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A variational approach for mixed elliptic problems involving the p-Laplacian with two parameters

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Abstract

By exploiting an abstract critical-point result for differentiable and parametric functionals, we show the existence of infinitely many weak solutions for nonlinear elliptic equations with nonhomogeneous boundary conditions. More accurately, we determine some intervals of parameters such that the treated problem admits either an unbounded sequence of solutions or a pairwise distinct sequence of solutions that strongly converges to zero. No symmetric condition on the nonlinear term is considered.

MSC: 35J25; 35J20

Keywords: Mixed problem; Critical points; Weak solutions; Infinitely many solutions

1 Introduction

The purpose of this paper is to discuss the mixed elliptic problem involving the p-Laplacian

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \mu g(u) & \text{on } \Gamma_2, \end{cases} \tag{$M_{\lambda,\mu}$}$$

where Ω is a nonempty bounded open subset of the Euclidean space $(\mathbb{R}^n, |\cdot|)$, $n \geq 3$, with a boundary of class C^1, Γ_1 and Γ_2 are two smooth $(n-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \Sigma$, with Σ a smooth $(n-2)$-dimensional submanifold of $\partial\Omega$, $\lambda > 0$ and $\mu \geq 0$ are real parameters, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > n$, $q \in L^\infty(\Omega)$ with $q_0 := \operatorname{ess\,inf}_\Omega q > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, and ν is the outer unit normal to $\partial\Omega$.

Elliptic differential problems with mixed boundary conditions of Dirichlet–Neumann type can be exploited to characterize many concrete situations, acting as models in applied sciences. Solidification and melting of a material in industrial processes as well as wave phenomena and heat transfer are classical examples of fields in which mixed conditions are important. In particular, a conversant example is presented by an iceberg partially submerged in water, for which mixed conditions must be constrained on its boundary. To be precise, in the portion under the water, one constrains a Dirichlet boundary condition, while in the remaining part of the boundary that is in contact with the air, Neumann conditions are used.

In the literature, the existence, multiplicity, and regularity of solutions for elliptic problems with mixed boundary conditions have been considered in the last decades; see for instance the papers [1, 4–6, 8–10, 13] and the references therein. In particular, the authors in [5], by using a smooth version of [7, Theorem 2.1], established the existence of infinitely many solutions for the following mixed boundary value problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2. \end{cases}$$

Recently, Bonanno and D'Aguì [4] studied problem ($M_{\lambda,\mu}$). They obtained the existence of at least two nontrivial and nonnegative weak solutions for problem ($M_{\lambda,\mu}$) by applying a two nonzero critical-point theorem, which is obtained in [3].

Further, Bonanno *et al.* in [6], by means of three critical-point theorems, studied the existence of at least three nonzero and nonnegative weak solutions for problem ($M_{\lambda,\mu}$).

Motivated by the above works, the goal of the present work is to acquire sufficient conditions to suggest that problem ($M_{\lambda,\mu}$) has infinitely many nontrivial and nonnegative weak solutions. For this, we need that the primitive F of f assures an appropriate oscillatory behavior either at infinity (for finding unbounded solutions) or at the origin (for obtaining arbitrarily small solutions), while G, the primitive of g, has an adequate growth (see Theorems 3.1 and 3.7). Our analysis is based on the last critical-point theorem of Bonanno and Molica Bisci [7] and is included in Lemma 2.3 below.

Finally, we cite the papers [11, 14, 15] for some relevant contributions related to the subject of this work.

2 Preliminaries

In the present section, we first give the notion of weak solutions, the variational setting of the problem and some classical definitions and the results that we will use in the rest of the paper.

Let X be a subset of the Sobolev space $W^{1,p}(\Omega)$, by which we mean

$$X = W^{1,p}_{0,\Gamma_1}(\Omega) := \{u \in W^{1,p}(\Omega) : u|_{\Gamma_1} = 0\}$$

equipped with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p\,dx + \int_\Omega q(x)|u(x)|^p\,dx\right)^{1/p}.$$

Definition 2.1 A weak solution of problem ($M_{\lambda,\mu}$) is any $u \in X$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)v(x)\,dx$$

$$= \lambda \int_{\Omega} f\bigl(x, u(x)\bigr) v(x)\, dx + \mu \int_{\Gamma_2} g\bigl(\gamma\bigl(u(x)\bigr)\bigr) \gamma\bigl(v(x)\bigr)\, d\sigma,$$

for all $v \in X$, where $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace operator.

