Ibrahim et al. *Boundary Value Problems* (2022) 2022:98
https://doi.org/10.1186/s13661-022-01680-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On the analytical and numerical study for fractional q-integrodifferential equations

Amira Abd-Elall Ibrahim[1], Afaf A.S. Zaghrout[2], K.R. Raslan[2] and Khalid K. Ali[2*]

[*]Correspondence:
khalidkaram2012@azhar.edu.eg
[2]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt
Full list of author information is available at the end of the article

Abstract

In this paper, we give some basic concepts of q-calculus that will be needed in this paper. Then, we built the q-nonlocal condition that ensures the solution existence and uniqueness of the fractional q-integrodifferential equation. Also, we introduce the continuous dependence of the solution. We find the numerical solution using the finite-difference-Trapezoidal and the cubic B-spline-Trapezoidal methods. Finally, we give three examples to illustrate the validity of our main results.

Keywords: q-integrodifferential equation; Existence and uniqueness of solution; Numerical solutions

1 Introduction

The q-calculus has many applications in various fields such as electronics, mathematics, and physics. Hence, many researchers have paid much attention to study it. The differential equations were developed using the q-calculus to describe several unique physical processes seen in quantum dynamics, discrete dynamical systems, discrete stochastic processes, and so on. The existence, uniqueness, and numerical solutions for the different types of differential equations have been under consideration by many researchers. El-Sayed et al. investigated the existence, uniqueness, and some properties of solutions to a variety of nonlocal integrodifferential equations [1–3]. Ibrahim et al. discussed the existence of a unique solution to the nonlinear integrodifferential equations of the first and second order with the initial and nonlocal conditions [4, 5]. Tair et al. used two numerical treatments for solving the linear integrodifferential Fredholm equation with a weakly singular kernel [6]. Zhao and Corless used the compact finite-difference method to find the numerical solution of the Fredholm integrodifferential equation [7]. Dehghan and Saadatmandi used the Chebyshev finite-difference method to solve the Fredholm integrodifferential equation [8]. Raftari introduced the numerical solutions of the linear Volterra integrodifferential equations using the homotopy perturbation method and the finite-difference method [9]. Ishak and Ahmed used the Trapezoidal method to find the numerical solution of the Volterra integrodifferential equation [10]. Garba and Bichi used the finite-difference-composite Simpson's method to find the numerical solution of the first-order Fredholm integrodifferential equation [11]. Pandey in [12], Saadati et al. in [13] and Ahmed in [14] used the finite-difference-Trapezoidal method to find the solutions to

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various integrodifferential equations. Mittal and Jain in [15], Gholamian, Nadjaf in [16], Hamzah in [17], and Mirzaee, and Alipour in [18] used the cubic B-spline to solve the different types of integrodifferential equations.

Now, we suggest the fractional q-integrodifferential equation as follows:

$$u''(t) = g\big(t, u(t), I_q^{\sigma} f\big(t, u'(t)\big)\big), \quad t \in (0,1], \tag{1}$$

with the q-nonlocal condition

$$(1-q)\tau \sum_{i=0}^{n} q^i u\big(q^i \tau\big) = \alpha, \qquad u'(0) = \beta, \quad \tau \in (0,1], \tag{2}$$

where α and β are constants, and f and g are given functions.

Our paper is organized as follows: In Sect. 2, we give some basic concepts of q-calculus. In Sect. 3, we give the integral representation. The existence of the solution $u(t)$ will be investigated in Sect. 4. In Sect. 5, we discuses the solution uniqueness. We show the continuous dependence of the solution on α in Sect. 6. In Sect. 7, we give a brief explanation of the derivation of the finite-difference-Trapezoidal and the cubic B-spline-Trapezoidal methods. In Sect. 8, we give three examples and satisfy the assumptions of the existence theorem on them and solve it numerically using the finite-difference-Trapezoidal and the cubic B-spline-Trapezoidal methods. Finally, we give the conclusions in Sect. 9.

2 Some q-calculus notations and definitions

Now, we go over some basic q-calculus definitions that will be used in this work.

Definition 2.1 ([19]) For any number c

$$[c]_q = \frac{1-q^c}{1-q},$$

where $q \in (0,1)$.

Definition 2.2 ([19]) The q-derivative of $f(t)$ can be defined as

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt},$$

$$\lim_{q \to 1} D_q f(t) = \frac{df(t)}{dt}.$$

Definition 2.3 ([20]) A q-analog of the common Pochhammer symbol that is called a q-shifted factorial is defined by

$$(c;q)_n = \begin{cases} 1, & n = 0, \\ \prod_{i=0}^{n-1}(1 - cq^i), & n \in \mathbb{N}^*, \end{cases}$$

Also,

$$(c;q)_\infty = \prod_{i=0}^{\infty}\big(1 - cq^i\big), \quad n \in \mathbb{N}^*.$$

Definition 2.4 ([20]) The q-gamma function is defined as

$$\Gamma_q(c) = \frac{(q,q)_\infty}{(q^c;q)_\infty}(1-q)^{1-c},$$

and satisfies $\Gamma_q(c+1) = [c]_q\Gamma_q(c)$, $\Gamma_q(1) = 1$.

