb{R}^N.$$

We set $\overrightarrow{p} := (p_1, \dots, p_N)$ and $\overrightarrow{q} := (q_1, \dots, q_N)$ where

$$\begin{aligned} &1 < p_1, p_2, \dots, p_N, \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1, \\ &1 < q_1, q_2, \dots, q_N, \quad \sum_{i=1}^{N} \frac{1}{q_i} > 1. \end{aligned}$$

Let $\overline{p}$ and $\overline{q}$ denote the harmonic means $\overline{p} = N/(\sum_{i=1}^N \frac{1}{p_i})$ and $\overline{q} = N/(\sum_{i=1}^N \frac{1}{q_i})$, respectively, and define

$$\begin{aligned} &p^\star := \frac{N}{(\sum_{i=1}^N \frac{1}{p_i}) - 1} = \frac{N\overline{p}}{N - \overline{p}}, \qquad q^\star := \frac{N}{(\sum_{i=1}^N \frac{1}{q_i}) - 1} = \frac{N\overline{q}}{N - \overline{q}}, \\ &p_\infty := \max\{p_+, p^\star\} \quad \text{and} \quad p_+ := \max\{p_i : i = 1, \dots, N\}. \end{aligned}$$

We define an order as follows:

$$\overrightarrow{q} \leq \overrightarrow{p} \quad \text{if and only if} \quad q_i \leq p_i \quad \text{for all } i = 1, \dots, N. \tag{1.2}$$

Throughout the paper, we assume that

$$\overrightarrow{q} \leq \overrightarrow{p}, \qquad q_N < q^\star, \qquad p_N < p^\star \quad \text{and} \quad q^\star < p^\star. \tag{1.3}$$

Also, we assume

(H_1) $|f(x,t,\xi)| \leq \sigma(x) + c_1|t|^{p^+-1} + c_2\sum_{i=1}^{N}|\xi_i|^{p_i-1}$ for a.e. $x \in \Omega$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $\xi = (\xi_1, \dots, \xi_N)$, $\sigma \in L^{\gamma'}(\Omega)$ for $\gamma \in (1, p^+)$, $\gamma' = \frac{\gamma}{\gamma-1}$ and constants $c_1 \geq 0$, $c_2 \geq 0$, satisfying

$$\|\phi\|_{L^1(\mathbb{R}^N)}^{p^+-1} c_1 S_{p^+} + c_2 \Pi < 1, \tag{1.4}$$

where $\Pi = \max_{1 \leq i \leq N}\{S'_{p_i}\|\phi\|_{L^1(\mathbb{R}^N)}^{p_i-1}\}$ and S'_{p_i} is the Sobolev constant for the embedding $W_0^{1,p_i}(\Omega) \subset L^{p_i}(\Omega)$ for $i = 1, \dots, N$.

The differential operator in (1.1), i.e.,

$$u \to \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} - \mu\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\right)\frac{\partial u}{\partial x_i}$$

is the difference of the anisotropic degenerated p-Laplacian and q-Laplacian. In fact, the negative anisotropic ϱ-Laplacian (for $\varrho = p, q$)

$$-\Delta_{\overrightarrow{\varrho}} : W_0^{1,\overrightarrow{\varrho}}(\Omega) \to W^{-1,\overrightarrow{\varrho}'(\Omega)}$$

is expressed as

$$\langle -\Delta_{\overrightarrow{\varrho}} u, v\rangle = \sum_{i=1}^{N} \int_\Omega \frac{\partial}{\partial x_i} \left|\frac{\partial u}{\partial x_i}\right|^{\varrho_i - 2} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\, dx$$

for all $u, v \in W_0^{1,\overrightarrow{\varrho}}(\Omega)$, where $\overrightarrow{\varrho} := (\varrho_1, \dots, \varrho_N)$ and $\overrightarrow{\varrho}' := (\frac{\varrho_1}{\varrho_1 - 1}, \dots, \frac{\varrho_N}{\varrho_N - 1})$.

