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Analysis of multipoint impulsive problem of fractional-order differential equations

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Abstract

This manuscript is related to establishing appropriate results for the existence and uniqueness of solutions to a class of nonlinear impulsive implicit fractional-order differential equations (FODEs). It is remarkable that impulsive differential equations have attracted great popularity due to various important applications in the mathematical modeling of real-world phenomena/processes, particularly in biological or biomedical engineering domains as well as in control theory. The mentioned problem is considered under four-point nonlocal boundary conditions and the derivative is taken in the Caputo sense. Our results are based on fixed-point theorems due to Banach and Schaefer. To justify our results, two suitable examples are given.

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1 Introduction

The area of FODEs has attracted considerable attention from researchers due to their applications in various scientific and engineering disciplines. For instance, different applications of FODEs were investigated in epidemiology and control theory in [1] and [2], respectively. Some fundamental results and applications in engineering were discussed in [3]. Applications of fractional calculus in physics were discussed in [4]. A detailed theory and applications were given in [5]. Some interesting applications of the said area in nanotechnology were studied in [6]. Various real-world applications of FODEs in engineering and sciences were investigated in [7]. Further, in [8] and [9] the authors discussed some applications of FODEs in bioengineering and dynamical systems of hereditary mechanics, respectively. Various phenomena of damped structure related to viscoelasticity were studied by using fractional calculus in [10].

Furthermore, one of the supreme desirable research areas in the field of FODEs is the qualitative theory of solutions. Much research work has been framed in this regard. For a detailed study of basic theory and results, we refer to the book [11]. One of the important areas is known as the study of boundary value problems (BVPs) of fractional differential equations because most technical, physical, and dynamical problems are subject to some boundary conditions. Therefore, the mentioned area has been considered very well by

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researchers in the last three decades. The authors of [12, 13], and [14] established sufficient results for the existence of solutions to various nonlocal BVPs of FODEs. The authors of [15] and [16] studied some multipoint BVPs of FODEs for qualitative theory. A coupled system of impulsive BVPs of FODEs was studied in [17] by using fixed-point theory. Also, the authors of [18, 19], and [20] studied the existence and stability theory for different kinds of initial and BVPs of FODEs via fixed-point theory.

Impulsive differential equations (IDEs) constitute a very important class of the aforesaid area. The said class models those evolutionary processes that undergo abrupt changes. Additionally, these problems consist of a natural description of the evolutionary processes and are hence considered the best tools for understanding various real-world problems in applied sciences. Actually, the theory of IDEs is widely explored as compared to classical order problems. The mentioned area has many applications, for instance, a simple IDE can present several new phenomena like rhythmic beats, fusion solution, and the absence of continuity of a solution. For general theory and applications of IDEs, we mention the book [21]. Impulsive evolution systems were analyzed in [22]. The theory about impulsive differential equations was given in [23]. Impulsive dynamical systems and their theory and applications were given in [24]. Extremal solutions for first-order impulsive problems were studied in [25]. The authors of [26] studied existence criteria for impulsive FODEs.

On the other hand, the impulsive BVPs for nonlinear FODEs have not been addressed as extensively and many features of them need to be explored. Most of the problems in IDEs have been studied under two- or at least three-point boundary conditions, since differential equations under nonlocal boundary conditions have significant applications in engineering disciplines as well as dynamics and fluid mechanics. However, to the best of our knowledge, IDEs with fractional order under multipoint boundary conditions have not been properly investigated. Recently, some authors have investigated impulsive FODEs under initial or two-point boundary conditions. For instance, the authors of [27] studied hybrid impulsive BVPs of FODEs. Similarly, the authors of [28] established the existence and uniqueness results for integral BVPs of impulsive FODEs. The existence of mild solutions has been investigated in [29]. Impulsive neutral FODEs have been studied in [30] by using a fixed-point approach. The upper and lower solution method has been utilized to investigate impulsive FODEs by some authors [31]. The existence and uniqueness of a solution to antiperiodic BVPs and nonlocal BVPs of impulsive FODEs were studied in [32] and [33], respectively. The existence theory for three-point BVPs of impulsive FODEs was developed in [34].

Motivated by the aforesaid work, we investigate the existence and uniqueness of a four-point impulsive nonlocal boundary value problem by updating the problem studied in [35] to the implicit form investigated as

$$\begin{cases} {}^{C}\mathbf{D}_{x_k}^{\beta} w(x) = f(x, w(x), {}^{C}\mathbf{D}_{x_k}^{\beta} w(x)), & 1 < \beta \leq 2, x \in \mathcal{J}_1 = \mathcal{J} \setminus \{x_1, x_2, \dots, x_p\}, \\ \Delta w(x_k) = I_k(w(x_k^-)), \qquad \Delta w'(x_k) = \bar{I}_k(w(x_k^-)), & x_k \in (0,1), k = 1, 2, \dots, p, \\ w'(0) + cw(\eta) = 0, \qquad dw'(1) + w(\zeta) = 0, & \eta, \zeta \in (0,1), \end{cases} \tag{1}$$

where ${}^{C}\mathbf{D}_{x_k}$ is denoted the fractional Caputo derivative at the point x_k, $\mathcal{J} = [0,1]$, $f : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and $I_k, \bar{I}_k : \mathcal{R} \to \mathcal{R}$ are continuous functions. Further, $\Delta w(x_k) =$

$w(x_k^+) - w(x_k^-)$ with $w(x_k^+) = \lim_{h\to 0^+} w(x_k + h)$, $w(x_k^-) = \lim_{h\to 0^-} w(x_k + h)$, $k = 1, 2, \ldots, p$, for $0 = x_0 < x_1 < x_2 \cdots < x_{p+1} = 1$. To inaugurate the essential results, we apply Schaefer's fixed-point theorem to develop appropriate conditions for the existence of at least one solution to the problem under consideration (1). The mentioned theorem has been applied very well to establish the existence criteria for FODEs in various research work, for instance, see [36–38]. Additionally, the condition of uniqueness is obtained by using Banach's contraction theorem. For the demonstration of our results, we deliver some concrete problems. The authors of [18, 19] studied some problems of FODEs for analysis. Also, the authors of [20] studied Hyers–Ulam stability for the almost periodic solution of FODEs with impulse and fractional Brownian motion under nonlocal conditions.