Definition 2.5 ([21]) Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann–Liouville type of order $\sigma \geq 0$ is given by

$$\big(I_q^\sigma f\big)(t) = \begin{cases} f(t), & \sigma = 0, \\ \frac{1}{\Gamma_q(\sigma)} \int_0^t (t-qs)^{\sigma-1} f(s)\,d_qs. \end{cases} \tag{3}$$

Lemma 2.6 ([21]) *For* $\sigma > 0$, *using* q*-integration by parts, we have*

$$\big(I_q^\sigma 1\big)(t) = \frac{t^{(\sigma)}}{\Gamma_q(\sigma+1)}. \tag{4}$$

3 Integral representation

Consider the fractional q-integrodifferential problem (1) and (2) with the following assumptions:

1. $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Caratheodory condition. There exist a function $G(t) \in L^1[0,1]$ and a positive constant $c_1 > 0$, such that

$$\big|g(t,u,\chi)\big| \leq G(t) + c_1|u| + c_1|\chi|.$$

2. $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition. There exist a function $\nu(t) \in L^1[0,1]$ and a positive constant $c_2 > 0$, such that

$$\big|f(t,\nu)\big| \leq \nu(t) + c_2|\nu|.$$

3. There exist positive constancies M_i for i = 1, 2 such that:

$$\sup_{t\in(0,1]} \int_0^t G(\theta)\,d\theta \leq M_1, \qquad \sup_{t\in(0,1]} \int_0^t I_q^\sigma \nu(\theta)\,d\theta \leq M_2.$$

4. $2c_1 + \frac{c_1c_2}{(\sigma+1)\Gamma_q(\sigma+1)} < 1$.

Lemma 3.1 *The solution of the fractional* q*-integrodifferential problem* (1) *and* (2), *if it exists, can be represented by the* q*-integral equation as follows*:

$$u(t) = \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\left[\alpha - (1-q)\tau\sum_{i=0}^n q^i \int_0^{q^i\tau} \nu(\theta)\,d\theta\right] + \int_0^t \nu(\theta)\,d\theta, \quad \forall t \in (0,1], \tag{5}$$

where

$$\begin{aligned} v(t) = \beta + \int_0^t g\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha - (1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau} v(\theta)\,d\theta\Bigg] + \int_0^\theta v(s)\,ds, \\ I_q^\sigma f\big(\theta, v(\theta)\big)\Bigg)\,d\theta. \end{aligned} \tag{6}$$

Proof Integrating both sides of (1) we obtain

$$u'(t) = u'(0) + \int_0^t g\big(\theta, u(\theta), I_q^\sigma f\big(\theta, u'(\theta)\big)\big)\,d\theta, \quad t \in (0,1]. \tag{7}$$

Let $u'(t) = v(t)$ in (7), we obtain

$$v(t) = \beta + \int_0^t g\big(\theta, u(\theta), I_q^\sigma f\big(\theta, v(\theta)\big)\big)\,d\theta, \quad t \in (0,1], \tag{8}$$

where

$$u(t) = u(0) + \int_0^t v(\theta)\,d\theta, \quad t \in (0,1], \tag{9}$$

using the q-nonlocal condition (2), we obtain

$$(1-q)\tau\sum_{i=0}^n q^i u\big(q^i\tau\big) = u(0)(1-q)\tau\sum_{i=0}^n q^i + (1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau} v(\theta)\,d\theta. \tag{10}$$

Then,

$$u(0) = \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha - (1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau} v(\theta)\,d\theta\Bigg]. \tag{11}$$

Using (8), (9), and (11), we obtain (5) and (6). This complete the proof. □

4 Solution existence

Theorem 4.1 *Let assumptions* 1–4 *be satisfied. Then, the q-integral equation* (6) *has at least one solution.*

Proof Let us define the operator F associated with the q-integral equation (6) as

$$\begin{aligned} Fv(t) = \beta + \int_0^t g\Bigg(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\Bigg[\alpha - (1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau} v(\theta)\,d\theta\Bigg] + \int_0^\theta v(s)\,ds, \\ I_q^\sigma f\big(\theta, v(\theta)\big)\Bigg)\,d\theta, \quad \forall t \in (0,1]. \end{aligned}$$

Let $Q_r = \{\nu(t) \in \mathbb{R} : \|\nu\|_C \leq r\}$, where $r = \frac{|\beta|+M_1+\frac{c_1|\alpha|}{(1-q)\tau\sum_{i=0}^{n} q^i}+c_1M_2}{1-(2c_1+\frac{c_1c_2}{(\sigma+1)\Gamma_q(\sigma+1)})}$. Then, we have for $\nu(t) \in Q_r$