Since $1 < q_1, \overrightarrow{q} < \overrightarrow{p}, p_N < \infty$, the continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow W_0^{1,\overrightarrow{q}}(\Omega)$ holds and the operator $-\Delta_{\overrightarrow{p}} + \mu\Delta_{\overrightarrow{q}}$ is well defined on $W_0^{1,\overrightarrow{p}}(\Omega)$.

The sign of $-\Delta_{\overrightarrow{p}} + \mu\Delta_{\overrightarrow{q}}$ for $\mu > 0$ and sufficiently large is different from $\mu > 0$ and sufficiently small. This makes it difficult to study (1.1). We owe essential ideas to [20] to overcome the lack of ellipticity, monotonicity, and variational structure in problem (1.1) (see [18–20, 22]). Therefore, for problem (1.1), the existence of a solution is proved by Theorem 1.1.

Theorem 1.1 *Suppose that (H_1) holds. Then, there exists a generalized solution to problem* (1.1). *In particular, if $\mu \le 0$, there exists a weak solution to problem* (1.1).

The rest of the paper is organized as follows: In Sect. 2, the suitable function spaces and some lemmas are recalled. In Sect. 3, the associated Nemytskij operator is introduced and then we show the anisotropic competing (p,q)-Laplacian (1.1) has a solution, i.e., the proof of Theorem 1.1 is presented.

2 Function space

Consider the anisotropic Sobolev spaces $W^{1,\overrightarrow{p}}(\Omega)$, with the norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} := \int_\Omega \left|u(x)\right| dx + \sum_{i=1}^{N} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}},$$

and $W_0^{1,\overrightarrow{p}}(\Omega)$ with the norm

$$\begin{aligned} \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} &:= \sum_{i=1}^{N} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}} \\ &= \sum_{i=1}^{N} \|u\|_{W_0^{1,p_i}(\Omega)}. \end{aligned}$$

Note that $W_0^{1,\overrightarrow{p}}(\Omega)$ is a reflexive and uniformly convex Banach space (see [26–28] and references therein for more details or more literature in [2, 4, 8–14, 30]). Here, is an embedding theorem [15, Theorem 1].

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. If*

$$p_i > 1, \quad \textit{for all } i = 1,\dots,N, \qquad \sum_{i=1}^{N} \frac{1}{p_i} > 1,$$

then for all $r \in [1, p_\infty]$, there is a continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset L^r(\Omega)$. For $r < p_\infty$, the embedding is compact.

Note that the Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ is embedded in $W^{1,\overrightarrow{p}}(\mathbb{R}^N)$ by identifying every $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ with its extension equal to zero outside Ω. Thus, one can define the convolution $\phi \star u$ of $\phi \in L^1(\mathbb{R}^N)$ with $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ (see [3, Sect. 4.4 and Sect. 9.1]) by

$$\phi \star u(x) = \int_{\mathbb{R}^N} \phi(x-y)u(y)\,dy \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Also,

$$\frac{\partial}{\partial x_i}(\phi \star u) = \phi \star \frac{\partial u}{\partial x_i} \in L^{p_i}\big(\mathbb{R}^N\big), \text{for all} i = 1, 2, \dots, N.$$

Remark 2.2 Assume $\phi \in L^1(\mathbb{R}^N)$ with $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, then

(i)

$$\|\phi \star u\|_{L^r(\mathbb{R}^N)} \le \|\phi\|_{L^1(\mathbb{R}^N)} \|u\|_{L^r(\Omega)} \tag{2.1}$$

whenever $r \in [1, p^\star]$;

(ii)

$$\left\|\phi \star \frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\mathbb{R}^N)} \le \|\phi\|_{L^1(\mathbb{R}^N)} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} \tag{2.2}$$

for all $i = 1, \dots, N$;