We note that, since $p > n$, $W^{1,p}(\Omega)$ is embedded in $C_0(\overline{\Omega})$, X is embedded in $C_0(\overline{\Omega})$. Thus, by setting

$$k := \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{(\int_\Omega |\nabla u(x)|^p\, dx + \int_\Omega q(x)|u(x)|^p\, dx)^{1/p}},$$

one has

$$\|u\|_\infty \le k\|u\|, \tag{2.1}$$

where $\|\cdot\|_\infty$ is the usual norm in $L^\infty(\Omega)$.

Note that if Ω is convex, an explicit upper bound for the constant k is

$$k_1 := 2^{\frac{p-1}{p}} \max\left\{ \left(\frac{1}{\int_\Omega q(x)\, dx} \right)^{\frac{1}{p}}, \frac{\operatorname{diam}(\Omega)}{n^{\frac{1}{p}}} \left(\frac{p-1}{p-n} \operatorname{meas}(\Omega) \right)^{\frac{p-1}{p}} \frac{\|q\|_\infty}{\int_\Omega q(x)\, dx} \right\},$$

where $\operatorname{diam}(\Omega)$ is the diameter of Ω, $\operatorname{meas}(\Omega)$ is the Lebesgue measure of Ω, and, explicitly, $k \le k_1$ (see [2, Remark 1].

Now, denoting the Euler function by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z}\, dz, \quad \forall t > 0,$$

we define

$$\sigma(p,n) := \inf_{\mu \in]0,1[} \frac{1-\mu^n}{\mu^n(1-\mu)^p},$$

and consider $\overline{\mu} \in]0,1[$ such that $\sigma(p,n) = \frac{1-\overline{\mu}^n}{\overline{\mu}^n(1-\overline{\mu})^p}$. Further, set

$$R := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega).$$

Easy computations show that there is $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$, and, for $\overline{\mu} \in]0,1[$, one has $B(x_0, \overline{\mu}R) \subset B(x_0, R)$. Put

$$g_{\overline{\mu}}(p,n) := \overline{\mu}^n + \frac{1}{(1-\overline{\mu})^p} n B_{(\overline{\mu},1)}(n, p+1),$$

where $B_{(\overline{\mu},1)}(n,p+1)$ denotes the generalized incomplete beta function defined as follows

$$B_{(\overline{\mu},1)}(n,p+1) := \int_{\overline{\mu}}^1 t^{n-1}(1-t)^{(p+1)-1}\, dt.$$

We also denote by

$$\omega_R := \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} R^n,$$

the measure of the n-dimensional ball of radius R, and

$$a(\Gamma_2) := \int_{\Gamma_2} d\sigma.$$

The following lemma is taken from [4, Lemma 2.3].

Lemma 2.2 *If we assume $f(x,0) \geq 0$ for a.e. $x \in \Omega$, then the weak solutions of problem $(M_{\lambda,\mu})$ are nonnegative.*

We will establish our results by exploiting the following smooth version of Theorem 2.1 of [7], which is a more exact version of Ricceri's Variational Principle [12, Theorem 2.5].

Lemma 2.3 *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \text{and} \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, the following properties hold:

(a) *If $\gamma < +\infty$, then for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either*
  - (a_1) *$I_\lambda := \Phi - \lambda\Psi$ possesses a global minimum, or*
  - (a_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(b) *If $\delta < +\infty$, then for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*
  - (b_1) *there is a global minimum of Φ that is a local minimum of I_λ, or*
  - (b_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that weakly converges to a global minimum of Φ, with $\lim_{n \to +\infty} \Phi(u_n) = \inf_X \Phi$.*

3 Main results

Before presenting the main result, we define some notation. We set

$$\delta := \frac{R}{k(\omega_R[\overline{\mu}^n \sigma(p,n) + \|q\|_\infty R^p g_{\overline{\mu}}(p,n)])^{\frac{1}{p}}},$$

$$A_\infty := \liminf_{\xi \to +\infty} \frac{\int_\Omega \max_{|t| \leq \xi} F(x,t)\,dx}{\xi^p},$$

$$B^\infty := \limsup_{\xi \to +\infty} \frac{\int_{B(x_0, \overline{\mu}R)} F(x,\xi)\,dx}{\xi^p},$$

where the constants R, k, ω_R, $\overline{\mu}$, $\sigma(p,n)$, and $g_{\overline{\mu}}(p,n)$ have been defined in Sect. 2, and F is the potential of f defined by

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

We assume throughout, and without further mention, that $f(x,0) \geq 0$ for a.e. $x \in \Omega$.

