Hajiaghasi and Azami *Boundary Value Problems* (2022) 2022:104
https://doi.org/10.1186/s13661-022-01687-x

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Gradient estimate of a Poisson equation under the almost Ricci solitons

Sakineh Hajiaghasi[1*] and Shahroud Azami[1]

*Correspondence:
S.Hajiaghasi@edu.ikiu.ac.ir
[1]Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran

Abstract

In this paper, we consider an n-dimensional manifold M^n endowed with an almost Bakry–Émery Ricci curvature and study a special case of gradient estimate for the positive solutions of $\Delta u - X.u = f$, for a smooth function f and a smooth vector field X under the almost Ricci solitons condition.

MSC: 53C99; 53C21; 53E99

Keywords: Sobolev constant; Gradient estimate; Almost Ricci soliton

1 Introduction

Gradient estimates for the solutions of the Poisson equation and the heat equation are very powerful tools in geometry and analysis, as shown for example in [2, 4, 18]. One of the most important works in this area is [13], where Li and Yau studied the parabolic kernel of the Schrödinger operator and proved an estimation for the solution of the heat equation $(\Delta - \frac{\partial}{\partial t})u(x,t) = 0$ with the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial M \times (0,\infty)$. Using this estimate, they deduced a Harnack inequality and stated how to establish various upper and lower bounds for the heat kernel from the boundary for both the Dirichlet and Neumann conditions. Then, Wang [16] generalized their results for a compact Riemannian manifold with a nonconvex boundary. In [17], Zhang obtained a sharpened local Li–Yau gradient estimate and showed that global and local Li–Yau estimates are identical, therefore he used the Nash method to obtain the upper bound for the fundamental solution of the following equation:

$$\begin{cases} \Delta u - Ru - \partial_t u = 0, \\ \frac{\partial}{\partial t} g_{ij} = 2R_{ij}, \end{cases} \tag{1.1}$$

here, R is the scalar curvature and R_{ij} is the Ricci curvature. In fact, Perelman [14], introduced an equation like this and gained the lower bound for the solution to it. For more studies and related research see [6, 7, 9–12]. Recently, Zhang et al. [18], stated the elliptic and parabolic gradient estimates for a Riemannian manifold M with Ricci curvature that was bounded from below and therefore they achieved Gaussian upper and lower bounds for the heat kernel and extended the maximum principle that was stated by Petersen and

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Wai in [15]; also as a result they constructed a kind of cut-off function that had been used for proving the volume convergence and cone rigidity for Gromov–Hausdorff limits (see also [5]). In addition, Bamler [3], produced a new version of these works and obtained bounds for the heat kernel on a Ricci flow background. In this paper, we study the gradient estimate for the solution of a new equation $L_2u = f$, in which $L_2u = \Delta u - Xu$, under the almost Ricci soliton condition. Here, X is a smooth vector field on a manifold.

2 Gradient estimate under an almost Ricci soliton

We say that a Riemannian manifold (M^n, g) is an almost Ricci soliton if there exist a vector field X and a soliton function $\lambda : M^n \longrightarrow \mathbb{R}$ satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = -\lambda g, \tag{2.1}$$

where Ric and $\mathcal{L}$ stand, respectively, for the Ricci tensor and the Lie derivative. It is called shrinking, steady or expanding, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. When the vector field X is a gradient of a differentiable function $h : M^n \longrightarrow \mathbb{R}$ the manifold is called a gradient almost Ricci soliton; in this case the preceding equation turns out to be

$$\mathrm{Ric} + \nabla^2 h = -\lambda g, \tag{2.2}$$

where ∇^2 denotes the Hessian of h. Moreover, when either the vector field X is trivial or the potential h is constant, the almost Ricci soliton will be called trivial. We note that when $n \geq 3$ and X is a Killing vector field, an almost Ricci soliton will be simply a Ricci soliton, since in this case we have an Einstein manifold, which implies that λ is a constant.