(iii) By (2.2), we have

$$\begin{aligned}
\|\phi \star u\|_{W_0^{1\overrightarrow{p}}(\mathbb{R}^N)} &= \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} \left| \frac{\partial(\phi \star u)}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}} \\
&= \sum_{i=1}^{N} \left\| \frac{\partial(\phi \star u)}{\partial x_i} \right\|_{L^{p_i}(\mathbb{R}^N)} \\
&\le \sum_{i=1}^{N} \|\phi\|_{L^1(\mathbb{R}^N)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\mathbb{R}^N)} \\
&= \|\phi\|_{L^1(\mathbb{R}^N)} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\mathbb{R}^N)} \\
&= \|\phi\|_{L^1(\mathbb{R}^N)} \|u\|_{W_0^{1\overrightarrow{p}}(\mathbb{R}^N)}.
\end{aligned} \tag{2.3}$$

Before ending this section we require a generalized solution for (1.1).

Definition 2.3 A function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is called a generalized solution to problem (1.1) if there exists a sequence $\{u_n\}_{n\geq 1}$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ such that

(I) $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ as $n \to \infty$;
(II) $-\Delta_{\overrightarrow{p}} u_n + \mu \Delta_{\overrightarrow{q}} u_n - f(\cdot, \phi \star u_n(\cdot), \nabla(\phi \star \nabla u)(\cdot)) \rightharpoonup 0$ in $W^{-1,\overrightarrow{p}'}(\Omega)$ as $n \to \infty$;
(III) $\lim_{n\to\infty} \langle -\Delta_{\overrightarrow{p}} u_n + \mu \Delta_{\overrightarrow{q}} u_n, u_n - u\rangle = 0$.

Remark 2.4 Assume u is a weak solution of (1.1), i.e., u satisfies

$$\bigl\langle (-\Delta_{\overrightarrow{p}} + \mu \Delta_{\overrightarrow{q}})(u), v\bigr\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} = \int_\Omega f\bigl(x, \phi \star u(x), \nabla\bigl(\phi \star u(x)\bigr)\bigr) v(x)\, dx$$

for all $v \in W_0^{1,\overrightarrow{p}}(\Omega)$. Set $u_n = u$ for all n, then any weak solution is a generalized solution to problem (1.1).

3 Weak and generalized solutions

Here, we study the behavior of the Nemytskij operator and construct a sequence (by the Galerkin basis of the space) that converges strongly to the generalized (weak) solution of (1.1) when $\mu \geq 0$ ($\mu < 0$). First, we recall an embedding result.

Since $\overrightarrow{q} < \overrightarrow{p}$ and Ω is bounded then

$$\begin{aligned} &W_0^{1\overrightarrow{p}}(\Omega) \quad \text{is continuously embedded in } W_0^{1\overrightarrow{q}}(\Omega) \quad \text{and} \\ &W^{-1,\overrightarrow{q}'}(\Omega) \quad \text{is continuously embedded in } W^{-1,\overrightarrow{p}'}(\Omega). \end{aligned} \tag{3.1}$$

Assume the operator $A : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ (see (1.1)) is defined by

$$\bigl\langle A(u), v\bigr\rangle = \langle -\Delta_{\overrightarrow{p}} u + \mu \Delta_{\overrightarrow{q}} u, v\rangle - \int_\Omega f\bigl(x, \phi \star u(x), \nabla(\phi \star u)(x)\bigr) v(x)\, dx. \tag{3.2}$$

Lemma 3.1 *The operator A defined by* (3.2) *is continuous, when* (H_1) *holds.*

Proof Define the operator

$$T : W_0^{1,\overrightarrow{p}}(\Omega) \to L^{p^*}(\Omega) \times L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega)$$

by $T(u) = (\phi \star u|_\Omega, \nabla(\phi \star u)|_\Omega)$. Relations (2.1) and (2.3) imply that T is linear and continuous. By (H_1) and Krasnoselskii's theorem [16], the Nemytskii operator