We formulate our main result as follows.

Theorem 3.1 *Assume that*

(A1) $F(x,t) \geq 0$ *for each* $(x,t) \in (B(x_0,R) \setminus B(x_0,\overline{\mu}R)) \times \mathbb{R}^+$;

(A2) $A_\infty < \delta^p B^\infty$.

Then, for every $\lambda \in \Lambda :=]\frac{1}{p(k\delta)^p B^\infty}, \frac{1}{pk^p A_\infty}[$ *and for every arbitrary nonnegative continuous function* $g : \mathbb{R} \to \mathbb{R}$, *whose potential* $G(t) := \int_0^t g(\xi)\,d\xi$ *for all* $t \in \mathbb{R}$, *satisfying the condition*

$$G^\infty := \limsup_{\xi \to +\infty} \frac{G(\xi)}{\xi^p} < +\infty, \tag{3.1}$$

if we put

$$\mu_{G,\lambda} := \frac{1}{pk^p a(\Gamma_2) G^\infty}\bigl(1 - \lambda pk^p A_\infty\bigr),$$

where $\mu_{G,\lambda} = +\infty$ *when* $G^\infty = 0$, *then problem* $(M_{\lambda,\mu})$ *has an unbounded sequence of nonnegative weak solutions for every* $\mu \in [0, \mu_{G,\lambda}[$ *in* X.

Proof Our purpose is to apply Lemma 2.3(a) to problem $(M_{\lambda,\mu})$. For this, fix $\bar{\lambda} \in \Lambda$ and g satisfying our hypotheses. Since $\bar{\lambda} < \frac{1}{pk^p A_\infty}$, one has

$$\mu_{G,\bar{\lambda}} = \frac{1}{pk^p a(\Gamma_2) G^\infty}\bigl(1 - \bar{\lambda} pk^p A_\infty\bigr) > 0.$$

Now, fix $\bar{\mu} \in [0, \mu_{G,\bar{\lambda}}[$. For every $u \in X$, let the functionals $\Phi, \Psi_{\bar{\lambda},\bar{\mu}} : E^\alpha \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{p}\|u\|^p,$$

$$\Psi_{\bar{\lambda},\bar{\mu}}(u) := \int_\Omega F\bigl(x, u(x)\bigr)\,dx + \frac{\bar{\mu}}{\bar{\lambda}} \int_{\Gamma_2} G\bigl(\gamma\bigl(u(x)\bigr)\bigr)\,d\sigma,$$

and set

$$I_{\bar{\lambda},\bar{\mu}}(u) := \Phi(u) - \bar{\lambda}\Psi_{\bar{\lambda},\bar{\mu}}(u), \quad u \in X.$$

It is known that $\Phi, \Psi_{\bar{\lambda},\bar{\mu}} \in C^1(X,\mathbb{R})$ and they satisfy all regularity hypotheses requested in Lemma 2.3. In particular, for each $u, v \in X$ we have

$$\Phi'(u)(v) = \int_\Omega \bigl|\nabla u(x)\bigr|^{p-2}\nabla u(x) \cdot \nabla v(x)\,dx + \int_\Omega q(x)\bigl|u(x)\bigr|^{p-2}u(x)v(x)\,dx,$$

$$\Psi'_{\bar{\lambda},\bar{\mu}}(u)(v) = \int_\Omega f\big(x,u(x)\big)v(x)\,dx + \frac{\bar{\mu}}{\bar{\lambda}}\int_{\Gamma_2} g\big(\gamma\big(u(x)\big)\big)\gamma\big(v(x)\big)\,d\sigma.$$

Hence, the critical points of $I_{\bar{\lambda},\bar{\mu}}$ are weak solutions of problem $(M_{\bar{\lambda},\bar{\mu}})$.