We organize our work as follows: We give some detailed literature in Sect. 1. Elementary results are recalled in Sect. 2. In Sect. 3, we provide our main results. Section 4 is devoted to pertinent examples to demonstrate our results. Finally, Sect. 5 presents a brief conclusion.

2 Background materials

In this part of our article, we present some valuable results and deliver some fundamental definitions and lemmas from the existing literature, which we need in this article. Let $\mathcal{J} = [0, 1]$, further, we explain the space of all the piecewise-continuous functions as $PC(\mathcal{J}, \mathcal{R}) = \{w : \mathcal{J} \to \mathcal{R}; w \in C((x_k, x_{k+1}], \mathcal{R}), k = 0, 1, 2, \ldots, p + 1\}$ and $w(x_k^+)$ and $w(x_k^-)$ exist with $w(x_k^-) = w(x_k)$, $k = 1, 2, \ldots, p$, and

$$PC^1(\mathcal{J}, \mathcal{R}) = \bigl\{w' \in PC(\mathcal{J}, \mathcal{R}); w'\bigl(x_k^+\bigr), w'\bigl(x_k^-\bigr) \text{ exist and } w' \text{ is left continuous at } x_k, \\ \text{for } k = 1, 2, \ldots, p\bigr\}.$$

Note that $PC^1(\mathcal{J}, \mathcal{R})$ is a Banach space with norm $\|w\| = \sup_{x\in\mathcal{J}} |w(x)|$.

Definition 2.1 ([3]) The integral of the function $w \in L^1([0, \mathrm{T}], \mathcal{R}_+)$ of order $\beta > 0$ is defined by

$$ {}_0\mathbf{I}_x^\beta w(x) = \int_0^x \frac{(x - s)^{\beta-1}}{\Gamma(\beta)} w(s)\, ds, \tag{2}$$

such that the right-hand side of the above equation is pointwise defined on $\mathcal{R}^+$.

Definition 2.2 ([5]) For arbitrary order $\beta > 0$, the usual Caputo derivative for the function $w : (0, \infty) \to \mathcal{R}$ is defined by

$$ {}_0^C\mathbf{D}_x^\beta w(x) = \int_0^x \frac{(x - s)^{n-\beta-1}}{\Gamma(n - \beta)} w^{(n)}(s)\, ds, \tag{3}$$

where $n = [\beta] + 1$ stands for the integral part of β.

Lemma 2.1 ([36]) *Let $w \in C(0, 1) \cap L^1(0, 1)$, then the general solution of the problem*

$$ {}_0^C\mathbf{D}_x^\beta w(x) = 0,$$

for $\beta > 0$ is given by

$$w(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1},$$

where $a_i \in \mathcal{R}$, $i = 0, 1, 2, \ldots, n-1$ ($n = [\beta] + 1$).

In view of Lemma 2.1, we recall the following result.

Lemma 2.2 ([36]) *For $\beta > 0$, $w \in C(0,1) \cap L^1(0,1)$, and $h \in L^1[0,1]$, the solution of the problem*

$${}_0^C\mathbf{D}_x^{\beta} w(x) = h(x)$$

is given by

$$w(x) = {}_0\mathbf{I}_x^{\beta} h(x) + a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}, \quad \text{where } n = [\beta] + 1.$$

3 Main results

To convert the problem (1) into the corresponding integral form, we establish the following result.

Lemma 3.1 *Let $w \in PC^1(\mathcal{J}, \mathcal{R})$ be a solution with $q \in L[0,1]$ of the given problem*

$$\begin{cases} {}_0^C\mathbf{D}_{x_k}^{\beta} w(x) = q(x), \quad 1 < \beta \leq 2, x \in \mathcal{J}_1 = \mathcal{J} \setminus \{x_1, x_2, \ldots, x_p\}, \\ \Delta w(x_k) = I_k(w(x_k^-)), \qquad \Delta w'(x_k) = \bar{I}_k(w(x_k^-)), \quad x_k \in (0,1), k = 1, 2, \ldots, p, \\ w'(0) + c w(\eta) = 0, \qquad d w'(1) + w(\zeta) = 0, \quad \eta, \zeta \in (0,1), \end{cases} \tag{4}$$

if and only if w is a solution of the impulsive fractional integral equation as

$$w(x) = \begin{cases} \int_0^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + \frac{c(d+\zeta-x)}{[1-c(d+\zeta-\eta)]} \int_{x_k}^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds \\ \quad - \frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]} [\int_{x_k}^{\zeta} \frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + d \int_{x_p}^{1} \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds] \\ \quad - \frac{1-c(d+\zeta+x-\eta)}{[1-c(d+\zeta-\eta)]} \sum_{i=1}^{p} d(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds + \bar{I}_i(w(x_i^-))) \\ \quad - \sum_{i=1}^{k} (\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + I(w(x_i^-))) \\ \quad - \sum_{i=1}^{k} (\frac{cx(\eta-\zeta)+(\zeta-cd\eta)}{[1-c(d+\zeta-\eta)]} - x_i)(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds + \bar{I}_i(w(x_i^-))), \\ \quad x \in [0, x_1], \\ \int_{x_k}^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + \frac{c(d+\zeta-x)}{[1-c(d+\zeta-\eta)]} \int_{x_k}^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds \\ \quad - \frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]} [\int_{x_k}^{\zeta} \frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + d \int_{x_p}^{1} \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds] \\ \quad - \frac{1-c(d+\zeta+x-\eta)}{[1-c(d+\zeta-\eta)]} \sum_{i=1}^{p} d(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds + \bar{I}_i(w(x_i^-))) \\ \quad + \sum_{i=1}^{k} (\frac{x(1-cd)+cd\eta-\zeta}{[1-c(d+\zeta-\eta)]})(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds + \bar{I}_i(w(x_i^-))), \\ \quad x \in (x_k, x_{k+1}]. \end{cases} \tag{5}$$

Proof Suppose w is a solution to problem (4), hence we use Lemmas 2.2 and 2.1, for some constants $a_0, a_1 \in \mathcal{R}$, following the same procedure as was used in [35], we have

$$w(x) = -a_0 - a_1 x + \int_0^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds, \quad x \in [0, x_1]. \tag{6}$$

Using constants $d_0, d_1 \in \mathcal{R}$, we have

$$w(x) = -d_0 - d_1(x - x_1) + \int_{x_1}^{x} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds, \quad x \in (x_1, x_2]. \tag{7}$$