Let M^n be an almost Ricci soliton with the following conditions

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g, \tag{2.3}$$

and

$$|V|(y) \leq \frac{K}{d(y, O)^{\alpha}}, \tag{2.4}$$

for a smooth function λ with an upper bound N, a smooth vector field V, and any $y \in M$. Here, $d(y, O)$ denotes the distance from O to y, K is the positive constant and $0 \leq \alpha < 1$. In particular, we consider one more condition named the volume noncollapsing condition when $\alpha \neq 0$,

$$\mathrm{Vol}\big(B(x, 1)\big) \geq \rho, \tag{2.5}$$

for all $x \in M$ and some constant $\rho > 0$.

Proving our main results, first we need to obtain the Sobolev inequalities for an almost Ricci soliton and we state the volume-comparison theorem for an almost Ricci soliton from [1].

Theorem 2.1 (Volume comparison [1]) *Assume that for an n-dimension almost Ricci soliton* (2.3) *and* (2.4) *hold. Moreover, consider a positive constant N as an upper bound for* λ.

Suppose, in addition, that the volume noncollapsing condition (2.5) *holds for positive constants* $\rho > 0$, $K \geq 0$ *and* $0 \leq \alpha < 1$, *then for any* $0 < r_1 < r_2 \leq 1$, *we have the volume ratio bound as follows*

$$\frac{\mathrm{Vol}(B(x,r_2))}{r_2^n} \leq e^{C(n,N,K,\alpha,\rho)[N(r_2^2-r_1^2)+K(r_2-r_1)^{1-\alpha}]} \cdot \frac{\mathrm{Vol}(B(x,r_1))}{r_1^n}, \tag{2.6}$$

where $C = C(n,N,K,\alpha,\rho)$ *is a constant that depends on* (n,N,K,α,ρ) *and* $B(x,r)$ *is a ball centered at* x *with radius* r. *In particular, this result is true by considering the gradient soliton vector field* $V = \nabla f$.

Now, we use the volume-comparison result and follow the technique and arguments in [8] to prove the Sobolev inequality on manifolds under the almost Ricci soliton condition.

Theorem 2.2 (Sobolev inequality) *Under the same conditions as in the above theorem, we have the following Sobolev inequalities.*

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}}\,dg\right)^{\frac{n-1}{n}} \leq C(n)r \oint_{B(x,r)} |\nabla f|\,dg, \tag{2.7}$$

and

$$\left(\oint_{B(x,r)} |f|^{\frac{2n}{n-2}}\,dg\right)^{\frac{n-2}{n}} \leq C(n)r^2 \oint_{B(x,r)} |\nabla f|^2\,dg. \tag{2.8}$$

Moreover, for the case that $X = \nabla f$ *for some smooth function* f, *we obtain*

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}}\,dg\right)^{\frac{n-1}{n}} \leq C(n)r \oint_{B(x,r)} |\nabla f|\,dg. \tag{2.9}$$

Proof Because of the similarity of the method for proving this theorem to the case of integral Ricci curvature [8] and also considering the Bakry–Émery Ricci condition [18], we only describe the general path of the proof here.

First, let (2.3), (2.4), and (2.5) hold for an almost Ricci soliton M^n. It follows from the above volume-comparison theorem, that for $r_0 = r_0(n,N,K,\alpha,\rho) < 1$, we have

$$e^{C(n,N,K,\alpha,\rho)(Nr_0^2+Kr_0^{1-\alpha})} \leq \frac{3}{2},$$

so for any $x \in M^n$ and $0 < r_1 < r_2 \leq r_0$, one has

$$\frac{\mathrm{Vol}(B(x,r_1))}{\mathrm{Vol}(B(x,r_2))} \geq \frac{2}{3}\frac{r_1^n}{r_2^n}.$$

It is clear that we could have the following just like Proposition 3.1 in [18],

$$\frac{\mathrm{Vol}(B(x,\delta r))}{\mathrm{Vol}(B(x,r))} \leq \frac{1}{2},$$

for $\delta = \delta(n)$ and $r \leq r_0$.

Now, Let H be any hypersurface dividing M into two parts M_1 and M_2, $B(x,r)$ be the geodesic ball that is divided equally by H, then we infer

$$\mathrm{Vol}\big(B(x,r)\big) \le 2^{n+3} r \mathrm{Vol}\big(H \cap B(x,2r)\big).$$

Following the proof of Theorem 1.1 in [8], we obtain the isoperimetric inequality as follows:

$$\mathrm{ID}_n^*\big(B(x,r)\big) \le C(n)r,$$

for any $r \le r_0$. Here, $r_0 = r_0(n,N,K,\alpha,\rho)$. This is equivalent to the Sobolev inequality stated in the theorem. □

Without reducing the generality of the issue, here we work on a compact manifold M that enables us to consider positive bounded functions on M.

