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The inverse problem for the heat equation with reflection of the argument and with a complex coefficient

Elmira Mussirepova[1], Abdissalam Sarsenbi[1*] and Abdizhahan Sarsenbi[1]

*Correspondence:
abdisalam.sarsenbi@auezov.edu.kz
[1]Scientific Center for Theoretical and Applied Mathematics and Department of Mathematics, M. Auezov South Kazakhstan University, Tauke Khan, 5, 160012, Shymkent, Kazakhstan

Abstract

The paper is devoted to finding a solution and restoring the right-hand side of the heat equation with reflection of the argument in the second derivative, with a complex-valued variable coefficient. We prove a theorem on the Riesz basis property for eigenfunctions of the second-order differential operator with involution in the second derivative. We establish the existence and uniqueness of the solution of the studied problems by the method of separation of variables

Keywords: Heat equation with involution; Nonlocal heat equation; Eigenfunctions; Boundary value problem; Reflection of the argument; Riesz basis; Inverse problem

1 Introduction

In this paper, we consider the equation of the type

$$u_t(x,t) - u_{xx}(x,t) + \alpha u_{xx}(-x,t) + q(x)u(x,t) = f(x), \quad (x,t) \in \Omega, \tag{1.1}$$

with a complex-valued coefficient $q(x) = q_1(x) + iq_2(x)$, where $\Omega = \{-1 < x < 1, 0 < t < T\}$, $-1 < \alpha < 1$. Equation (1.1) contains a linear transformation of the involution

$$(Sf)(x) = f(-x)$$

in the second derivative. A transformation S is called an involution if

$$(S^2 f)(x) = f(x)$$

for any function $f \in L_2(-1,1)$. For $\alpha = 0$, equation (1.1) is a classical heat conduction equation. If $\alpha \neq 0$, then equation (1.1) relates the values of the second derivatives in two different points and becomes a nonlocal equation.

Equations with involutions have been studied by many researchers. An extensive bibliography can be found in monographs [1–3]. A number of papers are devoted to the solvability of direct and inverse problems for partial differential equations with involution (see, for example, [4–12] and references therein). In [4–7], inverse problems for equations with involution with constant coefficients were considered by the method of separation of variables. In [9], inverse problems were studied for a parabolic equation containing an arbitrary linear positive self-conjugate operator with discrete spectrum. Some partial differential equations of hyperbolic and elliptic types [13, 14] and equations of subharmonic oscillations [15] were studied using the involution transformation.

Our work is based on the spectral properties of the second-order nonself-conjugate differential operators with involution. Note that in the last decade, there were many papers devoted to spectral problems for differential operators with involution (see, for example, [16–19] and references therein). The basis property of eigenfunctions of the first-order differential operators with involution was studied in [16, 17] (see also references therein), and in [18–24] the cases of the second-order operators were considered. In [25, 26] the problems with operators containing an involution in lower terms are considered. For results concerning nonclassical spectral problems, we refer the reader to [27–30].

The paper consists of four sections. In Sect. 2 the problem statement and necessary definitions are considered. Section 3 is devoted to the study of properties of eigenvalues and eigenfunctions of the second-order differential operators with involution and with a variable complex coefficient. In Sect. 4, we prove a theorem on the existence and uniqueness of the solution of inverse problems for the heat equation with involution.

2 Statements of problems

The section is devoted to the main aspects of the Fourier method for equation (1.1). Let us introduce a nonself-adjoint second-order differential operator $L_{\alpha q}: D(L_{\alpha q}) \subset L_2(-1,1) \to L_2(-1,1)$ by the formula

$$L_{\alpha q} y = -y''(x) + \alpha y''(-x) + q(x)y(x)$$

with the domain of definition

$$D(L_{\alpha q}) = \bigl\{y(x) \in C^2[-1,1] : U_i(y) = a_{i1}y'(-1) + a_{i2}y'(1) + a_{i3}y(-1) + a_{i4}y(1) = 0, \quad i = 1,2\bigr\},$$

where a_{ij} are given complex numbers. We assume that the linear forms $U_1(u)$ and $U_2(u)$ are linearly independent. Let us write equation (1.1) as

$$u_t(x,t) + L_{\alpha q} u(x,t) = f(x), \quad (x,t) \in \Omega, \tag{2.1}$$

and further consider the differential operator $L_{\alpha q}$ with domain generated by one of the following four types of boundary conditions:

$$U_1(y) = y(-1) = 0, \qquad U_2(y) = y(1) = 0; \tag{D}$$

$$U_1(y) = y'(-1) = 0, \qquad U_2(y) = y'(1) = 0; \tag{N}$$

$$U_1(y) = y(-1) - y(1) = 0, \qquad U_2(y) = y'(-1) - y'(1) = 0; \tag{P}$$

$$U_1(y) = y(-1) + y(1) = 0, \qquad U_2(y) = y'(-1) + y'(1) = 0. \tag{AP}$$

Consider the following problem: Find a pair of functions $u(x,t)$ and $f(x)$ satisfying equation (2.1) and conditions

$$u(x,0) = \varphi(x), \qquad u(x,T) = \psi(x), \quad -1 \le x \le 1. \tag{2.2}$$