First, we take derivatives of (6) and (7) to produce

$$w'(x) = -a_1 + \int_0^x \frac{(x-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds, \quad x \in [0, x_1], \tag{8}$$

$$w'(x) = -d_1 + \int_{x_1}^x \frac{(x-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds, \quad x \in (x_1, x_2]. \tag{9}$$

Using the impulsive conditions $\Delta w(x_1) = w(x_1^+) - w(x_1^-) = I_1(w(x_1^-))$ and $\Delta w'(x_1) = w'(x_1^+) - w'(x_1^-) = \bar{I}_1(w(x_1^-))$, we find from (8) and (9)

$$-d_0 = \int_0^{x_1} \frac{(x_1-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds - a_0 - a_1 x_1 + I_1\big(w(x_1^-)\big),$$

$$-d_1 = \int_0^{x_1} \frac{(x_1-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds - a_1 + \bar{I}_1\big(w(x_1^-)\big).$$

Thus, putting the values in (7), we have

$$\begin{aligned} w(x) &= \int_{x_1}^{x} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + \int_0^{x_1} \frac{(x_1-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds - a_0 - a_1 x + I_1\big(w(x_1^-)\big) \\ &\quad + (x - x_1)\left[\int_0^{x_1} \frac{(x_1-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds + \bar{I}_1\big(w(x_1^-)\big)\right], \quad x \in (x_1, x_2]. \end{aligned}$$

Repeating the above process, the obtained solution $w(x)$ for $x \in (x_k, x_{k+1}]$ has the following expression

$$\begin{aligned} w(x) &= \int_{x_k}^{x} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds - a_0 - a_1 x \\ &\quad + \sum_{i=1}^{k}\left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + I_i\big(w(x_i^-)\big)\right) \\ &\quad + \sum_{i=1}^{k}\left[(x - x_i)\left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds + \bar{I}_i\big(w(x_i^-)\big)\right)\right], \quad x \in (x_k, x_{k+1}]. \end{aligned} \tag{10}$$

Now, using the boundary conditions $w'(0) + cw(\eta) = 0$, $dw'(1) + w(\zeta) = 0$, $0 < \eta \leq \zeta < 1$ in (10), we obtain the values of a_0, a_1 as

$$\begin{aligned} a_0 &= -\frac{c(d+\zeta)}{[1 - c(d + \zeta - \eta)]} \int_{x_k}^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds \\ &\quad + \sum_{i=1}^{k}\left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + I_i\big(w(x_i^-)\big)\right) \\ &\quad + \frac{1 + c\eta}{[1 - c(d + \zeta - \eta)]}\left[\int_{x_k}^{\zeta} \frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)} q(s)\,ds + d\int_{x_p}^{1} \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} q(s)\,ds\right] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\frac{(1+c\eta)d}{[1-c(d+\zeta-\eta)]}\biggl(\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}q(s)\,ds-\bar{I}_i\bigl(w\bigl(x_i^-\bigr)\bigr)\biggr)\\
&+\sum_{i=1}^{k}\biggl[\biggl(\frac{(\zeta-cd\eta)}{[1-c(d+\zeta-\eta)]}-x_i\biggr)\biggl(\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}q(s)\,ds+\bar{I}_i\bigl(w\bigl(x_i^-\bigr)\bigr)\biggr)\biggr],\\
a_1={}&\frac{(c)}{[1-c(d+\zeta-\eta)]}\biggl[\int_{x_k}^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}q(s)\,ds-\int_{x_k}^{\zeta}\frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)}q(s)\,ds\\
&-d\int_{x_p}^{1}\frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)}q(s)\,ds-\sum_{i=1}^{p}d\biggl(\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}q(s)\,ds-\bar{I}_i\bigl(w\bigl(x_i^-\bigr)\bigr)\biggr)\\
&+\sum_{i=1}^{k}(\eta-\zeta)\biggl(\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}q(s)\,ds-\bar{I}_i\bigl(w\bigl(x_i^-\bigr)\bigr)\biggr)\biggr].
\end{aligned}
$$

Substituting the values of a_0, a_1 into (6) and (10), we obtain (5). Conversely, suppose w is a solution of the impulsive fractional-integral equation (5). It follows from a direct calculation that (5) satisfies the problem (4). □

For simplicity, we define

$$
\begin{aligned}
\sigma_1&=\sup_{x\in[0,1]}\biggl|\frac{c(d+\zeta-x)}{[1-c(d+\zeta-\eta)]}\biggr|, &\sigma_2&=\sup_{x\in[0,1]}\biggl|\frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]}\biggr|,\\
\sigma_3&=\sup_{x\in[0,1]}\biggl|\frac{1-c(d+\zeta+x-\eta)}{[1-c(d+\zeta-\eta)]}\biggr|, &\sigma_4&=\sup_{x\in[0,1]}\biggl|\frac{(1-cd)x+cd\eta-\zeta}{[1-c(d+\zeta-\eta)]}\biggr|.
\end{aligned}
$$

To derive our main results of the existence and uniqueness of solution, we need to define an operator T as $T:PC(\mathcal{J},\mathcal{R})\to PC(\mathcal{J},\mathcal{R})$ by

$$
\begin{aligned}
Tw(x)={}&\int_{x_k}^{x}\frac{(x-s)^{\beta-1}}{\Gamma(\beta)}f\bigl(s,w(s),{}_0^C\mathbf{D}_{s_k}^{\beta}w(s)\bigr)\,ds\\
&+\frac{c(d+\zeta-x)}{[1-c(d+\zeta-\eta)]}\int_{x_k}^{\eta}\frac{(\eta-t)^{\beta-1}}{\Gamma(\beta)}f\bigl(s,w(s),{}_0^C\mathbf{D}_{s_k}^{\beta}w(s)\bigr)\,ds\\
&-\frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]}\biggl[\int_{x_k}^{\zeta}\frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)}f\bigl(s,w(s),{}_0^C\mathbf{D}_{s_k}^{\beta}w(s)\bigr)\,ds\\
&+d\int_{x_p}^{1}\frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)}f\bigl(s,w(s),{}_0^C\mathbf{D}_{s_k}^{\beta}w(s)\bigr)\,ds\biggr]\\
&-\frac{1-c(d+\zeta+x-\eta)}{[1-c(d+\zeta-\eta)]}\\
&\times\sum_{i=1}^{p}d\biggl(\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}f\bigl(s,w(s),{}_0^C\mathbf{D}_{s_k}^{\beta}w(s)\bigr)\,ds+\bar{I}_i\bigl(w\bigl(x_i^-\bigr)\bigr)\biggr)\\
&+\sum_{i=1}^{k}\biggl(\frac{x(1-cd)+cd\eta-\zeta}{[1-c(d+\zeta-\eta)]}\biggr)\\
&\times\biggl(\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}f\bigl(s,w(s),{}_0^C\mathbf{D}_{s_k}^{\beta}w(s)\bigr)\,ds+\bar{I}_i\bigl(w\bigl(x_i^-\bigr)\bigr)\biggr).
\end{aligned}
\tag{11}
$$

