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An inverse boundary value problem for transverse vibrations of a bar

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Abstract
In this article, we study an inverse problem (IP) for a fourth-order hyperbolic equation with nonlocal boundary conditions. This IP is reduced to the not self-adjoint boundary value problem (BVP) with corresponding boundary condition. Then, we use the separation of variables method, to reduce the not self-adjoint BVP to an integral equation. The existence and uniqueness of the integral equation are established by the contraction mappings principle and it is concluded that this solution is unique for a not-adjoint BVP. The existence and uniqueness of a nonlocal BVP with integral condition is proved. In addition, the fourth-order hyperbolic PDE is discretized using a collocation technique based on the quintic B-spline (QnB-spline) functions and reformed by the Tikhonov regularization function. The noise and analytical data are considered. The numerical outcome for a standard numerical example is discussed. Furthermore, the stability of the discretized system is also analyzed. The rate of convergence (ROC) of the method is also obtained.

Keywords: Inverse problem; Fourth-order hyperbolic equation; Stability analysis; Nonlinear optimization; Tikhonov regularization

1 Introduction

In modern technology, it is necessary to regulate vibration processes in one-dimensional distributed systems, and the relevance of these problems is increasing. In aircraft, such elements are formed simultaneously by bending and torsional vibrations. One of the objectives of the project is to prevent the use of shaft vibrations with an adjustable speed [4, 15]. For such problems, mathematical models of transverse vibrations of rods are built on the basis of a refined theory and such problems are called inverse problems of mathematical physics. Inverse problems for hyperbolic equations of fourth order receive great attention due to the necessity of the generalization for the classical problems [1]. Inverse problems for PDEs in numerous settings have been examined by various authors, e.g., Tikhonov [26], Lavrentiev [18], and Ivanov [16]. This type of problem has various applications such as in biology, medicine, mineral investigation, geophysics, computer tomography, filtration theory, etc. [8, 21, 25]. During computational modeling of specific processes, a condition may occur when the region's boundary of the real process is challenging for measurements, but it is probable to obtain some further knowledge regarding the phenomena under investigation at the region's interior points. From the mathematics viewpoint, this condition

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leads to a new nonlocal problem with integral conditions. In [1–3, 5, 7], the authors studied third-order PDE with integral conditions for unique solvability.

Very few investigations were found in the literature for the numerical computations of the time and/or spacewise coefficients for the IP of the fourth-order equations. For example, Huntul et al. [12, 13] reconstructed the potential coefficient in pseudoparabolic and pseudohyperbolic equations of order four, respectively, from additional measurements. The authors of [11, 20] identified the unknown potential coefficient in Boussinesq and Boussinesq-type equations of order four.

Recently, Huntul et al. [14, 22] obtained results about the numerical solutions of the inverse problem for a higher-order pseudoparabolic equation. The existence and uniqueness of the solution of an inverse boundary value problem for a third order in time pseudoparabolic equation were proved by using analytical and operatortheoretic means, the Fourier method, and the contraction principle. In [10], the authors numerically identify the time-dependent potential coefficient in a fourth-order pseudoparabolic equation with nonlocal initial and boundary conditions supplemented by nonlocal integral observations by applying the quintic B-spline collocation, finite-difference method and the Tikhonov regularization method.

In the domain $D_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$, we consider an IP of the hyperbolic equation in fourth order

$$u_{tt}(x,t) + a^2 u_{xxxx}(x,t) = p(t)u(x,t) + q(t)g(x,t) + f(x,t), \quad (x,t) \in D_T, \tag{1.1}$$

with ICs

$$u(x,0) = \varphi(x), \qquad u_t(x,0) + \delta u(x,T) = \psi(x), \quad x \in [0,1], \tag{1.2}$$

the BCs

$$u(1,t) = 0, \qquad u_x(0,t) = u_x(1,t), \qquad u_{xx}(1,t) = 0, \quad t \in [0,T], \tag{1.3}$$

the nonlocal integral condition

$$\int_0^1 u(x,t)\,dx = 0, \quad t \in [0,T], \tag{1.4}$$

and the additional conditions

$$u(0,t) = h_1(t), \qquad u\left(\frac{1}{2},t\right) = h_2(t), \quad t \in [0,T], \tag{1.5}$$

where a, δ are given numbers, functions $\varphi(x)$, $\psi(x)$, $h_i(t)$, $i = 1,2$, $f(x,t)$, $g(x,t)$ are given, while $u(x,t)$, $p(t)$, and $q(t)$ are the desired functions.

The paper is divided into two parts. Part I discusses theory and proofs that contains Sects. 1, 2, and 3, while Part II investigates numerical experiments that contains Sects. 4, 5, 6, and 7. In Sect. 2, the IP reduces to an equivalent auxiliary IP. Section 3 proves the existence and uniqueness. In Sect. 4, the discretization of the forward problem is solved by using the QnB-spline collocation method. The stability has been analyzed in Sect. 5.

Section 6 describes the numerical process of the nonlinear Tikhonov regularization functional. The outcomes for an example are discussed in Sect. 7. Finally, Sect. 8 reveals some concluding remarks.

2 Preliminary results and reduction of problem to an auxiliary IP

We propose the following definition and lemma:

Definition 2.1 We call the triplet $\{u(x,t), p(t), q(t)\}$ the classic solution of IP, if the subsequent conditions are met:

1) the functions $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$, $u_t(x,t)$, $u_{tt}(x,t)$ are continuous in D_T;
2) the functions $p(t)$, $q(t)$ are continuous on $[0,T]$;
3) the problem (1.1)–(1.5) is assured in the ordinary sense.

Alongside (1.1)–(1.5), consider the subsequent ODEs:

$$y''(t) = p(t)y(t), \quad t \in [0,T], \tag{2.1}$$

$$y(0) = 0, \qquad y'(0) + \delta y(T) = 0, \tag{2.2}$$

where δ is a given number, function $p(t) \in C[0,T]$ is given, $y = y(t)$ is a desired function, if $y(t)$ is the solution of (2.1) and (2.2), then $y(t)$ and all its derivatives are continuous in $[0,T]$.

It is not difficult to determine that the problem

$$y''(t) = 0, \qquad y(t) = 0, \qquad y'(0) + \delta y(T) = 0 \tag{2.3}$$

has a trivial solution, if $\delta \geq 0$. Then, it is known [24] that (2.3) has only one Green function $G(t,\tau)$, given as

$$G(t,\tau) = \begin{cases} -\frac{\delta(T-\tau)t}{1+\delta T}, & t \in [0,\tau], \\ \frac{(1+\delta\tau)t}{1+\delta T} - \tau, & t \in [\tau, T]. \end{cases}$$

The subsequent Lemma is proved.

Lemma 2.2 *Let* $a(t) \in C[0,T]$, *and*

$$\|a(t)\|_{C[0,T]} \leq R \equiv const,$$

further $\delta \geq 0$ *and*

$$R\left(\frac{1}{2} + \frac{1}{1+\delta T} + \frac{\delta T}{2(1+\delta T)}\right)T^2 < 1. \tag{2.4}$$

Then, BVP (2.1) *and* (2.2) *has only a trivial solution.*

Proof It is evident from [24] that the BVP (2.1) and (2.2) is equivalent to the integral equation

$$y(t) = \int_0^T G(t,\tau)p(\tau)y(\tau)\,d\tau. \tag{2.5}$$

Having denoted

$$Ay = \int_0^T G(t,\tau)p(\tau)y(\tau)\,d\tau, \tag{2.6}$$

we can write equation (2.5) as:

$$y = Ay. \tag{2.7}$$

We will examine equation (2.7) in $C[0,T]$. It can be easily seen that the operator A is continuous in $C[0,T]$. Let us prove that A is a contraction mapping in $C[0,T]$. Surely, for any $y_1(t), y_2(t)$ from $C[0,T]$

$$\begin{aligned} &\|A(y_1) - A(y_2)\|_{C[0,T]} \\ &\quad \le \left(\frac{1}{2} + \frac{1}{1+\delta T} + \frac{\delta T}{2(1+\delta T)}\right) T^2 \|a(t)\|_{C[0,T]} \|y_1 - y_2\|_{C[0,T]}. \end{aligned} \tag{2.8}$$

Then, using (2.4) in (2.8), we find A is a contraction mapping in C[0,T]. Thus, in $C[0,T]$, A has a single fixed point $y = \{y_1, y_2\}$, which is a solution of (2.7). Thus, (2.6) has a unique solution in $C[0,T]$, and so, the boundary value problem (2.1) and (2.2) also has a unique solution in $C[0,T]$. As $y(t) = 0$ the boundary value problem (2.1) and (2.2) has only a trivial solution. The lemma is proved. Along with (1.1)–(1.5), we choose the subsequent auxiliary IP. It is needed to find the triple $\{u(x,t), p(t), q(t)\}$ of $u(x,t)$, $p(t)$, and $q(t)$ with properties 1) and 2) of the definition of the classical solution of BVP (1.1)–(1.5), from (1.1)–(1.3)

$$u_x(0,t) = u_x(1,t), \quad t \in [0,T], \tag{2.9}$$

$$h_1'' + a^2 u_{xxxx}(0,t) = p(t)h_1(t) + q(t)g(0,t) + f(0,t), \quad t \in [0,T], \tag{2.10}$$

$$h_2'' + a^2 u_{xxxx}\left(\frac{1}{2},t\right) = p(t)h_2(t) + q(t)g\left(\frac{1}{2},t\right) + f\left(\frac{1}{2},t\right), \quad t \in [0,T], \tag{2.11}$$

and

$$h(t) \equiv h_1(t)g\left(\frac{1}{2},t\right) - h_2(t)g(0,t) \ne 0, \quad t \in [0,T]. \tag{2.12}$$

□

The following theorem is valid.

Theorem 2.3 *Let* $\varphi(x), \psi(x) \in C[0,1]$, $h_i(t) \in C^2[0,T]$, $i = 1,2$, $h(t) \equiv h_1(t)g(\frac{1}{2},t) - h_2(t)g(0,t) \ne 0$, $t \in [0,T]$, $f(x,t), g(x,t) \in C(D_T)$, $\int_0^1 f(x,t)\,dx = 0$, $\int_0^1 g(x,t)\,dx = 0$, $t \in [0,T]$, *and the consistency conditions*

$$\int_0^1 \varphi(x)\,dx = 0, \qquad \int_0^1 \psi(x)\,dx = 0, \tag{2.13}$$

$$\varphi(0) = h_1(0), \qquad \varphi\left(\frac{1}{2}\right) = h_2(0), \tag{2.14}$$

$$\psi(0) = h_1'(0) + \delta h_1(T), \qquad \psi\left(\frac{1}{2}\right) = h_2' + \delta h_2(T) \tag{2.15}$$

be satisfied. Then, we have

1. *Every classical solution* $\{u(x,t), p(t), q(t)\}$ *of* (1.1)–(1.5) *is the solution of* (1.1)–(1.3) *and* (2.9)–(2.11);
2. *Every solution* $\{u(x,t), p(t), q(t)\}$ *of* (1.1)–(1.3) *and* (2.9)–(2.11) *is a classical solution of* (1.1)–(1.5), *if*

$$\|\rho(t)\|_{C[0,T]}\left(\frac{1}{2} + \frac{1}{1+\delta T} + \frac{\delta T}{2(1+\delta T)}\right)T^2 < 1. \tag{2.16}$$

Proof Let $\{u(x,t), p(t), q(t)\}$ be a solution of (1.1)–(1.5). Now, integrating (1.1) over x from 0 to 1, we obtain

$$\begin{aligned} &\frac{d^2}{dt^2}\int_0^1 u(x,t)\,dx + a_2\big(u_{xxx}(1,t) - u_{xxx}(0,t)\big) \\ &\quad = p(t)\int_0^1 u(x,t)\,dx + q(t)\int_0^1 g(x,t)\,dx + \int_0^1 f(x,t)\,dx, \quad t \in [0,T]. \end{aligned} \tag{2.17}$$

Assuming that

$$\int_0^1 f(x,t)\,dx = 0, \qquad \int_0^1 g(x,t)\,dx = 0, \quad t \in [0,T],$$

in view of (1.4), we arrive at the fulfillment of (2.9).

Using $x = 0$ and $x = \frac{1}{2}$ in (1.1), respectively, we obtain

$$u_{tt}(0,t) + a^2 u_{xxxx}(0,t) = p(t)u(0,t) + q(t)g(0,t) + f(0,t), \quad t \in [0,T], \tag{2.18}$$

$$\begin{aligned} &u_{tt}\left(\frac{1}{2},t\right) + a^2 u_{xxxx}\left(\frac{1}{2},t\right) = p(t)u\left(\frac{1}{2},t\right) + a_2(t)g\left(\frac{1}{2},t\right) + f\left(\frac{1}{2},t\right), \\ &\quad t \in [0,T]. \end{aligned} \tag{2.19}$$

Under the assumption $h_i(t) \in C^2[0,T]$, $i = 1,2$ and differentiating (1.5) twice, we obtain

$$u_t(0,t) = h_1'(t), \qquad u_{tt}(0,t) = h_1''(t), \quad t \in [0,T], \tag{2.20}$$

$$u_t\left(\frac{1}{2},t\right) = h_2'(t), \qquad u_{tt}\left(\frac{1}{2},t\right) = h_2''(t), \quad t \in [0,T]. \tag{2.21}$$

Considering these relations, from (2.18) and (2.19), taking into account (1.5), the fulfillment of (2.10) and (2.11) follows, respectively.

Now, let $\{u(x,t), p(t), q(t)\}$ be a solution to (1.1)–(1.3) and (2.9)–(2.11), and (2.16) is fulfilled. Then from (2.17), in view of (2.9), we obtain

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)\,dx = p(t)\int_0^1 u(x,t)\,dx, \quad t \in [0,T]. \tag{2.22}$$

From (1.2) and (2.13), we have

$$\int_0^1 u(x,0)\,dx = \int_0^1 \varphi(x)\,dx = 0, \tag{2.23}$$

$$\int_0^1 u_t(x,0)\,dx + \delta\int_0^1 u(x,T)\,dx = \int_0^1 \psi(x)\,dx = 0. \tag{2.24}$$

Since, by Theorem 2.3, problem (2.22) and (2.24) has only a trivial solution, $\int_0^1 u(x,t)\,dx = 0$, $t \in [0,T]$, i.e., conditions (1.4) are satisfied. Further, from (2.10), (2.18), (2.11), and (2.19), we obtain

$$\frac{d^2}{dt^2}\big(u(0,t) - h_1(t)\big) = p(t)\big(u(0,t) - h_1(t)\big), \quad t \in [0,T], \tag{2.25}$$

$$\frac{d^2}{dt^2}\left(u\left(\frac{1}{2},t\right) - h_2(t)\right) = p(t)\left(u\left(\frac{1}{2},t\right) - h_2(t)\right), \quad t \in [0,T]. \tag{2.26}$$

From (1.2) and (2.15), we have

$$u(0,0) - h_1(0) = \varphi(0) - h_1(0) = 0, \tag{2.27}$$

$$u_t(0,0) - h_1'(0) + \delta\big(u(0,T) - h_1(T)\big) = \psi(0) - h_1'(0) - \delta h_1(T) = 0, \tag{2.28}$$

$$u\left(\frac{1}{2},0\right) - h_2(0) = \varphi\left(\frac{1}{2}\right) - h_2(0) = 0, \tag{2.29}$$

$$u_t\left(\frac{1}{2},0\right) - h_2'(0) + \delta\left(u\left(\frac{1}{2},T\right) - h_2(T)\right) = \psi\left(\frac{1}{2}\right) - h_2'(0) - \delta h_2(T) = 0. \tag{2.30}$$

From (2.25), (2.30), and Lemma 2.2 the condition (1.5) is obtained. The theorem is proved. □

