Kakumani and Yadav *Boundary Value Problems* (2022) 2022:95
https://doi.org/10.1186/s13661-022-01674-2

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Decay estimate in a viscoelastic plate equation with past history, nonlinear damping, and logarithmic nonlinearity

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Abstract

In this article, we consider a viscoelastic plate equation with past history, nonlinear damping, and logarithmic nonlinearity. We prove explicit and general decay rate results of the solution to the viscoelastic plate equation with past history. Convex properties, logarithmic inequalities, and generalized Young's inequality are mainly used to prove the decay estimate.

Keywords: Viscoelasticity; Local existence; Convexity; Decay estimate; Logarithmic nonlinearity

1 Introduction

In this article, we consider the decay rate results of the solution to a viscoelastic plate equation with past history. Let Ω be a smooth bounded domain of $\mathbb{R}^n$ and u denotes the transverse displacement of waves. Assume u_0, u_1 are given initial data, then the partial differential equation is governed by the plate equation and is given by:

$$\begin{cases} |u_t|^\rho u_{tt} + \Delta^2 u + \Delta^2 u_{tt} + u - \int_0^\infty b(s)\Delta^2 u(t-s)\,ds + h(u_t) = ku\ln|u|, \\ \quad \text{in } \Omega \times (0,\infty), \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad \text{in } \partial\Omega \times (0,\infty), \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), \quad \text{in } \Omega, t \geq 0, \end{cases} \tag{1}$$

where ν is the outer unit normal to $\partial\Omega$, b, h are functions (defined later) and ρ is a positive constant ($\rho > 0$ if $n \geq 2$ and $0 < \rho \leq \frac{2}{n-2}$ if $n \geq 3$).

Viscoelasticity takes into account time-independent solid behavior (namely, elastic) and time-dependent fluid behavior (namely, viscosity). Some properties of viscoelastic materials are similar to those of elastic solids and some with Newtonian viscous fluids. Due to the significant advancements in the rubber and plastics industries, the importance of material viscoelastic characteristics has been recognized. This type of problem has many applications in several branches of physics, such as quantum mechanics, nuclear physics, supersymmetric field theories, and optics.

According to classical mechanics, any mathematical model that aims to capture the behavior of suspension bridges must have a sufficient number of degrees of freedom and incorporate some nonlinearity. Nonlinearities enable the observation of some hidden phenomena, as would be predicted (see [1] for more details). Logarithmic expression changes the shape of the distribution; it makes the sample less skewed, and in some cases, it reduces the skewness of data. The occurrence of torsional oscillation can be explained by nonlinearity since the behavior of a suspension bridge is nonlinear. Moreover, the amplitude of the oscillation will decrease if we employ logarithmic nonlinearity.

We begin our review with Dafermos' pioneer paper [2], in which the author presented the following one-dimensional viscoelastic problem:

$$\begin{cases} \rho u_{tt} = cu_{xx} - \int_{-\infty}^{t} g(t-\tau)u_{xx}\,\mathrm{d}\tau, & x \in [0,1], \\ u(0,t) = u(1,t) = 0, & t \in (-\infty,\infty). \end{cases}$$

Dafermos proved that for smooth monotonic decreasing relaxation functions, the solutions go to zero as t tends to infinity, and he also established an existence result. However, no rate of decay was specified. Since then, many authors have tried to prove the local/global existence results and also to calculate the explicit decay result. In [3] the authors considered the viscoelastic wave equation with infinite memory given by the equation:

$$u_{tt} - \Delta u + \int_0^{\infty} g(s)\Delta u(t-s)\,\mathrm{d}s = 0, \quad \text{in } \Omega \times \mathbb{R}^+,$$

and established energy-decay results without making any assumption on the boundedness of initial data and any growth constraint on the damping term. Piskin and Polat Pata [4] considered the following nonlinear Petrovsky equation:

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{m-1}u_t = |u|^{p-1}u.$$

They estimated the decay result by using the Nakaos inequality. In [5] the authors considered the following hyperbolic equation with logarithmic nonlinearity:

$$u_{tt} - M\big(\|\nabla u\|^2\big)\Delta u + |u_t|^{k-2}u_t = |u|^{p-2}u\ln|u|.$$