$$\begin{aligned} &\mathcal{N} : L^{p^*}(\Omega) \times \bigl(L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega)\bigr) \to L^{p^{*'}}(\Omega) \\ &\quad (v, , w_1, \dots, w_N) \mapsto f\bigl(\cdot, v(\cdot), w_1(\cdot), \dots, w_N(\cdot)\bigr) \end{aligned}$$

is well defined and continuous and so the composition operator

$$W_0^{1,\overrightarrow{p}}(\Omega) \to L^{p^{*'}}(\Omega), \qquad u \mapsto f\bigl(\cdot, \phi \star u(\cdot), \nabla(\phi \star u)(\cdot)\bigr) \tag{3.3}$$

is continuous. Note that $L^{p^{*'}}(\Omega)$ is continuously embedded in $W^{-1,p^{*'}}(\Omega)$.

The operator $-\Delta_{\overrightarrow{\varrho}}: W_0^{1,\overrightarrow{\varrho}}(\Omega) \to W^{-1,\overrightarrow{\varrho}'}(\Omega)$ (for $\varrho = p, q$) is continuous. Therefore, embedding (3.1) implies $-\Delta_{\overrightarrow{p}} + \mu\Delta_{\overrightarrow{q}}: W_0^{1\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ is continuous and finally the operator A is continuous. □

Assume $\{X_n\}$ (vector subspaces of $W_0^{1,\overrightarrow{p}}(\Omega)$) is a Galerkin basis for the separable Banach space $W_0^{1,\overrightarrow{p}}(\Omega)$, i.e.,

(i) $\dim(X_n) < \infty$, for all n;

(ii) $X_n \subset X_{n+1}$, for all n;

(iii) $\overline{\underset{n}{\cup} X_n} = W_0^{1,\overrightarrow{p}}(\Omega)$.

A consequence of Brouwer's fixed-point theorem will resolve each approximate problem on X_n. Due to this, we construct a sequence $\{u_n\}$ by the next Proposition.

Proposition 3.2 *Assume (H_1) holds. Then, for each $n \geq 1$ there exists $u_n \in X_n$ such that*

$$\langle(-\Delta_{\overrightarrow{p}} + \mu\Delta_{\overrightarrow{q}})(u_n), v\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} = \int_\Omega f\big(x, \phi \star u_n(x), \nabla\big(\phi \star u_n(x)\big)\big)v(x)\,dx \tag{3.4}$$

for all $v \in X_n$. In addition, $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$.

Proof We define $A_n : X_n \to X_n^\star$ by

$$\begin{aligned}
&\langle A_n(u), v\rangle_{X_n} \\
&\quad = \langle(-\Delta_{\overrightarrow{p}} + \mu\Delta_{\overrightarrow{q}})(u), v\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} - \int_\Omega f\big(x, \phi \star u(x), \nabla\big(\phi \star u(x)\big)\big)v(x)\,dx
\end{aligned}$$

for all $u, v \in X_n$ and all $n \in \mathbb{N}$. The operator A_n is continuous (by Lemma 3.1) and

$$\begin{aligned}
&\langle A_n(v), v\rangle_{X_n} \\
&\quad = \sum_{i=1}^N \int_\Omega \left(\left|\frac{\partial v}{\partial x_i}\right|^{p_i} - \mu\left|\frac{\partial v}{\partial x_i}\right|^{q_i}\right)dx - \int_\Omega f\big(x, \phi \star v(x), \nabla\big(\phi \star v(x)\big)\big)v(x)\,dx \\
&\quad \geq \sum_{i=1}^N \|v\|_{W_0^{1,p_i}(\Omega)}^{p_i} - \mu\sum_{i=1}^N |\Omega|^{\frac{p_i - q_i}{p_i}}\|v\|_{W_0^{1,p_i}(\Omega)}^{q_i} - \|\sigma\|_{L^{\gamma'}(\Omega)}\|v\|_{L^\gamma(\Omega)} \\
&\qquad - c_1\|\phi \star v\|_{L^{p^+}(\Omega)}^{p^+-1}\|v\|_{L^{p^+}(\Omega)} - c_2\sum_{i=1}^N \|\phi \star v\|_{W_0^{1,p_i}(\Omega)}^{p_i-1}\|v\|_{L^{p_i}(\Omega)}
\end{aligned}$$