2.1 Auxiliary facts

It is understood that the sequence of the functions

$$X_0(x) = 2(1-x), \dots, X_{2k-1}(x) = 4(1-x)\cos\lambda_k x, \qquad X_{2k}(x) = 4\sin\lambda_k x, \dots, \tag{2.31}$$

$$Y_0(x) = 1, \dots, Y_{2k-1}(x) = \cos\lambda_k x, \qquad Y_{2k}(x) = x\sin\lambda_k x, \dots \tag{2.32}$$

form in $L_2(0,1)$ a biorthogonal system and system (2.31) forms a basis in $L_2(0,1)$, where $\lambda_k = 2k\pi$, $k = 1,2,\dots$, [17, 23]. They are also Riesz bases in $L_2(0,1)$, see [24]. Then, any function $g(x) \in L_2(0,1)$ can be expanded as a biorthogonal series

$$g(x) = g_0 X_0(x) + \sum_{k=1}^{\infty} g_{2k-1}X_{2k-1}(x) + \sum_{k=1}^{\infty} g_{2k}X_{2k}(x), \tag{2.33}$$

where the coefficients g_0, g_{2k-1}, g_{2k}, are calculated by the following formulas

$$g_0 = \int_0^1 g_x Y_0(x)\,dx, \qquad g_{2k-1} = \int_0^1 g_x Y_{2k-1}(x)\,dx, \qquad g_{2k} = \int_0^1 g_x Y_{2k}(x)\,dx.$$

From (2.32), we have

$$\begin{aligned} &Y_0'(x) = 0, \qquad Y_{2k-1}'(x) = -\lambda_k \sin\lambda_k x, \\ &Y_{2k}'(x) = \lambda_k x\cos\lambda_k x + \sin\lambda_k x, \quad k = 1,2,\ldots, \end{aligned} \tag{2.34}$$

$$Y_0^{2i}(x) = 0, \qquad Y_{2k-1}^{2i}(x) = (-1)^i\lambda_k^{2i} Y_{2k-1}(x), \quad k = 1,2,\ldots, \tag{2.35}$$

$$Y_{2k}^{2i}(x) = (-1)^i\lambda_k^{2i} Y_{2k}(x) + 2i(-1)^{i+1}\lambda_k^{2i-1} Y_{2k-1}(x), \quad i \geq 0, k = 1,2,\ldots, \tag{2.36}$$

and from above equation, we obtain

$$Y_k^{2i}(0) = Y_k^{2i}(1), \qquad Y_k^{2i+1}(0) = 0, \quad k = 1,2,\ldots. \tag{2.37}$$

Under assumptions

$$\begin{aligned} &g(x) \in C^{2i-1}[0,1], \qquad g^{2i}(x) \in L_2(0,1), \\ &g^{(2s)}(1) = 0, \qquad g^{(2s+1)}(0) = g^{(2s+1)}(1), \quad s = \overline{0, i-1}, \end{aligned} \tag{2.38}$$

and using integration by parts and taking into account (2.37) and (2.38), we obtain

$$\int_0^1 g^{(2i)}(x) Y_k(x)\,dx = \int_0^1 g(x) Y_k^{(2i)}(x)\,dx, \quad k = 1,2,\ldots. \tag{2.39}$$

Equation (2.36) implies

$$Y_{2k-1}(x) = \frac{(-1)^i}{\lambda_k^{2i}} Y_{2k-1}^{2i}(x), \quad i \geq 0, k = 1,2,\ldots, \tag{2.40}$$

$$\begin{aligned} Y_{2k}(x) &= \frac{(-1)^i}{\lambda_k^{2i}} \left\{ Y_{2k}^{2i}(x) + 2i(-1)^i \lambda_k^{2i-1}(x) \right\} \\ &= \frac{(-1)^i}{\lambda_k^{2i}} Y_{2k}^{2i}(x) + 2i\frac{(-1)^i}{\lambda_k^{2i+1}} Y_{2k-1}^{2i}(x), \quad i \geq 0, k = 1,2,\ldots. \end{aligned} \tag{2.41}$$

Then, from (2.39) and (2.41), we find

$$g_{2k-1} = \frac{(-1)^i}{\lambda_k^{2i}} \int_0^1 g^{(2i)}(x)\cos\lambda_k x\,dx, \tag{2.42}$$

$$g_{2k} = \frac{(-1)^i}{\lambda_k^{2i}} \int_0^1 g^{(2i)}(x)x + 2ig^{2i-1}(x)\sin\lambda_k x\,dx. \tag{2.43}$$

Thus, we have

$$\sum_{k=1}^{\infty} \left(\lambda_k^{2i} g_{2k-1}\right)^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left(\int_0^1 g^{(2i)}(x)\sqrt{2}\cos\lambda_k x\,dx\right)^2 \leq \frac{1}{2}\left\| g^{(2i)}(x)\right\|_{L_2(0,1)}, \tag{2.44}$$

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i}g_{2k}\right)^2\leq\frac{1}{2}\left\|g^{(2i)}(x)x+2ig^{2i-1}(x)\right\|_{L_2(0,1)}. \tag{2.45}$$

Under the assumptions $g(x)\in C^{2i}[0,1]$, $g^{2i+1}(x)\in L_2(0,1)$, $g^{2s}(1)=0$, $g^{2s-1}(0)=g^{2s-1}(1)$, $i\geq 1$, $s=\overline{0,i}$, the following are valid

$$g_{2k-1}=\int_0^1 g(x)Y_{2k-1}(x)\,dx=\frac{(-1)^i}{\lambda_k^{2i+1}}\int_0^1 g^{(2i+1)}(x)\cos\lambda_k x\,dx, \tag{2.46}$$

$$g_{2k}=\int_0^1 g(x)Y_{2k}(x)\,dx=\frac{(-1)^{i+1}}{\lambda_k^{2i+1}}\int_0^1\left(g^{(2i+1)}(x)x+(2i+1)g^{(2i)}(x)\right)\sin\lambda_k x\,dx. \tag{2.47}$$

From the above equations, we find

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i+1}g_{2k-1}\right)^2\leq\frac{1}{2}\sum_{k=1}^{\infty}\left(\int_0^1 g^{(2i+1)}(x)\sqrt{2}\cos\lambda_k x\,dx\right)\leq\left\|g^{2i+1}(x)\right\|_{L_2(0,1)}^2, \tag{2.48}$$

$$\sum_{k=1}^{\infty}\left(\lambda_k^{2i+1}g_{2k}\right)^2\leq\frac{1}{2}\leq\left\|g^{2i+1}(x)x+(2i+1)g^{2i}(x)\right\|_{L_2(0,1)}^2,\quad i\geq 1. \tag{2.49}$$

Now, consider the subsequent spaces.

1. $B_{2,T}^5$ [21] can be illustrated as consisting of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x), \tag{2.50}$$

on D_T, where $u_k(T)\in C[0,T]$, $k=0,1,\ldots$, and

$$J_T(u)\equiv\left\|u_0(t)\right\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_{2k}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}<\infty. \tag{2.51}$$

The norm is given by the formula

$$\left\|u(x,t)\right\|_{B_{2,T}^5}=J_T(u). \tag{2.52}$$

2. E_T^5 can be illustrated as consisting of a vector by the formula

$$B_{2,T}^5\times C[0,T]\times C[0,T].$$

The norm of $z=\{u,p,q\}$ is obtained

$$\|z\|_{E_T^5}=\|u\|_{B_{2,T}^5}+\left\|p(t)\right\|_{C[0,T]}+\left\|q(t)\right\|_{C[0,T]}. \tag{2.53}$$

It is obvious that $B_{2,T}^5$, E_T^5 are Banach spaces.

3 Existence and uniqueness of a classical solution of the IP

Since the system (2.31) forms the Riesz basis in $L_2(0,1)$, then each solution of (1.1)–(1.3) and (2.9)–(2.11) can be written as

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x), \tag{3.1}$$

where

$$u_k(t)=\int_0^1 u(x,t)Y_k(x)\,dx,\quad k=0,1,\dots, \tag{3.2}$$

where $X_k(x)$ and $Y_k(x)$ are described by (2.31) and (2.32), respectively.

Using the variable-separation method to find the desired functions $u_k(t)$, $k=0,1,\dots,$ from (1.1), (1.2), we obtain

$$u_0''(t)=p(t)u_0(t)+q(t)g_0(t)+f_0(t), \tag{3.3}$$

$$u_{2k-1}''(t)+a_2\lambda_k^4u_{2k-1}(t)=p(t)u_{2k-1}(t)+q(t)g_{2k-1}(t)+f_{2k-1}(t),\quad k=1,2,\dots, \tag{3.4}$$

$$u_{2k}''(t)+a_2\lambda_k^4u_{2k}(t)=p(t)u_{2k}(t)+q(t)g_{2k}(t)+4a_2\lambda_k^3u_{2k-1}(t)+f_{2k-1}(t),$$
$$k=1,2,\dots, \tag{3.5}$$

$$u_k(0)=\varphi_k,\qquad u_k'(0)+\delta u_k(T)=\psi_k,\quad k=0,1,2,\dots, \tag{3.6}$$

where

$$f_k(t)=\int_0^1 f(x,t)Y_k(x)\,dx,\qquad \varphi_k=\int_0^1\varphi(x)Y_k(x)\,dx,$$
$$\psi_k=\int_0^1\psi(x)Y_k(x)\,dx,\quad k=0,1,\dots.$$

Solving (3.3)–(3.6), we obtain

$$\begin{aligned}u_0(t)={}&\frac{1+\delta(T-t)}{1+\delta T}\varphi_0+\frac{t}{1+\delta T}\psi_0\\&-\frac{\delta t}{1+\delta T}\int_0^T(T-\tau)F_0(\tau;u,p,q)\,d\tau+\int_0^t(t-\tau)F_0(\tau;u,p,q)\,d\tau,\end{aligned} \tag{3.7}$$

$$\begin{aligned}u_{2k-1}(t)={}&\frac{\beta_k\cos\beta_kt+\delta\sin\beta_k(T-t)}{\beta_k+\delta\sin\beta_kT}\varphi_{2k-1}+\frac{\sin\beta_kt}{\beta_k+\delta\sin\beta_kT}\psi_{2k-1}\\&-\frac{\delta\sin\beta_kt}{\beta_k(\beta_k+\delta\sin\beta_kT)}\int_0^TF_{2k-1}(\tau;u,p,q)\sin\beta_k(T-\tau)\,d\tau\\&+\frac{1}{\beta_k}\int_0^tF_{2k-1}(\tau;u,p,q)\sin\beta_k(t-\tau)\,d\tau,\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
u_{2k}(t) = {} & \frac{\beta_k \cos\beta_k t + \delta \sin\beta_k (T-t)}{\beta_k + \delta\sin\beta_k T}\varphi_{2k} + \frac{\sin\beta_k t}{\beta_k + \delta \sin\beta_k T}\psi_{2k} \\
& - \frac{\delta\sin\beta_k t}{\beta_k(\beta_k + \delta\sin\beta_k T)}\int_0^T F_{2k}(\tau;u,p,q)\sin\beta_k(T-\tau)\,d\tau \\
& + \frac{1}{\beta_k}\int_0^t F_{2k}(\tau;u,p,q)\sin\beta_k(t-\tau)\,d\tau + \frac{2\beta_k}{\lambda_k(\beta_k+\delta\sin\beta_k T)}\varphi_{2k-1} \\
& \times \biggl[-\frac{\delta\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\biggl(\beta_k T\sin\beta_k T + \delta\biggl(T - \frac{\sin 2\beta_k T}{2\beta_k}\biggr)\biggr) \\
& + \beta_k t\sin\beta_k t + \delta\biggl(T\cos\beta_k(T-t) - \frac{\sin 2\beta_k t\cos\beta_k T}{\beta_k}\biggr)\biggr] \\
& \times \frac{2\beta_k}{\lambda_k(\beta_k+\delta\sin\beta_k T)}\psi_{2k-1}\biggl[-\frac{\delta\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\biggl(\frac{2}{\beta_k}\sin\beta_k T - T\cos\beta_k T\biggr) \\
& + \frac{2}{\beta_k}\sin\beta_k t - t\cos\beta_k t\biggr] \\
& + \frac{2\delta}{\lambda_k(\beta_k+\delta\sin\beta_k T)}\int_0^t F_{2k-1}(\tau;u,p,q_2)\sin\beta_k(t-\tau)\,d\tau \\
& \times \biggl[\frac{\delta\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\biggl(\frac{2}{\beta_k}\sin\beta_k T - T\cos\beta_k T\biggr) \\
& - \biggl(\frac{2}{\beta_k}\sin\beta_k t - \frac{t}{2}\cos\beta_k t\biggr)\biggr] \\
& + \frac{4}{\lambda_k}\biggl[-\frac{\delta\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\int_0^T\biggl(\int_0^\tau F_{2k-1}(\tau;u,p,q)\sin\beta_k(\tau-\xi)\,d\xi\biggr) \\
& \times \sin\beta_k(t-\tau)\,d\tau \\
& + \int_0^t\biggl(\int_0^\tau F_{2k-1}(\tau;u,p,q)\sin\beta_k(\tau-\xi)\,d\xi\biggr)\sin\beta_k(t-\tau)\,d\tau\biggr],
\end{aligned} \tag{3.9}
$$

where

$$
\beta_k = a\lambda_k^2, \qquad F_k(t;u,p,q) = p(t)u_k(t) + q(t)g_k(t) + f_k(t), \quad k = 0,1,2,\dots.
$$

After substituting expressions $u_0(t)$, $u_{2k-1}(t)$, $u_{2k}(t)$, respectively, from (3.7), (3.8), and (3.9) into (3.1), to find $u(x,t)$ of the solution of (1.1)–(1.3) and (2.9)–(2.11), we obtain

$$
\begin{aligned}
u(x,t) = {} & \biggl\{\frac{1+\delta(T-t)}{1+\delta T}\varphi_0 + \frac{t}{1+\delta T}\psi_0 - \frac{\delta t}{1+\delta T}\int_0^T (T-\tau)F_0(\tau;u,p,q)\,d\tau \\
& + \int_0^t (t-\tau)F_0(\tau;u,p,q)\,d\tau\biggr\}X_0(x) \\
& + \sum_{k=1}^{\infty}\biggl\{\frac{\beta_k\cos\beta_k t + \delta\sin\beta_k(T-t)}{\beta_k+\delta\sin\beta_k T}\varphi_{2k-1} + \frac{\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\psi_{2k-1} \\
& - \frac{\delta\sin\beta_k t}{\beta_k(\beta_k+\delta\sin\beta_k T)}\int_0^T F_{2k-1}(\tau;u,p,q)\sin\beta_k(T-\tau)\,d\tau \\
& + \frac{1}{\beta_k}\int_0^t F_{2k-1}(\tau;u,p,q)\sin\beta_k(t-\tau)\,d\tau\biggr\}X_{2k-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}\left\{\frac{\beta_k\cos\beta_k t+\delta\sin\beta_k(T-t)}{\beta_k+\delta\sin\beta_k T}\varphi_{2k}+\frac{\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\psi_{2k}\right.\\
&-\frac{\delta\sin\beta_k t}{\beta_k(\beta_k+\delta\sin\beta_k T)}\int_0^T F_{2k}(\tau;u,p,q)\sin\beta_k(T-\tau)\,d\tau\\
&+\frac{1}{\beta_k}\int_0^t F_{2k}(\tau;u,p,q)\sin\beta_k(t-\tau)\,d\tau+\frac{2\beta_k}{\lambda_k(\beta_k+\delta\sin\beta_k T)}\varphi_{2k-1}\\
&\times\left[-\frac{\delta\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\left(\beta_k T\sin\beta_k T+\delta\left(T-\frac{\sin 2\beta_k T}{2\beta_k}\right)\right)\right.\\
&\left.+\beta_k t\sin\beta_k t+\delta\left(T\cos\beta_k(T-t)-\frac{\sin 2\beta_k t\cos\beta_k T}{\beta_k}\right)\right]\\
&+\frac{2\beta_k}{\lambda_k(\beta_k+\delta\sin\beta_k T)}\psi_{2k-1}\left[-\frac{\delta\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\left(\frac{2}{\beta_k}\sin\beta_k T-T\cos\beta_k T\right)\right.\\
&\left.+\frac{2}{\beta_k}\sin\beta_k t-t\cos\beta_k t\right]+\frac{2\delta}{\lambda_k(\beta_k+\delta\sin\beta_k T)}\\
&\times\int_0^t F_{2k-1}(\tau;u,p,q_2)\sin\beta_k(t-\tau)\,d\tau\\
&\times\left[\frac{\delta\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\left(\frac{2}{\beta_k}\sin\beta_k T-T\cos\beta_k T\right)-\left(\frac{2}{\beta_k}\sin\beta_k t-\frac{t}{2}\cos\beta_k t\right)\right]\\
&+\frac{4}{\lambda_k}\left[-\frac{\delta\sin\beta_k t}{\beta_k+\delta\sin\beta_k T}\int_0^T\left(\int_0^\tau F_{2k-1}(\tau;u,p,q)\sin\beta_k(\tau-\xi)\,d\xi\right)\right.\\
&\times\sin\beta_k(t-\tau)\,d\tau\\
&\left.\left.+\int_0^t\left(\int_0^\tau F_{2k-1}(\tau;u,p,q)\sin\beta_k(\tau-\xi)\,d\xi\right)\sin\beta_k(t-\tau)\,d\tau\right]\right\}X_{2k}(x). \qquad (3.10)
\end{aligned}
$$