They established the blow-up of solutions in finite time for negative initial energy under a few assumptions on M. For some more recent works in logarithmic nonlinearity, see [6] and [7]. In [8, 9], the authors considered the problem of the type:

$$u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} \mu(s)Au(t-s)\,ds = 0$$

and discussed the decay properties of the semigroup, where A is a strictly positive self-adjoint linear operator, and the memory kernel μ is a decreasing function. They also established necessary and sufficient conditions for the exponential stability under some suitable assumptions on α, β. Giorgi et al. [10] considered the semilinear hyperbolic equation:

$$u_{tt} - k(0)\Delta u - \int_0^{+\infty} k'(s)\Delta u(t-s)\,ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}^+,$$

where the memory term is bounded and $k(0) > 0$, $k(\infty) > 0$, $k'(s) \le 0$. The behavior of solutions over time was examined. Particularly in the autonomous situation, the existence of a global attractor for solutions was achieved. Conti and Pata [11] considered the following semilinear hyperbolic equation:

$$u_{tt} + \alpha u(t) - k(0)\Delta u - \int_0^{+\infty} k'(s)\Delta u(t-s)\,ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}.$$

The authors considered the memory term as a convex decreasing smooth function and g represents a nonlinear term of at most cubic growth satisfying some conditions. Under past history setup, the authors proved the existence of a regular global attractor. Guesmia [12] considered the following problem:

$$u_{tt} + Au - \int_0^{+\infty} h(s)Bu(t-s)\,ds = 0, \quad \text{for } t > 0,$$

with a class of infinite history kernels satisfying $\int_0^{+\infty} \frac{h(s)}{H^{-1}(-h'(s))}\,ds + \sup_{s\in\mathbb{R}_+} \frac{h(s)}{H^{-1}(-h'(s))} < +\infty$, where H is a strictly increasing convex function with $H(0)$, $H'(0) = 0$ and $\lim_{t\to+\infty} H'(t) = +\infty$. By using the properties of the convex function and Young's inequality the authors proved the more general decay result. Guesmia and Messaoudi [13] considered:

$$u_{tt} - \Delta u + \int_o^t g_1(t-s)\operatorname{div}\bigl(a_1(x)\nabla u(s)\bigr)\,\mathrm{d}s + \int_0^\infty g_2(s)\operatorname{div}\bigl(a_2(x)\nabla u(t-s)\bigr)\,\mathrm{d}s = 0,$$

where g_1 and g_2 are two positive nonincreasing functions defined on $\mathbb{R}^+$, and a_1, a_2 are nonnegative bounded function defined on Ω. The authors in [13] proved the general decay result. Later, Mahdi [14] considered the problem (1) with $h = 0$ and established an explicit and general decay rate result. In [15], Al-Mahdi *et al.* considered the memory-type Timoshenko system with Dirichlet boundary conditions and the system is given by

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\varphi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \int_0^{+\infty} g(s)\psi_{xx}(t-s)\,ds = 0. \end{cases}$$

They established a few decay rate results on the energy function under the unboundedness of the initial data. For more work related to past history problems, refer to [12, 16–20] (and the references therein). Throughout this paper, we consider the following hypothesis:

(H1) Let $b : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing C^1-function that satisfies:

$$0 < b(0), 1 - \int_0^\infty b(\tau)\,d\tau = l > 0.$$

(H2) Assume that $B : (0,\infty) \to (0,\infty)$ is a C^1 function that is linear or a strictly convex C^2 function and strictly increasing on $(0, r_1]$, where $r_1 \le b(0)$, $B(0) = B'(0) = 0$, $\lim_{s\to\infty} B'(s) = +\infty$, $s \mapsto sB'(s)$ and $s \mapsto s(B')^{-1}(s)$ are convex on $(0, r_1]$ and B satisfies

$$b'(t) \le -\xi(t)B\bigl(b(t)\bigr), \quad \forall t \ge 0,$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing positive function with $\xi(0) > 0$.

(H3) Let $h : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function that satisfies (for some c_1, c_2, ϵ are positive constants):

$$\tilde{h}(|t|) \leq |h(t)| \leq \tilde{h}^{-1}(|t|), \quad \forall |t| \leq \epsilon,$$
$$c_1|t| \leq |h(t)| \leq c_2|t|, \quad \forall |t| \geq \epsilon,$$

where $\tilde{h} \in C^1(\mathbb{R}^+)$ with $\tilde{h}(0) = 0$, which is a strictly increasing function. When $\tilde{h}$ is nonlinear, define H to be a strictly convex C^2 function in $(0, r_2]$, where $r_2 > 0$ such that $H(t) = \sqrt{t}\tilde{h}(\sqrt{t})$.

Remark 1 Since B is strictly convex on $(0, r_1]$ and $B(0) = 0$, then

$$B(\theta t) \leq \theta B(t), \quad 0 \leq \theta \leq 1 \text{ and } t \in (0, r_1].$$

In this article, we give certain notations and declare existence results in Sect. 2. In addition, we express a couple of Lemmas in Sect. 2 that will be useful later. We state and establish the decay rate estimate in Sect. 3, as well as present a few examples to demonstrate the decay rate.