Theorem 2.3 *Suppose that on a compact Riemannian manifold M^n, (2.3), (2.4), and (2.5) hold. For $p > \frac{n}{2}$, if $\lambda \le N$, $|X| \le L$ and $u \le \theta$ is a positive bounded function on $B(x,r)$ and $u = 0$ on $\partial B(x,r)$ that satisfies*

$$L_2 u = f, \tag{2.10}$$

where $L_2 u = \Delta u - X.u$, then there exists a positive constant $r_0 = r_0(n,N,K,\alpha,\rho,L,\theta)$ such that for any $x \in M$ and $0 < r \le r_0$ we have

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \le C(n,N,K,\alpha,\rho,L,\theta)\Big[\big(\|f\|^*_{2q,B(x,r)}\big)^2 + r^{-2}\big(\|u\|^*_{2,B(x,r)}\big)^2\Big].$$

Now, by the same idea of [18] we prove this theorem.

Proof We consider a positive function $v = |\nabla u|^2 + \|f^2\|^*_{q,B(x,r)}$, where

$$\|f\|^*_{q,B(x,r)} = \left(\oint_{B(x,r)} |f|^q\right)^{\frac{1}{q}}.$$

The Bochner formula gives

$$\frac{1}{2}\Delta|\nabla u|^2 = \big|\nabla^2 u\big|^2 + \langle \nabla u, \Delta \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u),$$

hence,

$$\Delta v = 2\big|\nabla^2 u\big|^2 + 2\langle \nabla u, \nabla \Delta u\rangle + 2\mathrm{Ric}(\nabla u, \nabla u). \tag{2.11}$$

Applying the condition stated in the theorem for an almost Ricci soliton, we obtain

$$\begin{aligned} \Delta v &\ge 2\big\langle \nabla(X.u), \nabla u\big\rangle + 2\langle \nabla u, \nabla f\rangle + 2\mathrm{Ric}(\nabla u, \nabla u) \\ &\ge 2u_i f_i - 2\lambda v - (\mathcal{L}_V g)_{ij} u_i u_j + 2\big\langle \nabla(X.u), \nabla u\big\rangle, \end{aligned} \tag{2.12}$$

and so for arbitrary positive p,

$$\begin{aligned}\Delta v^p &= pv^{p-1}\Delta v + p(p-1)v^{p-2}|\nabla v|^2 \\ &\geq 2pv^{p-1}u_i f_i - 2\lambda p v^p - pv^{p-1}(\mathcal{L}_V g)_{ij}u_i u_j - 2pv^{p-1}\langle\nabla(X.u),\nabla u\rangle \\ &\quad + \frac{p-1}{p}v^{-p}\left|\nabla v^p\right|^2.\end{aligned} \tag{2.13}$$

Then, we infer that

$$\begin{aligned}\int_B \left|\nabla\left(\eta v^p\right)\right|^2 \leq \int_B & v^{2p}|\nabla\eta|^2 - 2p\eta^2 v^{2p-1}u_i f_i + 2\lambda p\eta^2 v^{2p} + p\eta^2 v^{2p-1}(\mathcal{L}_V g)_{ij}u_i u_j \\ & + 2p\eta^2 v^{2p-1}\langle\nabla(X.u),\nabla u\rangle,\end{aligned} \tag{2.14}$$

for any $\eta \in C_0^\infty(B_x(1))$ and $p \geq 1$. We know that $(\mathcal{L}_V g)_{ij} = \nabla_i V_j + \nabla_j V_i$, thus we obtain

$$\begin{aligned}\frac{1}{2}\int_B \eta^2 v^{2p-1}(\mathcal{L}_V g)_{ij}u_i u_j = -\int_B & 2\eta v^{2p-1}\eta_j V_i u_i u_j + (2p-1)\eta^2 v^{2p-2}v_j V_i u_i u_j \\ & + \eta^2 v^{2p-1}V_i u_{ij}u_j + \eta^2 v^{2p-1}V_i u_i u_{jj}.\end{aligned} \tag{2.15}$$