Definition 2.1 The pair of functions $u(x,t)$ and $f(x)$ is called a solution to problem (2.1)–(2.2) if the following three conditions are satisfied:

(1) the functions $u(x,t)$ and $u_x(x,t)$ are continuous in a closed domain $\bar{\Omega}$, $f(x) \in C[-1,1]$;

(2) in the domain Ω the function $u(x,t)$ is continuously differentiable with respect to t and has a continuous second-order derivative with respect to x;

(3) it satisfies equation (2.1) and conditions (2.2) in the general sense.

To prove the existence and uniqueness of a solution to the problem posed, we use the Fourier method. In this regard, we have to solve the problem of convergence of expansions of functions from a certain class in terms of eigenfunctions of the following spectral problem:

$$L_{\alpha q}X(x) = \lambda X(x). \tag{2.3}$$

3 Spectral properties of problem (2.3)

The convergence of expansions of eigenfunctions of the operator $L_{\alpha q}$ is easier to solve if the system of eigenfunctions $\{X_k(x)\}$ forms a Riesz basis in the class $L_2(-1,1)$. Therefore, in this section, we study the basis property of the eigenfunctions of the differential operator $L_{\alpha q}$. The differential operator $L_{\alpha q}$ is not a self-adjoint operator. The adjoint spectral problem is written as

$$L^*_{\alpha q}Z(x) = \bar{\lambda}Z(x),$$

where $L^*_{\alpha q}Z(x) = -Z''(x) + \alpha Z''(-x) + \bar{q}(x)Z(x)$ is the operator adjoint to the operator $L_{\alpha q}$. The domain of the adjoint operator $L^*_{\alpha q}$ is given by one of the same boundary conditions (D), (N), (P), or (AP), so that $D(L^*_{\alpha q}) = D(L_{\alpha q})$. We further assume that all eigenvalues of the operator $L_{\alpha q}$ are simple and zero is not an eigenvalue. Note that if the number $\lambda = 0$ is an eigenvalue, then we can consider the problem $L_{\alpha q}X(x) + \ell X(x) = \lambda X(x)$ for a fixed number ℓ. The number ℓ can be chosen so that the number $\lambda = 0$ is not an eigenvalue. In this case, the eigenfunctions will not change.

Let us denote the system of eigenfunctions of operator $L^*_{\alpha q}$ as $\{Z_n(x)\}$. The elements of the systems $\{X_k(x)\}$ and $\{Z_n(x)\}$ satisfy the biorthogonality condition [31]

$$(X_k, Z_j) = \int_{-1}^{1} X_k(x)\bar{Z}_j(x)\,dx = \delta_{kj},$$

where δ_{kj} the Kronecker symbol.

The eigenfunctions $X_k(x)$ satisfy the homogeneous equation (2.3), and therefore they can be considered normalized in the class $L_2(-1,1)$. For $q(x) \equiv 0$, we denote the operator $L_{\alpha q}$ as $L_{\alpha 0}$, $D(L_{\alpha 0}) = D(L_{\alpha q})$. Explicit forms of eigenvalues and eigenfunctions of the operator $L_{\alpha 0}$ are presented in [23, 24]. The boundary value problem $L_{\alpha 0}X(x) = \lambda X(x)$ has the Green's function $G_0(x,t,\lambda)$ if $\lambda = \rho^2$, where $\arg\sqrt{\rho} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, is not an eigenvalue [23, 24]. The simple poles of the Green's function $G_0(x,t,\lambda)$ are the numbers $\sqrt{\lambda_{0k}}$. Let the eigenvalues λ_{0k}, λ_k of operators $L_{\alpha 0}$ and $L_{\alpha q}$ be numbered so that $|\lambda_{0k}| < |\lambda_{0k+1}|$ and $|\lambda_k| < |\lambda_{k+1}|$ for all k, and $c_0 < |\sqrt{\lambda_{0k+1}}| - |\sqrt{\lambda_{0k}}|$ for a positive number c_0. Around every point $\sqrt{\lambda_{0k}}$, draw the circumference $O_{c_0}(\sqrt{\lambda_{0k}}) = \{\sqrt{\lambda} : |\sqrt{\lambda_{0k}} - \sqrt{\lambda}| = \frac{c_0}{2}\}$ of radius $\frac{c_0}{2}$ with center at this point. Then circumferences $P_k = \{\sqrt{\lambda} : |\sqrt{\lambda}| = |\sqrt{\lambda_{0k}}| + \frac{c_0}{2}\}$ centered at the origin of the complex $\sqrt{\lambda}$-plane and radius $|\sqrt{\lambda_k}| + \frac{c_0}{2}$ do not intersect circumferences $O_{c_0}(\sqrt{\lambda_{0k}})$. Let the function $G_q(x,t,\lambda)$ be the Green's function of the boundary value (2.3). We denote the partial sums of expansions of an arbitrary function $f \in L_1(-1,1)$ in terms of eigenfunctions of operators $L_{\alpha 0}$ and $L_{\alpha q}$ as