for all $v \in X_n$, by (H_1) and the Hölder inequality. Now (2.1), (2.3), and Sobolev embedding show that

$$\begin{aligned}
&\langle A_n(v), v\rangle_{X_n} \\
&\quad = \sum_{i=1}^N \int_\Omega \left(\left|\frac{\partial v}{\partial x_i}\right|^{p_i} - \mu\left|\frac{\partial v}{\partial x_i}\right|^{q_i}\right)dx \\
&\qquad - \int_\Omega f\big(x, \phi \star v(x), \nabla\big(\phi \star v(x)\big)\big)v(x)\,dx
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=1}^{N}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)} - \mu\sum_{i=1}^{N}|\Omega|^{\frac{p_i-q_i}{p_i}}\|v\|^{q_i}_{W_0^{1,p_i}(\Omega)} - \|\sigma\|_{L^{\gamma'}(\Omega)}\|v\|_{L^{\gamma}(\Omega)} \\
&\quad - c_1\|\phi\|^{p^+-1}_{L^1(\mathbb{R}^N)}\|v\|^{p^+}_{L^{p^+}(\Omega)} - c_2\sum_{i=1}^{N}\|\phi\|^{p_i-1}_{L^1(\mathbb{R}^N)}\|v\|^{p_i-1}_{W_0^{1,p_i}(\Omega)}\|v\|_{L^{p_i}(\Omega)} \qquad (3.5)\\
&\geq \sum_{i=1}^{N}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)} - \mu\sum_{i=1}^{N}|\Omega|^{\frac{p_i-q_i}{p_i}}\|v\|^{q_i}_{W_0^{1,p_i}(\Omega)} - S_\gamma\|\sigma\|_{L^{\gamma'}(\Omega)}\|v\|_{W_0^{1,\vec{p}}(\Omega)} \\
&\quad - c_1S_{p^+}\|\phi\|^{p^+-1}_{L^1(\mathbb{R}^N)}\sum_{i=1}^{N}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)} - c_2\sum_{i=1}^{N}S'_{p_i}\|\phi\|^{p_i-1}_{L^1(\mathbb{R}^N)}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)} \\
&\geq \sum_{i=1}^{N}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)} - \mu\sum_{i=1}^{N}|\Omega|^{\frac{p_i-q_i}{p_i}}\|v\|^{q_i}_{W_0^{1,p_i}(\Omega)} - S_\gamma\|\sigma\|_{L^{\gamma'}(\Omega)}\|v\|_{W_0^{1,\vec{p}}(\Omega)} \\
&\quad - \bigl(\|\phi\|^{p^+-1}_{L^1(\mathbb{R}^N)}c_1S_{p^+} + c_2\Pi\bigr)\sum_{i=1}^{N}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)},
\end{aligned}$$

for all $x \in X_n$, where $|\Omega|$ is the Lebesgue measure of Ω.

Assume $\lambda_{1,\vec{p}} > 0$ denotes the first eigenvalue of the negative anisotropic p-Laplacian on $W_0^{1,\vec{p}}(\Omega)$ that is given by

$$\lambda_{1,\vec{p}} = \min\left\{\frac{\sum_{i=1}^{N}\int_\Omega|\frac{\partial u}{\partial x_i}|^{p_i}\,dx}{\|u\|^{p^+}_{L^{p^+}(\Omega)}} : u \in W_0^{1,\vec{p}}(\Omega)\backslash\{0\}\right\}. \qquad (3.6)$$