As we know $v_j = 2u_{jj}u_j$, hence, (2.15) becomes

$$\begin{aligned}&\frac{1}{2}\int_B \eta^2 v^{2p-1}(\mathcal{L}_V g)_{ij}u_i u_j \\ &\quad\leq \int_B v^{2p}|\nabla\eta|^2 + \eta^2 v^{2p-2}|V|^2|\nabla u|^4 - \frac{2p-1}{p}\eta v^{p-1}V_i u_i u_j\left[\left(\eta v^p\right)_j - v^p\eta_j\right] \\ &\quad - \frac{1}{2}\eta^2 v^{2p-1}V_i v_i + \frac{1}{2}\eta^2 v^{2p-2}f^2|\nabla u|^2 + \frac{1}{2}\eta^2 v^{2p}|V|^2 - \eta^2 v^{2p-1}V_i X.u.\end{aligned} \tag{2.16}$$

From the definition of v, it is obvious that $|\nabla u|^4 \leq v^2$, therefore

$$\begin{aligned}&\frac{1}{2}\int_B \eta^2 v^{2p-1}(\mathcal{L}_V g)_{ij}u_i u_j \\ &\quad\leq \int_B \frac{8p-1}{4p}v^{2p}|\nabla\eta|^2 + \frac{2(2p-1)^2+5p}{2p}\eta^2 v^{2p}|V|^2 + \frac{1}{2p}\left|\nabla\left(\eta v^p\right)\right|^2 + \frac{1}{2}\eta^2 v^{2p-1}f^2 \\ &\quad - \eta^2 v^{2p-1}V_i X.u,\end{aligned} \tag{2.17}$$

and

$$-\int_B \eta^2 v^{2p-1}u_i f_i \leq \int_B \frac{4(2p-1)^2+1}{2p}\eta^2 v^{2p-1}f^2 + \frac{1}{2p}v^{2p}|\nabla\eta|^2 + \frac{1}{8p}\left|\nabla\left(\eta v^p\right)\right|^2. \tag{2.18}$$

By (2.16) and (2.18), (2.14) becomes

$$\begin{aligned}2\int_B \left|\nabla\left(\eta v^p\right)\right|^2 \leq \int_B & 4v^{2p}|\nabla\eta|^2 + 8\lambda p\eta^2 v^{2p} + (8p-1)v^{2p}|\nabla\eta|^2 \\ & + \left(4(2p-1)^2 + 10p\right)\eta^2 v^{2p}|V|^2 + 2p\eta^2 v^{2p-1}f^2 \\ & + \left(16(2p-1)^2 + 4\right)\eta^2 v^{2p-1}f^2 + 4v^{2p}|\nabla\eta|^2\end{aligned}$$

$$-p\eta^2 v^{2p-1} V_i X.u + 2p\eta^2 v^{2p-1}\langle\nabla(X.u),\nabla u\rangle.$$

Define a cut function $\varphi_i(s)$ so that $\eta_i(y)=\varphi_i(s)$, such that for $r_i=(\frac{1}{2}+\frac{1}{2^{i+2}})r$, $i=0,1,2,\dots$, $\varphi_i(t)\equiv 1$ for $t\in[0,r_{i+1}]$, supp$\varphi_i\subseteq[0,r_i]$ and $-\frac{52^i}{r}\le\varphi_i'\le 0$. Hence,

$$\oint_{B(x,r_i)}\left|\nabla\left(\eta_i v^p\right)\right|^2 \le \oint_{B(x,r_i)} 8\lambda p\eta_i^2 v^{2p} + 16pv^{2p}|\nabla\eta_i|^2 + 70p^2\eta_i^2 v^{2p-1}f^2 + 30p^2\eta_i^2 v^{2p}|V|^2 - p\eta_i^2 v^{2p-1}V_i X.u + 2p\eta_i^2 v^{2p-1}\langle\nabla(X.u),\nabla u\rangle. \tag{2.19}$$