$$\sigma_m(f) = -\frac{1}{2\pi i}\int_{P_m}\left(\int_{-1}^{1} G_0\big(x,t,\rho^2\big)f(t)\,dt\right)\rho\,d\rho,$$

$$S_m(f) = -\frac{1}{2\pi i}\int_{P_m}\left(\int_{-1}^{1} G_q\big(x,t,\rho^2\big)f(t)\,dt\right)\rho\,d\rho,$$

respectively.

Definition 3.1 We say that the sequence $S_m(f)$ is equiconvergent with the $\sigma_m(f)$ on the interval $-1 \le x \le 1$ if $S_m - \sigma_m \to 0$ uniformly on the interval as $m \to \infty$.

The following equiconvergence theorem is valid.

Theorem 3.2 *Let the following three conditions be satisfied*:

(1) *all eigenvalues of the operators* $L_{\alpha 0}$ *and* $L_{\alpha q}$ *are simple*;

(2) *the complex-valued coefficient* $q(x)$ *belong to the class* $L_1(-1,1)$, *and in the case of problems* (P) *and* (AP), *we additionally require that* $\alpha \neq 0$;

(3) *for any* $\sqrt{\lambda}$ *beyond the circles* $O_{c_0}(\sqrt{\lambda_{0k}})$, *the Green's function* $G_0(x,t,\lambda)$ *satisfies the estimate*

$$\big|G_0(x,t,\lambda)\big| \le c_1(\alpha,c_0)\big|\sqrt{\lambda}\big|^{-1} r(x,t,\sqrt{\lambda}), \tag{3.1}$$

where $r(x,t,\sqrt{\lambda}) = (e^{-\alpha_2|\operatorname{Im}\sqrt{\lambda}|(2-|x|-|t|)} + e^{-\alpha_2|\operatorname{Im}\sqrt{\lambda}||x|-|t||})$, $\alpha_2 = \min\{\alpha_1,\alpha_0\}$, *and* $\alpha_0 = \sqrt{\frac{1}{1-\alpha}}$, $\alpha_1 = \sqrt{\frac{1}{1+\alpha}}$.

Then for any function $f \in L_1(-1,1)$, *the sequences* $S_m(f)$ *and* $\sigma_m(f)$ *equiconverge on the interval* $-1 \le x \le 1$.

Proof Note that for problems (N) and (P), estimates (3.1) were obtained and theorems on equiconvergence were proved [23, 24]. An analysis of the proofs of the equiconvergence theorems shows that they are based only on estimate (3.1) and do not depend on the type of the boundary conditions. Therefore the proof of the theorem is a word-for-word repetition of the proof of those results.

For problems (D) and (AP), as in [23, 24], the validity of estimate (3.1) can be shown. Therefore we suppose that estimate (3.1) is valid for all the problems under consideration. The operator L_{00} with boundary conditions (P) and (AP) has an infinite number of multiple eigenvalues. For problems of this type, the question of the Riesz basis property of the eigenfunctions of the operator L_{0q} with an arbitrary differentiable complex-valued coefficient $q(x)$ remains unsolved. Therefore we assume that $\alpha \neq 0$. Regarding the conditions of Theorem 3.2, note the following. If $q(x) \equiv 0$, then the operator $L_{\alpha 0}$ with periodic boundary conditions has two series of eigenvalues $\lambda_{k1} = (1+\alpha)k^2\pi^2$ and $\lambda_{k2} = (1-\alpha)k^2\pi^2$, which are simple if $\sqrt{\frac{1+\alpha}{1-\alpha}} \neq m_0$ and $\sqrt{\frac{1-\alpha}{1+\alpha}} \neq n_0$ for all integers m_0, n_0. The corresponding eigenfunctions have the form

$$X_{k1}(x) = \sin k\pi x, \quad k = 1, 2, \dots; \qquad X_{k2}(x) = \cos k\pi x, \quad k = 0, 1, 2, \dots,$$

and form a complete orthonormal system in $L_2(-1,1)$. □

Let us formulate a theorem on the basis property of the eigenfunctions of the operator $L_{\alpha q}$.