See [15, Theorem 3] or [17, Theorem 2] for more details. By (1.4) (recall that $S_{p^+} = \lambda_{1,\vec{p}}^{-\frac{1}{p^+}}$) and $p_i > q_i > 1$ and $p^+ > q^+ > 1$, for $i = 1,\dots,N$, for $R = R(n) > 0$ sufficiently large we obtain

$$\langle A_n(v), v\rangle_{X_n} \geq 0 \quad \text{whenever } v \in X_n \text{ with } \|v\|_{W_0^{1,\vec{p}}(\Omega)} = R.$$

As a consequence of Brouwer's fixed-point theorem (see, e.g., [29, p. 37]) (since X_n is a finite-dimensional space) there exists $u_n \in X_n$ solving the equation $A_n(u_n) = 0$ and this shows that $u_n \in X_n$ is a solution for problem (3.4).

$\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,\vec{p}}(\Omega)$. To show this, let $v = u_n \in X_n$ in (3.5), then

$$\begin{aligned}
&\sum_{i=1}^{N}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)} - c_1S_{p^+}\|\phi\|^{p^+-1}_{L^1(\mathbb{R}^N)}\sum_{i=1}^{N}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)} \\
&\quad - c_2\sum_{i=1}^{N}S'_{p_i}\|\phi\|^{p_i-1}_{L^1(\mathbb{R}^N)}\|v\|^{p_i}_{W_0^{1,p_i}(\Omega)} \\
&\quad \leq \mu\sum_{i=1}^{N}|\Omega|^{\frac{p_i-q_i}{p_i}}\|v\|^{q_i}_{W_0^{1,p_i}(\Omega)} + S_\gamma\|\sigma\|_{L^{\gamma'}(\Omega)}\|v\|_{W_0^{1,\vec{p}}(\Omega)}.
\end{aligned}$$

Since $p_i > q_i > 1$ and $p^+ > q^+ > 1$, for $i = 1,\dots,N$, then (1.4) shows that $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,\vec{p}}(\Omega)$. □

Now, we can prove the existence of the solution of problem (1.1), i.e., we present the proof of Theorem 1.1.

Proof Assume $\{u_n\}_{n\geq 1} \subset W_0^{1,\overrightarrow{p}}(\Omega)$ is given by Proposition 3.2 that is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ and the reflexively, there exists a subsequence still denoted by $\{u_n\}_{n\geq 1}$ that is bounded and

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\overrightarrow{p}}(\Omega) \tag{3.7}$$

with some $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. The continuity of the operator in (3.3), shows that the sequence $\{f(\cdot, \phi \star u_n, \nabla(\phi \star u_n))\}_{n\geq 1}$ is bounded in $L^{\overrightarrow{p}'}$. Suppose

$$-\Delta_{\overrightarrow{p}} u_n + \mu \Delta_{\overrightarrow{q}} u_n - f\big(\cdot, \phi \star u_n, \nabla(\phi \star u_n)\big) \rightharpoonup \eta \quad \text{in } W^{-1,\overrightarrow{p}'}(\Omega) \tag{3.8}$$

with some $\eta \in W^{-1,\overrightarrow{p}'}(\Omega)$, by the reflexivity of $W^{-1,\overrightarrow{p}'}(\Omega)$.

Assume $v \in \bigcup_{n\geq 1} X_n$. Fix an integer $m \geq 1$ such that $v \in X_m$. Proposition 3.2 provides that (3.4) holds for all $n \geq m$. Letting $n \to \infty$ in (3.4), by means of (3.8) we obtain

$$\langle \eta, v\rangle \geq 0 \quad \text{for all} v \in \bigcup_{n\geq 1} X_n.$$

By the density of $\bigcup_{n\geq 1} X_n$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ (see (iii) in the definition of the Galerkin basis), it turns out that $\eta = 0$ and so in $W^{-1,\overrightarrow{p}'}(\Omega)$ we have

$$-\Delta_{\overrightarrow{p}} u_n + \mu \Delta_{\overrightarrow{q}} u_n - f\big(\cdot, \phi \star u_n, \nabla(\phi \star u_n)\big) \rightharpoonup 0. \tag{3.9}$$