2 Preliminaries and existence results

In this section, we review Dafermos' theory (see [2]) and define the energy functional that is relevant to our problem. We also state a local existence result and a couple of additional Lemmas that will be useful later. We introduce the function η as follows:

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \quad \forall s,t \geq 0, x \in \Omega, \tag{2}$$

then the initial and boundary conditions are obtained as follows:

$$\begin{cases} \eta^t(x,0) = 0, \quad \forall t \geq 0, x \in \Omega, \\ \eta^t(x,s) = 0, \quad \forall s,t \geq 0, x \in \partial\Omega, \\ \eta^0(x,s) = \eta_0(x,s) = u_0(x,0) - u_0(x,s), \quad \forall s \geq 0, x \in \Omega. \end{cases}$$

Observe that, (2) implies

$$\eta^t_s(x,s) + \eta^t_t(x,s) = u_t(x,t). \tag{3}$$

After combining (1) and (2), we obtain the following system

$$\begin{cases} |u_t|^\rho u_{tt} + l\Delta^2 u + \Delta^2 u_{tt} + u + \int_0^\infty b(s)\Delta^2\eta^t\, ds + h(u_t) = ku\ln|u|, \\ \quad \forall t \geq 0, x \in \Omega, \\ \eta^t_s(x,s) + \eta^t_t(x,s) - u_t(x,t) = 0, \quad \forall s,t \geq 0, x \in \Omega, \end{cases} \tag{4}$$

with the following initial and boundary data

$$\begin{cases} u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), \quad \forall t \geq 0, \forall x \in \Omega, \\ \eta^0(x,s) = \eta_0(x,s) = u_0(x,0) - u_0(x,s), \qquad \eta^t(x,0) = 0, \quad \forall s,t \geq 0, \forall x \in \Omega, \\ u(x,t) = 0, \qquad \eta^t(x,s) = 0 \quad \text{in } \partial\Omega, \forall s,t \geq 0. \end{cases} \tag{5}$$

The energy functional associated with the system (4) is given by

$$\begin{aligned} E(t) = {}& \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{l}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\Delta u_t\|_2^2 - \frac{k}{2}\int_\Omega u^2 \ln|u|\,\mathrm{d}x \\ &+ \frac{k+2}{4}\|u\|_2^2 + \frac{1}{2}\big(b\circ\Delta\eta^t\big), \end{aligned} \tag{6}$$

where $(b\circ\Delta n^t)(t) = \int_0^\infty\int_\Omega b(s)|\Delta\eta^t|^2\,ds\,dx$ and $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$. Differentiating $E(t)$ with t and making use of system (4), we obtain

$$E'(t) \le \frac{1}{2}\big(b'\circ\Delta\eta^t\big) - \int_\Omega u_t h(u_t)\,dx \le 0. \tag{7}$$

Remark 2 For any $\epsilon_0 \in (0,1)$, we obtain that

$$\big(b\circ\Delta\eta^t\big) = \big(b\circ\Delta\eta^t\big)^{\frac{\epsilon_0}{1+\epsilon_0}}\big(b\circ\Delta\eta^t\big)^{\frac{1}{1+\epsilon_0}} \le c\big(b\circ\Delta\eta^t\big)^{\frac{1}{1+\epsilon_0}}.$$

Theorem 1 *Let* $(u_0(\cdot,0),u_1) \in H_0^2(\Omega)\times H_0^2(\Omega)$, *and assume that the hypothesis* $(H1)$–$(H3)$ *holds. Then, the problem* (1) *has weak solution on* $[0,T]$.

The proof of the above theorem can be obtained by following similar lines to those given in [21].

Lemma 2.1 (Cf. [14], Lemma 3.1) *There exists a constant* $M > 0$ *such that*

$$\int_t^\infty\int_\Omega b(s)\big(\Delta\eta^t(x,s)\big)^2\,ds\,dx \le Mf_1(t),$$

where $f_1(t) = \int_0^\infty b(s+t)(1+\|\Delta u_0(s)\|_2^2)\,ds$.

Lemma 2.2 (Cf. [21], Lemma 9) *Assume that* $(H1)$–$(H3)$ *hold, for some* $\epsilon_0 \in (0,1)$ *and* $0 < E(0) < d$.[1] *Define the functionals*

$$\begin{aligned} \psi_1(t) &:= \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx + \int_\Omega \Delta u\Delta u_t\,dx, \\ \psi_2(t) &:= -\int_\Omega\left(\Delta^2 u_t + \frac{1}{\rho+1}|u_t|^\rho u_t\right)\left(\int_0^\infty b(s)\eta^t(s)\,ds\right)dx, \end{aligned}$$

and $L(t) := mE(t) + \epsilon\psi_1(t) + \psi_2(t)$, *where* $m,\epsilon \ge 0$, *then* L *satisfies the following*:

(I) $L \sim E$ (*i.e.*, $\alpha_1 E(t) \le L(t) \le \alpha_2 E(t)$, *for some* $\alpha_1,\alpha_2 > 0$),

(II) $L'(t) \le -mE(t) + c(b\circ\Delta\eta^t)(t) + c(b\circ\Delta\eta^t)^{\frac{1}{1+\epsilon_0}}(t) + c\int_\Omega h^2(u_t)\,dx$.