Theorem 3.3 *Let conditions* (1) *and* (2) *of Theorem* 3.2 *be satisfied. Then the system of eigenfunctions of the operator* $L_{\alpha q}$ *forms a basis in the space* $L_2(-1,1)$.

Proof It is known [23, 24] that the system of eigenfunctions of the operator $L_{\alpha 0}$ forms an orthonormal basis of the space $L_2(-1,1)$. For any function $f \in L_2(-1,1)$, the inequality

$$\left|f(x) - S_m(f)\right| \le \left|f(x) - \sigma_m(f)\right| + \left|\sigma_m(f) - S_m(f)\right|$$

is satisfied. Using Theorem 3.2 and the basis property of the eigenfunctions of the operator $L_{\alpha 0}$, we obtain the statement of the theorem. The theorem is proved. □

Theorem 3.3 does not answer the question of unconditional basis or Riesz basis property of the eigenfunctions of the operator $L_{\alpha q}$. The solution to this question is given by the following theorem.

Theorem 3.4 *Let all the conditions of Theorem* 3.3 *be satisfied, and let for all eigenvalues* λ_k *of the operator* $L_{\alpha q}$, *the inequalities* $|\operatorname{Im} \lambda_k| \le const$ *be fulfilled.*

Then the system of eigenfunctions of operator $L_{\alpha q}$ *forms a Riesz basis in the space* $L_2(-1,1)$, *and therefore for any function* $f(x) \in L_2(-1,1)$, *we have the relations* [32]

$$\begin{aligned} c_2\|f\|_{L_2}^2 &\le \sum_{k=1}^{\infty} \left| \int_{-1}^{1} f(x)\bar{X}_k(x)\,dx \right|^2 \le c_3\|f\|_{L_2}^2; \\ c_4\|f\|_{L_2}^2 &\le \sum_{k=1}^{\infty} \left| \int_{-1}^{1} f(x)\bar{Z}_k(x)\,dx \right|^2 \le c_5\|f\|_{L_2}^2. \end{aligned} \tag{3.2}$$

Proof Under conditions of the theorem, the system of eigenfunctions $\{Z_k(x)\}$ of the adjoint operator $L_{\alpha q}^*$ also forms a basis in $L_2(-1,1)$ [32]. Since any basis is a uniformly minimal system, the following condition [33] is satisfied for any number k:

$$\|X_k\|_{L_2}\|Z_k\|_{L_2} \le c_6, \quad c_6 > 0, \tag{3.3}$$

in the sense of the norm in $L_2(-1,1)$. It was shown in [19] that condition (3.3) is necessary and sufficient for the unconditional basis property in $L_2(-1,1)$ for each of the systems

$\{X_k(x)\}$ and $\{Z_k(x)\}$ if all eigenvalues λ of the operator $L_{\alpha q}$ satisfy the conditions

$$\sup|\operatorname{Im}\sqrt{\lambda}|<\infty;\qquad \sup_{\beta\geq 1}\sum_{|\operatorname{Re}\sqrt{\lambda}-\beta|\leq 1}1<\infty. \tag{3.4}$$

Estimates (3.4) are satisfied by the condition of the theorem. Therefore, by the main theorem in [19], each of the systems $\{X_k(x)\}$ and $\{Z_k(x)\}$ forms an unconditional basis. Since the system $\{X_k(x)\}$ is normalized and

$$1=(X_k,Z_k)\leq\|X_k\|_{L_2}\|Z_k\|_{L_2}\leq c_6,$$

the system $\{Z_k(x)\}$ is almost normalized: $1\leq\|Z_k\|_{L_2}\leq c_6$. Since an almost normalized unconditional basis is a Riesz basis [34], the theorem is proved. □

In the case of positive self-adjoint operators, the eigenvalues are real and positive. In the case of nonself-adjoint operators, the eigenvalues can be complex numbers. We must study the conditions for the nonnegativity of their real parts. The eigenvalues λ_k of the operator $L_{\alpha q}$ have the following properties.