Now, to obtain an equation for $p(t)$, $q(t)$ of the solution $\{u(x,t),p(t),q(t)\}$ of (1.1)–(1.3) and (2.9)–(2.11), from (2.10) and (2.11), taking into account (3.1), we have

$$
h_1''(t)+4\sum_{k=1}^{\infty}\lambda_k^2 u_{2k-1}(t)=p(t)h_1(t)+q(t)g(0,t)+f(0,t),\quad t\in[0,T], \qquad (3.11)
$$

$$
h_2''(t)+4\sum_{k=1}^{\infty}\lambda_k^2(-1)^k u_{2k-1}(t)=p(t)h_2(t)+q(t)g\left(\frac{1}{2},t\right)+f\left(\frac{1}{2},t\right),\quad t\in[0,T]. \qquad (3.12)
$$

Assume that

$$
h(t)=\begin{vmatrix} h_1(t) & g(0,t)\\ h_2(t) & g(\frac{1}{2},t)\end{vmatrix}\neq 0\quad \text{if } t\in[0,T]. \qquad (3.13)
$$

Then, from (3.11) and (3.12), we obtain

$$
\begin{aligned}
p(t)=\left[h(t)\right]^{-1}&\left\{\left(h_1''(t)-f(0,t)\right)g\left(\frac{1}{2},t\right)-\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)g(0,t)\right.\\
&\left.+4\sum_{k=1}^{\infty}\left(g\left(\frac{1}{2},t\right)-(-1)^k g(0,t)\right)\lambda_k^4 u_{2k-1}(t)\right\}, \qquad (3.14)
\end{aligned}
$$

$$q(t) = \left[h(t)\right]^{-1}\left\{\left(h_2''(t) - f\left(\frac{1}{2},t,t\right)\right)h_1(t) - \left(h_2''(t) - f(0,t)\right)h_2(t) + 4\sum_{k=1}^{\infty}\left((-1)^k h_1(t) - h_2(t)\right)\lambda_k^4 u_{2k-1}(t)\right\}. \tag{3.15}$$

Further, after substituting the expression $u_{2k-1}(t)$ from (3.8) into (3.14) and (3.15), respectively, we have

$$\begin{aligned} p(t) = \left[h(t)\right]^{-1}\Bigg\{&\left(h_1''(t) - f(0,t)\right)g\left(\frac{1}{2},t\right) - \left(h_2''(t) - f\left(\frac{1}{2},t\right)\right)g(0,t) \\ &+ 4\sum_{k=1}^{\infty}\left(g\left(\frac{1}{2},t\right) - (-1)^k g(0,t)\right)\lambda_k^4\Bigg[\frac{\beta_k\cos\beta_k t + \delta\sin\beta_k(T-t)}{\beta_k + \delta\sin\beta_k T}\varphi_{2k-1} \\ &+ \frac{\sin\beta_k t}{\beta_k + \delta\sin\beta_k T}\psi_{2k-1} \\ &- \frac{\delta\sin\beta_k t}{\beta_k(\beta_k + \delta\sin\beta_k T)}\int_0^T F_{2k-1}(\tau;u,p,q)\sin\beta_k(T-\tau)\,d\tau \\ &+ \frac{1}{\beta_k}\int_0^t F_{2k-1}(\tau;u,p,q)\sin\beta_k(t-\tau)\,d\tau\Bigg\}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} q(t) = \left[h(t)\right]^{-1}\Bigg\{&\left(h_2''(t) - f\left(\frac{1}{2},t,t\right)\right)h_1(t) - \left(h_2''(t) - f(0,t)\right)h_2(t) \\ &+ 4\sum_{k=1}^{\infty}\left((-1)^k h_1(t) - h_2(t)\right)\lambda_k^4\Bigg[\frac{\beta_k\cos\beta_k t + \delta\sin\beta_k(T-t)}{\beta_k + \delta\sin\beta_k T}\varphi_{2k-1} \\ &+ \frac{\sin\beta_k t}{\beta_k + \delta\sin\beta_k T}\psi_{2k-1} \\ &- \frac{\delta\sin\beta_k t}{\beta_k(\beta_k + \delta\sin\beta_k T)}\int_0^T F_{2k-1}(\tau;u,p,q)\sin\beta_k(T-\tau)\,d\tau \\ &+ \frac{1}{\beta_k}\int_0^t F_{2k-1}(\tau;u,p,q)\sin\beta_k(t-\tau)\,d\tau\Bigg\}. \end{aligned} \tag{3.17}$$

Thus, the solution of (1.1)–(1.3) and (2.9)–(2.11) was reduced to the solution of (3.10), (3.16), and (3.17) with respect to the unknowns $u(x,t)$, $p(t)$, and $q(t)$.

To study the question of the uniqueness of the solution of (1.1)–(1.3) and (2.9)–(2.11), the following plays an essential role.

Lemma 3.1 *If* $\{u(x,t), p(t), q(t)\}$ *is any solution of* (1.1)–(1.3) *and* (2.9)–(2.11), *then* $u_k(t)$, $k = 0,1,\dots$, *defined by* (3.2), *satisfy the system* (3.7), (3.8), *and* (3.9) *on* $[0,T]$.

Proof Let $\{u(x,t), p(t), q(t)\}$ be any solution of (1.1)–(1.3) and (2.9)–(2.11). Then, multiplying in equation (1.1) by $Y_k(x)$, $k = 0,1,2,\dots$, and then integrating from 0 to 1 using:

$$\int_0^1 u_{tt}(x,t)Y_k(x)\,dx = \frac{d^2}{dt^2}\int_0^1 u(x,t)Y_k(x)\,dx = u_k''(t),$$

$$\int_0^1 u_{xxxx}(x,t)Y_0(x)\,dx = \int_0^1 u(x,t)Y_0''(x)\,dx = 0,$$
$$\int_0^1 u_{xxxx}(x,t)Y_{2k-1}(x)\,dx = \lambda_k^4 u_{2k-1}(t), \quad k = 1,2,\dots,$$
$$\int_0^1 u_{xxxx}(x,t)Y_{2k}(x)\,dx = \lambda_k^4 u_{2k}(t) - 4\lambda_k^3 u_{2k-1}(t), \quad k = 1,2,\dots,$$

we obtain that (3.3)–(3.5) are satisfied.

Similarly, from (1.2), we find that condition (3.6) is satisfied. Thus, $u_k(t)$, $k = 0,1,2,\dots$, is a solution of (3.3)–(3.6). Hence, it immediately follows that $u_k(t)$, $k = 0,1,2,\dots$, satisfy the system (3.7)–(3.9) on $[0,T]$. The lemma is proved. □

Obviously, if

$$u_k(t) = \int_0^1 u(x,t)Y_k(x)\,dx, \quad k = 0,1,2,\dots$$

is a solution to (3.7)–(3.9), then the triple $\{u(x,t),p(t),q(t)\}$ of $u(x,t) = \sum_{k=0}^{\infty} u_k(t)X_k(x)$, $p(t)$ and $q(t)$ is a solution to (3.7)–(3.9). From Lemma 2.2 it follows that.

Corollary 3.1 *Let systems* (3.10), (3.16), *and* (3.17) *have a unique solution. Then* (1.1)–(1.3) *and* (2.9)–(2.11) *have at most one solution, i.e., if* (1.1)–(1.3) *and* (2.9)–(2.11) *has a solution, then it is unique.*

Consider the operator

$$\Phi(u,p,q) = \{\Phi_1(u,p,q), \Phi_2(u,p,q), \Phi_3(u,p,q)\}, \tag{3.18}$$

in E_T^5, where

$$\Phi_1(u,p,q) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t)X_k(x), \qquad \Phi_2(u,p,q) = \tilde{p}(t), \qquad \Phi_3(u,p,q) = \tilde{q}(t),$$

and $\tilde{u}_0(t)$, $\tilde{u}_{2k}(t)$, $\tilde{u}_{2k-1}(t)$, $\tilde{p}(t)$, and $\tilde{q}(t)$ are equal, respectively, to the right sides of (3.7)–(3.9), (3.16), and (3.17). Let $0 \le \delta < 2\pi a$. It is not difficult to see that

$$\beta_k + \delta \sim \lambda_k T \ge \beta_k - \delta > 2\pi a - \delta > 0.$$

Considering these relations, we have

$$\begin{aligned}\|\tilde{u}_0(t)\|_{C[0,T]} \le{}& |\varphi_0| + T|\psi_0| + (1+\delta T)T\sqrt{T}\left(\int_0^T |f_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}} \\ &+ (1+\delta T)T^2\|p(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]} \\ &+ (1+\delta T)T\sqrt{T}\|q(t)\|_{C[0,T]}\left(\int_0^T |g_0(\tau)\,d\tau|^2\right)^{\frac{1}{2}},\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|\widetilde{u}_{2k-1}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\quad\leq \sqrt{5}(1+\delta)\sup_k \frac{\beta_k}{\beta_k-\delta}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{2k-1}|\right)^2\right)^{\frac{1}{2}} \\
&\qquad+\frac{\sqrt{5}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\psi_{2k-1}|\right)^2\right)^{\frac{1}{2}} \\
&\qquad+\frac{\sqrt{5}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_{2k-1}(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}\right. \\
&\qquad+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\qquad\left.+\sqrt{T}\left\|q(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|g_{2k-1}(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}\right],
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|\widetilde{u}_{2k}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\quad\leq \sqrt{10}(1+\delta)\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{2k}|\right)^2\right)^{\frac{1}{2}} \\
&\qquad+\frac{\sqrt{10}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\psi_{2k}|\right)^2\right)^{\frac{1}{2}}+\frac{\sqrt{10}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right) \\
&\qquad\times\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_{2k}(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}\right. \\
&\qquad+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_{2k}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\qquad\left.+\sqrt{T}\left\|q(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|g_{2k}(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}\right] \\
&\qquad+2a\sqrt{10}(T+\delta T+\delta)\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(1+\sup_k\frac{\beta_k}{\beta_k-\delta}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^6|\varphi_{2k-1}|\right)^2\right)^{\frac{1}{2}} \\
&\qquad+\frac{4\pi a\sqrt{10}}{2\pi a-\delta}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(\frac{1}{\pi a}+T\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^4|\psi_{2k-1}|\right)^2\right)^{\frac{1}{2}} \\
&\qquad+4\sqrt{10}\left(T+\frac{\delta}{2\pi a-\delta}\left(T+\frac{1}{\pi a}\right)\right)\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right) \\
&\qquad\times\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^4\left|f_{2k-1}(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&+\sqrt{T}\|q(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^4|g_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\Bigg], \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
&\|\widetilde{p}(t)\|_{C[0,T]} \\
&=\|[h(t)]^{-1}\|_{C[0,T]}\left\{\left\|\left(h_1''(t)-f(0,t)\right)g\left(\frac{1}{2},t\right)-\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)g(0,t)\right\|_{C[0,T]}\right. \\
&+4\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\||g(0,t)|+\left|g\left(\frac{1}{2},t\right)\right|\right\|_{C[0,T]} \\
&\times\left[(1+\delta)\sup_k\frac{\beta_k}{\beta_k-\delta}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{2k-1}|\right)^2\right)^{\frac{1}{2}}\right. \\
&+\frac{1}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\psi_{2k-1}|\right)^2\right)^{\frac{1}{2}}+\frac{1}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right) \\
&\times\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|f_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right. \\
&+T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\left.\left.\left.+\sqrt{T}\|q(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|g_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right]\right]\right\}, \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
&\|\widetilde{q}(t)\|_{C[0,T]} \\
&=\|[h(t)]^{-1}\|_{C[0,T]}\left\{\left\|\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)h_1(t)-\left(h_2''(t)-f(0,t)\right)h_2(t)\right\|_{C[0,T]}\right. \\
&+4\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\||h_2(t)|+|h_1(t)|\right\|_{C[0,T]}\left[(1+\delta)\sup_k\frac{\beta_k}{\beta_k-\delta}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{2k-1}|\right)^2\right)^{\frac{1}{2}}\right. \\
&+\frac{1}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\psi_{2k-1}|\right)^2\right)^{\frac{1}{2}}+\frac{1}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right) \\
&\times\left[\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|f_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right. \\
&+T\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\left.\left.\left.+\sqrt{T}\|q(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|g_{2k-1}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}\right]\right]\right\}. \qquad (3.23)
\end{aligned}
$$

Let the data of (1.1)–(1.3) and (2.9)–(2.11) fulfill the subsequent conditions:

(Q_1) $\varphi(x) \in C^5[0,1]$, $\varphi^6(x) \in L_2(0,1)$, $\varphi(1) = \varphi''(1) = \varphi^{(4)}(1) = 0$, $\varphi'(0) = \varphi'(1)$, $\varphi'''(0) = \varphi'''(1)$, $\varphi^{(5)}(0) = \varphi^{(5)}(1)$;

(Q_2) $\psi(x) \in C^3[0,1]$, $\varphi^4(x) \in L_2(0,1)$, $\psi(1) = \psi''(1) = 0$, $\psi'(0) = \psi'(1)$, $\psi'''(0) = \psi'''(1)$;

(Q_3) $f(x,t), f_x(x,t), f_{xx}(x,t), f_{xxx}(x,t) \in C(D_T)$, $f_{xxxx}(x,t) \in L_2(D_T)$, $f(1,t) == f_{xx}(1,t) = 0$, $f_x(0,t) = f_x(1,t)$, $f_{xxx}(0,t) = f_{xxx}(1,t)$, $t \in [0,T]$;

(Q_4) $g(x,t), g_x(x,t), g_{xx}(x,t), g_{xxx}(x,t) \in C(D_T)$, $g_{xxxx}(x,t) \in L_2(D_T)$, $g(1,t) == g_{xx}(1,t) = 0$, $g_x(0,t) = g_x(1,t)$, $g_{xxx}(0,t) = g_{xxx}(1,t)$, $t \in [0,T]$;

(Q_5) $h_i \in C^2[0,T]$, $i = 1,2$, $h(t) \equiv h_1(t)(\frac{1}{2},t) - h_2(t)g(0,t) \neq 0$, $t \in [0,T]$, $0 \le \delta \le 2\pi$.