Using Lemma 3.1 with $z_w(x) = f(x, w(x), {}_0^C\mathbf{D}_{x_k}^{\beta} w(x))$, problem (1) is reduced to a fixed-point problem $Tw(x) = w(x)$, where T is given by (11). Therefore, problem (1) has a solution if and only if the operator T has a fixed point, where $z_w(x) = f(x, w(x), z_w(x))$ and $z_w(x) = {}_0^C\mathbf{D}_{x_k}^{\beta} w(x)$. We assume that the following hypotheses are satisfied:

(A_1) The function $f : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is continuous;

(A_2) There exist constant $K_f > 0$ and $0 < L_f < 1$ such that

$$\left|f\left(x, w(x), z_w(x)\right) - f\left(x, \bar{w}(x), \bar{z}_w(x)\right)\right| \le K_f\left|w(x) - \bar{w}(x)\right| + L_f\left|z_w(x) - \bar{z}_w(x)\right|,$$

for any $w, \bar{w}, z_w, \bar{z}_w \in \mathrm{PC}(\mathcal{J}, \mathcal{R})$, and $x \in \mathcal{J}$;

(A_3) There exists a constant $M > 0$, such that

$$\left|\bar{I}_i\left(w(x)\right) - \bar{I}_i\left(\bar{w}(x)\right)\right| \le M\left|w(x) - \bar{w}(x)\right|,$$

for each $w, \bar{w} \in PC(\mathcal{J}, \mathcal{R})$ and $i = 1, 2, 3, \ldots, p$.

Theorem 3.1 *Under the hypotheses (A_1), (A_2), and (A_3) and if the condition*

$$\left[\frac{K_f}{1 - L_f}\left(\frac{\sigma_1 + \sigma_2 + 1}{\Gamma(\beta + 1)} + \frac{(\sigma_2 + \sigma_3 p)|d| + \sigma_4 p}{\Gamma(\beta)}\right) + \left(\sigma_3|d| + \sigma_4\right)pM\right] < 1, \tag{12}$$

holds, then there exists a unique solution for problem (1) on $\mathcal{J}$.

Proof suppose $w, \bar{w}, z_w, \bar{z}_w \in PC(\mathcal{J}, \mathcal{R})$, for some $x \in \mathcal{J}$ we have

$$\begin{aligned}
\left|Tw(x) - T\bar{w}(x)\right| \le{}& \int_{x_k}^{x} \frac{(x - s)^{\beta-1}}{\Gamma(\beta)}\left|z_w(s) - \bar{z}_w(s)\right| ds \\
&+ \left|\frac{c(d + \zeta - x)}{[1 - c(d + \zeta - \eta)]}\right| \int_{x_k}^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}\left|z_w(s) - \bar{z}_w(s)\right| ds \\
&+ \left|\frac{1 + c(\eta - x)}{[1 - c(d + \zeta - \eta)]}\right| \left[\int_{x_k}^{\zeta} \frac{(\zeta - s)^{\beta-1}}{\Gamma(\beta)}\left|z_w(s) - \bar{z}_w(s)\right| ds \right.\\
&\left.+ |d| \int_{x_p}^{1} \frac{(1 - s)^{\beta-2}}{\Gamma(\beta - 1)}\left|z_w(s) - \bar{z}_w(s)\right| ds\right] \\
&+ \left|\frac{1 - c(d + \zeta + x - \eta)}{[1 - c(d + \zeta - \eta)]}\right| \sum_{i=1}^{p} |d| \left(\int_{x_{i-1}}^{x_i} \frac{(x_i - s)^{\beta-2}}{\Gamma(\beta - 1)}\left|z_w(s) - \bar{z}_w(s)\right| ds \right.\\
&\left.+ \left|\bar{I}_i\left(w(x_i^-)\right) - \bar{I}_i\left(\bar{w}(x_i^-)\right)\right|\right) \\
&+ \sum_{i=1}^{k} \left|\left(\frac{x(1 - cd) + cd\eta - \zeta}{[1 - c(d + \zeta - \eta)]}\right)\right| \left(\int_{x_{i-1}}^{x_i} \frac{(x_i - s)^{\beta-2}}{\Gamma(\beta - 1)}\left|z_w(s) - \bar{z}_w(s)\right| ds \right.\\
&\left.+ \left|\bar{I}_i\left(w(x_i^-)\right) - \bar{I}_i\left(\bar{w}(x_i^-)\right)\right|\right),
\end{aligned}$$

and by using (A_2), we have

$$\begin{aligned}
\left|z_w(x) - \bar{z}_w(x)\right| &= \left|f\left(x, w(x), z_w(x)\right) - f\left(x, \bar{w}(x), \bar{z}_w(x)\right)\right| \\
&\le K_f\left|w(x) - \bar{w}(x)\right| + L_f\left|z_w(x) - \bar{z}_w(x)\right|.
\end{aligned}$$

Repeating this process one has

$$|z_w(x) - \bar{z}_w(x))| \le \frac{K_f}{1-L_f}\left|w(x) - \bar{w}(x)\right|.$$