First, we prove that $\bar{\lambda} < 1/\gamma$. Hence, assume that $\{\xi_n\}$ is a sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n = +\infty$ and

$$\lim_{n\to+\infty}\frac{\int_\Omega \max_{|t|\le\xi_n} F(x,t)\,dx}{\xi_n^p} = A_\infty.$$

Set $r_n := \frac{1}{p}(\frac{\xi_n}{k})^p$ for every $n\in\mathbb{N}$. Then, for all $v\in X$ with $\Phi(v) < r_n$, taking (2.1) into account, we have $\|v\|_\infty \le \xi_n$. Clearly, $\Phi(0) = \Psi_{\bar{\lambda},\bar{\mu}}(0) = 0$. Thus, for every $n\in\mathbb{N}$,

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)}\frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi_{\bar{\lambda},\bar{\mu}}(v)) - \Psi_{\bar{\lambda},\bar{\mu}}(u)}{r_n - \Phi(u)}\\ &\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi_{\bar{\lambda},\bar{\mu}}(v)}{r_n}\\ &\le pk^p\left(\frac{\int_\Omega \max_{|t|\le\xi_n}F(x,t)\,dx}{\xi_n^p} + \frac{\bar{\mu}}{\bar{\lambda}}\frac{a(\Gamma_2)G(\xi_n)}{\xi_n^p}\right).\end{aligned}$$

Therefore, from hypothesis (A2) and situation (3.1), we obtain

$$\gamma \le \liminf_{n\to+\infty}\varphi(r_n) \le pk^p\left(A_\infty + \frac{\bar{\mu}}{\bar{\lambda}}a(\Gamma_2)G^\infty\right) < +\infty.$$

It follows from $\bar{\mu}\in[0,\mu_{G,\bar{\lambda}}[$ that

$$\gamma \le pk^p\left(A_\infty + \frac{\bar{\mu}}{\bar{\lambda}}a(\Gamma_2)G^\infty\right) < pk^pA_\infty + \frac{1-pk^p\bar{\lambda}A_\infty}{\bar{\lambda}}.$$

Hence,

$$\bar{\lambda} = \frac{1}{pk^pA_\infty + (1-pk^p\bar{\lambda}A_\infty)/\bar{\lambda}} < \frac{1}{\gamma}.$$

Assume that $\bar{\lambda}$ is fixed. We claim that the functional $I_{\bar{\lambda},\bar{\mu}}$ is unbounded from below. Since $1/\bar{\lambda} < p(k\delta)^pB^\infty$, there is a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty}\eta_n = +\infty$ and

$$1/\bar{\lambda} < \tau < p(k\delta)^p\frac{\int_{B(x_0,\overline{\mu}R)}F(x,\eta_n)\,dx}{\eta_n^p} \tag{3.2}$$

for every $n\in\mathbb{N}$ large enough. For all $n\in\mathbb{N}$, define $w_n\in X$ by

$$w_n(x) := \begin{cases} 0, & \text{if } x\in\Omega\setminus B(x_0,R),\\ \frac{\eta_n}{R(1-\overline{\mu})}(R-|x-x_0|), & \text{if } x\in B(x_0,R)\setminus B(x_0,\overline{\mu}R),\\ \eta_n, & \text{if } x\in B(x_0,\overline{\mu}R).\end{cases} \tag{3.3}$$

Hence, one has

$$\begin{aligned}\|w_n\|^p &= \int_\Omega \left|\nabla w_n(x)\right|^p dx + \int_\Omega q(x)\left|w_n(x)\right|^p dx \\ &= \frac{\eta_n^p}{R^p(1-\overline{\mu})^p}\left(\int_{B(x_0,R)\setminus B(x_0,\overline{\mu}R)} \left(1 + q(x)\left|R - |x-x_0|\right|^p\right) dx\right) \\ &\quad + \eta_n^p \int_{B(x_0,\overline{\mu}R)} q(x)\, dx.\end{aligned}$$

Taking into account that

$$\int_{B(x_0,R)\setminus B(x_0,\overline{\mu}R)} \left|R - |x-x_0|\right|^p dx = n\omega_R R^p B_{(\overline{\mu},1)}(n,p+1),$$

for any fixed $n \in \mathbb{N}$, we deduce that

$$\Phi(w_n) \le \frac{1}{p}\left(\frac{\eta_n}{R}\right)^p \omega_R\left[\overline{\mu}^n \sigma(p,n) + \|q\|_\infty R^p g_{\overline{\mu}(p,n)}\right] = \frac{1}{p(k\delta)^p}\eta_n^p. \tag{3.4}$$

On the other hand, bearing (A1) in mind, we obtain

$$\begin{aligned}\Psi_{\bar{\lambda},\bar{\mu}}(w_n) &= \int_{B(x_0,R)\setminus B(x_0,\overline{\mu}R)} F\left(x, \frac{\eta_n}{R(1-\overline{\mu})}\left(R - |x-x_0|\right)\right) dx + \int_{B(x_0,\overline{\mu}R)} F(x,\eta_n)\, dx \\ &\ge \int_{B(x_0,\overline{\mu}R)} F(x,\eta_n)\, dx.\end{aligned} \tag{3.5}$$