Then, from (3.19)–(3.23), taking into account (2.44)–(2.49), we obtain

$$\|\tilde{u}_0(t)\|_{C[0,T]} \le A_1(T) + B_1(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} + C_1(T)\|q(t)\|_{C[0,T]}, \tag{3.24}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|\tilde{u}_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le A_2(T) + B_2(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} + C_2(T)\|q(t)\|_{C[0,T]}, \tag{3.25}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|\tilde{u}_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le A_3(T) + B_3(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} + C_3(T)\|q(t)\|_{C[0,T]}, \tag{3.26}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \le A_4(T) + B_4(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} + C_4(T)\|q(t)\|_{C[0,T]}, \tag{3.27}$$

$$\|\tilde{q}(t)\|_{C[0,T]} \le A_5(T) + B_5(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} + C_5(T)\|q(t)\|_{C[0,T]}, \tag{3.28}$$

where

$$A_1(T) = \|\varphi_0(x)\|_{L_2(0,1)} + T\|\psi(x)\|_{L_2(0,1)} + (1+\delta T)T\sqrt{T}\|f(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = (1+\delta T)T^2, \qquad C_1(T) = (1+\delta T)T\sqrt{T}\|g(x,t)\|_{L_2(D_T)},$$

$$A_2(T) = \sqrt{2,5}(1+\delta T)\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\|\varphi^{(5)}(x)\|_{L_2(0,1)} + \frac{\sqrt{2,5}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\|\psi'''(x)\|_{L_2(0,1)} + \frac{\sqrt{2,5T}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\|f_{xxx}(x,t)\|_{L_2(D_T)},$$

$$B_2(T) = \frac{\sqrt{2,5}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)T, \qquad C_2(T) = \frac{\sqrt{2,5T}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\|g_{xxx}(x,t)\|_{L_2(D_T)},$$

$$A_3(T) = \sqrt{5}(1+\delta T)\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\|\varphi^{(5)}(x)x + 5\varphi^{(4)}(x)\|_{L_2(0,1)} + \frac{\sqrt{5}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\|\psi'''(x)x + 3\psi''(x)\|_{L_2(0,1)} + \frac{\sqrt{5T}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\|f_{xxx}(x,t)x + 3f_{xxx}(x,t)\|_{L_2(D_T)} + 2a\sqrt{5}(T+\delta T+\delta)\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(1+\sup_k\frac{\beta_k}{\beta_k-\delta}\right)\|\varphi^{(6)}(x)\|_{L_2(0,1)}$$

$$
\begin{aligned}
&+\frac{4\pi a\sqrt{5}}{2\pi a-\delta}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left(\frac{1}{\pi a}+T\right)\left\|\psi^{(4)}(x)\right\|_{L_2(0,1)}\\
&+4\sqrt{5T}\left(T+\frac{\delta}{2\pi a-\delta}\left(T+\frac{1}{\pi a}\right)\right)\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},\\
B_3(T)&=\sqrt{10}\left(\frac{1}{a}s+4\left(T+\frac{\delta}{2\pi a-\delta}\left(T+\frac{1}{\pi a}\right)\right)\right)\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)T,\\
C_3(T)&=\sqrt{5T}\Big[\frac{1}{a}\left\|g_{xxx}(x,t)x+3g_{xxx}(x,t)\right\|_{L_2(D_T)}\\
&+4\left(T+\frac{\delta}{2\pi a-\delta}\left(T+\frac{1}{\pi a}\right)\right)\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right),\\
A_4(T)&=\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\Big\{\left\|\left(h_1''(t)-f(0,t)\right)g\left(\frac{1}{2},t\right)-\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)g(0,t)\right\|_{C[0,T]}\\
&+2\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\|\left|g(0,t)\right|+\left|g\left(\frac{1}{2},t\right)\right|\right\|_{C[0,T]}\\
&\times\Big[(1+\delta)\sup_k\frac{\beta_k}{\beta_k-\delta}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}\\
&+\frac{1}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left\|\psi'''(x)\right\|_{L_2(0,1)}+\frac{1}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},\\
B_4(T)&=\frac{4T}{a}\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\|\left|g(0,t)\right|+\left|g\left(\frac{1}{2},t\right)\right|\right\|_{C[0,T]}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right),\\
C_4(T)&=\frac{4T}{a}\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\|\left|g(0,t)\right|+\left|g\left(\frac{1}{2},t\right)\right|\right\|_{C[0,T]}\\
&\times\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)},\\
A_5(T)&=\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\Big\{\left\|\left(h_2''(t)-f\left(\frac{1}{2},t\right)\right)h_1(t)-\left(h_2''(t)-f(0,t)\right)h_2(t)\right\|_{C[0,T]}\\
&+2\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\|\left|h_2(t)\right|+\left|h_1(t)\right|\right\|_{C[0,T]}\Big[(1+\delta)\sup_k\frac{\beta_k}{\beta_k-\delta}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}\\
&+\frac{1}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left\|\psi'''(x)\right\|_{L_2(0,1)}+\frac{\sqrt{T}}{a}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},\\
B_5(T)&=\frac{4T}{a}\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\|\left|h_2(t)\right|+\left|h_1(t)\right|\right\|_{C[0,T]}\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right),\\
C_5(T)&=\frac{4\sqrt{T}}{a}\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\|\left|h_2(t)\right|+\left|h_1(t)\right|\right\|_{C[0,T]}\\
&\times\sup_k\left(\frac{\beta_k}{\beta_k-\delta}\right)\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}.
\end{aligned}
$$

From (3.24)–(3.28), we conclude that

$$\begin{aligned}
&\|\widetilde{u}(x,t)\|_{B^5_{2,T}} + \|\widetilde{p}(t)\|_{C[0,T]} + \|\widetilde{q}(t)\|_{C[0,T]} \\
&\quad \equiv \|\widetilde{u}_0(t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\left(\lambda_k^6\|\widetilde{u}_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\
&\qquad + \left(\sum_{k=1}^{\infty}\left(\lambda_k^6\|\widetilde{u}_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \|\widetilde{p}(t)\|_{C[0,T]} + \|\widetilde{q}(t)\|_{C[0,T]} \\
&\quad \le A(T) + B(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} + C(T)\|q(t)\|_{C[0,T]},
\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}
&A(T) = A_1(T) + A_2(T) + A_3(T) + A_3(T) + A_5(T), \\
&B(T) = B_1(T) + B_2(T) + B_3(T) + B_3(T) + B_5(T), \\
&C(T) = C_1(T) + C_2(T) + C_3(T) + C_3(T) + C_5(T).
\end{aligned}$$

Hence, we can prove the subsequent theorem:

Theorem 3.2 *Let conditions* (Q_1)–(Q_5) *be fulfilled, and*

$$\big(A(T)+2\big)\big(\big(A(T)+2\big)B(T)+C(T)\big) < 1. \tag{3.30}$$

Then, (1.1)–(1.3) *and* (2.9)–(2.11) *has a unique solution in the sphere* $K = K_R(\|z\|_{E^5_T} \le R = A(T)+2)$ *of* E^5_T.

Proof In E^5_T, we consider

$$z = \Phi z, \tag{3.31}$$

where $z = \{u,p,q\}$ the components $\Phi_i(u,p,q)$, $i = 1,2,3$, of the operator $\Phi(u,p,q)$ are determined by the RHS of equations (3.10), (3.16), and (3.17). Consider $\Phi(u,p,q)$ in $K_R = K$ from E^5_T. Similiar to (3.17), we obtain that for any $z, z_1, z_2 \in K_R$ the following estimates are valid.

$$\begin{aligned}
\|\Phi z\|_{E^5_T} &\le A(T) + B(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}} + C(T)\|q(t)\|_{C[0,T]} \\
&\le A(T) + B(T)R^2 + C(T)R \le A(T) + B(T)\big(A(T)+2\big)^2 \\
&\quad + C(T)\big(A(T)+2\big),
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\|\Phi z_1 - \Phi z_2\|_{E^5_T} &\le \big(A(T)+B(T)\big)\big(\|p_1(t)-p_2(t)\|_{C[0,T]} + \|q_1(t)-q_2(t)\|_{C[0,T]} \\
&\qquad + \|u_1(x,t)-u_2(x,t)\|_{B^5_{2,T}}\big) \\
&\le \big(B(T)\big(A(T)+2\big) + C(T)\big)\|z_1 - z_2\|_{E^5_T}.
\end{aligned} \tag{3.33}$$

Then, using (3.31) and (3.33), it follows from (3.30) that Φ acts in $K_R = K$ and it is a contraction mapping. Therefore, in $K_R = K$, the operator has a unique fixed point $\{u, p, q\}$ that is a solution of equation (3.31).

The $u(x,t)$, as the element of $B^5_{2,T}$, has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$, $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$ in D_T. Now, from (3.33)–(3.35), we obtain

$$\begin{aligned}\left\|u_0''(t)\right\|_{C[0,T]} &\le \left\|a_1(t)\right\|_{C[0,T]}\left\|u_0(t)\right\|_{C[0,T]} \\ &\quad + \left\|a_2(t)\right\|_{C[0,T]}\left\|u_0'(t)\right\|_{C[0,T]} + \left\|\left\|f(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)},\end{aligned} \tag{3.34}$$

$$\begin{aligned}&\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_{2k-1}''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &\quad\le \sqrt{2}a^2\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &\qquad + \left\|\left\|p(t)u_x(x,t) + q(t)g_x(x,t) + f_x(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)},\end{aligned} \tag{3.35}$$

$$\begin{aligned}&\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_{2k-1}''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &\quad\le \sqrt{3}a^2\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_{2k}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \\ &\qquad + 4\sqrt{3}a^2\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_{2k-1}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \frac{\sqrt{6}}{2}\left\|\left\|p(t)\left(u_x(x,t)x + u(x,t)\right)\right.\right. \\ &\qquad \left.\left. + q(t)\left(g_x(x,t)x + g(x,t)\right) + f_x(x,t)x + f(x,t)\right\|_{C[0,T]}\right\|_{L_2(0,1)}.\end{aligned} \tag{3.36}$$

Hence, it follows that $u_{tt}(x,t)$ is continuous in D_T.

It is easy to validate that (1.1)–(1.3) and (2.9)–(2.11) are fulfilled in the ordinary sense. Therefore, $\{u(x,t), p(t), q(t)\}$ is a solution of (1.1)–(1.3) and (2.9)–(2.11), and by Lemma 3.1, it is unique in the ball $K_R = K$. The theorem is proved. □

The subsequent theorem is proved by Lemma 2.2.

Theorem 3.3 *Let all the conditions of Theorem 3.2 be fulfilled:*

$$\begin{aligned}&\int_0^1 g(x,t)\,dx = 0, \qquad \int_0^1 f(x,t)\,dx = 0, \quad t \in [0,T], \\ &\int_0^1 \varphi(x)\,dx = 0, \qquad \int_0^1 \psi(x)\,dx = 0, \\ &\varphi(0) = h_1(0), \qquad \varphi\left(\frac{1}{2}\right) = h_2(0), \\ &\psi(0) = h_1'(0) + \delta h_1(T), \qquad \psi\left(\frac{1}{2}\right) = h_0' + \delta h_2(T), \\ &\left(A(T) + 2\right)\left(\frac{1}{2} + \frac{1}{1+\delta T}\frac{5\delta T}{2(1+\delta T)}\right)T^2 < 1.\end{aligned}$$

Then, in $K = K_R(\|z\|_{E_T^5} \le R = A(T) + 2)$ *of* E_T^5, (1.1)–(1.5) *has a unique classical solution.*

4 Discretization of the direct problem

We consider the IBVP (1.1)–(1.4), when a, $q(t)$, $p(t)$, $g(x,t)$, and $f(x,t)$ are given. First, we divide $[0,l]$ into a mesh of equal size $h = x_{i+1} - x_i$, $i = 0, 1, \dots, M$. The discrete form of the direct problem is as follows. We denote $u(x_i,t_j) = u_i^j$, $p(t_j) = p^j$, $q(t_j) = q^j$, $g(x_i,t_j) = g_i^j$ and $f(x_i,t_j) = f_i^j$, where $x_i = ih$, $t_j = jk$, $h = \Delta x = \frac{l}{M}$ and $k = \Delta t = \frac{T}{N}$ for $i = 0, 1, \dots, M$ and $j = 0, 1, \dots, N$. The QnB-spline $QB_i(x)$, $i = -2, -1, \dots, M+1, M+2$ are given by [6, 9]:

$$QB_i(x) = \frac{1}{h^5}\begin{cases} (x - x_{i-3})^5, & x \in [x_{i-3}, x_{i-2}), \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5, & x \in [x_{i-2}, x_{i-1}), \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5, & x \in [x_{i-1}, x_i), \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5 + 15(x_{i+1} - x)^5, & x \in [x_i, x_{i+1}), \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5, & x \in [x_{i+1}, x_{i+2}), \\ (x_{i+3} - x)^5, & x \in [x_{i+2}, x_{i+3}), \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

The values of $QB_i(x)$, $QB_i'(x)$, $QB_i''(x)$, $QB_i'''(x)$, and $QB_i^{iv}(x)$ are given in Table 1, where $\xi_1 = \frac{5}{h}$, $\xi_2 = \frac{20}{h^2}$, $\xi_3 = \frac{60}{h^3}$, and $\xi_4 = \frac{120}{h^4}$.

We suppose that $u(x,t)$ at the point (x, t_j) is expressed as:

$$u(x,t_j) = \sum_{k=-2}^{M+2} C_k(t_j) QB_k(x). \tag{4.2}$$

The variation of the $U_M(x,t)$ is expressed as

$$u(x,t_j) = \sum_{k=i-2}^{i+2} C_k(t_j) QB_k(x). \tag{4.3}$$

Using (4.3), we obtain u, u_x, u_{xx}, u_{xxx}, u_{xxxx} as:

$$u_i^j = C_{i+2}^j + 26C_{i+1}^j + 66C_i^j + 26C_{i-1}^j + C_{i-2}^j, \tag{4.4}$$

$$(u_x)_i^j = \xi_1\big(C_{i+2}^j + 10C_{i+1}^j - 10C_{i-1}^j - C_{i-2}^j\big), \tag{4.5}$$

$$(u_{xx})_i^j = \xi_2\big(C_{i+2}^j + 2C_{i+1}^j - 6C_i^j + 2C_{i-1}^j + C_{i-2}^j\big), \tag{4.6}$$

$$(u_{xxx})_i^j = \xi_3\big(C_{i+2}^j - 2C_{i+1}^j + 2C_{i-1}^j - C_{i-2}^j\big), \tag{4.7}$$

$$(u_{xxxx})_i^j = \xi_4\big(C_{i+2}^j - 4C_{i+1}^j + 6C_i^j - 4C_{i-1}^j + C_{i-2}^j\big), \tag{4.8}$$

Table 1 The $QB_i(x)$, $QB_i'(x)$, $QB_i''(x)$, and $QB_i'''(x)$

	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
$QB_i(x)$	0	1	26	66	26	1	0
$QB_i'(x)$	0	ξ_1	$10\,\xi_1$	0	$-10\xi_1$	$-\xi_1$	0
$QB_i''(x)$	0	ξ_2	$2\xi_2$	$-6\xi_2$	$2\xi_2$	ξ_2	0
$QB_i'''(x)$	0	ξ_3	$-2\xi_3$	0	$2\xi_3$	$-\xi_3$	0
$QB_i^{iv}(x)$	0	ξ_4	$-2\xi_4$	$6\xi_4$	$-2\xi_4$	ξ_4	0

where $C_i^j = C_i(t_j)$. Now, we discretize equation (1.1) as:

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} + a^2\left(\frac{(u_{xxxx})_i^{j+1} + (u_{xxxx})_i^j}{2}\right) = \frac{p^{j+1}u_i^{j+1} + p^j u_i^j}{2} + \frac{q^{j+1}g_i^{j+1} + q^j g_i^j}{2} + \frac{f_i^{j+1} + f_i^j}{2}, \quad i = 0, 1, \dots, M, j = 0, 1, \dots, N, \tag{4.9}$$

which implies

$$\big(1 - \bar{A}^{j+1}\big)u_i^{j+1} + \bar{B}(u_{xxxx})_i^{j+1} = \big(2 + \bar{A}^j\big)u_i^j - \bar{B}(u_{xxxx})_i^j - u_i^{j-1} + R_i^j, \quad i = 0, 1, \dots, M, j = 0, 1, \dots, N, \tag{4.10}$$

where

$$\bar{A}^j = \frac{(\Delta t)^2}{2}p^j, \qquad \bar{B} = a^2\frac{(\Delta t)^2}{2}, \qquad R_i^j = \frac{q^{j+1}g_i^{j+1} + q^j g_i^j}{2} + \frac{f_i^{j+1} + f_i^j}{2}.$$