Therefore, for each $x \in \mathcal{J}$, we have

$$\begin{aligned}
&\left|Tw(x) - T\bar{w}(x)\right| \\
&\quad\le \frac{K_f}{1-L_f}\int_{x_k}^{x}\frac{(x-s)^{\beta-1}}{\Gamma(\beta)}\left|w(s)-\bar{w}(s)\right|ds \\
&\qquad + \left|\frac{c(d+\zeta-x)}{[1-c(d+\zeta-\eta)]}\right|\frac{K_f}{1-L_f}\int_{x_k}^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left|w(s)-\bar{w}(s)\right|ds \\
&\qquad + \left|\frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]}\right|\frac{K_f}{1-L_f}\int_{x_k}^{\zeta}\frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)}\left|w(s)-\bar{w}(s)\right|ds \\
&\qquad + \left|\frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]}\right||d|\frac{K_f}{1-L_f}\int_{x_k}^{1}\frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)}\left|w(s)-\bar{w}(s)\right|ds \\
&\qquad + \left|\frac{1-c(d+\zeta+x-\eta)}{[1-c(d+\zeta-\eta)]}\right|\sum_{i=1}^{p}|d|\left[\frac{K_f}{1-L_f}\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}\left|w(s)-\bar{w}(s)\right|ds\right. \\
&\qquad \left. + M\left|w(s)-\bar{w}(s)\right|\right] \\
&\qquad + \left|\left(\frac{x(1-cd)+cd\eta-\zeta}{[1-c(d+\zeta-\eta)]}\right)\right|\sum_{i=1}^{k}\left[\frac{K_f}{1-L_f}\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}\left|w(s)-\bar{w}(s)\right|ds\right. \\
&\qquad \left. + M\left|w(s)-\bar{w}(s)\right|\right] \\
&\quad\le \frac{K_f}{1-L_f}\frac{(x-x_k)^{\beta}}{\Gamma(\beta+1)}\|w-\bar{w}\|\,ds + \sigma_1\frac{K_f}{1-L_f}\frac{(\eta-x_k)^{\beta}}{\Gamma(\beta+1)}\|w-\bar{w}\|\,ds \\
&\qquad + \sigma_2\frac{K_f}{1-L_f}\frac{(\zeta-x_k)^{\beta}}{\Gamma(\beta+1)}\|w-\bar{w}\|\,ds + \sigma_2|d|\frac{K_f}{1-L_f}\frac{(1-x_k)^{\beta-1}}{\Gamma(\beta)}\|w-\bar{w}\|\,ds \\
&\qquad + \sigma_3 p|d|\left[\frac{K_f}{1-L_f}\frac{(x_i-x_{i-1})^{\beta-1}}{\Gamma(\beta)}\|w-\bar{w}\|\,ds + M\|w-\bar{w}\|\right] \\
&\qquad + \sigma_4 p\left[\frac{K_f}{1-L_f}\frac{(x_i-x_{i-1})^{\beta-1}}{\Gamma(\beta)}\|w-\bar{w}\|\,ds + M\|w-\bar{w}\|\right] \\
&\quad\le \left[\frac{K_f}{1-L_f}\frac{1}{\Gamma(\beta+1)} + \sigma_1\frac{K_f}{1-L_f}\frac{1}{\Gamma(\beta+1)} + \sigma_2\frac{K_f}{1-L_f}\frac{1}{\Gamma(\beta+1)} + \sigma_2|d|\frac{K_f}{1-L_f}\frac{1}{\Gamma(\beta)}\right. \\
&\qquad \left. + \sigma_3 p|d|\left(\frac{K_f}{1-L_f}\frac{1}{\Gamma(\beta)} + M\right) + \sigma_4 p\left(\frac{K_f}{1-L_f}\frac{1}{\Gamma(\beta)} + M\right)\right]\|w-\bar{w}\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|Tw - T\bar{w}\| \\
&\quad\le \left[\frac{K_f}{1-L_f}\left(\frac{\sigma_1+\sigma_2+1}{\Gamma(\beta+1)} + \frac{(\sigma_2+\sigma_3 p)|d| + \sigma_4 p}{\Gamma(\beta)}\right) + \left(\sigma_3|d| + \sigma_4\right)pM\right]\|w-\bar{w}\|.
\end{aligned} \tag{13}$$

From (12), the operator T is a contraction. Therefore, according to Banach's contraction principle, T has a unique fixed point that is the unique solution of problem (1). □

Our subsequent result is constructed on the Schaefer fixed-point theorem, consequently the following assumptions hold true:

(A_4) There exist $a, b, c \in PC(\mathcal{J}, \mathcal{R})$, with

$$a^* = \sup_{x\in[0,1]} a(x), \qquad b^* = \sup_{x\in[0,1]} b(x)$$

and

$$c^* = \sup_{x\in[0,1]} \left|c(x)\right| < 1$$

such that

$$\left|f(x, w, z_w)\right| \le a(x) + b(x)\left|w(x)\right| + c(x)\left|z_w(x)\right|,$$

for $x \in \mathcal{J}$, $w, z_w \in PC(\mathcal{J}, \mathcal{R})$.

(A_5) The function $\bar{I}_k : PC(\mathcal{J}, \mathcal{R}) \to \mathcal{R}$ is continuous and there exist constants $A^*, B^* > 0$, such that $|\bar{I}_k w(x)| \le A^*|w(x)| + B^*$ for every $w \in PC(\mathcal{J}, \mathcal{R})$, $k = 1, \ldots, p$.

Theorem 3.2 *If the hypotheses* (A_1), (A_2), (A_4), *and* (A_5) *hold, then the problem* (1) *has at least one solution.*

Proof We will use Schaefer's theorem to establish our main result. The required proof consists of the following steps.

Step 1. T is continuous.

Let $\{w_n\}$ be a sequence such that $w_n \to w$ on $PC(\mathcal{J}, \mathcal{R})$. For $x \in \mathcal{J}$, one has

$$\begin{aligned}
\left|Tw_n(x) - Tw(x)\right| \le{}& \int_{x_k}^{x} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} \left|z_w^{(n)}(s) - z_w(s)\right| ds \\
&+ \left|\frac{c(d+\zeta-x)}{[1-c(d+\zeta-\eta)]}\right| \left|\int_{x_k}^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \left|z_w^{(n)}(s) - z_w(s)\right| ds\right. \\
&+ \left|\frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]}\right| \left|\left[\int_{x_k}^{\zeta} \frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)} \left|z_w^{(n)}(s) - z_w(s)\right| ds\right.\right. \\
&+ |d| \int_{x_p}^{1} \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \left|z_w^{(n)}(s) - z_w(s)\right| ds \Bigg] \\
&+ \left|\frac{1-c(d+\zeta+x-\eta)}{[1-c(d+\zeta-\eta)]}\right| \left|\sum_{i=1}^{p} |d| \left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} \left|z_w^{(n)}(s) - z_w(s)\right| ds\right.\right. \\
&+ \left|\bar{I}_i\big(w_n(x_i^-)\big) - \bar{I}_i\big(w(x_i^-)\big)\right| \Bigg) \\
&+ \sum_{i=1}^{k} \left|\left(\frac{x(1-cd)+cd\eta-\zeta}{[1-c(d+\zeta-\eta)]}\right)\right| \left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} \left|z_w^{(n)}(s) - z_w(s)\right| ds\right.
\end{aligned} \tag{14}$$