It follows from (3.2), (3.4), and (3.5) that

$$I_{\bar{\lambda},\bar{\mu}}(w_n) \le \frac{1}{p(k\delta)^p}\eta_n^p - \bar{\lambda}\int_{B(x_0,\overline{\mu}R)} F(x,\eta_n)\, dx < \frac{1}{p(k\delta)^p}\eta_n^p(1 - \bar{\lambda}\tau),$$

for all $n \in \mathbb{N}$ large enough. Since $\bar{\lambda}\tau > 1$ and $\lim_{n\to+\infty} \eta_n = +\infty$, we have

$$\lim_{n\to+\infty} I_{\bar{\lambda},\bar{\mu}}(w_n) = -\infty.$$

Hence, our claim is established. Thus, $I_{\bar{\lambda},\bar{\mu}}$ has no global minimum and, by Lemma 2.3(a), there is a sequence $\{u_n\}$ of critical points of $I_{\bar{\lambda},\bar{\mu}}$ such that $\lim_{n\to+\infty} \|u_n\| = +\infty$. Hence, problem $(M_{\bar{\lambda},\bar{\mu}})$ has an unbounded sequence of weak solutions, and bearing Lemma 2.2 in mind, the weak solutions are nonnegative. The proof is complete. □

Remark 3.2 When f is a nonnegative function, hypothesis (A1) holds and hypothesis (A2) becomes

(A2′) $A'_\infty := \liminf_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)\,dx}{\xi^p} < \delta^p B^\infty$.

In this case, (A2′) certifies that for all $\lambda \in \left]\frac{1}{p(k\delta)^p B^\infty}, \frac{1}{pk^p A'_\infty}\right[$ and all $\mu \in [0, \frac{1}{pk^p a(\Gamma_2)G^\infty}(1 - pk^p\lambda A'_\infty)[$, problem $(M_{\lambda,\mu})$ admits an unbounded sequence of nonnegative weak solutions in X.

Here, we present some specific cases of the main result. The first one is in the autonomous case.

Corollary 3.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $F(t) := \int_0^t f(\xi)\,d\xi$ for all $t \in \mathbb{R}$, and assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 0 \quad \text{and} \quad 0 < B_\star^\infty := \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} \le +\infty.$$

Then, for every $\lambda > \frac{1}{p(k\delta)^p \overline{\mu}^n \omega_R B_\star^\infty}$, for every arbitrary nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential satisfying (3.1), and for each $\mu \in [0, \frac{1}{pk^p a(\Gamma_2)G^\infty}[$, the problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \mu g(u) & \text{on } \Gamma_2, \end{cases} \tag{$AM_{\lambda,\mu}$}$$

has an unbounded sequence of nonnegative weak solutions in X.

Corollary 3.4 *Assume that the hypothesis* (A1) *holds, and*

$$A_\infty < \frac{1}{pk^p} \quad \text{and} \quad B^\infty > \frac{1}{p(k\delta)^p}.$$

Then, for every arbitrary nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential G satisfies the condition (3.1), if we put

$$\mu_G := \frac{1}{pk^p a(\Gamma_2)G^\infty}\left(1 - pk^p A_\infty\right),$$

problem $(M_{1,\mu})$ has an unbounded sequence of nonnegative weak solutions for every $\mu \in [0, \mu_G[$ in X.

Now, we present the following existence result in which instead of the hypothesis (A2) a more general situation is considered.

Theorem 3.5 *Assume that the hypothesis* (A1) *holds, and there exist two sequences $\{a_n\}$ and $\{b_n\}$ in $[0, +\infty[$, with $\lim_{n\to\infty} b_n = +\infty$, such that*

(A3) *for some $n_0 \in \mathbb{N}$ one has $a_n < \delta b_n$ for each $n \ge n_0$;*

(A4) $\mathcal{A}_\infty := \lim_{n\to\infty} \frac{\int_\Omega \max_{|t|\le b_n} F(x,t)\,dx - \int_{B(x_0,\overline{\mu}R)} F(x,a_n)\,dx}{b_n^p - \frac{1}{\delta^p} a_n^p} < \delta^p B^\infty$.