Lemma 3.5 *Let $q\in C[-1,1]$. Then the inequality $|\operatorname{Im}\lambda_k|\leq\max|q_2(x)|$ is fulfilled for all numbers k. Under the additional condition $Req(x)=q_1(x)\geq 0$ in the interval $-1\leq x\leq 1$, the estimate $\operatorname{Re}\lambda_k>0$ is valid for all eigenvalues of the operator $L_{\alpha q}$.*

Proof Let us multiply both parts of equation (2.3) by the complex adjoint function $\bar{X}_k(x)$ and integrate the resulting equality over the interval $(-1,1)$. Each of the considered boundary conditions is self-adjoint. Therefore the resulting nonintegral terms vanish, and we get the equality

$$\int_{-1}^{1}\left|X'_k(x)\right|^2dx+\alpha\int_{-1}^{1}X'_k(-x)\bar{X}'_k(x)\,dx+\int_{-1}^{1}q(x)\left|X_k(x)\right|^2dx=\lambda_k\int_{-1}^{1}\left|X_k(x)\right|^2dx.$$

Writing out separately the real and imaginary parts of the last equality, we arrive at the following two relations:

$$\alpha\int_{-1}^{1}\operatorname{Im}\bigl[X'_k(-x)\bar{X}'_k(x)\bigr]\,dx+\int_{-1}^{1}q_2(x)\left|X_k(x)\right|^2dx=\operatorname{Im}\lambda_k\int_{-1}^{1}\left|X_k(x)\right|^2dx,$$
$$\int_{-1}^{1}\left|X'_k(x)\right|^2dx+\alpha\int_{-1}^{1}\operatorname{Re}\bigl\{X'_k(-x)\bar{X}'_k(x)\bigr\}\,dx+\int_{-1}^{1}q_1(x)\left|X_k(x)\right|^2dx$$
$$=\operatorname{Re}\lambda_k\int_{-1}^{1}\left|X_k(x)\right|^2dx.$$

As $\operatorname{Im}[X'_k(-x)\bar{X}'_k(x)]=(2i)^{-1}[X'_k(-x)\bar{X}'_k(x)-\bar{X}'_k(-x)\bar{\bar{X}}'_k(x)]$, the first integral in the first relation is equal to zero, and we get the inequality $|\operatorname{Im}\lambda_k|\leq\max|q_2(x)|$. The continuity of the coefficient q implies the first statement of the lemma.

To prove the second statement of the lemma, assume the contrary. Let there be a subsequence $\{\lambda_{k_n}\}$ satisfying the condition $\operatorname{Re}\lambda_{k_n}<0$. Then the second relation implies the inequality

$$\int_{-1}^{1} |X'_{k_n}(x)|^2\,dx + \alpha \int_{-1}^{1} \operatorname{Re}\{X'_{k_n}(-x)\bar{X}'_{k_n}(x)\}\,dx + \int_{-1}^{1} q_1(x)|X_{k_n}(x)|^2\,dx$$
$$= \operatorname{Re}\lambda_{k_n} \int_{-1}^{1} |X_{k_n}(x)|^2\,dx < 0,$$

from which we obtain the estimate

$$\int_{-1}^{1} |X'_{k_n}(x)|^2\,dx + \int_{-1}^{1} q_1(x)|X_{k_n}(x)|^2\,dx < -\alpha \int_{-1}^{1} \operatorname{Re}\{X'_{k_n}(-x)\bar{X}'_{k_n}(x)\}\,dx.$$

The left side of the last relation is positive. Then the right side of this relation is also positive. Therefore we can apply the inequality $2|ab| \le |a|^2 + |b|^2$ to the right side of the obtained relation. After simple transformations, we arrive at the estimate

$$(1-|\alpha|) \int_{-1}^{1} |X'_{k_n}|^2\,dx + \int_{-1}^{1} q_1(x)|X_{k_n}|^2\,dx < 0.$$

As $-1 < \alpha < 1$ and $q_1(x) \ge 0$, we come to a contradiction, which proves the lemma. □

Note that the lemma just proved is also valid for an arbitrary $q \in C[-1,1]$. In this case, $\operatorname{Re}\lambda_k > 0$ starting from some number k_0, so that $\operatorname{Re}\lambda_k \ge |\min q_1(x)|$, $k \ge k_0$, if $\min q_1(x) < 0$.

Theorem 3.4 and the first statement of Lemma 3.5 imply the following.

Corollary 3.6 *Let the following two conditions be satisfied*:

(1) *all eigenvalues of the operators* $L_{\alpha 0}$ *and* $L_{\alpha q}$ *are simple*;

(2) *the complex-valued coefficient* q *belongs to the class* $C[-1,1]$, *and in the case of problems* (P) *and* (AP), *we additionally require that* $\alpha \neq 0$.