Now, using (4.4)–(4.8) and simplifying the terms, we obtain

$$\begin{aligned}
&\big(1 - \bar{A}^{j+1} + \xi_4\bar{B}\big)C_{i-2}^{j+1} + \big(26 - 26\bar{A}^{j+1} - 4\xi_4\bar{B}\big)C_{i-1}^{j+1} \\
&\quad + \big(66 - 66\bar{A}^{j+1} + 6\xi_4\bar{B}\big)C_i^{j+1} + \big(26 - 26\bar{A}^{j+1} - 4\xi_4\bar{B}\big)C_{i+1}^{j+1} \\
&\quad + \big(1 - \bar{A}^{j+1} + \xi_4\bar{B}\big)C_{i+2}^{j+1} = \big(2 + \bar{A}^j - \xi_4\bar{B}\big)C_{i-2}^j + \big(52 + 26\bar{A}^j + 4\xi_4\bar{B}\big)C_{i-1}^j \\
&\quad + \big(132 + 66\bar{A}^j - 6\xi_4\bar{B}\big)C_i^j + \big(52 + 26\bar{A}^j + 4\xi_4\bar{B}\big)C_{i+1}^j \\
&\quad + \big(2 + \bar{A}^j - \xi_4\bar{B}\big)C_{i+2}^j - C_{i+2}^{j-1} - 26C_{i+1}^{j-1} - 66C_i^{j-1} - 26C_{i-1}^{j-1} - C_{i-2}^{j-1} + R_i^j, \\
&i = 2, 3, \dots, M-2, j = 1, 2, \dots, N.
\end{aligned} \tag{4.11}$$

The above equation can be written as

$$\begin{aligned}
&\hat{A}_1^{j+1}C_{i-2}^{j+1} + \hat{A}_2^{j+1}C_{i-1}^{j+1} + \hat{A}_3^{j+1}C_i^{j+1} + \hat{A}_2^{j+1}C_{i+1}^{j+1} + \hat{A}_1^{j+1}C_{i+2}^{j+1} \\
&\quad = \hat{B}_1^j C_{i-2}^j + \hat{B}_2^j C_{i-1}^j \\
&\qquad + \hat{B}_3^j C_i^j + \hat{B}_2^j C_{i+1}^j + \hat{B}_1^j C_{i+2}^j - C_{i+2}^{j-1} - 26C_{i+1}^{j-1} - 66C_i^{j-1} - 26C_{i-1}^{j-1} - C_{i-2}^{j-1} + R_i^j, \\
&i = 2, 3, \dots, M-2, j = 1, 2, \dots, N,
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
&\hat{A}_1^{j+1} = 1 - \bar{A}^{j+1} + \xi_4\bar{B}, \qquad \hat{A}_2^{j+1} = 26 - 26\bar{A}^{j+1} - 4\xi_4\bar{B}, \\
&\hat{A}_3^{j+1} = 66 - 66\bar{A}^{j+1} + 6\xi_4\bar{B}, \qquad \hat{B}_1^j = 2 + \bar{A}^j - \xi_4\bar{B}, \\
&\hat{B}_2^j = 52 + 26\bar{A}^j + 4\xi_4\bar{B}, \quad \text{and} \quad \hat{B}_3^j = 132 + 66\bar{A}^j - 6\xi_4\bar{B}.
\end{aligned}$$

Now, we discretize the initial conditions (1.2) as

$$u(x, 0) = \varphi_1(x) \quad \Longrightarrow \quad u_i^0 = \varphi_1(x_i), \quad i = 0, 1, \dots, M, \tag{4.13}$$

$$u_t(x,0)=\varphi_2(x) \quad\Longrightarrow\quad u_i^{-1}=u_i^1-2\Delta t\varphi_2(x_i),\quad i=0,1,\dots,M. \tag{4.14}$$

For $j=0$, using the IC (4.14) in (4.10), we obtain

$$\left(2-\bar{A}^1\right)u_i^1+\bar{B}(u_{xxxx})_i^1=\left(2+\bar{A}^0\right)u_i^0-\bar{B}(u_{xxxx})_i^0+\bar{R}_i^0,\quad i=0,1,\dots,M, \tag{4.15}$$

where

$$\bar{R}_i^0=R_i^0+2\Delta t\varphi(x_i),\quad i=0,1,\dots,N.$$

Using the approximated values of u and u_{xxxx} in (4.15) and simplifying the terms, we obtain

$$\begin{aligned}
&\left(2-\bar{A}^1+\xi_4\bar{B}\right)C_{i-2}^1+\left(52-26\bar{A}^1-4\xi_4\bar{B}\right)C_{i-1}^1+\left(132-66\bar{A}^1+6\xi_4\bar{B}\right)C_i^1\\
&\quad+\left(52-26\bar{A}^1-4\xi_4\bar{B}\right)C_{i+1}^1+\left(2-\bar{A}^1+\xi_4\bar{B}\right)C_{i+2}^1\\
&=\left(2+\bar{A}^0-\xi_4\bar{B}\right)C_{i-2}^0+\left(52+26\bar{A}^0+4\xi_4\bar{B}\right)C_{i-1}^0\\
&\quad+\left(132+66\bar{A}^0-6\xi_4\bar{B}\right)C_i^0+\left(52+26\bar{A}^0+4\xi_4\bar{B}\right)C_{i+1}^0\\
&\quad+\left(2+\bar{A}^0-\xi_4\bar{B}\right)C_{i+2}^0+\bar{R}_i^0,\quad i=2,3,\dots,M-2,
\end{aligned} \tag{4.16}$$

which can be written as

$$\begin{aligned}
&A_1^*C_{i-2}^1+A_2^*C_{i-1}^1+A_3^*C_i^1+A_2^*C_{i+1}^1+A_1^*C_{i+2}^1\\
&=B_1^*C_{i-2}^0+B_2^*C_{i-1}^0+B_3^*C_i^0+B_2^*C_{i+1}^0+B_1^*C_{i+2}^0+\bar{R}_i^0,\quad i=2,3,\dots,M-2,
\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}
&A_1^*=2-\bar{A}^1+\xi_4\bar{B}, && A_2^*=52-26\bar{A}^1-4\xi_4\bar{B}, && A_3^*=132-66\bar{A}^1+6\xi_4\bar{B},\\
&B_1^*=2+\bar{A}^0-\xi_4\bar{B}, && B_2^*=52+26\bar{A}^0+4\xi_4\bar{B}, && B_3^*=132+66\bar{A}^0-6\xi_4\bar{B}.
\end{aligned}$$

The system (4.17) has unknowns $(C_{-2}^0,C_{-1}^0,C_0^0,\dots,C_{M+1}^0,C_{M+2}^0)$, where C_{-2}^0, C_{-1}^0, C_{M+1}^0, and C_{M+2}^0 are outside the domain. For a unique solution, we need to remove these quantities. For this purpose, we use the BCs $u(1,t)=0$, $u_x(0,t)=u_x(1,t)$, $u_{xx}(1,t)=0$, and $\int_0^1 u(x,t)\,dx=0$, which give us following equations:

$$C_{M-2}^j+26C_{M-1}^j+66C_M^j+26C_{M+1}^j+C_{M+2}^j=0, \tag{4.18}$$

$$-C_{-2}^j-10C_{-1}^j+10C_1^j+C_2^j=-C_{M-2}^j-10C_{M-1}^j+10C_{M+1}^j+C_{M+2}^j, \tag{4.19}$$

$$C_{M-2}^j+2C_{M-1}^j-6C_M^j+2C_{M+1}^j+C_{M+2}^j=0, \tag{4.20}$$

and

$$\begin{aligned}
&C_{-2}^j+28C_{-1}^j+120C_0^j+212C_1^j+239C_2^j+240\sum_{k=3}^{M-3}C_k^j+238C_{M-2}^j\\
&\quad+186C_{M-1}^j+54C_M^j+2C_{M+1}^j=0,
\end{aligned} \tag{4.21}$$

where, $j=0,1,\dots,N$.

Solving the above equations, we obtain

$$C_{-1}^{j}=-\frac{20}{3}C_{0}^{j}-\frac{37}{3}C_{1}^{j}-\frac{40}{3}\sum_{k=2}^{M-2}C_{k}^{j}-\frac{34}{3}C_{M-1}^{j}-\frac{11}{3}C_{M}^{j}, \tag{4.22}$$

$$\begin{aligned}C_{-2}^{j}={}&\frac{200}{3}C_{0}^{j}+\frac{400}{3}C_{1}^{j}+\frac{403}{3}C_{2}^{j}+\frac{400}{3}\sum_{k=3}^{M-3}C_{k}^{j}\\&+\frac{406}{3}C_{M-2}^{j}+\frac{400}{3}C_{M-1}^{j}+\frac{164}{3}C_{M}^{j},\end{aligned} \tag{4.23}$$

$$C_{M+1}^{j}=-C_{M-1}^{j}-3C_{M}^{j}, \tag{4.24}$$

$$C_{M+2}^{j}=-C_{M-2}^{j}+12C_{M}^{j}. \tag{4.25}$$

For $i=0$, using (4.22) and (4.23) in (4.17), we obtain

$$\begin{aligned}&\bar{A}_{1}^{*}C_{0}^{1}+\bar{A}_{2}^{*}C_{1}^{1}+\bar{A}_{3}^{*}C_{2}^{1}+\bar{A}_{4}^{*}\sum_{k=3}^{M-3}C_{k}^{1}+\bar{A}_{5}^{*}C_{M-2}^{1}+\bar{A}_{6}^{*}C_{M-1}^{1}+\bar{A}_{7}^{*}C_{M}^{1}\\&\quad=\bar{B}_{1}^{*}C_{0}^{0}+\bar{B}_{2}^{*}C_{1}^{0}+\bar{B}_{3}^{*}C_{2}^{0}+\bar{B}_{4}^{*}\sum_{k=3}^{M-3}C_{k}^{0}+\bar{B}_{5}^{*}C_{M-2}^{0}+\bar{B}_{6}^{*}C_{M-1}^{0}+\bar{B}_{7}^{*}C_{M}^{0}+\bar{R}_{0}^{0},\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}&\bar{A}_{1}^{*}=\frac{200}{3}A_{1}^{*}-\frac{20}{3}A_{2}^{*}+A_{3}^{*},\qquad \bar{A}_{2}^{*}=\frac{400}{3}A_{1}^{*}-\frac{37}{3}A_{2}^{*}+A_{2}^{*},\\&\bar{A}_{3}^{*}=\frac{403}{3}A_{1}^{*}-\frac{40}{3}A_{2}^{*}+A_{1}^{*},\qquad \bar{A}_{4}^{*}=\frac{400}{3}A_{1}^{*}-\frac{40}{3}A_{2}^{*},\qquad \bar{A}_{5}^{*}=\frac{406}{3}A_{1}^{*}-\frac{40}{3}A_{2}^{*},\\&\bar{A}_{6}^{*}=\frac{400}{3}A_{1}^{*}-\frac{34}{3}A_{2}^{*},\qquad \bar{A}_{7}^{*}=\frac{164}{3}A_{1}^{*}-\frac{11}{3}A_{2}^{*},\qquad \bar{B}_{1}^{*}=\frac{200}{3}B_{1}^{*}-\frac{20}{3}B_{2}^{*}+B_{3}^{*},\\&\bar{B}_{2}^{*}=\frac{400}{3}B_{1}^{*}-\frac{37}{3}B_{2}^{*}+B_{2}^{*},\qquad \bar{B}_{3}^{*}=\frac{403}{3}B_{1}^{*}-\frac{40}{3}B_{2}^{*}+B_{1}^{*},\qquad \bar{B}_{4}^{*}=\frac{400}{3}B_{1}^{*}-\frac{40}{3}B_{2}^{*},\\&\bar{B}_{5}^{*}=\frac{406}{3}B_{1}^{*}-\frac{40}{3}B_{2}^{*},\qquad \bar{B}_{6}^{*}=\frac{400}{3}B_{1}^{*}-\frac{34}{3}B_{2}^{*},\qquad \bar{B}_{7}^{*}=\frac{164}{3}B_{1}^{*}-\frac{11}{3}B_{2}^{*}.\end{aligned}$$

For $i=1$, using (4.22) in (4.17), we obtain

$$\begin{aligned}&A_{1}^{**}C_{0}^{1}+A_{2}^{**}C_{1}^{1}+A_{3}^{**}C_{2}^{1}+A_{4}^{**}C_{3}^{1}+A_{5}^{**}\sum_{k=4}^{M-2}C_{k}^{1}+A_{6}^{**}C_{M-1}^{1}+A_{7}^{**}C_{M}^{1}\\&\quad=B_{1}^{**}C_{0}^{0}+B_{2}^{**}C_{1}^{0}+B_{3}^{**}C_{2}^{0}+B_{4}^{**}C_{3}^{0}+B_{5}^{**}\sum_{k=4}^{M-2}C_{k}^{0}+B_{6}^{**}C_{M-1}^{0}+B_{7}^{**}C_{M}^{0}+\bar{R}_{1}^{0},\end{aligned} \tag{4.27}$$

where

$$\begin{aligned}&A_{1}^{**}=-\frac{20}{3}A_{1}^{*}+A_{2}^{*},\qquad A_{2}^{**}=-\frac{37}{3}A_{1}^{*}+A_{3}^{*},\qquad A_{3}^{**}=-\frac{40}{3}A_{1}^{*}+A_{2}^{*},\\&A_{4}^{**}=-\frac{40}{3}A_{1}^{*}+A_{1}^{*},\qquad A_{5}^{**}=-\frac{40}{3}A_{1}^{*},\qquad A_{6}^{**}=-\frac{34}{3}A_{1}^{*},\qquad A_{7}^{**}=-\frac{11}{3}A_{1}^{*},\\&B_{1}^{**}=-\frac{20}{3}B_{1}^{*}+B_{2}^{*},\qquad B_{2}^{**}=-\frac{37}{3}B_{1}^{*}+B_{3}^{*},\qquad B_{3}^{**}=-\frac{40}{3}B_{2}^{*}+B_{2}^{*},\end{aligned}$$

$$B_4^{**} = -\frac{40}{3}B_1^* + B_1^*, \qquad B_5^{**} = -\frac{40}{3}B_1^*, \qquad B_6^{**} = -\frac{34}{3}B_1^*, \qquad B_7^{**} = -\frac{11}{3}B_1^*.$$

For $i = M-1$, using (4.24) in (4.17), we obtain

$$\begin{aligned} &A_1^* C_{M-3}^1 + A_2^* C_{M-2}^1 + \left(A_3^* - A_1^*\right) C_{M-1}^1 + \left(A_2^* - 3A_1^*\right) C_M^1 \\ &\quad = B_1^* C_{M-3}^0 + B_2^* C_{M-2}^0 + \left(B_3^* - B_1^*\right) C_{M-1}^0 + \left(B_2^* - 3B_1^*\right) C_M^0 + R_{M-1}^0. \end{aligned} \tag{4.28}$$

For $i = M$, using (4.25) in (4.17), we obtain

$$\left(A_3^* - 3A_2^* + 12A_1^*\right) C_M^1 = \left(B_3^* - 3B_2^* + 12B_1^*\right) C_M^0 + R_M^0. \tag{4.29}$$