$$+ \left|\bar{I}_i\left(x_n\left(x_i^-\right)\right) - \bar{I}_i\left(w\left(x_i^-\right)\right)\right|\Big),$$

where $\mathrm{z}_w^{(n)}(x), \mathrm{z}_w(x)) \in PC(\mathcal{J}, \mathcal{R})$ are given by

$$\mathrm{z}_w^{(n)}(x) = f\left(x, w_n(x), \mathrm{z}_w^{(n)}(x)\right).$$

Now, from assumption (A_2), we have

$$\begin{aligned}\left|\mathrm{z}_w^{(n)}(x) - \mathrm{z}_w(x)\right| &= \left|f\left(x, w_n(x), \mathrm{z}_w^{(n)}(x)\right) - f(x, w(x), \mathrm{z}_w(x)\right| \\ &\le K_f \|w_n - w\| + L_f \left|\mathrm{z}_w^{(n)}(x) - \mathrm{z}_w(x)\right|.\end{aligned}$$

Repeating this process, $|\mathrm{z}_w^{(n)}(x) - \mathrm{z}_w(x)| \le \frac{K_f}{1-L_f}\|w_n - w\|$, since $w_n \to w$, $\mathrm{z}_w^{(n)}(x) \to \mathrm{z}_w(x)$ as n tends to ∞ for each $x \in \mathcal{J}$. Since the sequence is convergent and bounded, then there is $\xi > 0$ such that for each $x \in \mathcal{J}$, we have $|\mathrm{z}_w^{(n)}(x)| \le \xi$ and $|\mathrm{z}_w(x)| \le \xi$. Then,

$$(x-s)^{\beta-1}\left|\mathrm{z}_w^{(n)}(x) - \mathrm{z}_w(x)\right| \le (x-s)^{\beta-1}\left[\left|\mathrm{z}_w^{(n)}(x)\right| + \left|\mathrm{z}_w(x)\right|\right] \le 2\xi(x-s)^{\beta-1}$$

and

$$(x_k-s)^{\beta-1}\left|\mathrm{z}_w^{(n)}(x) - \mathrm{z}_w(x)\right| \le (x_k-s)^{\beta-1}\left[\left|\mathrm{z}_w^{(n)}(x)\right| + \left|\mathrm{z}_w(x)\right|\right] \le 2\xi(x_k-s)^{\beta-1},$$

for every $x \in \mathcal{J}$ the functions $s \to 2\xi(x-s)^{\beta-1}$ and $s \to 2\xi(t_k-s)^{\beta-1}$ are integrable on $[0,1]$. Using these facts and the Lebesgue dominated convergence theorem in (14) and using the assumptions (A_4) and (A_5), we see that

$$\left|Tw_n(x) - Tw(x)\right| \to 0 \quad \text{as } n \text{ tends to } \infty,$$

and hence

$$\|Tw_n - Tw\| \to 0, \quad n \text{ tends to } \infty.$$

Therefore, an operator T is continuous.

Step 2. The operator T sends bounded sets into bounded sets of $PC(\mathcal{J}, \mathcal{R})$. We prove that for any $\eta^* > 0$ there exists a positive constant R^*, such that for every $w \in B = \{w \in PC(\mathcal{J}, \mathcal{R}), \|w\| \le \eta^*\}$, we have $\|Tw\| \le R^*$. To derive this result for each $x \in \mathcal{J}$, we have

$$\begin{aligned}\left|Tw(x)\right| \le{}& \int_{x_k}^{x} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)}\left|\mathrm{z}_w(s)\right| ds \\ &+ \left|\frac{c(d+\zeta-x)}{[1-c(d+\zeta-\eta)]}\right| \left|\int_{x_k}^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left|\mathrm{z}_w(s)\right| ds \right. \\ &+ \left|\frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]}\right| \left[\int_{x_k}^{\zeta} \frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)}\left|\mathrm{z}_w(s)\right| ds \right. \\ &\left. + |d| \int_{x_p}^{1} \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)}\left|\mathrm{z}_w(s)\right| ds \right]\end{aligned} \qquad (15)$$

$$+\left|\frac{1-c(d+\zeta+x-\eta)}{[1-c(d+\zeta-\eta)]}\right|\sum_{i=1}^{p}|d|\left(\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}\left|z_w(s)\right|ds\right.$$
$$\left.+\left|\bar{I}_i\left(w\left(x_i^-\right)\right)\right|\right)+\sum_{i=1}^{k}\left|\left(\frac{x(1-cd)+cd\eta-\zeta}{[1-c(d+\zeta-\eta)]}\right)\right|\left(\int_{x_{i-1}}^{x_i}\frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)}\left|z_w(s)\right|ds\right.$$
$$\left.+\left|\bar{I}_i\left(w\left(t_i^-\right)\right)\right|\right).$$

From (A_4), for every $x\in\mathcal{J}$, we have

$$\begin{aligned}\left|z_w(x)\right| &= \left|f\left(x,w(x),z_w(x)\right)\right|\\ &\le a(x)+b(x)\left|w(x)\right|+c(x)\left|z_w(x)\right|\\ &\le a(x)+b(x)\|w\|+c(x)\left|z_w(x)\right|\\ &\le a(x)+b(x)\eta^*+c(x)\left|z_w(x)\right|\\ &\le a^*+b^*\eta^*+c^*\left|z_w(x)\right|.\end{aligned}$$