Lemma 2.3 (Cf. [22], Lemma 4.1) *Let* h *satisfy* (H3). *Then, the solution of* (1) *satisfies*

$$\int_\Omega h^2(u_t)\,dx \le c\big(H^{-1}\big(G(t)\big) - E'(t)\big),$$

where $G(t) := \frac{1}{|\Omega_1|}\int_\Omega u_t h(u_t)\,dx \le -cE'(t)$ *and* $\Omega_1 = \{x\in\Omega : |u_t| \le \epsilon\}$ *for some* $c,\epsilon > 0$.

[1] where d denotes the depth of the potential well, we refer to [21] for such a construction of d.

Lemma 2.4 (Cf. [17], Lemma 3.3) *Let* $\delta_0 > 0$ *and assume that* $(H1)$ *and* $(H2)$ *hold, then we have* $\forall t \geq 0$,

$$\int_0^t b(s)\left|\Delta\eta^t\right|^2 ds \leq \left(\frac{1+t}{\delta_0}\right)B^{-1}\left(\frac{\delta_0\mu(1+t)}{t\xi(t)}\right),$$

where

$$\mu(t) := \int_0^t b'(s)\left|\Delta\eta^t\right|^2 ds \leq -cE'(t).$$

3 Decay result

We state and prove the major result in this section. We also provide an example to demonstrate the decay rate result. We introduce a few notations and a function for this purpose:

$$W_1(t) := \int_t^1 \frac{1}{sW'(s)}\,ds, \qquad W_2(t) = tW'(t), \qquad W_3(t) = t\left(W'\right)^{-1}(t), \tag{8}$$

where $W(t) = ((B^{-1}(t))^{\frac{1}{1+\epsilon_0}} + H^{-1})^{-1}$. Further, denote $\mathcal{S}$ to be the class of functions $\chi : (0,\infty) \to [0,\infty)$ and C^1 a function satisfying $\chi \leq 1$ and $\chi' \leq 0$. Also, for fixed $C_1, C_2 > 0$, assume that χ satisfies the estimate:

$$C_2 W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}(t)\right] \leq C_1\left(W_2\left(\frac{W_4(t)}{\chi(t)}\right) - \frac{W_2(W_4(t))}{\chi(t)}\right), \tag{9}$$

where W_3^* (defined in (14)) is the convex conjugate of W_3. Let $\delta_1, c > 0$ be generic constants, $q(t) = (t+1)^{-\frac{1}{1+\epsilon_0}}$ and

$$W_4(t) = W_1^{-1}\left(C_1\int_0^t \xi(s)\,ds\right). \tag{10}$$

Note: For ϵ small enough with $0 < \epsilon \leq 1$, $\epsilon W_4(s) \in \mathcal{S}$. Hence, the set $\mathcal{S}$ is nonempty.

Theorem 2 *Under the hypothesis of Lemma* 2.2, *there exists a constant* $C > 0$ *such that the solution to problem* (1) *satisfies*,

$$E(t) \leq \frac{CW_4(t)}{\chi(t)q(t)}, \quad \forall t \geq 0. \tag{11}$$

Proof Using Lemmas (2.2)–(2.4), note that

$$L'(t) \leq -mE(t) + \left[\left(\frac{t+1}{\delta_0}\right)B^{-1}\left(\frac{\delta_0\mu(t)}{(t+1)\xi(t)}\right)\right]^{\frac{1}{1+\epsilon_0}} + \left[c\int_0^\infty b(t+s)\left(1+\left\|\Delta u_0(s)\right\|_2^2\right)ds\right]^{\frac{1}{1+\epsilon_0}} + cH^{-1}\left(G(t)\right) - cE'(t).$$

Assume $L_1(t) := (L+cE)(t)$ and using Lemma (2.1), the above inequality can be written as

$$L_1'(t) \leq -mE(t) + c\left[\left(\frac{t+1}{\delta_0}\right)B^{-1}\left(\frac{\delta_0\mu(t)}{(t+1)\xi(t)}\right)\right]^{\frac{1}{1+\epsilon_0}} + \left[cf_1(t)\right]^{\frac{1}{1+\epsilon_0}} + cH^{-1}\left(G(t)\right).$$