$$\begin{aligned}
\|F\nu(t)\|_C &\leq |\beta| + \int_0^t \left| g\left(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[\alpha - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} \nu(\theta)\,d\theta\right]\right.\right. \\
&\quad \left.\left. + \int_0^\theta \nu(s)\,ds, I_q^\sigma f\big(\theta,\nu(\theta)\big)\right)\right| d\theta \\
&\leq |\beta| + \int_0^t \left[G(\theta) + c_1\left|\frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[\alpha - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} \nu(\theta)\,d\theta\right]\right.\right. \\
&\quad \left.\left. + \int_0^\theta \nu(s)\,ds\right| + c_1 I_q^\sigma \left|f\big(\theta,\nu(\theta)\big)\right|\right] d\theta \\
&\leq |\beta| + M_1 + \int_0^t \left[\frac{c_1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[|\alpha| + (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} \left|\nu(\theta)\right|\,d\theta\right]\right. \\
&\quad \left. + c_1\int_0^\theta \left|\nu(s)\right|\,ds + c_1 I_q^\sigma \left(\nu(\theta) + c_2\left|\nu(\theta)\right|\right)\right] d\theta \\
&\leq |\beta| + M_1 + \int_0^t \left[\frac{c_1}{(1-q)\tau\sum_{i=0}^{n} q^i}|\alpha| + c_1\|\nu\| + c_1\|\nu\| + c_1M_2\right. \\
&\quad \left. + c_1c_2\|\nu\|\frac{\theta^\sigma}{\Gamma_q(\sigma+1)}\right] d\theta \\
&\leq |\beta| + M_1 + \frac{c_1|\alpha|}{(1-q)\tau\sum_{i=0}^{n} q^i} + 2c_1r + c_1M_2 + \frac{c_1c_2r}{(\sigma+1)\Gamma_q(\sigma+1)} = r.
\end{aligned}$$

This proves that $F : Q_r \to Q_r$ and the class of functions $\{F\nu(t)\}$ is uniformly bounded in Q_r.

Now, let t_1 and $t_2 \in (0,1]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
&\left|F\nu(t_2) - F\nu(t_1)\right| \\
&\quad = \left|\beta + \int_0^{t_2} g\left(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[\alpha - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} \nu(\theta)\,d\theta\right]\right.\right. \\
&\qquad \left. + \int_0^\theta \nu(s)\,ds, I_q^\sigma f\big(\theta,\nu(\theta)\big)\right) d\theta \\
&\qquad - \beta - \int_0^{t_1} g\left(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[\alpha - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} \nu(\theta)\,d\theta\right]\right. \\
&\qquad \left.\left. + \int_0^\theta \nu(s)\,ds, I_q^\sigma f\big(\theta,\nu(\theta)\big)\right) d\theta\right| \\
&\quad \leq \int_{t_1}^{t_2} \left| g\left(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[\alpha - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} \nu(\theta)\,d\theta\right] + \int_0^\theta \nu(s)\,ds,\right.\right. \\
&\qquad \left.\left. I_q^\sigma f\big(\theta,\nu(\theta)\big)\right)\right| d\theta
\end{aligned}$$

$$\leq \int_{t_1}^{t_2} G(\theta)\,d\theta + \frac{c_1(t_2-t_1)|\alpha|}{(1-q)\tau\sum_{i=0}^{n} q^i} + c_1 r(t_2-t_1) + c_1 r(t_2-t_1)$$
$$+ c_1 \int_{t_1}^{t_2} I_q^\sigma v(\theta)\,d\theta + c_1 c_2 r \int_{t_1}^{t_2} \frac{\theta^\sigma}{\Gamma_q(\sigma+1)}\,d\theta.$$

This means that the class of functions $\{Fv(t)\}$ is equicontinuous in Q_r. □

Let $v_m(t) \in Q_r$, $v_m(t) \to v(t)(m \to \infty)$, then from the continuity of the two functions g and f, we obtain $g(t, u_m, \chi_m) \to g(t, u, \chi)$ and $f(t, v_m) \to f(t, v)$ as $m \to \infty$. Also,

$$\lim_{m\to\infty} Fv_m(t) = \lim_{m\to\infty}\left[\beta + \int_0^t g\left(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[\alpha - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} v_m(\theta)\,d\theta\right]\right.\right.$$
$$\left.\left. + \int_0^\theta v_m(s)\,ds, I_q^\sigma f\big(\theta, v_m(\theta)\big)\right)d\theta\right].$$

Using the Lebesgue dominated convergence theorem [22], and assumptions 1 and 2, we obtain

$$\lim_{m\to\infty} Fv_m(t) = \beta + \int_0^t \lim_{m\to\infty} g\left(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\left[\alpha - (1-q)\tau\sum_{i=0}^{n} q^i \int_0^{q^i\tau} v_m(\theta)\,d\theta\right]\right.$$
$$\left. + \int_0^\theta v_m(s)\,ds, I_q^\sigma f\big(\theta, v_m(\theta)\big)\right)d\theta = Fv(t).$$

Then, $Fv_m(t) \to Fv(t)$ as $m \to \infty$. This implies that the operator F is continuous. By using the Schauder fixed-point theorem [23], the q-integral equation (6) has at least one solution $v(t) \in C[0,1]$. Thus, the fractional q-integrodifferential equation (1) and (2) has a solution $u(t) \in C[0,1]$ from Lemma 3.1.