Note that

$$2p\oint_{B(x,r_i)}\eta_i^2 v^{2p-1}\langle\nabla(X.u),\nabla u\rangle = -2p\oint_{B(x,r_i)}\eta_i^2 v^{2p-1}X.uu_{ij} + 2\eta_i v^{2p-1}X.uu_i\eta_{ij} + \eta_i^2(2p-1)v^{2p-2}v_i X.uu_i. \tag{2.20}$$

Also,

$$\begin{aligned}-\oint_{B(x,r_i)}\eta_i^2 v^{2p-1}X.uu_{ij} &= -\oint_{B(x,r_i)}\eta_i^2 v^{2p-1}fX.u - \oint_{B(x,r_i)}\eta_i^2 v^{2p-1}|X.u|^2\\ &\le \frac{1}{2}\oint_{B(x,r_i)}\eta_i^2 v^{2p}f^2 + \frac{1}{2}\oint_{B(x,r_i)}\eta_i^2 v^{2p-2}|X.u|^2,\end{aligned} \tag{2.21}$$

$$-\oint_{B(x,r_i)}\eta_i v^{2p-1}X.uu_i\eta_{ij} \le -\oint_{B(x,r_i)} v^{2p}|\nabla\eta|^2 + \eta_i^2 v^{2p-2}|X|^2|\nabla u|^4, \tag{2.22}$$

and

$$\begin{aligned}-2p-1\oint_{B(x,r_i)}\eta_i^2 v^{2p-2}v_i X.uu_i &= -\frac{2p-1}{p}\oint_{B(x,r_i)}\eta_i v^{p-1}X.uu_i\left[\left(\eta v^p\right)_j - v^p\eta_j\right]\\ &\le \oint_{B(x,r_i)}\frac{1}{4p}\left|\nabla\left(\eta v^p\right)\right|^2 + \frac{(2p-1)^2}{p}\eta_i^2 v^{2p}|X|^2\\ &\quad + \frac{2p-1}{2p}v^{2p}|\nabla\eta|^2 + \frac{2p-1}{2p}\eta_i^2 v^{2p}|X|^2.\end{aligned} \tag{2.23}$$

Hence, substituting (2.21), (2.22), and (2.23) into (2.19), we obtain

$$\begin{aligned}&\frac{1}{2}\oint_{B(x,r_i)}\left|\nabla\left(\eta_i v^p\right)\right|^2\\ &\le \oint_{B(x,r_i)} 8\lambda p\eta_i^2 v^{2p} + 20pv^{2p}|\nabla\eta_i|^2 + 70p^2\eta_i^2 v^{2p-1}f^2\\ &\quad + 30p^2\eta_i^2 v^{2p}|V|^2 - p\eta_i^2 v^{2p-1}V_i X.u + p\eta_i^2 v^{2p}f^2 + 9p^2\eta_i^2 v^{2p}|X|^2,\end{aligned} \tag{2.24}$$

by a simple computation, we have

$$-\oint_{B(x,r_i)} p\eta_i^2 v^{2p-1}V_i X.u \le \frac{1}{2}\oint_{B(x,r_i)} p^2\eta_i^2 v^{2p}|V|^2 + \eta_i^2 v^{2p-1}|X|^2. \tag{2.25}$$

On the other hand, using Young's inequality we obtain

$$xy \le \epsilon x^{\gamma} + \epsilon^{-\frac{\gamma^*}{\gamma}} y^{\gamma^*}, \quad \forall x, y > 0, \gamma > 1, \frac{1}{\gamma} + \frac{1}{\gamma^*} = 1,$$

and using the volume-comparison theorems from [1], for $\frac{r}{2} \le r_i \le \frac{3}{4}r$, we obtain

$$p^2 \oint_{B(x,r_i)} \eta_i^2 v^{2p-1} f^2 \le \epsilon \left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \tag{2.26}$$

$$+ \epsilon^{-\frac{a}{1-a}} C^{\frac{2q}{2q-n}} p^{\frac{4q}{2q-n}} \oint_{B(x,r_i)} \eta_i^2 v^{2p}, \tag{2.27}$$

here $C = C(n, N, K, \alpha, \rho)$ by considering the upper bound N for λ. If $q \in (\frac{n}{2}, \frac{n}{2\alpha})$, then

$$\begin{aligned} &p^2 \oint_{B(x,r_i)} \eta_i^2 v^{2p} |V|^2 \\ &\quad \le \epsilon r_i^{-2\alpha} \left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ &\qquad + \epsilon^{-\frac{a}{1-a}} p^{\frac{2}{1-a}} C^{\frac{1}{1-a}} r_i^{-2\alpha} \oint_{B(x,r_i)} \eta_i^2 v^{2p}. \end{aligned} \tag{2.28}$$