Then,

$$\left|z_w(x)\right|\le\frac{a^*+b^*\eta^*}{1-c^*}:=M^*. \tag{16}$$

Thus, (15) implies

$$\begin{aligned}\left|Tw(x)\right|\le{}&\frac{M^*}{\Gamma(\beta+1)}+\frac{\sigma_1M^*}{\Gamma(\beta+1)}+\frac{\sigma_2M^*}{\Gamma(\beta+1)}+\frac{\sigma_2|d|M^*}{\Gamma(\beta)}\\ &+\frac{\sigma_3p|d|M^*}{\Gamma(\beta)}+\sigma_3|d|\left(A^*\eta^*+B^*\right)+\frac{p\sigma_4M^*}{\Gamma(\beta)}+p\sigma_4\left(A^*\eta^*+B^*\right).\end{aligned}$$

Hence, one has

$$\begin{aligned}\|Tw\|&\le\frac{M^*(1+\sigma_1+\sigma_2)}{\Gamma(\beta+1)}+\frac{|d|M^*(\sigma_2+\sigma_3p)+\sigma_4pM^*}{\Gamma(\beta)}+\left(\sigma_3|d|+\sigma_4p\right)\left(A^*\eta^*+B^*\right)\\ &:=R^*,\\ \|Tw\|&\le R^*.\end{aligned}$$

Therefore, T is bounded.

Step 3. T assigns bounded sets to equicontinuous sets of $PC(\mathcal{J},\mathcal{R})$. Let $x_1,x_2\in\mathcal{J}$, $x_1<x_2$, and B be a bounded set as in *Step 2*, and suppose $w\in B$, then

$$\begin{aligned}\left|Tw(x_2)-Tw(x_1)\right|\le{}&\left|\int_{x_k}^{x_2}\frac{(x_2-s)^{\beta-1}}{\Gamma(\beta)}z_w(s)\,ds-\int_{x_k}^{x_1}\frac{(x_1-s)^{\beta-1}}{\Gamma(\beta)}z_w(s)\,ds\right|\\ &+\left|\frac{(c(d+\zeta-x_2))-(c(d+\zeta-x_1))}{[1-c(d+\zeta-\eta)]}\right|\int_{x_k}^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left|z_w(s)\right|ds\\ &+\left|\frac{(1+c(\eta-x_1))-(1+c(\eta-x_2))}{[1-c(d+\zeta-\eta)]}\right|\left[\int_{x_k}^{\zeta}\frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)}\left|z_w(s)\right|ds\right.\end{aligned}$$

$$
\begin{aligned}
&+ |d| \int_{x_p}^{1} \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \left|z_w(s)\right| ds \Bigg] \\
&+ \left| \frac{(1 - c(d+\zeta+x_1-\eta)) - (1 - c(d+\zeta+x_2-\eta))}{[1 - c(d+\zeta-\eta)]} \right| \\
&\times \left(\sum_{i=1}^{p} |d| \left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} \left|z_w(s)\right| ds + \left|\bar{I}_i\left(w\left(x_i^-\right)\right)\right| \right) \right) \\
&+ \sum_{i=1}^{k} \left| \left(\frac{(x_2(1-cd) + cd\eta - \zeta) - (x_1(1-cd) + cd\eta - \zeta)}{[1 - c(d+\zeta-\eta)]} \right) \right| \\
&\times \left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} \left|z_w(s)\right| ds + \left|\bar{I}_i\left(w\left(x_i^-\right)\right)\right| \right).
\end{aligned} \tag{17}
$$

Using (16) in (17), we obtain

$$
\begin{aligned}
&\left|Tw(x_2) - Tw(x_1)\right| \\
&\quad\le \left[\frac{M^*}{\Gamma(\beta+1)} \right] \left((x_2-x_k)^\beta - (x_1-x_k)^\beta \right) + \left[\frac{cM^*(\eta-x_k)^\beta}{[1-c(d+\zeta-\eta)]\Gamma(\beta+1)} \right] (x_2-x_1) \\
&\qquad + \left[\frac{cM^*[(\zeta-x_k)^\beta + d(1-x_p)^\beta]}{\Gamma(\beta+1)} \right] (x_2-x_1) + \left[\frac{cM^*(\eta-x_k)^\beta}{\Gamma(\beta+1)} \right] (x_2-x_1) \\
&\qquad + \frac{1}{1-c(d+\zeta-\eta)} \left[\frac{M^* cd \sum_{i=1}^{p} (x_i-x_{i-1})^\beta}{\Gamma(\beta-1)} + c\left(A^*\eta^* + B^*\right) \right] (x_2-x_1) \\
&\qquad + \frac{(1-cd)}{1-c(d+\zeta-\eta)} \left[\frac{M^* \sum_{i=1}^{k} (x_i-x_{i-1})^{\beta-1}}{\Gamma(\beta-1)} + A^*\eta^* + B^* \right] (x_2-x_1).
\end{aligned} \tag{18}
$$

Similarly, we can see that the right-hand side of the inequality (18) tends to 0 when $x_1 \to x_2$. Thus, $|Tw(x_2) - Tw(x_1)| \to 0$ as $x_1 \to x_2$. As, T is bounded,

$$
\left\| Tw(x_2) - Tw(x_1) \right\| \to 0 \quad \text{as } x_1 \to x_2.
$$

Hence, T is uniformly continuous and is relatively compact. Thus, in view of the Arzelà–Ascoli theorem, the operator $T : PC(\mathcal{J},\mathcal{R}) \to PC(\mathcal{J},\mathcal{R})$ is completely continuous.

Step 4. Finally, we will show that the set $E = \{w \in PC(\mathcal{J},\mathcal{R}) : w = \phi T(w), \text{for some } 0 < \phi < 1\}$ is bounded. Suppose that $w \in E$; then $w = \phi T(w)$ for some $0 < \phi < 1$. Therefore, for each $x \in \mathcal{J}$, we have

$$
\begin{aligned}
\left|w(x)\right| &= \left|\phi T\left(w(x)\right)\right| \\
&= \Bigg| \phi \int_{x_k}^{x} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} z_w(s)\, ds + \phi \left(\frac{c(d+\zeta-x)}{[1-c(d+\zeta-\eta)]} \right) \int_{x_k}^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} z_w(s)\, ds \\
&\quad - \phi \left(\frac{1+c(\eta-x)}{[1-c(d+\zeta-\eta)]} \right) \left[\int_{x_k}^{\zeta} \frac{(\zeta-s)^{\beta-1}}{\Gamma(\beta)} z_w(s)\, ds + d \int_{x_p}^{1} \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} z_w(s)\, ds \right] \\
&\quad - \phi \left(\frac{1-c(d+\zeta+x-\eta)}{[1-c(d+\zeta-\eta)]} \right) \sum_{i=1}^{p} d \left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} z_w(s)\, ds + \bar{I}_i\left(w\left(x_i^-\right)\right) \right) \\
&\quad + \sum_{i=1}^{k} \phi \left(\frac{x(1-cd) + cd\eta - \zeta}{[1-c(d+\zeta-\eta)]} \right) \left(\int_{x_{i-1}}^{x_i} \frac{(x_i-s)^{\beta-2}}{\Gamma(\beta-1)} z_w(s)\, ds + \bar{I}_i\left(w\left(x_i^-\right)\right) \right) \Bigg|.
\end{aligned} \tag{19}
$$