Then, for each $\lambda \in \left]\frac{1}{p(k\delta^p)B^\infty}, \frac{1}{pk^p \mathcal{A}_\infty}\right[$, the problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2, \end{cases} \tag{M_λ}$$

has an unbounded sequence of nonnegative weak solutions in X.

Proof Obviously, from (A4) we obtain (A2), by choosing $a_n = 0$ for all $n \in \mathbb{N}$. Further, if we assume (A4) instead of (A2) and set $r_n := \frac{1}{p}(\frac{b_n}{k})^p$ for every $n \in \mathbb{N}$, by the same reasoning as in the proof of Theorem 3.1 with $\mu = 0$, we obtain

$$
\begin{aligned}
\varphi(r_n) &\leq \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{\int_\Omega \max_{|t|\leq b_n} F(x,t)\,dx - \int_\Omega F(x,u(x))\,dx}{\frac{1}{p}(\frac{b_n}{k})^p - \frac{1}{p}\|u\|^p} \\
&\leq \frac{\int_\Omega \max_{|t|\leq b_n} F(x,t)\,dx - \int_{B(x_0,\overline{\mu}R)} F(x,a_n)\,dx}{\frac{1}{p}(\frac{b_n}{k})^p - \frac{1}{p(k\delta)^p}a_n^p},
\end{aligned}
$$

for every $n \geq n_0$, where w_n is as in (3.3) with a_n instead of η_n. Hence, we have a favorable conclusion. □

The next result is an outcome of Theorem 3.5 and suggests the existence of infinitely many solutions to (M_λ) for every λ that lies in a precise half-line.

Corollary 3.6 *Assume that the hypothesis* (A1) *holds, and there exist two sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* $[0,+\infty[$, *with* $\lim_{n\to\infty} b_n = +\infty$, *such that* (A3) *holds and*

(A5) $\int_{B(x_0,\overline{\mu}R)} F(x,a_n)\,dx = \int_\Omega \max_{|t|\leq b_n} F(x,t)\,dx$ *for each* $n \in \mathbb{N}$.

If $B^\infty > 0$, *then, for each* $\lambda > \frac{1}{p(k\delta)^p B^\infty}$, *problem* ($M_\lambda$) *has an unbounded sequence of nonnegative weak solutions in* X.

Proof By (A5) we obtain $\mathcal{A}_\infty = 0$. Thus, since $B^\infty > 0$, hypothesis (A4) of Theorem 3.5 holds and the proof is complete. □

Set

$$
\begin{aligned}
A_0 &:= \liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\leq\xi} F(x,t)\,dx}{\xi^p}, \\
B^0 &:= \limsup_{\xi\to 0^+} \frac{\int_{B(x_0,\overline{\mu}R)} F(x,\xi)\,dx}{\xi^p}.
\end{aligned}
$$

In a similar way as in the proof of Theorem 3.1 but using conclusion (b) of Lemma 2.3 instead of (a), we will obtain the following result.

Theorem 3.7 *Let the hypotheses* (A1) *and*

(A6) $A_0 < \delta^p B^0$,

be satisfied. Then, for every $\lambda \in \,]\frac{1}{p(k\delta)^p B^0}, \frac{1}{pk^p A_0}[$ *and for every arbitrary nonnegative continuous function* $g : \mathbb{R} \to \mathbb{R}$, *whose potential satisfies the condition*

$$
G^0 := \limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^p} < +\infty, \tag{3.6}
$$

if we put

$$
\widetilde{\mu}_{G,\lambda} := \frac{1}{pk^p a(\Gamma_2) G^0}\bigl(1 - \lambda p k^p A_0\bigr),
$$

where $\widetilde{\mu}_{G,\lambda} = +\infty$ *when* $G^0 = 0$, *for every* $\mu \in [0, \widetilde{\mu}_{G,\lambda}[$, *then problem* ($M_{\lambda,\mu}$) *has a sequence of nonnegative weak solutions, which strongly converges to zero in* X.

Theorem 3.8 *Assume that the hypothesis* (A1) *holds, and there exist two sequences* $\{c_n\}$ *and* $\{d_n\}$ *in* $[0,+\infty[$, *with* $\lim_{n\to\infty} d_n = 0$, *such that*

(A7) *for some* $n_0 \in \mathbb{N}$ *one has* $c_n < \delta d_n$ *for each* $n \geq n_0$;

(A8) $\mathcal{A}_0 := \lim_{n\to\infty} \frac{\int_\Omega \max_{|t|\leq d_n} F(x,t)\,dx - \int_{B(x_0,\overline{\mu}R)} F(x,c_n)\,dx}{d_n^p - \frac{1}{\delta^p} c_n^p} < \delta^p B^0$.