5 Solution uniqueness

Let g and f satisfy the following assumptions:

(i) $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in t for any $u, \chi \in \mathbb{R}$ and satisfies the Lipschitz condition

$$\big|g(t,u,\chi) - g(t,u_1,\chi_1)\big| \leq c_1|u-u_1| + c_1|\chi-\chi_1|.$$

(ii) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $v \in \mathbb{R}$ and satisfies the Lipschitz condition

$$\big|f(t,v) - f(t,w)\big| \leq c_2|v-w|.$$

Theorem 5.1 *Let the assumptions* (i)–(ii) *be satisfied. Then, the q-integral equation* (6) *has a unique solution.*

Proof Let v and w be two solutions of the q-integral equation (6). Then,

$$\begin{aligned}
&\bigl|v(t)-w(t)\bigr| \\
&\leq \int_0^t \biggl| g\biggl(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\biggl[\alpha-(1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau} v(\theta)\,d\theta\biggr] \\
&\quad + \int_0^\theta v(s)\,ds, I_q^\sigma f\bigl(\theta, v(\theta)\bigr)\biggr) \\
&\quad - g\biggl(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^n q^i}\biggl[\alpha-(1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau} w(\theta)\,d\theta\biggr] \\
&\quad + \int_0^\theta w(s)\,ds, I_q^\sigma f\bigl(\theta, w(\theta)\bigr)\biggr)\biggr|\,d\theta \\
&\leq \int_0^t \biggl[c_1\biggl|\frac{1}{(1-q)\tau\sum_{i=0}^n q^i}(1-q)\tau\sum_{i=0}^n q^i\int_0^{q^i\tau}\bigl(w(\theta)-v(\theta)\bigr)\,d\theta + \int_0^\theta \bigl(v(s)-w(s)\bigr)\,ds\biggr| \\
&\quad + c_1 I_q^\sigma\bigl|f\bigl(\theta, v(\theta)\bigr)-f\bigl(\theta, w(\theta)\bigr)\bigr|\biggr]\,d\theta \\
&\leq c_1\int_0^t \biggl[\bigl|w(\theta)-v(\theta)\bigr| + \bigl|v(s)-w(s)\bigr| + c_2\frac{\theta^\sigma}{\Gamma_q(\sigma+1)}\bigl|v(\theta)-w(\theta)\bigr|\biggr]\,d\theta \\
&\leq c_1\|w-v\|_C + c_1\|w-v\|_C + c_1c_2\frac{1}{(\sigma+1)\Gamma_q(\sigma+1)}\|v-w\|_C \\
&\leq \biggl(2c_1 + \frac{c_1c_2}{(\sigma+1)\Gamma_q(\sigma+1)}\biggr)\|w-v\|_C.
\end{aligned}$$

Hence,

$$\biggl[1-\biggl(2c_1+\frac{c_1c_2}{(\sigma+1)\Gamma_q(\sigma+1)}\biggr)\biggr]\|w-v\|_C \leq 0.$$

Since $2c_1 + \frac{c_1c_2}{(\sigma+1)\Gamma_q(\sigma+1)} < 1$, then $w(t) = v(t)$ and the solution of the integral equation (6) is unique. Thus, from Lemma 3.1, the fractional q-integrodifferential problem (1) with the q-nonlocal condition (2) possesses a unique solution $u(t) \in C[0,1]$. □

6 Continuous dependence

In this section, we present the continuous dependence for a solution on a constant α.

6.1 Continuous dependence on α

Definition 6.1 The solution $u(t) \in C[0,1]$ of the q-nonlocal problem (1) and (2) depends continuously on α, if

$$\forall\epsilon > 0,\ \exists\delta(\epsilon) \quad \text{s.t.} \quad \bigl|\alpha-\alpha^*\bigr| < \delta \quad \Rightarrow \quad \bigl\|u-u^*\bigr\| < \epsilon,$$

where u^* is the solution of the q-nonlocal problem

$$u^{*\prime\prime}(t) = g\bigl(t, u^*(t), I_q^\sigma f\bigl(t, u^{*\prime}(t)\bigr)\bigr), \quad t \in (0,1], \tag{12}$$

with the q-nonlocal condition

$$(1-q)\tau\sum_{i=0}^{n} q^i u^*\bigl(q^i\tau\bigr) = \alpha, \qquad u^{*\prime}(0) = \beta. \tag{13}$$

Theorem 6.2 *Let the assumptions of the theorem* (5.1) *be satisfied. Then, the solution of the fractional q-integrodifferential equation* (1) *with the q-nonlocal condition* (2) *is continuously dependent on* α.

Proof Let $u(t)$ and $u^*(t)$ be two solutions of the q-nonlocal problem (1) and (2), and (12) and (13), respectively. Then,