Using Remark 1 with $\theta = \frac{1}{t+1}$ for all $t > 0$, we obtain

$$B^{-1}\left(\frac{\delta_0\mu(t)}{(t+1)\xi(t)}\right) \le B^{-1}\left(\frac{\delta_0 q(t)\mu(t)}{\xi(t)}\right).$$

Hence,

$$\begin{aligned} L_1'(t) &\le -mE(t) + c\left(\frac{t+1}{\delta_0}\right)^{\frac{1}{1+\epsilon_0}} B^{-1}\left(\frac{\delta_0 q(t)\mu(t)}{\xi(t)}\right)^{\frac{1}{1+\epsilon_0}} + \left[cf_1(t)\right]^{\frac{1}{1+\epsilon_0}} \\ &\quad + \frac{c}{\delta_0 q(t)} H^{-1}\big(G(t)q(t)\big). \end{aligned} \tag{12}$$

Denote $\beta(t) := \max(\frac{\delta_0 q(t)\mu(t)}{\xi(t)}, G(t)q(t))$ and recall $W(t) = ((B^{-1})^{\frac{1}{1+\epsilon_0}} + H^{-1})^{-1}(t)$.
Then (12) becomes, for any $t \ge 0$ and $\epsilon_0 \in (0,1)$,

$$L_1'(t) \le -mE(t) + c\frac{1}{\delta_0 q(t)} W^{-1}\big(\beta(t)\big) + cf_1^{\frac{1}{1+\epsilon_0}}.$$

Let $0 < \epsilon_1 < r := \min\{r_1, r_2\}$, define the functional L_2 as

$$L_2(t) := W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right)L_1(t),$$

then it is easy to see that $L_2 \sim E$, also

$$\begin{aligned} L_2'(t) &\le -mE(t)W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + c\frac{1}{\delta_0 q(t)} W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) W^{-1}\big(\beta(t)\big) \\ &\quad + cf_1^{\frac{1}{1+\epsilon_0}} W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right). \end{aligned} \tag{13}$$

Let the convex conjugate of W be denoted by W^* and be defined as

$$W^*(\tau) = \tau\big(W'\big)^{-1}(\tau) - W\big[\big(W'\big)^{-1}(\tau)\big], \quad \tau \in (0, W'(r)]. \tag{14}$$

Using the Generalized Young's inequality, W^* satisfies the following estimate

$$\tilde{A}\tilde{B} \le W^*(\tilde{A}) + W(\tilde{B}), \quad \tilde{A} \in \big(0, W'(r)\big], \text{ and } \tilde{B} \in (0, r]. \tag{15}$$

Therefore, with $\tilde{A} = W'(\epsilon_1 q(t)\frac{E(t)}{E(0)})$ and $\tilde{B} = W^{-1}(\beta(t))$, (13) leads to

$$\begin{aligned} L_2'(t) &\le -mE(t)W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + \frac{c}{\delta_0 q(t)} W^*\left(W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right)\right) \\ &\quad + c\frac{\beta(t)}{\delta_0 q(t)} + cf_1^{\frac{1}{1+\epsilon_0}} W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right). \end{aligned}$$

Multiplying the above equation by $\xi(t)$, we obtain

$$\begin{aligned} \xi(t)L_2'(t) &\le -m\xi(t)E(t)W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + \frac{c\xi(t)}{\delta_0 q(t)} W^*\left(W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right)\right) \\ &\quad + c\xi(t)\frac{\beta(t)}{\delta_0 q(t)} + c\xi(t)f_1^{\frac{1}{1+\epsilon_0}} W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right). \end{aligned}$$

Define a functional $L_3 := \xi L_2 + cE \sim E$. Since $\xi(t)\beta(t) \le -cE'(t)$ and $W^*(W'(t)) \le tW'(t)$, we obtain

$$\begin{aligned}
L_3'(t) &\le -mE(t)\xi(t)W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + \frac{c}{\delta_0}\epsilon_1\xi(t)\frac{E(t)}{E(0)}W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) \\
&\quad + c\xi(t)f_1^{\frac{1}{1+\epsilon_0}}W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) \\
&\le -\left(\frac{mE(0)}{\epsilon_1} - \frac{c}{\delta_0}\right)\epsilon_1\xi(t)\frac{E(t)}{E(0)}W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + c\xi(t)f_1^{\frac{1}{1+\epsilon_0}}W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right).
\end{aligned}$$