At time $t = 0$, the equations (4.26), (4.27), (4.17), (4.28), and (4.29) form a system:

$$\begin{aligned} &\begin{pmatrix} \bar{A}_1^* & \bar{A}_2^* & \bar{A}_3^* & \bar{A}_4^* & \bar{A}_4^* & \cdots & \bar{A}_4^* & \bar{A}_5^* & \bar{A}_6^* & \bar{A}_7^* \\ A_1^{**} & A_2^{**} & A_3^{**} & A_4^{**} & A_5^{**} & A_5^{**} & \cdots & A_5^{**} & A_6^{**} & A_7^{**} \\ A_1^* & A_2^* & A_3^* & A_2^* & A_1^* & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_1^* & A_2^* & A_3^* & A_2^* & A_1^* & 0 & \cdots & 0 & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & A_1^* & A_2^* & A_3^* & A_2^* & A_1^* \\ 0 & 0 & 0 & 0 & 0 & 0 & A_1^* & A_2^* & A_3^* - A_1^* & A_2^* - 3A_1^* \end{pmatrix} \\ &\quad \times \begin{pmatrix} C_0^1 \\ C_1^1 \\ C_2^1 \\ \vdots \\ C_{M-2}^1 \\ C_{M-1}^1 \\ C_M^1 \end{pmatrix} = \begin{pmatrix} \hat{R}_0^0 \\ \hat{R}_1^0 \\ \hat{R}_2^0 \\ \vdots \\ \hat{R}_{M-2}^0 \\ \hat{R}_{M-1}^0 \\ \hat{R}_M^0 \end{pmatrix}, \end{aligned} \tag{4.30}$$

where

$$\hat{R}_0^0 = \bar{B}_1^* C_0^0 + \bar{B}_2^* C_1^0 + \bar{B}_3^* C_2^0 + \bar{B}_4^* \sum_{k=3}^{M-3} C_k^0 + \bar{B}_5^* C_{M-2}^0 + \bar{B}_6^* C_{M-1}^0 + \bar{B}_7^* C_M^0 + \bar{R}_0^0,$$

$$\hat{R}_1^0 = B_1^{**} C_0^0 + B_2^{**} C_1^0 + B_3^{**} C_2^0 + B_4^{**} C_3^0 + B_5^{**} \sum_{k=4}^{M-2} C_k^0 + B_6^{**} C_{M-1}^0 + B_7^{**} C_M^0 + \bar{R}_1^0,$$

$$\hat{R}_i^0 = B_1^* C_{i-2}^0 + B_2^* C_{i-1}^0 + B_3^* C_i^0 + B_2^* C_{i+1}^0 + B_1^* C_{i+2}^0 + \bar{R}_i^0, \quad i = 2, 3, \dots, M-2,$$

$$\hat{R}_{M-1}^0 = B_1^* C_{M-3}^0 + B_2^* C_{M-2}^0 + \left(B_3^* - B_1^*\right) C_{M-1}^0 + \left(B_2^* - 3B_1^*\right) C_M^0 + R_{M-1}^0,$$

$$\hat{R}_M^0 = \left(B_3^* - 3B_2^* + 12B_1^*\right) C_M^0 + R_M^0.$$

Finally, for $i = 0$, using (4.22) and (4.23) in (4.12), we obtain

$$\begin{aligned} &\hat{A}_1^{*j+1} C_0^{j+1} + \hat{A}_2^{*j+1} C_1^{j+1} + \hat{A}_3^{*j+1} C_2^{j+1} + \hat{A}_4^{*j+1} \sum_{k=3}^{M-3} C_k^{j+1} + \hat{A}_5^{*j+1} C_{M-2}^{j+1} \\ &\quad + \hat{A}_6^{*j+1} C_{M-1}^{j+1} + \hat{A}_7^{*j+1} C_M^{j+1} \end{aligned}$$

$$= \hat{B}_1^{*j} C_0^j + \hat{B}_2^{*j} C_1^j + \hat{B}_3^{*j} C_2^j + \hat{B}_4^{*j} \sum_{k=3}^{M-3} C_k^j + \hat{B}_5^{*j} C_{M-2}^j + \hat{B}_6^{*j} C_{M-1}^j + \hat{B}_7^{*j} C_M^j - u_0^{j-1} + R_0^j, \quad j = 1, 2, \ldots, N, \tag{4.31}$$

where

$$\begin{aligned}
&\hat{A}_1^{*j+1} = \frac{200}{3}\hat{A}_1^{j+1} - \frac{20}{3}\hat{A}_2^{j+1} + \hat{A}_3^{j+1}, \qquad \hat{A}_2^{*j+1} = \frac{400}{3}\hat{A}_1^{j+1} - \frac{37}{3}\hat{A}_2^{j+1} + \hat{A}_2^{j+1}, \\
&\hat{A}_3^{*j+1} = \frac{403}{3}\hat{A}_1^{j+1} - \frac{40}{3}\hat{A}_2^{j+1} + \hat{A}_1^{j+1}, \qquad \hat{A}_4^{*j+1} = \frac{400}{3}\hat{A}_1^{j+1} - \frac{40}{3}\hat{A}_2^{j+1}, \\
&\hat{A}_5^{*j+1} = \frac{406}{3}\hat{A}_1^{j+1} - \frac{40}{3}\hat{A}_2^{j+1}, \hat{A}_6^{*j+1} = \frac{400}{3}\hat{A}_1^{j+1} - \frac{34}{3}\hat{A}_2^{j+1}, \\
&\hat{A}_7^{*j+1} = \frac{164}{3}\hat{A}_1^{j+1} - \frac{11}{3}\hat{A}_2^{j+1}, \qquad \hat{B}_1^{*j+1} = \frac{200}{3}\hat{B}_1^{j+1} - \frac{20}{3}\hat{B}_2^{j+1} + \hat{B}_3^{j+1}, \\
&\hat{B}_2^{*j+1} = \frac{400}{3}\hat{B}_1^{j+1} - \frac{37}{3}\hat{B}_2^{j+1} + \hat{B}_2^{j+1}, \qquad \hat{B}_3^{*j+1} = \frac{403}{3}\hat{B}_1^{j+1} - \frac{40}{3}\hat{B}_2^{j+1} + \hat{B}_1^{j+1}, \\
&\hat{B}_4^{*j+1} = \frac{400}{3}\hat{B}_1^{j+1} - \frac{40}{3}\hat{B}_2^{j+1}, \qquad \hat{B}_5^{*j+1} = \frac{406}{3}\hat{B}_1^{j+1} - \frac{40}{3}\hat{B}_2^{j+1}, \\
&\hat{B}_6^{*j+1} = \frac{400}{3}\hat{B}_1^{j+1} - \frac{34}{3}\hat{B}_2^{j+1}, \qquad \hat{B}_7^{*j+1} = \frac{164}{3}\hat{B}_1^{j+1} - \frac{11}{3}\hat{B}_2^{j+1}.
\end{aligned}$$

For $i = 1$, using (4.22) in (4.12), we obtain

$$\begin{aligned}
&\hat{A}_1^{**j+1} C_0^{j+1} + \hat{A}_2^{**j+1} C_1^{j+1} + \hat{A}_3^{**j+1} C_2^{j+1} + \hat{A}_4^{**j+1} C_3^{j+1} \\
&\quad + \hat{A}_5^{**j+1} \sum_{k=4}^{M-2} C_k^{j+1} + \hat{A}_6^{**j+1} C_{M-1}^{j+1} + \hat{A}_7^{**j+1} C_M^{j+1} = \hat{B}_1^{**j} C_0^j + \hat{B}_2^{**j} C_1^j \\
&\quad + \hat{B}_3^{**j} C_2^j + \hat{B}_4^{**j} C_3^j + \hat{B}_5^{**j} \sum_{k=4}^{M-2} C_k^j + \hat{B}_6^{**j} C_{M-1}^j + \hat{B}_7^{**j} C_M^j - u_1^{j-1} + R_1^j, \\
&j = 1, 2, \ldots, N,
\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}
&\hat{A}_1^{**j+1} = -\frac{20}{3}\hat{A}_1^{j+1} + \hat{A}_2^{j+1}, \qquad \hat{A}_2^{**j+1} = -\frac{37}{3}\hat{A}_1^{j+1} + \hat{A}_3^{j+1}, \\
&\hat{A}_3^{**j+1} = -\frac{40}{3}\hat{A}_1^{j+1} + \hat{A}_2^{j+1}, \qquad \hat{A}_4^{**j+1} = -\frac{40}{3}\hat{A}_1^{j+1} + \hat{A}_1^{j+1}, \qquad \hat{A}_5^{**j+1} = -\frac{40}{3}\hat{A}_1^{j+1}, \\
&\hat{A}_6^{**j+1} = -\frac{34}{3}\hat{A}_1^{j+1}, \qquad \hat{A}_7^{**j+1} = -\frac{11}{3}\hat{A}_1^{j+1}, \qquad \hat{B}_1^{**j} = -\frac{20}{3}\hat{B}_1^{j+1} + \hat{B}_2^{j+1}, \\
&\hat{B}_2^{**j} = -\frac{37}{3}\hat{B}_1^{j+1} + \hat{B}_3^{j+1}, \qquad \hat{B}_3^{**j} = -\frac{40}{3}\hat{B}_1^{j+1} + \hat{B}_2^{j+1}, \qquad \hat{B}_4^{**j} = -\frac{40}{3}\hat{B}_1^{j+1} + \hat{B}_1^{j+1}, \\
&\hat{B}_5^{**j} = -\frac{40}{3}\hat{B}_1^{j+1}, \qquad \hat{B}_6^{**j} = -\frac{34}{3}\hat{B}_1^{j+1}, \qquad \hat{B}_7^{**j} = -\frac{11}{3}\hat{B}_1^{j+1}.
\end{aligned}$$

For $i = M-1$, using (4.24) in (4.11), we obtain

$$\begin{aligned}
&\hat{A}_1^{j+1}C_{M-3}^{j+1} + \hat{A}_2^{j+1}C_{M-2}^{j+1} + \left(\hat{A}_3^{j+1} - \hat{A}_1^{j+1}\right)C_{M-1}^{j+1} + \left(\hat{A}_2^{j+1} - 3\hat{A}_1^{j+1}\right)C_M^{j+1} \\
&\quad = \hat{B}_1^j C_{M-3}^j + \hat{B}_2^j C_{M-2}^j + \left(\hat{B}_3^j - \hat{B}_1^j\right)C_{M-1}^j + \left(\hat{B}_2^j - 3\hat{B}_1^j\right)C_M^j + R_{M-1}^j, \\
&\quad j = 1,2,\dots,N.
\end{aligned} \tag{4.33}$$

For $i = M$, using (4.25) in (4.12), we obtain

$$\left(\hat{A}_3^{j+1} - 3\hat{A}_2^{j+1} + 12\hat{A}_1^{j+1}\right)C_M^{j+1} = \left(\hat{B}_3^j - 3\hat{B}_2^j + 12\hat{B}_1^j\right)C_M^j - u_M^{j-1} + R_M^j. \tag{4.34}$$

At the time t_j, $j = 1,2,\dots,N$, the equations (4.31), (4.32), (4.12), (4.33), and (4.34) form a system:

$$\begin{pmatrix}
\hat{A}_1^{*j+1} & \hat{A}_2^{*j+1} & \hat{A}_3^{*j+1} & \hat{A}_4^{*j+1} & \cdots & \hat{A}_4^{*j+1} & \hat{A}_5^{*j+1} & \hat{A}_6^{*j+1} & \hat{A}_7^{*j+1} \\
\hat{A}_1^{**j+1} & \hat{A}_2^{**j+1} & \hat{A}_3^{**j+1} & \hat{A}_4^{**j+1} & \hat{A}_5^{**j+1} & \cdots & \hat{A}_5^{**j+1} & \hat{A}_6^{**j+1} & \hat{A}_7^{**j+1} \\
\hat{A}_1^{j+1} & \hat{A}_2^{j+1} & \hat{A}_3^{j+1} & \hat{A}_2^{j+1} & \hat{A}_1^{j+1} & 0 & \cdots & 0 & 0 \\
0 & \hat{A}_1^{j+1} & \hat{A}_2^{j+1} & \hat{A}_3^{j+1} & \hat{A}_2^{j+1} & \hat{A}_1^{j+1} & 0 & \cdots & 0 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
0 & 0 & 0 & 0 & \hat{A}_1^{j+1} & \hat{A}_2^{j+1} & \hat{A}_3^{j+1} & \hat{A}_2^{j+1} & \hat{A}_1^{j+1} \\
0 & 0 & 0 & 0 & 0 & \hat{A}_1^{j+1} & \hat{A}_2^{j+1} & \hat{A}_3^{j+1} - \hat{A}_1^{j+1} & \hat{A}_2^{j+1} - 3\hat{A}_1^{j+1}
\end{pmatrix}$$

$$\times \begin{pmatrix} C_0^{j+1} \\ C_1^{j+1} \\ C_2^{j+1} \\ \vdots \\ C_{M-2}^{j+1} \\ C_{M-1}^{j+1} \\ C_M^{j+1} \end{pmatrix} = \begin{pmatrix} \bar{R}_0^j \\ \bar{R}_1^j \\ \bar{R}_2^j \\ \vdots \\ \bar{R}_{M-2}^j \\ \bar{R}_{M-1}^j \\ \bar{R}_M^j \end{pmatrix},$$

where

$$\bar{R}_0^j = \hat{B}_1^{*j}C_0^j + \hat{B}_2^{*j}C_1^j + \hat{B}_3^{*j}C_2^j + \hat{B}_4^{*j}\sum_{k=3}^{M-3}C_k^j + \hat{B}_5^{*j}C_{M-2}^j + \hat{B}_6^{*j}C_{M-1}^j + \hat{B}_7^{*j}C_M^j - u_0^{j-1} + R_0^j,$$

$$\bar{R}_1^j = \hat{B}_1^{**j}C_0^j + \hat{B}_2^{**j}C_1^j + \hat{B}_3^{**j}C_2^j + \hat{B}_4^{**j}C_3^j + \hat{B}_5^{**j}\sum_{k=4}^{M-2}C_k^j + \hat{B}_6^{**j}C_{M-1}^j + \hat{B}_7^{**j}C_M^j - u_1^{j-1} + R_1^j,$$

$$\begin{aligned}
&\bar{R}_i^0 = \hat{B}_1^j C_{i-2}^j + \hat{B}_2^j C_{i-1}^j + \hat{B}_3^j C_i^j + \hat{B}_2^j C_{i+1}^j + \hat{B}_1^j C_{i+2}^j - u_i^{j-1} + R_i^j, \quad i = 2,3,\dots,M-2, \\
&\bar{R}_{M-1}^0 = \hat{B}_1^j C_{M-3}^j + \hat{B}_2^j C_{M-2}^j + \left(\hat{B}_3^j - \hat{B}_1^j\right)C_{M-1}^j + \left(\hat{B}_2^j - 3\hat{B}_1^j\right)C_M^j + R_{M-1}^j, \\
&\bar{R}_M^0 = \left(\hat{B}_3^j - 3\hat{B}_2^j + 12\hat{B}_1^j\right)C_M^j - u_M^{j-1} + R_M^j.
\end{aligned}$$

In order to solve the systems (4.30) and (4.31), we need to determine the initial vector $(C_0^0, C_1^0, \dots, C_M^0)$ from the ICs. To remove the C_{-2}^0, C_{-1}^0, C_{M+1}^0, and C_{M+2}^0, we use

$$u_x(0,0) = \kappa_1, \qquad u_x(1,0) = \kappa_2, \tag{4.35}$$

$$u_{xx}(0,0) = \kappa_3, \qquad u_{xx}(1,0) = 0. \tag{4.36}$$

Using (4.5) and (4.6) in (4.35) and (4.36) and eliminating the unknowns C^0_{-2}, C^0_{-1}, C^0_{M+1}, and C^0_{M+2}, we obtain the $(M+1)\times(M+1)$ system:

$$\begin{pmatrix} 54 & 60 & 6 & 0 & 0 & 0 & \cdots & 0 \\ \frac{101}{4} & \frac{135}{2} & \frac{105}{4} & 1 & 0 & 0 & \cdots & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & \cdots & 0 & 0 & 1 & \frac{105}{4} & \frac{135}{2} & \frac{101}{4} \\ 0 & \cdots & 0 & 0 & 0 & 6 & 60 & 54 \end{pmatrix} \begin{pmatrix} C^0_0 \\ C^0_1 \\ C^0_2 \\ C^0_2 \\ \vdots \\ C^0_{M-2} \\ C^0_{M-1} \\ C^0_M \end{pmatrix}$$

$$= \begin{pmatrix} \varphi(x_0) - \frac{1}{4\xi_1} - \frac{26}{8\xi_1}\kappa_1 - \frac{5}{4\xi_2} - \frac{26}{8\xi_2}\kappa_3 \\ \varphi(x_1) + \frac{1}{8\xi_1}\kappa_1 + \frac{1}{8\xi_2}\kappa_3 \\ \varphi(x_2) \\ \varphi(x_3) \\ \vdots \\ \varphi(x_{M-2}) \\ \varphi(x_{M-1}) - \frac{1}{8\xi_1}\kappa_2 \\ \varphi(x_M) - \frac{3}{\xi_1}\kappa_2 \end{pmatrix}.$$

5 Stability analysis

The von Neumann stability [13, 27] is analyzed in this section. For stability, we choose $f(x,t)=0$, $g(x,t)=0$ and assume $p(t)=\bar{p}$ is a local constant. We discretize the problem as follows:

$$(1-\bar{A})u_i^{j+1} + \bar{B}(u_{xxxx})_i^{j+1} = (2+\bar{A})u_i^j - \bar{B}(u_{xxxx})_i^j - u_i^{j-1},$$
$$i=0,1,\dots,M, j=0,1,\dots,N, \tag{5.1}$$

where

$$\bar{A} = \frac{(\Delta t)^2}{2}\bar{p}, \qquad \bar{B} = a^2\frac{(\Delta t)^2}{2}.$$