$$\begin{aligned}
&\bigl|v(t)-v^*(t)\bigr| \\
&\quad= \Biggl|\int_0^t \Biggl[g\Biggl(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\Biggl[\alpha-(1-q)\tau\sum_{i=0}^{n} q^i\int_0^{q^i\tau} v(\theta)\,d\theta\Biggr] \\
&\qquad+\int_0^\theta v(s)\,ds, I_q^\sigma f\bigl(\theta, v(\theta)\bigr)\Biggr) \\
&\qquad-g\Biggl(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\Biggl[\alpha^*-(1-q)\tau\sum_{i=0}^{n} q^i\int_0^{q^i\tau} v^*(\theta)\,d\theta\Biggr] \\
&\qquad+\int_0^\theta v^*(s)\,ds I_q^\sigma f\bigl(\theta, v^*(\theta)\bigr)\Biggr)\Biggr]\,d\theta\Biggr| \\
&\quad\le \int_0^t \Biggl|g\Biggl(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\Biggl[\alpha-(1-q)\tau\sum_{i=0}^{n} q^i\int_0^{q^i\tau} v(\theta)\,d\theta\Biggr] \\
&\qquad+\int_0^\theta v(s)\,ds, I_q^\sigma f\bigl(\theta, v(\theta)\bigr)\Biggr)\,d\theta \\
&\qquad-g\Biggl(\theta, \frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\Biggl[\alpha^*-(1-q)\tau\sum_{i=0}^{m} q^i\int_0^{q^i\tau} v^*(\theta)\,d\theta\Biggr] \\
&\qquad+\int_0^\theta v^*(s)\,ds, I_q^\sigma f\bigl(\theta, v^*(\theta)\bigr)\Biggr)\Biggr|\,d\theta \\
&\quad\le \int_0^t \Biggl[c_1\frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\bigl|\alpha-\alpha^*\bigr| + c_1\bigl|v^*(\theta)-v(\theta)\bigr| + c_1\bigl|v(s)-v^*(s)\bigr| \\
&\qquad+ c_1 I_q^\sigma\bigl|f\bigl(\theta, v(\theta)\bigr) - f\bigl(\theta, v^*(\theta)\bigr)\bigr|\Biggr]\,d\theta \\
&\quad\le \int_0^t \Biggl[c_1\frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\bigl|\alpha-\alpha^*\bigr| + 2c_1\bigl\|v-v^*\bigr\|_C + c_1c_2\frac{\theta^\sigma}{\Gamma_q(\sigma+1)}\bigl\|v-v^*\bigr\|_C\Biggr]\,d\theta \\
&\quad\le c_1\frac{1}{(1-q)\tau\sum_{i=0}^{n} q^i}\bigl|\alpha-\alpha^*\bigr| + 2c_1\bigl\|v-v^*\bigr\|_C + c_1c_2\frac{1}{(\sigma+1)\Gamma_q(\sigma+1)}\bigl\|v-v^*\bigr\|_C \\
&\quad\le \frac{c_1\delta}{(1-q)\tau\sum_{i=0}^{n} q^i} + \biggl(2c_1 + \frac{c_1c_2}{(\sigma+1)\Gamma_q(\sigma+1)}\biggr)\bigl\|v-v^*\bigr\|_C.
\end{aligned}$$

Hence,

$$\|v - v^*\|_C \le \frac{\frac{c_1 \delta}{(1-q)\tau \sum_{i=0}^{n} q^i}}{1 - (2c_1 + \frac{c_1 c_2}{(\sigma+1)\Gamma_q(\sigma+1)})}.$$

Therefore,

$$\begin{aligned}|u(t) - u^*(t)| &= \left| \frac{1}{(1-q)\tau \sum_{i=0}^{n} q^i} \left[\alpha - (1-q)\tau \sum_{i=0}^{n} q^i \int_0^{q^i \tau} v(\theta)\, d\theta \right] + \int_0^t v(\theta)\, d\theta \right. \\ &\quad \left. - \frac{1}{(1-q)\tau \sum_{i=0}^{n} q^i} \left[\alpha^* - (1-q)\tau \sum_{i=0}^{n} q^i \int_0^{q^i \tau} v^*(\theta)\, d\theta \right] + \int_0^t v^*(\theta)\, d\theta \right| \\ &\le \frac{1}{(1-q)\tau \sum_{i=0}^{n} q^i} |\alpha - \alpha^*| + 2\|v - v^*\|_C.\end{aligned}$$

Hence,

$$\|u - u^*\|_C \le \frac{\delta}{(1-q)\tau \sum_{i=0}^{n} q^i} + \frac{\frac{2c_1 \delta}{(1-q)\tau \sum_{i=0}^{n} q^i}}{1 - (2c_1 + c_1 c_2 \frac{1}{(\sigma+1)\Gamma_q(\sigma+1)})} = \epsilon.$$

From the above results, the solution to the q-nonlocal problem (1) and (2) is continuously dependent on α. □

7 Numerical technique methodology

In this part, we want to find the numerical solution of equation (1) under the q-nonlocal condition (2) using the finite-difference-Trapezoidal and the cubic B-spline-Trapezoidal methods. Before we start, let us introduce the problem (1) and (2) in the following

$$\begin{aligned}& u''(t) - c_1 \varphi(u(t)) = G(t) + c_1 I_q^{\sigma} f(t, u'(t)), \qquad (14) \\ & (1-q)\tau \sum_{i=0}^{n} q^i u(q^i \tau) = \alpha, \qquad u'(0) = \beta,\end{aligned}$$

where $f(t, u'(t)) = c_2(\rho(t) + \phi(u'(t))$. Then, by using (3), we can write (14) as

$$u''(t) - c_1 \varphi(u(t)) = G(t) + c_1 \frac{1}{\Gamma_q(\sigma)} \int_0^t c_2 (t - qs)^{\sigma-1} (\rho(s) + \phi(u'(s)))\, d_q s. \qquad (15)$$