Letting $v = u_n$ in (3.4), we obtain

$$\langle -\Delta_{\overrightarrow{p}} u_n + \mu \Delta_{\overrightarrow{q}} u_n, u_n\rangle - \int_\Omega f\big(\cdot, \phi \star u_n, \nabla(\phi \star u_n)\big)\, dx = 0 \tag{3.10}$$

for all $n \geq 1$, while (3.9) gives

$$\langle -\Delta_{\overrightarrow{p}} u_n + \mu \Delta_{\overrightarrow{q}} u_n, u_n\rangle - \int_\Omega f\big(\cdot, \phi \star u_n, \nabla(\phi \star u_n)\big)\, dx \to 0 \tag{3.11}$$

as $n \to \infty$. Together, (3.10) and (3.11) yield

$$\langle -\Delta_{\overrightarrow{p}} u_n + \mu \Delta_{\overrightarrow{q}} u_n, u_n - u\rangle - \int_\Omega f\big(\cdot, \phi \star u_n, \nabla(\phi \star u_n)\big)(u_n - u)\, dx \to 0 \tag{3.12}$$

as $n \to \infty$. Theorem 2.1 and (3.7) imply that $u_n \to u$ strongly in $L^p(\Omega)$, and since $\{f(\cdot, \phi \star u_n, \nabla(\phi \star u_n))\}$ is bounded, then

$$\lim_{n\to\infty} \int_\Omega f\big(\cdot, \phi \star u_n, \nabla(\phi \star u_n)\big)(u_n - u)\, dx = 0. \tag{3.13}$$

By inserting (3.13) into (3.12) we obtain

$$\lim_{n\to\infty} \langle -\Delta_{\overrightarrow{p}} u_n + \mu \Delta_{\overrightarrow{q}} u_n, u_n - u\rangle = 0. \tag{3.14}$$

Thus, the conditions of Definition 2.3 are satisfied and this implies that $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is a generalized solution to problem (1.1).

Now, we prove the existence of a weak solution in the case $\mu \leq 0$. Assume u is a generalized solution to problem (1.1) and $\{u_n\}_{n\geq 1}$ satisfy the conditions of Definition 2.3 with respect to u. We obtain

$$\begin{aligned}
&\langle -\Delta_{\overrightarrow{p}} u_n, u_n - u\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} \\
&\quad \leq \langle -\Delta_{\overrightarrow{p}} u_n, u_n - u\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} - \mu\langle -\Delta_{\overrightarrow{q}} u_n + \Delta_{\overrightarrow{q}} u, u_n - u\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} \\
&\quad = \langle -\Delta_{\overrightarrow{p}} u_n + \mu\Delta_{\overrightarrow{q}} u_n, u_n - u\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} - \mu\langle \Delta_{\overrightarrow{q}} u, u_n - u\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)}
\end{aligned}$$

by the monotonicity of $-\Delta_{\overrightarrow{q}}$ and hence,

$$\limsup_{n\to\infty}\langle \Delta_{\overrightarrow{p}} u_n, u_n - u\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} \leq 0.$$

Then, $u_n \to u$ strongly in $W^{1,\overrightarrow{p}}(\Omega)$ (see, e.g., [21, Proposition 2.72]). The continuity of A (Lemma 3.1), shows $A(u_n) \to A(u)$ in $W^{-1,\overrightarrow{p}'}(\Omega)$ and condition (II) of Definition 2.3, shows $A(u) = 0$. This shows that

$$\big\langle (-\Delta_{\overrightarrow{p}} + \mu\Delta_{\overrightarrow{q}})(u), v\big\rangle_{W_0^{1,\overrightarrow{p}}(\Omega)} = \int_\Omega f\big(x, \phi \star u(x), \nabla\big(\phi \star u(x)\big)\big)v(x)\,dx$$

for all $v \in W_0^{1,\overrightarrow{p}}(\Omega)$, which means u is a weak solution to problem (1.1). □

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