Then the system of eigenfunctions of the operator $L_{\alpha q}$ *forms a Riesz basis in the space* $L_2(-1,1)$.

Note that the proved Theorem 3.3 means that each of the expansions

$$f(x) = \sum_{k=1}^{\infty} (f, Z_k) X_k(x), \qquad f(x) = \sum_{k=1}^{\infty} (f, X_k) Z_k(x)$$

converges to the function $f(x)$ in $L_2(-1,1)$, where

$$(f, X_k) = \int_{-1}^{1} f(x)\bar{X}_k(x)\,dx, \qquad (f, Z_k) = \int_{-1}^{1} f(x)\bar{Z}_k(x)\,dx. \tag{3.5}$$

We need the following lemma.

Lemma 3.7 *Let all the conditions of Corollary 3.6 be satisfied. Then for any function* φ *from the domain of the operator* $L_{\alpha q}$, *each of the Fourier series*

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, Z_k) X_k(x), \qquad \varphi(x) = \sum_{k=1}^{\infty} (\varphi, X_k) Z_k(x) \tag{3.6}$$

converges uniformly for $-1 \le x \le 1$.

Proof Let us rewrite equation (2.3) in the form (the number $\lambda = 0$ is not an eigenvalue)

$$X_k(x) = \frac{-X_k''(x) + \alpha X_k''(-x) + q(x)X_k(x)}{\lambda_k}.$$

Then

$$\begin{aligned}(\varphi, X_k) &= \int_{-1}^{1} \varphi(x)\bar{X}_k(x)\,dx \\ &= \int_{-1}^{1} \varphi(x)\left[\frac{-\bar{X}''_k(x) + \alpha\bar{X}''_k(-x) + \bar{q}(x)\bar{X}_k(x)}{\bar{\lambda}_k}\right]dx \\ &= \frac{1}{\bar{\lambda}_k}\int_{-1}^{1}\left[-\varphi''(x) + \alpha\varphi''(-x) + \bar{q}(x)\varphi(x)\right]\bar{X}_k(x)\,dx.\end{aligned}$$

Using this relation, the second series in (3.6) can be written as

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{A_k}{\bar{\lambda}_k} Z_k(x), \tag{3.7}$$

where

$$A_k = \int_{-1}^{1}\left[-\varphi''(x) + \alpha\varphi''(-x) + \bar{q}(x)\varphi(x)\right]\bar{X}_k(x)\,dx.$$

On the other hand, it is well known that the adjoint spectral problem is equivalent to the integral equation

$$Z_k(x) = \bar{\lambda}_k \int_{-1}^{1} G^*(x,t)\bar{Z}_k(t)\,dt,$$

where $G^*(x,t)$ is the Green's function of the adjoint boundary value problem for $\lambda = 0$. By definition [22, 23], the Green's function $G^*(x,t)$ is continuous for $x \in [-1,1]$ and $t \in [-1,1]$ and therefore is bounded. Denote $C_k(x) = \int_{-1}^{1} G^*(x,t)\bar{Z}_k(t)\,dt$. Then equality (3.7) takes the form

$$\sum_{k=1}^{\infty} \lambda_k{}^{-1} A_k Z_k(x) = \sum_{k=1}^{\infty} A_k C_k(x).$$

As $|A_k||C_k(x)| \le \frac{1}{2}(|A_k|^2 + |C_k(x)|^2)$, we have

$$\sum_{k=1}^{\infty} \left|\lambda_k{}^{-1} A_k Z_k(x)\right| = \sum_{k=1}^{\infty} \left|A_k C_k(x)\right| \le \sum_{k=1}^{\infty} |A_k|^2 + \sum_{k=1}^{\infty} \left|C_k(x)\right|^2. \tag{3.8}$$

Since the quantities A_k are the Fourier coefficients of the expansion in the Riesz basis $\{Z_k(x)\}$, $k = 1, 2, \dots$, and $C_k(x)$ are the Fourier coefficients of the expansion of the Green's function $G(x,t)$ in the Riesz basis $\{X_k(x)\}$, by (3.2) both series on the right-hand side of inequality (3.8) converge, and

$$\sum_{k=1}^{\infty} \left|C_k(x)\right|^2 \le \int_{-1}^{1} \left|G^*(x,t)\right|^2 dt \le M_0, \quad \forall x \in [-1,1].$$

This implies the absolute and uniform convergence of the second series (3.6). The absolute and uniform convergence of the first series (3.6) is proved similarly. The lemma is proved. □

4 Solvability of the inverse problem for the heat equation with involution

Let us state a theorem on the solvability of the inverse problem (2.1)–(2.2). Recall that the domain $D(L_{\alpha q})$ of the operator $L_{\alpha q}$ is generated by one of the boundary conditions (D), (N), (P), or (AP) (in the case of problems (P) and (AP), we additionally require that $\alpha \neq 0$) and all its eigenvalues are simple.