Now, we divide the domain $[0, t]$ of equation (15) into m finite points as $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = mh$. We use a uniform step length $h = (t_i - a)/i$. By taking $u_i'' = u''(t_i)$, $\varphi(u_i) = \varphi(u(t_i))$, $\phi(u_j') = \phi(u'(s_j))$, $\rho(s_j) = \rho_j$, $G(t_i) = G_i$. Then, (15) can be written as

$$u_i'' - c_1 \varphi(u_i) = \gamma_i + \frac{c_1 c_2}{\Gamma_q(\sigma)} \int_0^{t_i} (t_i - q s_j)^{\sigma-1} \phi(u_j')\, d_q s, \qquad (16)$$

where

$$\gamma_i = G_i + \frac{c_1 c_2}{\Gamma_q(\sigma)} \int_0^{t_i} (t_i - q s_j)^{\sigma-1} \rho_j\, d_q s.$$

Let $k_{ij} = (t_i - qs_j)^{\sigma-1}$. Then, (16) can be written as

$$u_i'' - c_1\varphi(u_i) = \gamma_i + \frac{c_1c_2}{\Gamma_q(\sigma)}\int_0^{t_i} k_{ij}\phi(u_j')\,d_qs. \tag{17}$$

7.1 A brief review of the finite-difference-trapezoidal method

The idea is based on approximating the differential part of (17) using the finite-difference method and the integral part using the Trapezoidal rule [5] as follows:

1. The derivative part of (17) can be approximated using the central difference as follows

$$\begin{aligned} u_i'' &\approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \\ u_i' &\approx \frac{u_{i+1} - u_{i-1}}{2h}. \end{aligned} \tag{18}$$

2. The integral part of (17) can be approximated using the Trapezoidal rule as

$$\int_0^{t_i} k_{ij}\phi(u_j')\,d_qs \approx \frac{h}{2}\left[k_{i0}\phi(u_0') + 2\sum_{j=1}^{i-1}k_{ij}\phi(u_j') + k_{ii}\phi(u_i')\right], \quad i = 0,1,2,3,\dots,m.$$

3. Then, (17) can be written as

$$\begin{aligned} &\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - c_1\varphi(u_i) \\ &\quad = \gamma_i + \frac{c_1c_2}{\Gamma_q(\sigma)}\frac{h}{2}\left[k_{i0}\phi\left(\frac{u_1 - u_{-1}}{2h}\right) + 2\sum_{j=1}^{i-1}k_{ij}\phi\left(\frac{u_{j+1} - u_{j-1}}{2h}\right)\right. \\ &\qquad \left. + k_{ii}\phi\left(\frac{u_{i+1} - u_{i-1}}{2h}\right)\right], \quad i = 0,1,\dots,m. \end{aligned} \tag{19}$$

7.2 A brief review of the cubic B-spline-trapezoidal method

The interpolation function of the continuous function $u(t)$ on a set of points $\{t_i\}_{i=0}^{m}$ based on the cubic B-spline basis functions is defined as follows

$$u(t) = \sum_{i=-1}^{m+1} C_iB_i(t), \quad t \in (0,1], \tag{20}$$

where $\{C_i\}_{i=-1}^{m+1}$ are constants to be determined and $\{B_{-1}, B_0, B_1, \dots, B_{m+1}\}$ form a basis that was defined in [18]. Now, we combine the cubic B-spline with the Trapezoidal method to find the numerical solution of (17) by following these steps:

1. Using the cubic B-spline method, the solution $u(t)$ of the q-integral equation (17) can be approximated as

$$u(t_i) = u_i \approx C_{i-1} + 4C_i + C_{i+1},$$

also,

$$u'(t_i) = u'_i \approx \frac{3}{h}(C_{i+1} - C_{i-1}),$$

$$u''(t_i) = u''_i \approx \frac{6}{h^2}(C_{i-1} - 2C_i + C_{i+1}).$$

2. We use the Trapezoidal method to approximate the integral part of (17).
3. Therefore, (17) can be written as

$$\frac{6}{h^2}(C_{i-1} - 2C_i + C_{i+1}) - c_1\varphi(C_{i-1} + 4C_i + C_{i+1})$$
$$= \gamma_i + \frac{c_1 c_2}{\Gamma_q(\sigma)}\frac{3}{2}\left[k_{i0}\phi(C_1 - C_{-1}) + 2\sum_{j=1}^{i-1} k_{ij}\phi(C_{j+1} - C_{j-1}) + k_{ii}\phi(C_{i+1} - C_{i-1})\right],$$
$$i = 0, 1, \ldots, m.$$

8 Test problems

In this part, we satisfy the assumptions of the existence Theorem 4.1 on three examples of fractional q-integrodifferential equations and we solve them numerically by using the finite-difference-Trapezoidal method and the cubic B-spline-Trapezoidal method.

Test problem 1 In (15) we take $G(t) = (-0.111111t - 1.33333)\sin(t) + 0.111111\cos(t) - 0.111111$, $\rho(t) = \cos(t)$, $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{3}$, $\sigma = 2$, $q = 0.5$, $\tau = 0.2$, $n = 2$, $\alpha = 0.026108$, $\beta = 1$, $\varphi(u(t)) = u(t)$, $\phi(u'(t)) = u'(t)$, then $2c_1 + \frac{c_1 c_2}{(\sigma+1)\Gamma_q(\sigma+1)} = \frac{2}{3} + \frac{1}{9}\frac{1}{3\Gamma_{0.5}(3)} = 0.691358 < 1$. The exact solution of this problem is $u(t) = \sin(t)$.