Consequently, from (8) and choosing ϵ_1 such that $k := (\frac{mE(0)}{\epsilon_1} - c) > 0$, we obtain

$$\begin{aligned}
L_3'(t) &\le -k\epsilon_1\xi(t)\frac{E(t)}{E(0)}W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + c\xi(t)f_1^{\frac{1}{1+\epsilon_0}}W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) \\
&\le -k\frac{\xi(t)}{q(t)}W_2\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + c\xi(t)f_1^{\frac{1}{1+\epsilon_0}}W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right).
\end{aligned} \tag{16}$$

Since $W_2'(t) = W'(t) + tW''(t)$, using the property of W, we conclude that $W_2(t), W_2'(t) > 0$ on $(0, r]$. Making use of (15) with $\tilde{A} = W'(\epsilon_1 q(t)\frac{E(t)}{E(0)})$ and $\tilde{B} = [\frac{c}{\delta_1}f_1^{\frac{1}{1+\epsilon_0}}]$ where $\delta_1 > 0$, we obtain

$$\begin{aligned}
&cf_1^{\frac{1}{1+\epsilon_0}}W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) \\
&\quad = \frac{\delta_1}{q(t)}\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right]W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) \\
&\quad \le \frac{\delta_1}{q(t)}W_3\left(W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right)\right) + \frac{\delta_1}{q(t)}W^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right] \\
&\quad \le \frac{\delta_1}{q(t)}\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right)W'\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + \frac{\delta_1}{q(t)}W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right] \\
&\quad \le \frac{\delta_1}{q(t)}W_2\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + \frac{\delta_1}{q(t)}W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right].
\end{aligned} \tag{17}$$

Now, combining (16) and (17), and choosing δ_1 small enough so that $k_1 = (k - \delta_1) > 0$, we have

$$L_3'(t) \le -k_1\frac{\xi(t)}{q(t)}W_2\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + \frac{\delta_1\xi(t)}{q(t)}W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right], \tag{18}$$

using the nonincreasing property of W_2 and for some $\gamma_1, \gamma_2 > 0$ that satisfies $\gamma_1 L_3(t) \le E(t) \le \gamma_2 L_3(t)$. We have, for some $\alpha = \frac{\gamma_1}{E(0)} > 0$,

$$W_2\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) \ge W_2\alpha L_3(t)q(t).$$

Assume $L_4(t) := \alpha L_3(t)q(t)$, then using (18) we have

$$\begin{aligned} L_4'(t) &\le \alpha q(t)\left(-k_1\frac{\xi(t)}{q(t)}W_2\left(\epsilon_1 q(t)\frac{E(t)}{E(0)}\right) + \frac{\delta_1\xi(t)}{q(t)}W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right]\right) \\ &\le -C_1\xi(t)W_2\big(L_4(t)\big) + C_2\xi(t)W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right], \end{aligned} \tag{19}$$

where $C_1 = \alpha k_1 > 0$ and $C_2 = \alpha\delta_1 > 0$. Since, $L_3 \sim E$, we have $L_4(t) \le \alpha_0 E(t)q(t)$ for some $\alpha_0 > 0$. In order to establish the decay estimate on $E(t)$, we will divide the proof into two cases (like in [15]). From the definition of χ, we can estimate $E(t)$ by considering two cases:

Case I: If $\alpha_0 E(t)q(t) \le 2\frac{W_4(t)}{\chi(t)}$, then we obtain

$$E(t) \le \left(\frac{2}{\alpha_0}\right)\frac{W_4(t)}{\chi(t)q(t)}. \tag{20}$$

Case II: If $\alpha_0 E(t)q(t) \ge 2\frac{W_4(t)}{\chi(t)}$, then observe that for any $0 \le s \le t$, we have $\alpha_0 q(s)E(s) > 2\frac{W_4(s)}{\chi(s)}$. Therefore, we obtain

$$L_4(s) > 2\frac{W_4(s)}{\chi(s)}, \quad 0 \le s \le t. \tag{21}$$

Using Remark 1, $0 < \chi(t) \le 1$ and the property of W_2, we have for any $0 < \epsilon_2 \le 1$, and $0 \le s \le t$,

$$\begin{aligned} W_2\big(\epsilon_2\chi(s)L_4(s) - \epsilon_2 W_4(s)\big) &= W_2\left(\epsilon_2\chi(s)L_4(s) - \frac{\epsilon_2\chi(s)W_4(s)}{\chi(s)}\right) \\ &\le \epsilon_2\chi(s)W_2\left(L_4(s) - \frac{W_4(s)}{\chi(s)}\right) \\ &\le \epsilon_2\chi(s)\left(L_4(s) - \frac{W_4(s)}{\chi(s)}\right)W'\left(L_4(s) - \frac{W_4(s)}{\chi(s)}\right). \end{aligned}$$