Then, for every $\lambda \in \left]\frac{1}{p(k\delta)^p B^0}, \frac{1}{pk^p \mathcal{A}_0}\right[$, *problem* ($M_\lambda$) *has a sequence of nonnegative weak solutions that strongly converges to zero in* X.

The following theorem is an important consequence of Theorem 3.8.

Theorem 3.9 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $\inf_{t\geq 0} F(t) = 0$. *Moreover, let* $h \in C(\Omega)$ *with* $\min_{x\in\Omega} h(x) > 0$. *Suppose that there are two sequences* $\{c_n\}$ *and* $\{d_n\}$ *in* $[0,+\infty[$, *with* $c_n < d_n$ *for every* $n \geq \nu$, *and* $\lim_{n\to\infty} d_n = 0$, *such that*

(A9) $\lim_{n\to\infty} \frac{d_n}{c_n} = +\infty$;

(A10) $\max_{t\in[c_n,d_n]} f(t) \leq 0$ *for every* $n \geq \nu$;

(A11) $\frac{1}{p(k\delta)^p \int_{B(x_0,\overline{\mu}R)} h(x)\,dx} < \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} < +\infty$.

Then, the problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = h(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2, \end{cases}$$

has a sequence of nonnegative weak solutions that strongly converges to zero in X.

Proof Our purpose is to deal with Theorem 3.8. First, note that, by $\inf_{t\geq 0} F(t) = 0$ and $\min_{x\in\Omega} h(x) > 0$, hypothesis (A1) holds. Moreover, if $\{c_n\}$ and $\{d_n\}$ are two sequences in $[0,+\infty[$ satisfying our hypotheses, then there exists $n_0 \geq \nu$ such that $c_n < \delta d_n$ for all $n \geq n_0$. Thus, the hypothesis (A7) in Theorem 3.5 is checked. We will establish that

$$\mathcal{A}_0 := \lim_{n\to\infty} \frac{\|h\|_{L^1(\Omega)} \max_{|t|\leq d_n} F(t) - (\int_{B(x_0,\overline{\mu}R)} h(x)\,dx) F(c_n)}{d_n^p - \frac{1}{\delta^p} c_n^p} = 0.$$

For this, we define

$$h_n := \|h\|_{L^1(\Omega)} \frac{\max_{|t|\leq d_n} F(t)}{c_n^p} - \left(\int_{B(x_0,\overline{\mu}R)} h(x)\,dx\right) \frac{F(c_n)}{c_n^p},$$

for every $n \geq n_0$. At this step, observe that hypothesis (A10) yields

$$\max_{|t|\leq d_n} F(t) = \max_{|t|\leq c_n} F(t). \tag{3.7}$$

Therefore, since

$$\frac{\int_{B(x_0,\overline{\mu}R)} h(x)\,dx}{\|h\|_{L^1(\Omega)}} \leq 1 \quad \text{and} \quad F(c_n) \geq 0,$$

and bearing in mind (3.7), we can write

$$\frac{\max_{|t|\leq d_n} F(t)}{c_n^p} = \frac{\max_{|t|\leq c_n} F(t)}{c_n^p} \geq \frac{F(c_n)}{c_n^p} > \frac{\int_{B(x_0,\overline{\mu}R)} h(x)\,dx}{\|h\|_{L^1(\Omega)}} \frac{F(c_n)}{c_n^p}$$

for all $n \geq n_0$. Hence, since $h_n \geq 0$ for all $n \geq n_0$, we have

$$0 \leq \limsup_{n\to\infty} h_n.$$

Further, by (A11) one has

$$0 < \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} < +\infty, \tag{3.8}$$

and consequently (note that $c_n \searrow 0^+$ as $n \to \infty$) we obtain

$$0 \leq \limsup_{n\to\infty} \frac{F(c_n)}{c_n^2} < +\infty. \tag{3.9}$$

Now, let $\xi_n \in]0, c_n]$ be a sequence such that $F(\xi_n) := \max_{|t|\leq c_n} F(t)$ for all $n \geq n_0$. Thus,

$$\limsup_{n\to\infty} \frac{\max_{|t|\leq d_n} F(t)}{c_n^p} = \limsup_{n\to\infty} \frac{\max_{|t|\leq c_n} F(t)}{c_n^p} = \limsup_{n\to\infty} \frac{F(\xi_n)}{c_n^p} \leq \limsup_{n\to\infty} \frac{F(\xi_n)}{\xi_n^p}.$$