The assumptions 1–4 of theorem (4.1) are clearly satisfied, implying that the given q-nonlocal problem has a continuous solution. The numerical solution of this problem is now found using the finite-difference-Trapezoidal and the cubic-Trapezoidal approach with $m = 20$.

Table 1 and Fig. 1 give the comparison between the numerical solutions of the problem using the finite-Trapezoidal and the cubic-Trapezoidal methods and exact solutions of this problem. We can see from the previous comparison that the cubic-Trapezoidal method is better than the finite-Trapezoidal method and both methods are effective.

Table 1 The exact and numerical solutions of Test problem 1

t_i	Exact solutions	Finite-Trap.	Abs. error (finite-Trap.)	cubic-Trap.	Abs. error (cubic-Trap.)
0.1	0.099833	0.099813	2.0228E–5	0.099833	2.6953E–7
0.2	0.198669	0.198696	2.0337E–5	0.198669	2.1720E–7
0.3	0.295520	0.295579	5.9337E–5	0.295519	1.5071E–6
0.4	0.389418	0.389515	9.6192E–5	0.389414	3.9097E–6
0.5	0.479426	0.479556	1.3044E–4	0.479418	7.6246E–6
0.6	0.564642	0.564804	1.6178E–4	0.564629	1.2706E–5
0.7	0.644218	0.644408	1.9009E–4	0.644199	1.9032E–5
0.8	0.717356	0.717572	2.1550E–4	0.717329	2.6272E–5
0.9	0.783327	0.783565	2.3837E–4	0.783293	3.3862E–5

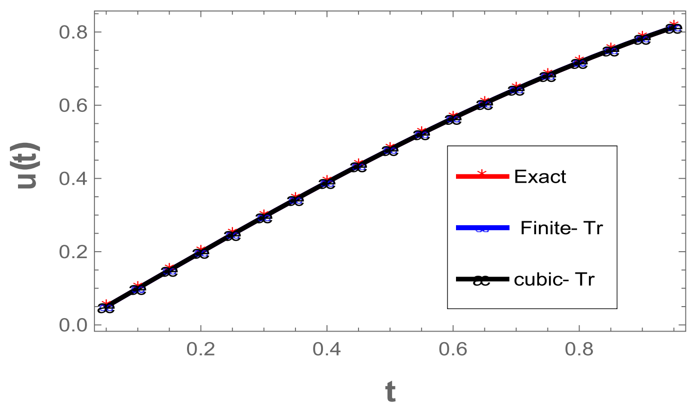

Figure 1 Comparison between the numerical and exact solutions of Test problem 1

Table 2 The exact and numerical solutions of Test problem 2

t_i	Exact solutions	Finite-Trap.	Abs. error (finite-Trap.)	cubic-Trap.	Abs. error (cubic-Trap.)
0.1	0.995004	0.995051	4.68397E–5	0.995708	7.03498E–4
0.2	0.980067	0.980110	4.38007E–5	0.980776	7.09112E–4
0.3	0.955336	0.955374	3.78139E–5	0.956054	7.17392E–4
0.4	0.921061	0.921086	2.46425E–5	0.921785	7.23787E–4
0.5	0.877583	0.877573	9.36794E–6	0.878297	7.14083E–4
0.6	0.825336	0.825237	9.86059E–5	0.825989	6.53025E–4
0.7	0.764842	0.764526	3.16339E–4	0.765308	4.66117E–4
0.8	0.696707	0.695906	8.01112E–4	0.696719	1.31684E–5
0.9	0.621609	0.619817	1.79268E–3	0.620662	9.47673E–4

Furthermore, we study the continuous dependence on α using the finite-difference-Trapezoidal method. If we take $|\alpha - \alpha^*| = 10^{-5} \Rightarrow |u(0.4) - u^*(0.4)| = 5.84198 \times 10^{-5}$. Therefore, $u(t)$ is continuously dependent on α.

Test problem 2 In (15) we take $G(t) = -\frac{1}{4}(5\cos(t))$, $\rho(t) = \sin(t)$, $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{5}$, $\sigma = \frac{5}{2}$, $q = 0.2$, $\tau = 0.5$, $n = 1$, $\alpha = 0.43063$, $\beta = 0$, $\varphi(u(t)) = u(t)$, $\phi(u'(t)) = u'(t)$, then $2c_1 + \frac{c_1c_2}{(\sigma+1)\Gamma_q(\sigma+1)} = \frac{2}{4} + \frac{1}{20}\frac{1}{\frac{7}{2}\Gamma_{0.2}(\frac{7}{2})} = 0.510707 < 1$. The exact solution of this problem is $u(t) = \cos(t)$.

The assumptions 1–4 of Theorem 4.1 are clearly satisfied, implying that the given q-nonlocal problem has a continuous solution. The numerical solution of this problem is now found using the finite-difference-Trapezoidal and the cubic-Trapezoidal approach with $m = 20$.

Table 2 and Fig. 2 give the comparison between the numerical solutions of the problem using the finite-Trapezoidal and the cubic-Trapezoidal methods and exact solutions of this problem. We can see from the previous comparison that the finite-Trapezoidal method is better than the cubic-Trapezoidal method and both methods are effective.