Theorem 4.1 *Let $q \in C[-1,1]$, $\varphi, \psi \in D(L_{\alpha q})$. Then problem (2.1)–(2.2) has a unique solution, which can be represented as*

$$u(x,t) = \varphi(x) + \sum_{k=1}^{\infty} \frac{(L_{\alpha q}\varphi, Z_k) - (L_{\alpha q}\psi, Z_k)}{\lambda_k(1-e^{-\lambda_k T})}\left(e^{-\lambda_k t} - 1\right)X_k(x),$$

$$f(x) = L_{\alpha q}\varphi(x) - \sum_{k=1}^{\infty} \frac{(L_{\alpha q}\varphi, Z_k) - (L_{\alpha q}\psi, Z_k)}{(1-e^{-\lambda_k T})}X_k(x).$$

Proof According to Theorem 3.4, each of the systems $\{X_k(x)\}$ and $\{Z_k(x)\}$ consisting of the eigenfunctions of operators $L_{\alpha q}$ and $L^*_{\alpha q}$, respectively, forms a Riesz basis in the space $L_2(-1,1)$. The functions $u(x,t)$ and $f(x)$ can be represented as

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x), \tag{4.1}$$

$$f(x) = \sum_{k=1}^{\infty} f_k X_k(x), \tag{4.2}$$

where $T_k(t)$ are unknown functions, and f_k are unknown constants. Substituting expressions (4.1) and (4.2) into equation (2.1), we obtain the equation

$$T'_k(t) + \lambda_k T_k(t) = f_k.$$

Solution to this equation has the form

$$T_k(t) = \frac{f_k}{\lambda_k} + C_k e^{-\lambda_k t}$$

with unknown constants f_k and C_k. Taking into account conditions (2.2), we obtain the following equalities:

$$T_k(0) = \frac{f_k}{\lambda_k} + C_k = \varphi_k, \qquad T_k(T) = \frac{f_k}{\lambda_k} + C_k e^{-\lambda_k T} = \psi_k,$$

where $\varphi_k = (\varphi, Z_k)$ and $\psi_k = (\psi, Z_k)$ are presented as in (3.5). From the last two equalities we find the constants

$$C_k = \frac{\varphi_k - \psi_k}{1-e^{-\lambda_k T}}, \qquad f_k = \lambda_k\varphi_k - \lambda_k C_k.$$

First, substituting the values $T_k(t)$ and f_k into (4.1) and (4.2), we obtain a formal solution to problem (2.1)–(2.2) in the form

$$u(x,t) = \varphi(x) + \sum_{k=1}^{\infty} C_k\left(e^{-\lambda_k t} - 1\right)X_k(x), \tag{4.3}$$

$$f(x) = L_{\alpha q}\varphi(x) - \sum_{k=1}^{\infty} \lambda_k C_k X_k(x). \tag{4.4}$$

In (4.3) the first term was obtained by virtue of Lemma 3.7. In (4.4), we used the equality $\lambda_k \varphi_k = \lambda_k(\varphi, Z_k) = (\varphi, L^*_{\alpha q} Z_k) = (L_{\alpha q}\varphi, Z_k)$ and the convergence of series (3.6), from which we obtain the convergence of the series $\sum_{k=1}^{\infty}(L_{\alpha q}\varphi, Z_k)X_k(x)$ to the function $L_{\alpha q}\varphi(x)$. The found values of the constants are transformed into the form

$$\begin{aligned} C_k &= \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \\ &= \frac{(\varphi, Z_k) - (\psi, Z_k)}{1 - e^{-\lambda_k T}} = \frac{(\varphi, L^*_{\alpha q} Z_k) - (\psi, L^*_{\alpha q} Z_k)}{\lambda_k(1 - e^{-\lambda_k T})} = \frac{(L_{\alpha q}\varphi, Z_k) - (L_{\alpha q}\psi, Z_k)}{\lambda_k(1 - e^{-\lambda_k T})}. \end{aligned}$$