Using equations (4.5) and (4.6) in the above equation, we obtain

$$\hat{A}_1 C_{i-2}^{j+1} + \hat{A}_2 C_{i-1}^{j+1} + \hat{A}_3 C_i^{j+1} + \hat{A}_2 C_{i+1}^{j+1} + \hat{A}_1 C_{i+2}^{j+1}$$
$$= \hat{B}_1 C_{i-2}^j + \hat{B}_2 C_{i-1}^j + \hat{B}_3 C_i^j$$
$$+ \hat{B}_2 C_{i+1}^j + \hat{B}_1 C_{i+2}^j - C_{i+2}^{j-1} - 26C_{i+1}^{j-1} - 66C_i^{j-1} - 26C_{i-1}^{j-1} - C_{i-2}^{j-1},$$
$$i=2,3,\dots,M-2, j=1,2,\dots,N, \tag{5.2}$$

where

$$\hat{A}_1 = 1-\bar{A}+\xi_4\bar{B}, \qquad \hat{A}_2 = 26-26\bar{A}-4\xi_4\bar{B}, \qquad \hat{A}_3 = 66-66\bar{A}+6\xi_4\bar{B},$$
$$\hat{B}_1 = 2+\bar{A}-\xi_4\bar{B}, \qquad \hat{B}_2 = 52+26\bar{A}+4\xi_4\bar{B}, \qquad \hat{B}_3 = 132+66\bar{A}-6\xi_4\bar{B}.$$

Now, we consider the trial solution $C_i^j = \delta^j e^{ki\Theta}$ at a given point x_i, where $\Theta = \theta h$, where $k = \sqrt{-1}$. Substituting C_i^j in (5.2), we obtain

$$\begin{aligned}&\bigl(2\hat{A}_1\cos(2\Theta) + 2\hat{A}_2\cos(\Theta) + \hat{A}_3\bigr)\delta^2\\&\quad - \bigl(2\hat{B}_1\cos(2\Theta) + 2\hat{B}_2\cos(\Theta) + \hat{B}_3\bigr)\delta + \bigl(2\cos(2\Theta) + 52\cos(\Theta) + 66\bigr) = 0,\end{aligned} \tag{5.3}$$

which can be written as

$$\Lambda_1\delta^2 - \Lambda_2\delta + \Lambda_3 = 0, \tag{5.4}$$

where

$$\begin{aligned}&\Lambda_1 = 2\hat{A}_1\cos(2\Theta) + 2\hat{A}_2\cos(\Theta) + \hat{A}_3, \qquad \Lambda_2 = 2\hat{B}_1\cos(2\Theta) + 2\hat{B}_2\cos(\Theta) + \hat{B}_3,\\&\Lambda_3 = 2\cos(2\Theta) + 52\cos(\Theta) + 66.\end{aligned}$$

Now, employing the Routh–Hurwitz criterion under $\delta = \frac{1+\varsigma}{1-\varsigma}$ in the above equation, we obtain

$$(\Lambda_1 + \Lambda_2 + \Lambda_3)\varsigma^2 + 2(\Lambda_1 - \Lambda_3)\delta + (\Lambda_1 - \Lambda_2 + \Lambda_3). \tag{5.5}$$

The necessary and sufficient conditions for $|\delta| \le 1$ are

$$\Lambda_1 + \Lambda_2 + \Lambda_3 \ge 0, \qquad \Lambda_1 - \Lambda_3 \ge 0, \quad \text{and} \quad \Lambda_1 - \Lambda_2 + \Lambda_3 \ge 0. \tag{5.6}$$

Substituting the values of Λ_1, Λ_2, and Λ_3 into equation (5.6), we have

$$\Lambda_1 + \Lambda_2 + \Lambda_3 = 16\cos^2\left(\frac{\Theta}{2}\right) + 416\cos^2\left(\frac{\Theta}{2}\right) + 64, \tag{5.7}$$

$$\begin{aligned}\Lambda_1 - \Lambda_3 &= 8(\Delta t)^2\left(\frac{120}{(\Delta x)^4} - \bar{p}\right)\bigl(2\cos^2\Theta + 6\bigr)\\&\quad - 4(\Delta t)^2\left(\frac{240}{(\Delta x)^4} + 13\bar{p}\right)\cos^2\left(\frac{\Theta}{2}\right),\end{aligned} \tag{5.8}$$

$$\begin{aligned}\Lambda_1 - \Lambda_2 + \Lambda_3 &= (a\Delta t)^2\frac{120}{(\Delta x)^4}\left(8\cos^2\Theta - 16\cos^2\left(\frac{\Theta}{2}\right) + 10\right)\\&\quad - (\Delta t)^2\bar{p}\left(8\cos^2\Theta + 104\cos^2\left(\frac{\Theta}{2}\right) + 10\right).\end{aligned} \tag{5.9}$$

It is obvious from equations (5.7)–(5.9) that $\Lambda_1 + \Lambda_2 + \Lambda_3 \ge 0$, $\Lambda_1 - \Lambda_3 \ge 0$, and $\Lambda_1 - \Lambda_2 + \Lambda_3 \ge 0$. Hence, the technique is unconditionally stable for the discretized problem.

6 Numerical algorithm for IP

We intend to obtain stable and accurate solutions of $p(t)$, $q(t)$ and $u(x,t)$ that assure (1.1)–(1.5). The considered problem is solved approximately by minimizing the subsequent regularized cost function

$$\mathbb{F}(p,q) = \bigl\|u(0,t) - h_1(t)\bigr\|^2 + \left\|u\left(\frac{1}{2},t\right) - h_2(t)\right\|^2 + \gamma\bigl(\bigl\|p(t)\bigr\|^2 + \bigl\|q(t)\bigr\|^2\bigr), \tag{6.1}$$

where u fulfills (1.1)–(1.4) with known $p(t)$, $q(t)$, and $\gamma > 0$ is a regularization parameter initiated for stabilizing the approximate solutions. The discretized form of (6.1) is

$$\begin{aligned}\mathbb{F}(\underline{p},\underline{q}) = {} & \sum_{j=1}^{N}\big(u(0,t_j) - h_1(t_j)\big)^2 + \sum_{j=1}^{N}\left(u\left(\frac{1}{2},t_j\right) - h_2(t_j)\right)^2 \\ & + \gamma\left(\sum_{j=1}^{N}\big(p^j\big)^2 + \sum_{j=1}^{N}\big(q^j\big)^2\right).\end{aligned} \tag{6.2}$$

Equation (6.2) is minimized by the MATLAB *lsqnonlin* tool [19].

7 Results and discussion

An example is considered in this section to examine the accuracy and stability. To validate the efficiency, we use the RMSE as:

$$\mathrm{RMSE}(p) = \left[\frac{T}{N}\sum_{j=1}^{N}\big(p^{\mathrm{numerical}}(t_j) - p^{\mathrm{exact}}(t_j)\big)^2\right]^{1/2}, \tag{7.1}$$

$$\mathrm{RMSE}(q) = \left[\frac{T}{N}\sum_{j=1}^{N}\big(q^{\mathrm{numerical}}(t_j) - q^{\mathrm{exact}}(t_n)\big)^2\right]^{1/2}. \tag{7.2}$$

We take $T = 1$. The lower and upper bounds for $p(t)$ and $q(t)$ are considered to be -10^2 and 10^2, respectively.

The BVP (1.1)–(1.5) is solved for both exact and perturbed data. The perturbed data is managed as

$$h_1^{\epsilon 1}(t_j) = h_1(t_j) + \epsilon 1_j, \quad j = \overline{1,N}, \tag{7.3}$$

$$h_2^{\epsilon 2}(t_j) = h_2(t_j) + \epsilon 2_j, \quad j = \overline{1,N}, \tag{7.4}$$

where $\epsilon 1_j$ and $\epsilon 2_j$ indicate the r.v.s and the subsequent S.D.s

$$\sigma 1 = \max_{0\le t\le T}\big|h_1(t)\big| \times p\%, \qquad \sigma 2 = \max_{0\le t\le T}\big|h_2(t)\big| \times p\%. \tag{7.5}$$

For the perturbed data (7.3), (7.4), $h_1(t_j)$, and $h_2(t_j)$ are replaced by $h_1^{\epsilon 1}(t_j)$ and $h_2^{\epsilon 2}(t_j)$ in (6.2).

Let us examine the BVP (1.1)–(1.5) with unknowns $p(t)$ and $q(t)$, from

$$u_{tt} + a^2 u_{xxxx} = p(t)u + g(x,t)q(t) + f(x,t), \quad (x,t) \in [0,1] \times [0,1], \tag{7.6}$$

the ICs

$$\begin{aligned}\varphi(x) &= u(x,0)\\ &= \frac{1}{5}\bigl(5 - 16x - 99x^2 + 176x^3 + 330x^4 - 528x^5\\ &\quad - 462x^6 + 1056x^7 - 495x^8 + 33x^10\bigr),\\ \psi(x) &= u_t(x,0) + \delta u(x,1)\\ &= \frac{1}{5}\bigl(-5 + 16x + 99x^2 - 176x^3 - 330x^4 + 528x^5 + 462x^6\\ &\quad - 1056x^7 + 495x^8 - 33x^{10}\bigr),\\ \delta &= 0,\end{aligned} \tag{7.7}$$

the BCs

$$\begin{aligned}&u(1,t) = 0, \qquad u_x(0,t) = -\frac{16e^{-t}}{5} = u_x(1,t), \qquad u_{xx}(1,t) = 0, \quad t \in [0,1],\\ &\int_0^1 u(x,t)\,dx = 0, \quad t \in [0,1],\end{aligned} \tag{7.8}$$

with

$$h_1(t) = e^{-t}, \qquad h_2(t) = -\frac{511e^{-t}}{1024}, \quad t \in [0,1], \tag{7.9}$$

and

$$\begin{aligned}&a^2 = 1, \qquad g(x,t) = e^{-t}\bigl(1 - 9x - 21x^2 + 122x^3 - 120x^4 + 6x^5 + 21x^6\bigr),\\ &f(x,t) = \frac{1}{5}e^{-t}\bigl(7920 - 63{,}331x - 166{,}314x^2 + 886{,}606x^3 - 830{,}670x^4 - 558x^5\\ &\qquad + 165{,}753x^6 + 1056x^7 - 495x^8 + 33x^{10} + t\bigl(-10 + 61x + 204x^2\\ &\qquad - 786x^3 + 270x^4 + 498x^5 + 357x^6 - 1056x^7 + 495x^8 - 33x^{10}\bigr)\bigr).\end{aligned} \tag{7.10}$$

We consider the exact $p(t)$, $q(t)$ and $u(x,t)$ as:

$$p(t) = t, \quad t \in [0,1], \tag{7.11}$$

$$q(t) = 1 + t, \quad t \in [0,1], \tag{7.12}$$

$$\begin{aligned}u(x,t) &= \frac{1}{5}e^{-t}\bigl(5 - 16x - 99x^2 + 176x^3 + 330x^4 - 528x^5 - 462x^6\\ &\quad + 1056x^7 - 495x^8 + 33x^{10}\bigr), \quad (x,t) \in [0,1] \times [0,1].\end{aligned} \tag{7.13}$$

Theorem 3.3 is fulfilled, which indicates that a unique solution is assured. First, when $p(t)$ and $q(t)$ are given by (7.11) and (7.12), the accuracy of (1.1)–(1.5) is validated using (7.7), (7.8), and (7.10). Figure 1 shows the exact (7.13) and numerical $u(x,t)$, as well as absolute errors, for $\Delta x = \Delta t \in \{\frac{1}{20}, \frac{1}{40}, \frac{1}{80}\}$. The approximate additional measurements $h_1(t)$ and $h_2(t)$ in (1.5) are compared to the exact solution (7.9) derived using the QnB-spline method

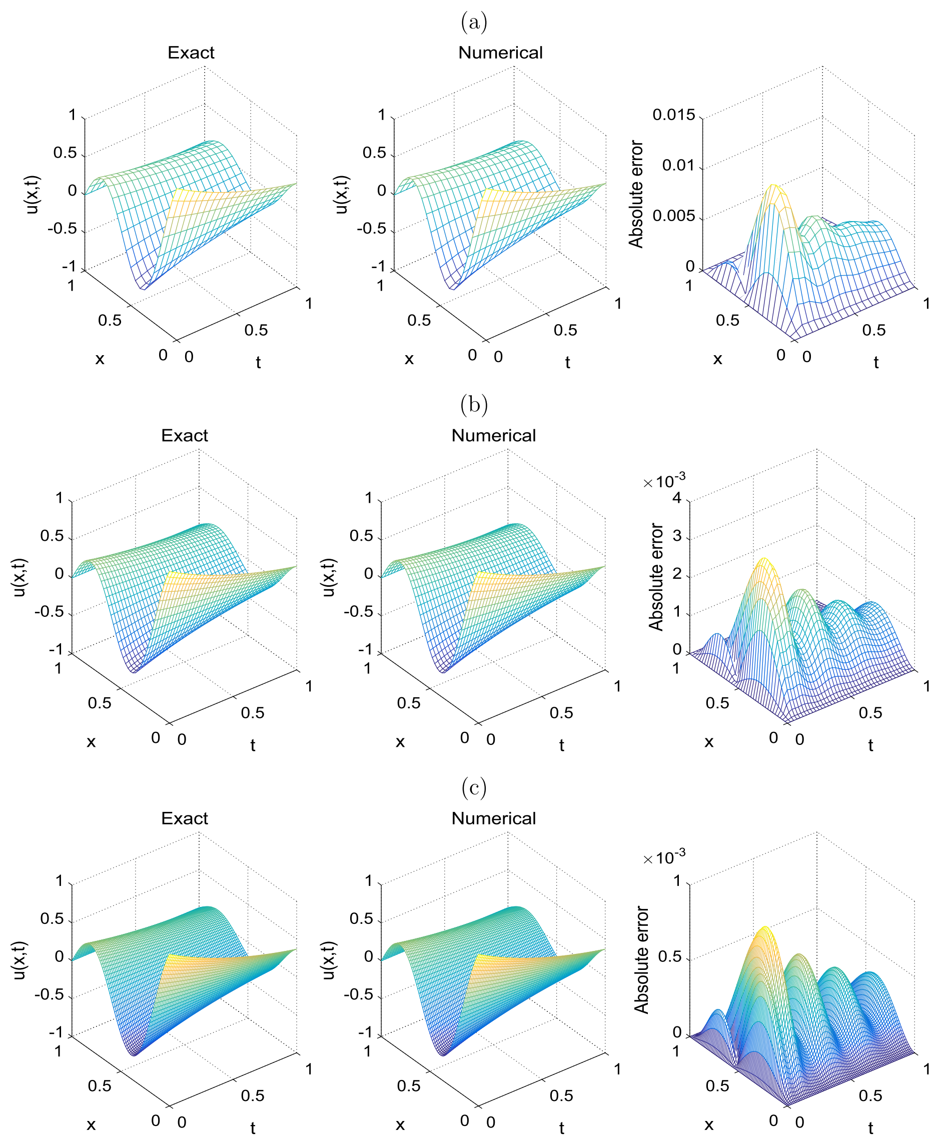

Figure 1 The exact (7.13) and approximate $u(x,t)$ with abs. errors for (**a**) $\Delta x = \Delta t = \frac{1}{20}$, (**b**) $\Delta x = \Delta t = \frac{1}{40}$, and (**c**) $\Delta x = \Delta t = \frac{1}{80}$, for the direct problem

Table 2 Rate of convergence with $\Delta t = 0.025$

Δx	L_∞	ROC	RMS	ROC
0.2	6.145e–02	–	1.297e–02	–
0.1	1.489e–02	2.04	2.423e–03	2.42
0.05	2.604e–03	2.52	3.149e–04	2.94
0.025	5.020e–04	2.37	4.272e–05	2.88

with $M = N \in \{20, 40, 80\}$ in Fig. 2. The rate of convergence of the method is checked with $\Delta t = 0.025$, which shows that the method is second-order convergent in space, see Table 2.