Using (21), we have for $0 \le s \le t$,

$$W_2\big(\epsilon_2\chi(t)L_4(s) - \epsilon_2 W_4(s)\big) \le \epsilon_2\chi(s)L_4(s)W'\big(L_4(s)\big) - \epsilon_2\chi(s)\frac{W_4(s)}{\chi(s)}W'\left(\frac{W_4(s)}{\chi(s)}\right). \tag{22}$$

Denote the functional L_5 as

$$L_5(t) := \epsilon_2\chi(t)L_4(t) - \epsilon_2 W_4(t), \tag{23}$$

where ϵ_2 is chosen in such a way that $L_5(0) \le 1$. Using (8), inequality (22) can be written as,

$$W_2\big(L_5(s)\big) \le \epsilon_2\chi(s)W_2\big(L_4(s)\big) - \epsilon_2\chi(s)W_2\left(\frac{W_4(s)}{\chi(s)}\right), \quad 0 \le s \le t. \tag{24}$$

Since, $L_5'(t) = \epsilon_2\chi'(t)L_4(t) + \epsilon_2\chi(t)L_4'(t) - \epsilon_2 W_4'(t)$, using (19) we obtain

$$L_5'(t) \le -C_1\xi(t)\epsilon_2\chi(t)W_2\big(L_4(t)\big) + C_2\epsilon_2\xi(t)\chi(t)W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right] - \epsilon_2 W_4'(t),$$

and applying (22) to the above inequality, we obtain

$$\begin{aligned} L_5'(t) \le & -C_1\xi(t)W_2\big(L_5(t)\big) + C_2\epsilon_2\xi(t)\chi(t)W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right] \\ & - C_1\epsilon_2\xi(t)\chi(t)W_2\left(\frac{W_4(t)}{\chi(t)}\right) - \epsilon_2 W_4'(t). \end{aligned} \tag{25}$$

From the definition of W_1 and W_4 we have $W_4'(t) = -C_1\xi(t)W_2(W_4(t))$, and using (9) we see that

$$\epsilon_2\xi(t)\chi(t)\left(C_2W_3^*\left[\frac{c}{\delta_1}q(t)f_1^{\frac{1}{1+\epsilon_0}}\right] - C_1W_2\left(\frac{W_4(t)}{\chi(t)}\right) + C_1\frac{W_2(W_4(t))}{\chi(t)}\right) \le 0,$$

therefore (25) leads to,

$$L_5'(t) \le C_1\xi(t)W_2\big(L_5(t)\big). \tag{26}$$

From (8) and (26) we obtain

$$C_1\xi(t) \le \big(W_1\big(L_5(t)\big)\big)'.$$

Integrating the above inequality over $[0,t]$, we observe that

$$W_1\big(L_5(t)\big) \ge C_1\int_0^t \xi(s) - W_1\big(L_5(0)\big).$$

Since W_1 is decreasing, $L_5(0) \le 1$ and $W_1(1) = 0$, we have

$$L_5(t) \le W_1^{-1}\left(C_1\int_0^t \xi(s)\,ds\right) = W_4(t).$$

From (23), we obtain

$$L_4(t) \le \left(\frac{1+\epsilon_2}{\epsilon_2}\right)\frac{W_4(t)}{\chi(t)}.$$

Similarly, recalling the definition of the functional L_4, we obtain

$$L_3(t) \le \left(\frac{1+\epsilon_2}{\alpha_0\epsilon_2}\right)\frac{W_4(t)}{\chi(t)q(t)}, \tag{27}$$

since $L_3 \sim E$, for some $c > 0$ we have $E(t) \le cL_3$, then (27) becomes

$$E(t) \le \left(\frac{c(1+\epsilon_2)}{\alpha_0\epsilon_2}\right)\frac{W_4(t)}{\chi(t)q(t)}, \tag{28}$$

and from (20) and (28), we conclude that

$$E(t) \le \frac{CW_4(t)}{\chi(t)q(t)}, \quad \text{where } C = \max\left(\frac{2}{\alpha_0}, \frac{c(1+\epsilon_2)}{\delta_0\epsilon_2}\right).$$

Hence, the theorem is proved. □

Remark 3 Assuming that χ satisfies (9) we have established the stability estimate of the energy function. In general, inequality (11) does not lead to the asymptotic stability $\lim_{t\to\infty} E(t) = 0$ (see [15] for more details). Hence, we have chosen χ in such a way that (9) and (11) give the best possible decay rate for E.