Now, considering $|X| \le L$ and $\lambda \le N$, for any $\epsilon > 0$ and $a = \frac{n}{2q}$ we can rewrite (2.24) as follows

$$\begin{aligned} &\oint_{B(x,r_i)} \left| \nabla (\eta_i v^p) \right|^2 \\ &\quad \le \oint_{B(x,r_i)} (16N + 9p^2 L^2 + L^2) \eta_i^2 v^{2p} + 40 p v^{2p} |\nabla \eta_i|^2 \\ &\qquad + 71\epsilon \left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ &\qquad + 71\epsilon^{-\frac{a}{1-a}} C^{\frac{2q}{2q-n}} p^{\frac{4q}{2q-n}} \oint_{B(x,r_i)} \eta_i^2 v^{2p} \\ &\qquad + \epsilon r_i^{-2\alpha} \left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ &\qquad + p^{\frac{4q}{2q-n}} \epsilon^{-\frac{a}{1-a}} C^{\frac{2q}{2q-n}} r_i^{-2\alpha} \oint_{B(x,r_i)} \eta_i^2 v^{2p}. \end{aligned} \tag{2.29}$$

Since $r_i \le r \le 1$ and $\alpha < 1$, using the Sobolev inequality from [18] and choosing ϵ small enough, the above inequality becomes

$$\begin{aligned} \left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} &\le C(n, N, K, \alpha, \rho, L) r_i^2 \oint_{B(x,r_i)} p v^{2p} |\nabla \eta_i|^2 \\ &\quad + p^2 \eta_i^2 v^{2p}. \end{aligned} \tag{2.30}$$

Using the volume-comparison theorem for $r_2 = r_{i+1}$ and $r_1 = r_i$, we infer that

$$\begin{aligned}\left(\oint_{B(x,r_{i+1})} (v^p)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\leq C(n,N,K,\alpha,\rho)\left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq C(n,N,K,\alpha,\rho,L)\oint_{B(x,r_i)} 2^{2i}pv^{2p} + p^2v^{2p}.\end{aligned}$$

Now, take $\mu = \frac{n}{n-2}$ and $p = \frac{\mu^i}{2}$ for $i = 0,1,2,\ldots$, therefore

$$\begin{aligned}\left(\oint_{B(x,r_{i+1})} v^{\mu^{i+1}}\right)^{\frac{n-2}{n}} &= \left(\oint_{B(x,r_{i+1})} (v^p)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq C(n,N,K,\alpha,\rho,L)\big(2^{2i-1}\mu^i + \mu^{2i}\big)\oint_{B(x,r_i)} v^{\mu^i} \\ &\leq C(n,N,K,\alpha,\rho,L)4^{2i}\oint_{B(x,r_i)} v^{\mu^i},\end{aligned}$$

which means that

$$\|V\|^*_{\mu^{i+1},B(x,r_{i+1})} \leq C^{\mu^{-i}}\left(4^{2i}\right)^{\mu^{-i}}\|v\|^*_{\mu^i,B(x,r_i)}. \tag{2.31}$$

Hence,

$$\sup_{B(x,\frac{1}{2}r)} v \leq C^{\Sigma_i\mu^{-i}}\left(4^{2i}\right)^{\Sigma_i\mu^{-i}}\|v\|^*_{1,B(x,\frac{3}{4}r)} \leq C(n,N,K,\alpha,\rho,L)\|v\|^*_{1,B(x,\frac{3}{4}r)}. \tag{2.32}$$

On the other hand, by considering $0 \leq u \leq \theta$, we have

$$\begin{aligned}\int_{B(x,r)} \eta^2|\nabla u|^2 &= \int_{B(x,r)} -\eta^2u(f + X.u) - 2\eta u\nabla_i u\nabla_i\eta \\ &\leq \int_{B(x,r)} \frac{1}{2}u^2\eta^2 + \frac{1}{2}f^2\eta^2 + \eta^2L\theta + \frac{1}{2}\eta^2|\nabla u|^2 + 2u^2|\nabla\eta|^2.\end{aligned}$$