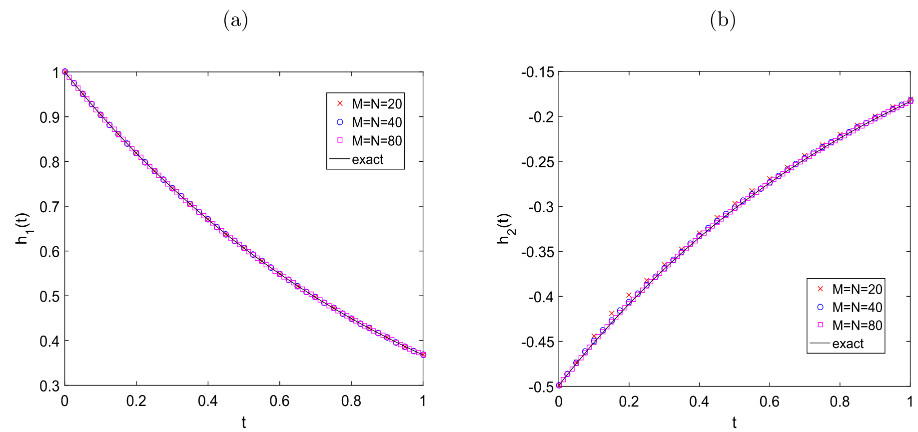

Figure 2 The exact (7.9) and approximate (**a**) $h_1(t)$ and (**b**) $h_2(t)$, for the direct problem with $M = N \in \{20, 40, 80\}$

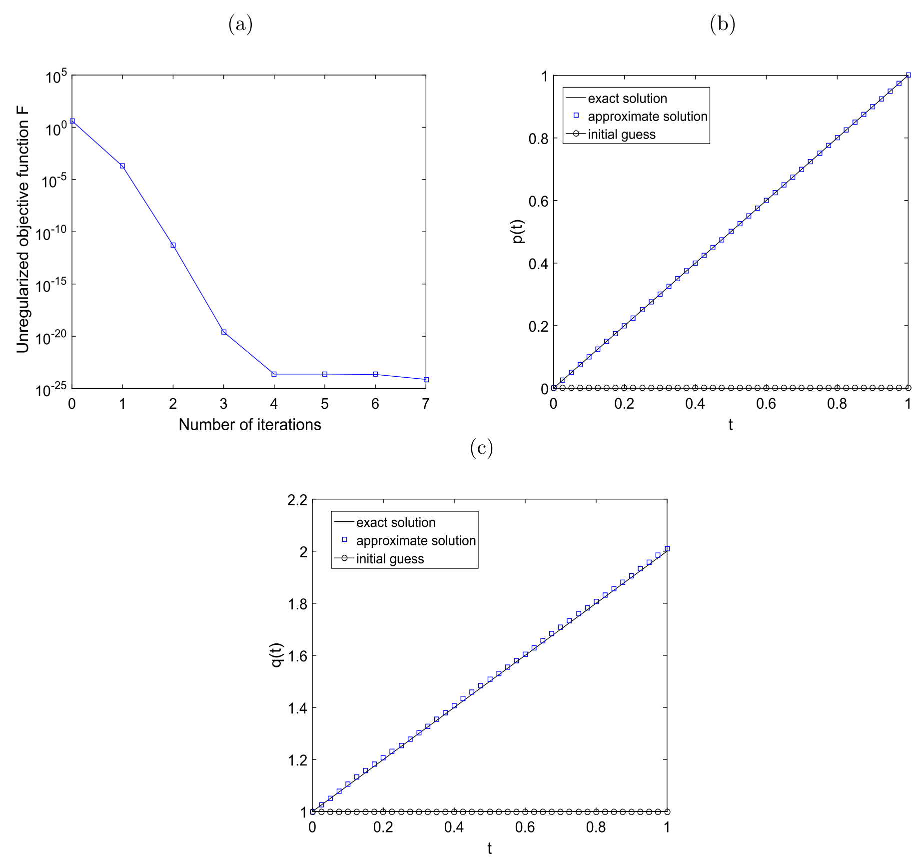

Figure 3 (**a**) The $\mathbb{F}$ (6.2), and the approximate and exact ((7.11) and (7.12)) for: (**b**) $p(t)$ and (**c**) $q(t)$, with $p\% = 0$ and $\gamma = 0$

In the IP (1.1)–(1.5), the initial guesses for p and q are taken as:

$$p^0(t_j) = p(0) = 0, \quad j = \overline{1, N}, \tag{7.14}$$

$$q^0(t_j) = q(0) = 1, \quad j = \overline{1, N}. \tag{7.15}$$

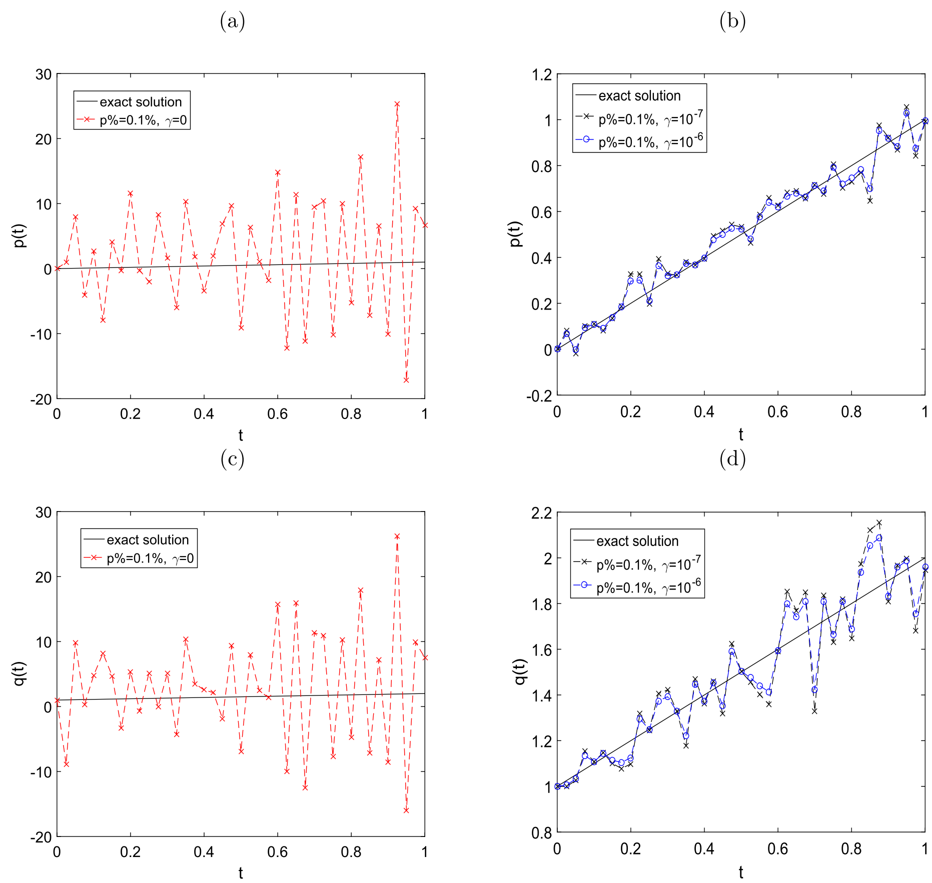

Figure 4 The approximate and exact ((7.11) and (7.12)) $p(t)$, and $q(t)$, for $p\% = 0.1\%$, with $\gamma \in \{0, 10^{-7}, 10^{-6}\}$

When $p\% = 0$ in (7.5), we use $\Delta x = \Delta t = \frac{1}{40}$ to start analyzing to recover $p(t)$, $q(t)$ and $u(x,t)$. The $\mathbb{F}$ in (6.2) is shown in Fig. 3(a), where a monotonic decrease in convergence is realized in 7 itrs for a given tolerance of $O(10^{-25})$. Figures 3(b) and 3(c) depict the exact ((7.11) and (7.12)) and approximate $p(t)$, $q(t)$ without regularization. An accepted and stable accurate $p(t)$ and $q(t)$, producing RMSE(p) = 1.0827E–3 and RMSE(q) = 6.6000E–3 can be seen.

Now, as in equation (7.5), we add $p\% \in \{0.1\%, 1\%\}$ to $h_1(t)$ and $h_2(t)$ through (1.5). In Figs. 4 and 5, $p(t)$ and $q(t)$ are depicted. As $p\%$ increases, the solutions begin oscillations with RMSE(p) $\in \{9.1265, 59.5488\}$, as seen in Figs. 4(a) and 5(a), and RMSE(q) $\in \{9.0045, 56.7323\}$, as seen in Figs. 4(c) and 5(c). Figures 4(b), 4(d), 5(b), and 5(d) show the recovered $p(t)$ and $q(t)$ for various γ, and the most accurate solution is attained for $\gamma \in \{10^{-7}, 10^{-6}\}$, producing RMSE($p$) $\in \{0.0692, 0.0519\}$, and RMSE(q) $\in \{0.1403, 0.1051\}$ for $p\% = 0.1\%$, and for $\gamma \in \{10^{-5}, 10^{-4}\}$, producing RMSE($p$) $\in \{0.1297, 0.0865\}$, and RMSE(q) $\in \{0.2633, 0.1754\}$ for $p\% = 1\%$, see Table 3. The abs. errors between the exact (7.13) and approximate $u(x,t)$ are shown in Fig. 6, where the impact of $\gamma > 0$ in minimizing the unstable behavior of the recovered u can be observed.

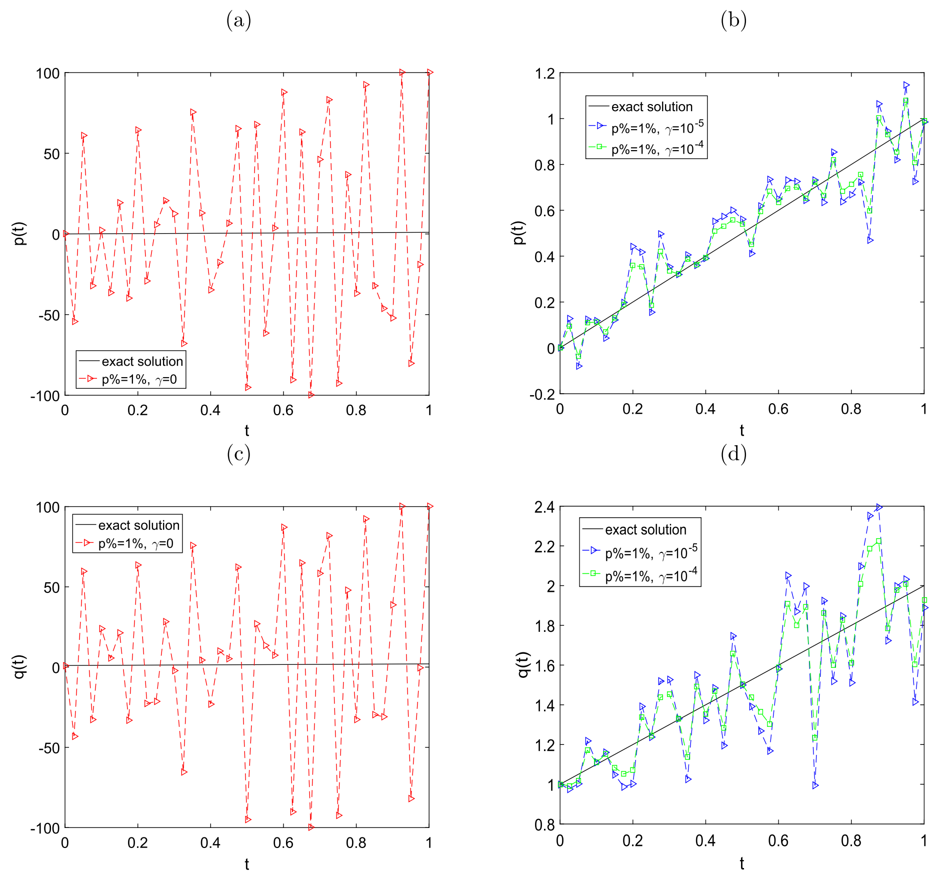

Figure 5 The approximate and exact ((7.11) and (7.12)) $p(t)$, and $q(t)$, for $p\% = 1\%$, with $\gamma \in \{0, 10^{-5}, 10^{-4}\}$

Table 3 RMSE values ((7.1) and (7.2)) with $p\% \in \{0.1\%, 1\%\}$ and various values of γ

$p\%$	γ	RMSE(p)	RMSE(q)	Iter
0.1%	$\gamma = 0$	9.1265	9.0045	20
	$\gamma = 10^{-8}$	0.1039	0.2106	10
	$\gamma = 10^{-7}$	0.0692	0.1403	10
	$\gamma = 10^{-6}$	0.0519	0.1051	10
	$\gamma = 10^{-5}$	0.0932	0.1895	10
1%	$\gamma = 0$	59.5488	56.7323	400
	$\gamma = 10^{-7}$	0.2595	0.5271	20
	$\gamma = 10^{-6}$	0.1557	0.3161	20
	$\gamma = 10^{-5}$	0.1297	0.2633	20
	$\gamma = 10^{-4}$	0.0865	0.1754	20
	$\gamma = 10^{-3}$	0.1329	0.2145	20

8 Conclusions

This article discusses the existence and uniqueness of an IP for a fourth-order PDE with nonlocal integral conditions. The spectral analysis technique is used to reduce the problem to an operator equation in a certain Banach space. Then, the principle of contraction maps is used to prove the existence and uniqueness. This work is novel and has never been investigated theoretically and/or numerically before. A collocation method based on QnB-splines is applied for the direct problem. The stability analysis is also discussed for

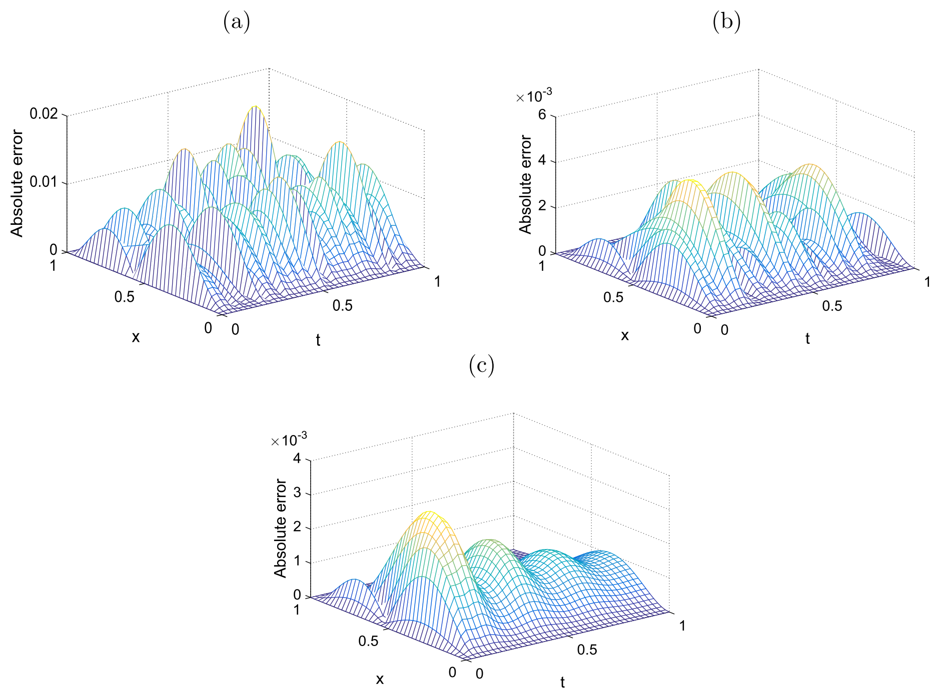

Figure 6 The abs. errors in the exact (7.13) and approximate $u(x,t)$ with: (**a**) $\gamma = 0$, (**b**) $\gamma = 10^{-5}$, and (**c**) $\gamma = 10^{-4}$, with $p\% = 1\%$

the discretized system. The MATLAB subroutine *lsqnonlin* is used to solve the resulting nonlinear optimization problem. To deal with stability and accuracy, Tikhonov regularization is employed. The numerical analysis revealed that accurate solutions are attained for $\gamma \in \{10^{-7}, 10^{-6}\}$ when $p\% = 0.1\%$ and for $\gamma \in \{10^{-5}, 10^{-4}\}$ when $p\% = 1\%$.

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