Example 3.1 Here, we present an example that demonstrates the main theorem of this paper. Assume that $\epsilon_0 \in (0,1)$, c is a positive generic constant and

$$b(t) = c_0(t+1)^{-\frac{1}{p-1}}, \qquad \xi(t) = \frac{c_0^{1-p}}{p-1}, \qquad B(t) = t^p, \qquad H(t) = t^{p(1+\epsilon_0)},$$

where $0 < c_0 < \frac{2-p}{p-1}$ and p is defined later. Moreover, assume that u_0 satisfies

$$\left[1 + \|\Delta u_0\|_2^2\right] \sim (1+t)^{\lambda}, \quad \text{for } \lambda < \frac{2-p}{p-1}.$$

Then, recalling the definitions from (8) and (10), we obtain:

$$W(t) = ct^{p(1+\epsilon_0)}, \qquad W_1(t) = c\left(t^{1-p(1+\epsilon_0)} - 1\right), \qquad W_2(t) = ct^{p(1+\epsilon_0)},$$
$$W_3(t) = ct^{\frac{p(1+\epsilon_0)}{p(1+\epsilon_0)-1}}, \qquad W_3^*(t) = ct^{p(1+\epsilon_0)}, \qquad W_4(t) = c(t+1)^{\frac{1}{1-p(1+\epsilon_0)}}.$$

Note that $q(t)f_1^{\frac{1}{(1+\epsilon_0)}}(t) \sim (t+1)^{(\frac{1}{1+\epsilon_0})(\lambda - \frac{1}{p-1})}$.

Now, choose $\chi(t) = (t+1)^{\gamma}$, where $\gamma < \min(0, -\frac{1}{p(1+\epsilon_0)-1} + \frac{1+\lambda-\lambda p}{(p-1)(1+\epsilon_0)})$. Then, it is easy to observe that $\chi(t)$ satisfies (9). Therefore (11) implies

$$E(t) \le \begin{cases} c(t+1)^{-\frac{1}{1+\epsilon_0}(\frac{2-p}{p-1}-\lambda)}, & \text{if } 0 < \lambda < \frac{2-p}{p-1} \text{ and } 1 < p < 2, \\ c(t+1)^{-(\frac{2-p+\epsilon_0(1-p)}{(1+\epsilon_0)(p-1+p\epsilon_0)})}, & \text{if } \lambda \le 0 \text{ and } 1 < p < \frac{3}{2}. \end{cases}$$

Hence, from the above estimate we conclude that $\lim_{t\to\infty} E(t) = 0$. From the above estimate on $E(t)$, we observe that as ϵ_0 converges to zero we obtain a faster rate of convergence when compared to ϵ_0 closer to 1.

Example 3.2 In this example, we assume that $\epsilon_0 \in (0,1)$ and b, ξ, B, and H are given by

$$b(t) = \frac{c_0}{(t+2)(\log^{\kappa}(t+2))}, \qquad \xi(t) = c_1, \qquad B(t) = t^2, \qquad H(t) = t^{2(1+\epsilon_0)},$$

where $0 < c_0 < (\kappa - 1)(\log^{\kappa-1}(2))$ and c_1 depends on c_0 and κ. Moreover, assume that u_0 satisfies

$$\left[1 + \|\Delta u_0\|_2^2\right] \sim \log^{\lambda}(t+2), \quad \text{for } \lambda < \kappa - 1.$$

From (8) and (10), we obtain:

$$W(t) = ct^{2(1+\epsilon_0)}, \qquad W_1(t) = c\left(t^{-1-2\epsilon_0} - 1\right), \qquad W_2(t) = ct^{2(1+\epsilon_0)},$$
$$W_3(t) = ct^{\frac{2(1+\epsilon_0)}{1+2\epsilon_0}}, \qquad W_3^*(t) = ct^{2(1+\epsilon_0)}, \qquad W_4(t) = c(t+1)^{\frac{-1}{1+2\epsilon_0}}.$$

It is easy to see that $q(t)f_1^{\frac{1}{(1+\epsilon_0)}}(t) \sim [(t+1)^{-1}\log^{\lambda-(\kappa-1)}(t+2)]^{(\frac{1}{1+\epsilon_0})}$.

Now, choose $\chi(t) = (\log^{\frac{(\kappa-1)-\lambda}{2(1+\epsilon_0)}}(t+2))((t+1)^{\frac{\epsilon_0}{(1+\epsilon_0)(1+2\epsilon_0)}})$, then $\chi(t)$ satisfies (9). Therefore, (11) implies

$$E(t) \leq c\log^{-[\frac{(\kappa-1)-\lambda}{2(1+\epsilon_0)}]}(t+2).$$

Since $\lambda < \kappa - 1$, we conclude that $\lim_{t\to\infty} E(t) = 0$.

Acknowledgements
The authors are very grateful to the anonymous reviewers for their comments and suggestions that greatly helped to improve this manuscript.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
All contributions of the authors are equal.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 September 2022 Accepted: 15 November 2022 Published online: 28 November 2022

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