From this we finally obtain the formal solution of problem (2.1)–(2.2) in the form

$$u(x,t) = \varphi(x) + \sum_{k=1}^{\infty} \frac{(L_{\alpha q}\varphi, Z_k) - (L_{\alpha q}\psi, Z_k)}{\lambda_k(1 - e^{-\lambda_k T})}\left(e^{-\lambda_k t} - 1\right)X_k(x), \tag{4.5}$$

$$f(x) = L_{\alpha q}\varphi(x) - \sum_{k=1}^{\infty} \frac{(L_{\alpha q}\varphi, Z_k) - (L_{\alpha q}\psi, Z_k)}{(1 - e^{-\lambda_k T})} X_k(x). \tag{4.6}$$

The convergence of series (4.6) follows from Lemma 3.7. From Lemma 3.5 ($\operatorname{Re}\lambda_k > 0$), we get the estimates $\lambda_k e^{-\lambda_k \tau} \to 0$, $k \to \infty$, and $|\lambda_k e^{-\lambda_k t}| \le |\lambda_k e^{-\lambda_k \tau}|$, $t \ge \tau > 0$, and starting from a number $k \ge k_0$, the inequality $|\lambda_k e^{-\lambda_k t}| \le |\lambda_k e^{-\lambda_k \tau}| < 1$, $t \ge \tau > 0$, is satisfied. These estimates allow us to prove the uniform convergence of series (4.5) and formally differentiated series corresponding to the functions $u_t(x,t)$, $u_{tt}(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$ in the domain $\bar{\Omega}$. The existence of a solution to problem (2.1)–(2.2) is proved.

Let us prove the uniqueness of the solution. Let there be two solutions $u_1(x), f_1(x)$ and $u_2(x), f_2(x)$ to problem (2.1)–(2.2). Then the functions $u(x,t) = u_1(x,t) - u_2(x,t)$ and $f(x) = f_1(x) - f_2(x)$ satisfy equation (2.1) and homogeneous conditions

$$u(0) = u(T) = 0. \tag{4.7}$$

Consider the following sequences:

$$u_k(t) = \int_{-1}^{1} u(x,t)\bar{X}_k(x)\,dx = \big(u(x,t), X_k(x)\big), \qquad f_k = \int_{-1}^{1} f(x)\bar{X}_k(x)\,dx = \big(f(x), X_k(x)\big).$$

Further, as $\bar{L}^*_{\alpha q} = L_{\alpha q}$ and $D(L^*_{\alpha q}) = D(L_{\alpha q})$, we have $X_k = \bar{Z}_k$. Then, taking into account the self-conjugation of the boundary conditions (D), (N), (P), and (AP), from equations (2.1) we obtain

$$\begin{aligned}\frac{d}{dt}u_k(t) &= \bigl(u_t(x,t), X_k(x)\bigr)\\ &= \bigl(L_{\alpha q}u(x,t) + f(x), X_k(x)\bigr)\\ &= \bigl(L_{\alpha q}u(x,t), X_k(x)\bigr) + f_k\\ &= \bigl(u(x,t), L^*_{\alpha q}X_k(x)\bigr) + f_k\\ &= \bigl(u(x,t), \bar{\lambda}_k X_k(x)\bigr) + f_k = \lambda_k u_k(t) + f_k.\end{aligned}$$

Thus for $u_k(t)$, we obtain the equation

$$\frac{d}{dt}u_k(t) = \lambda_k u_k(t) + f_k,$$

whose solution is

$$u_k(t) = A_k e^{\lambda_k t} - \frac{f_k}{\lambda_k},$$

where A_k and f_k are unknown constants. The function $u_k(t)$ satisfies homogeneous conditions (4.7):

$$u_k(0) = A_k - \frac{f_k}{\lambda_k} = 0, \qquad u_k(T) = A_k e^{\lambda_k T} - \frac{f_k}{\lambda_k} = 0.$$

Then from the equalities

$$A_k = \frac{f_k}{\lambda_k}, \qquad \frac{f_k}{\lambda_k}\bigl(e^{\lambda_k T} - 1\bigr) = 0$$

we get $f_k = A_k = 0$, from which it follows that

$$u_k(t) = \int_{-1}^{1} u(x,t)\bar{X}_k(x)\,dx \equiv 0.$$

Then the basis property of the system $\{X_k(x)\}$ implies the equality

$$f(x) \equiv 0, \qquad u(x,t) \equiv 0,$$

that is, $f_1(x) = f_2(x)$, $u_1(x,t) = u_2(x,t)$. The theorem is proved. □

5 Conclusions

Summarizing, we have proved the unique solvability of inverse problems for the heat equation with involution with a complex-valued variable coefficient. In the course of proving the solvability, we have proved an important theorem on the Riesz basis property of eigenfunctions of the second-order differential operator with involution in the second derivative. The results obtained can be useful for further development of the theory of solvability of mixed problems for partial differential equations with involution.

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