The above relations and (3.8) yield

$$0 \leq \limsup_{n\to\infty} \frac{\max_{|t|\leq d_n} F(t)}{c_n^p} \leq \limsup_{n\to\infty} \frac{F(\xi_n)}{\xi_n^p} < +\infty.$$

Hence, there exists a constant β such that

$$0 \leq \limsup_{n\to\infty} h_n = \beta. \tag{3.10}$$

Then, by (A9) and (3.10), we have

$$\mathcal{A}_0 = \limsup_{n\to\infty} \frac{h_n}{(\frac{d_n}{c_n})^p - \frac{1}{\delta^p}} = 0.$$

Consequently, hypothesis (A8) holds. Finally, bearing in mind hypothesis (A11), one has $1 \in]\frac{1}{p(k\delta)^p B^0}, +\infty[$. Thanks to Theorem 3.8, the proof is complete. □

The next result is a direct consequence of Theorem 3.9.

Proposition 3.10 *Let $h \in C(\Omega)$ satisfying $\min_{x\in\Omega} h(x) > 0$. Also, let $\{c_n\}$ and $\{d_n\}$ be two sequences in $[0, +\infty[$ such that $d_{n+1} < c_n < d_n$ for all $n \geq \nu$, $\lim_{n\to\infty} d_n = 0$, and $\lim_{n\to\infty} \frac{d_n}{c_n} = +\infty$. Moreover, let $\varphi \in C^1([0, 1])$ be a nonnegative function such that $\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0$ and*

$$\max_{s\in[0,1]} \varphi(s) > \frac{1}{p(k\delta)^p \int_{B(x_0,\overline{\mu}R)} h(x)\,dx}.$$

Further, let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(t) := \begin{cases} \varphi(\frac{t-d_{n+1}}{c_n-d_{n+1}}) & \text{if } t \in \bigcup_{n\geq\nu}[d_{n+1}, c_n], \\ 0 & \text{otherwise.} \end{cases}$$

Then, the problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = h(x)y(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2, \end{cases} \tag{M_1^{hy}}$$

where

$$y(u) := |u|^{p-1}\big(pg(u) + ug'(u)\big),$$

has a sequence of nonnegative weak solutions that strongly converges to zero in X.

Proof Assume that $\{c_n\}$ and $\{d_n\}$ are two positive sequences satisfying our hypotheses. We assert that all the hypotheses of Theorem 3.9 are verified. Indeed, we have

$$F(t) := \int_0^t y(\xi)\, d\xi = t^p g(t) \quad \text{for all } t \in \mathbb{R}^+.$$

Moreover, straightforward computations certify that

$$\max_{t\in[c_{n+1},d_{n+1}]} y(t) = 0,$$

for all $n \geq \nu$, and

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} = \limsup_{\xi\to 0^+} g(\xi) = \max_{s\in[0,1]} \varphi(s) > \frac{1}{p(k\delta)^p \int_{B(x_0,\overline{\mu}R)} h(x)\, dx}.$$

The conclusion follows by Theorem 3.9. □

Finally, we show a real example of the application of Proposition 3.10.

Example 3.11 Let $h \in C(\Omega)$ satisfying $\min_{x\in\Omega} h(x) > 0$ and take the positive real sequences

$$a_n := \frac{1}{n!n} \quad \text{and} \quad b_n := \frac{1}{n!},$$

for all $n \geq 2$. Now, define $\varphi \in C^1([0,1])$ as follows

$$\varphi(s) := \alpha e^{\frac{1}{s(s-1)}+4}, \quad \big(\forall s \in [0,1]\big)$$

and let

$$g(t) := \begin{cases} \varphi\left(\frac{t-1/(n+1)!}{1/(n!n)-1/(n+1)!}\right) & \text{if } t \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$A := \bigcup_{n\geq 2}\left[\frac{1}{(n+1)!}, \frac{1}{(n!n)}\right].$$

If

$$\alpha > \frac{1}{p(k\delta)^p \int_{B(x_0,\overline{\mu}R)} h(x)\,dx},$$

the problem (M_1^{hy}) has a sequence of nonnegative weak solutions that strongly converges to zero in X.

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