Test problem 3 In (15) we take $G(t) = -0.0389995t^{3/2} - 0.0118895t^{7/2} - \frac{t^2}{6}$, $\rho(t) = t^2$, $\varphi(u(t)) = u^2$, $\phi(u'(t)) = u'^2$, $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{4}$, $\sigma = \frac{3}{2}$, $q = 0.5$, $\tau = 0.2$, $\alpha = 0.025$, $\beta = 1$, $n = 1$, $2c_1 + c_1c_2\frac{1}{(\sigma+1)\Gamma_q(\sigma+1)} = \frac{2}{4} + \frac{1}{20}\frac{1}{\frac{5}{2}\Gamma_{0.5}(\frac{5}{2})} = 0.347332 < 1$. The exact solution of this problem is $u(t) = t$.

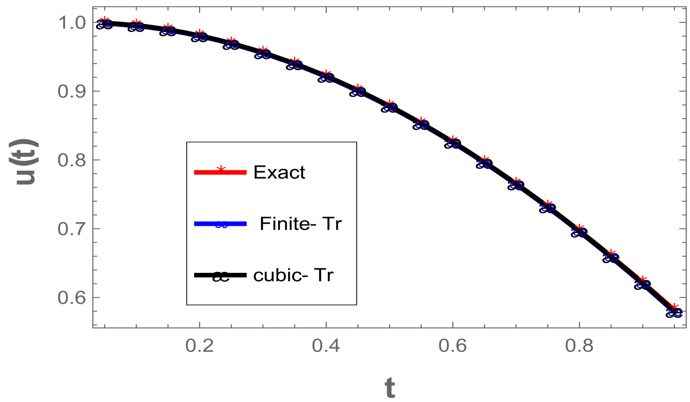

Figure 2 Comparison between the numerical and exact solutions of Test problem 2

Table 3 The exact and numerical solutions of Test problem 3

t_i	Exact solutions	Finite-Trap.	Abs. error (finite-Trap.)	cubic-Trap.	Abs. error (cubic-Trap.)
0.1	0.1	0.100001	7.6971E–7	0.100001	8.1913E–7
0.2	0.2	0.199999	3.8486E–7	0.199999	4.0956E–7
0.3	0.3	0.299996	4.3613E–6	0.299996	4.4824E–6
0.4	0.4	0.399987	1.3468E–5	0.399986	1.3704E–5
0.5	0.5	0.499970	3.0454E–5	0.499969	3.0822E–5
0.6	0.6	0.599942	5.8463E–5	0.599941	5.8978E–5
0.7	0.7	0.699899	1.0100E–4	0.699898	1.0168E–4
0.8	0.8	0.799838	1.6195E–4	0.799837	1.6281E–4
0.9	0.9	0.899754	2.4555E–4	0.899753	2.4661E–4

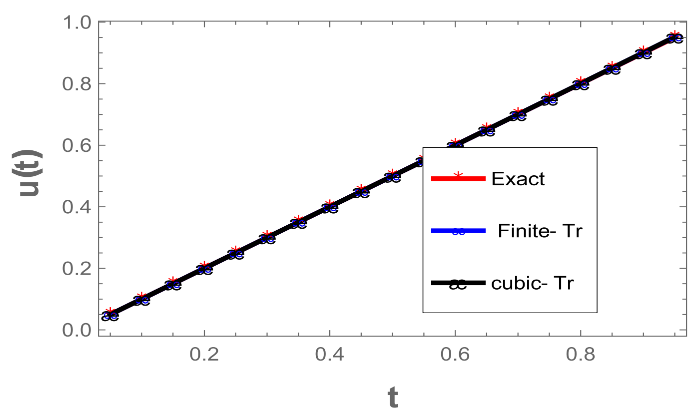

Figure 3 Comparison between the numerical and exact solutions of Test problem 3

The assumptions 1–4 of Theorem 4.1 are clearly satisfied, implying that the given q-nonlocal problem has a continuous solution. The numerical solution of this problem is now found using the finite-difference-Trapezoidal and the cubic-Trapezoidal approach with $m = 20$.

Table 3 and Fig. 3 give the comparison between the numerical solutions of the problem using the finite-Trapezoidal and the cubic-Trapezoidal methods and the exact solutions of this problem. We can see from the previous comparison that both methods are effective.

9 Conclusion

The existence and uniqueness of the solution for the fractional q-integrodifferential equation have been investigated under some conditions. We discussed the continuous dependence of the solution on α. We used the finite-difference-Trapezoidal and the cubic B-spline-Trapezoidal methods to find the numerical solution of the proposed problem. We give three examples to compare between the results of the finite-Trapezoidal, cubic-Trapezoidal methods, and exact solutions. The results illustrated that the two methods are effective.

Acknowledgements
The authors thank the editor-in-chief of the journal and all those in charge of it.

Funding
Open Access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB).

Availability of data and materials
Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate
The authors confirm that all the results they obtained are new and there is no conflict of interest with anyone.

Consent for publication
The paper has four authors who agree to publish.

Competing interests
The authors declare no competing interests.

Author contributions
If we look at the contribution of each author in this paper, we will find that each of them participated in the work from beginning to end in equal measure.

Author details
[1]October High Institute for Engineering and Technology, Giza, Egypt. [2]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt.

Publisher's Note

Received: 19 June 2022 Accepted: 26 November 2022 Published online: 07 December 2022

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