Due to the definition of η, we have

$$\begin{aligned}\oint_{B(x,r)} \eta^2|\nabla u|^2 &\leq 4\oint_{B(x,r)} u^2\eta^2 + f^2\eta^2 + \eta^2L\theta + u^2|\nabla\eta|^2 \\ &\leq 100r^{-2}\left(\|u|^*_{2,B(x,r)}\right)^2 + 4\left\|f^2\right\|^*_{q,B(x,r)} + L\theta.\end{aligned}$$

Subsequently, we infer that

$$\begin{aligned}\|v\|^*_{1,B(x,\frac{3}{4}r)} &\leq \frac{\mathrm{Vol}(B(x,r))}{\mathrm{Vol}(B(x,\frac{3}{4}r))}\oint_{B(x,r)} \eta^2\left(|\nabla u|^2 + \left\|f^2\right\|^*_{q,B(x,r)}\right) \\ &\leq C(n,N,K,\alpha,\rho,L,\theta)\left[r^{-2}\left(\|u\|^*_{2,B(x,r)}\right)^2 + \left(\|f\|^*_{2q,B(x,r)}\right)^2\right].\end{aligned} \tag{2.33}$$

Combining (2.33) and (2.32), we arrive at

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2$$

$$\leq \|v\|_{\infty,B(x,\frac{1}{2}r)} \leq C(n,N,K,\alpha,\rho,L,\theta)\big[r^{-2}\big(\|u\|^*_{2,B(x,r)}\big)^2 + \big(\|f\|^*_{2q,B(x,r)}\big)^2\big]. \qquad \square$$

By a similar argument, we have:

Corollary 2.4 *Let the following condition hold for a compact gradient almost Ricci soliton*

$$\mathrm{Ric} + \mathrm{Hess}h \geq -\lambda g,$$

and, moreover, we have two conditions for the potential function h as follows:

$$\big|h(y) - h(z)\big| \leq K_1 d(y,z)^\alpha \quad \text{and} \quad \sup_{x\in M, 0\leq r\leq 1}\big(r^\beta \|\nabla h\|^*_{q,B(x,r)}\big) \leq K_2.$$

Then, there is a constant $r_0 = r_0(n,N,K_1,K_2,\alpha,\beta,L,\theta)$, such that by the same conditions as the last theorem, the solution of (2.10) *for any $q > \frac{n}{2}$, satisfies*

$$\sup_{B(x,\frac{r}{2})} |\nabla u|^2 \leq C(n,N,K_1,K_2,\alpha,\beta,L,\theta)\big[r^{-2}\big(\|u\|^*_{2,B(x,r)}\big)^2 + \big(\|h\|^*_{2q,B(x,r)}\big)^2\big].$$

Corollary 2.5 *Suppose that all conditions in Theorem* 2.3 *hold. If $X = 0$, then*
(*i*) *If $\lambda \leq N$ holds, we obtain $r_0 = r_0(n,N,K,\alpha,\rho)$ such that*

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \leq C(n,N,K,\alpha,\rho)\big[\big(\|f\|^*_{2q,B(x,r)}\big)^2 + r^{-2}\big(\|u\|^*_{2,B(x,r)}\big)^2\big].$$

(*ii*) *If $\lambda = 0$, the constant coefficient changes as $C(n,K,\alpha,\rho)$.*

Remark 2.6 Note that if $X = 0$ and letting λ be a constant, then the results are the same as [18].

Corollary 2.7 *Let all assumptions in Theorem* 2.3 *and Corollary* 2.4 *hold. If $r \longrightarrow \infty$ and $X = 0$, then u is a constant.*

Acknowledgements
We thank the anonymous referees for their valuable comments, which improved the paper.

Funding
Not available.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
S.A. proposed the problem and S.H. analyzed the problem and proved main results. All authors read and approved the final manuscript.

Publisher's Note

Received: 15 May 2022 Accepted: 13 December 2022 Published online: 27 December 2022

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