Now, using assumptions (A_4), (A_5), and (16), we have

$$\|w\| \leq \frac{M^*(1+\sigma_1+\sigma_2)}{\Gamma(\beta+1)} + \frac{|d|M^*(\sigma_2+\sigma_3 p)+\sigma_4 pM^*}{\Gamma(\beta)} + \bigl(\sigma_3|d|+\sigma_4 p\bigr)\bigl(A^*\eta^*+B^*\bigr) := Z^*,$$

$$\|w\| \leq Z^*.$$

Consequently, set E is bounded. From Schaefer's theorem, we conclude that an operator T has a fixed point and hence the resultant problem (1) has at least one solution. □

4 Examples

We verify our results by considering the following examples.

Example 1 Let us consider the four-point impulsive nonlinear FODEs as

$$\begin{cases} {}^C_0\mathbf{D}^{\beta}_{x_1} w(x) = \frac{\sin(|w(x)|)+\sin(|{}^C_0\mathbf{D}^{\beta}_{x_1} w(x)|)}{60+e^x}, \quad 1<\beta\leq 2, x_1=\frac{1}{3}, x_1\neq x\in[0,1], \\ \Delta w(\frac{1}{3}) = I_1(w(\frac{1}{3})) = \frac{\cos(|w(\frac{1}{3})|)}{10}, \\ \Delta w'(\frac{1}{3}) = \bar{I}_1(w(\frac{1}{3})) = \frac{e^{-|w(\frac{1}{3})|}}{10}, \\ w'(0)+cw(\frac{1}{2}) = 0, \qquad dw'(1)+w(\frac{1}{3}) = 0, \end{cases} \tag{20}$$

where $\beta=\frac{3}{2}$, $c=1$, $d=2$, $p=1$, $\mathrm{z}_w(x) = {}^C_0\mathbf{D}^{\beta}_{x_1} w(x)$. Now, set

$$f(x,w,\mathrm{z}_w) = \frac{\sin(|w(x)|)+\sin(|\mathrm{z}_w(x)|)}{60+e^x}, \quad w,\mathrm{z}_w\in PC(\mathrm{J},\mathcal{R}), \text{ and } x\in[0,1].$$

Obviously, the function f is a jointly continuous function.

Hence, for each $w,\bar{w},\mathrm{z}_w,\bar{\mathrm{z}}_w \in PC(\mathcal{J},\mathcal{R})$, we have

$$\begin{aligned} \bigl|f(x,w,\mathrm{z}_w)-f(x,\bar{w},\bar{\mathrm{z}}_w)\bigr| &= \left|\frac{\sin(|w(x)|)+\sin(|\mathrm{z}_w(x)|)}{60+e^x} - \frac{\sin(|\bar{w}(x)|)+\sin(|\bar{\mathrm{z}}_w(x)|)}{60+e^x}\right| \\ &= \left|\frac{\sin(|w(x)|)-\sin(|\bar{w}(x)|)+\sin(|\mathrm{z}_w(x)|)-\sin(|\bar{\mathrm{z}}_w(x)|)}{60+e^x}\right| \\ &\leq \left|\frac{\sin|(w(x)|)-\sin(|\bar{w}(x)|)}{60+e^x}\right| + \left|\frac{\sin(|\mathrm{z}_w(x)|)-\sin(|\bar{\mathrm{z}}_w(x)|)}{60+e^x}\right|, \\ \bigl|f(x,w,\mathrm{z}_w)-f(x,\bar{w},\bar{\mathrm{z}}_w)\bigr| &\leq \frac{1}{60}\bigl(|w-\bar{w}|+|\mathrm{z}_w-\bar{\mathrm{z}}_w|\bigr), \end{aligned}$$

which satisfies condition (A_2) with $K_f = L_f = \frac{1}{60}$. Now, set

$$\Delta w'\left(\frac{1}{3}\right) = \bar{I}_1\left(w\left(\frac{1}{3}\right)\right) = \frac{e^{-|w(\frac{1}{3})|}}{10}, \quad w\in PC(\mathrm{J},\mathcal{R}).$$

Suppose that $w,\bar{w}\in PC(\mathcal{J},\mathcal{R})$, we have

$$\begin{aligned} \left|\bar{I}_1\left(w\left(\frac{1}{3}\right)\right) - \bar{I}_1\left(\bar{w}\left(\frac{1}{3}\right)\right)\right| &= \left|\frac{e^{-|w(\frac{1}{3}|)}}{10} - \frac{e^{-|\bar{w}(\frac{1}{3})|}}{10}\right| \\ &\leq \frac{1}{10}|w-\bar{w}|. \end{aligned}$$

Hence, with $M = \frac{1}{10}$, the condition (A_3) satisfies. Also, the condition

$$\left[\frac{K_f}{1-L_f}\left(\frac{\sigma_1+\sigma_2+1}{\Gamma(\beta+1)}+\frac{(\sigma_2+\sigma_3 p)|d|+\sigma_4 p}{\Gamma(\beta)}\right)+\left(\sigma_3|d|+\sigma_4\right)pM\right]$$
$$=\frac{1}{59}\left(\frac{56}{10\Gamma(\beta+1)}+\frac{7}{\Gamma(\beta)}\right)+\frac{52}{100}<1$$

satisfies with $\sigma_1 = 2\cdot 8$, $\sigma_2 = 1\cdot 8$, $\sigma_3 = 2\cdot 2$, $\sigma_4 = 0\cdot 8$, and $\beta = \frac{3}{2}$.

Thanks to theorem 3.1, problem (20) has at most one solution.

Example 2 Take another problem as

$$\begin{cases} {}^C_0\mathbf{D}^{\beta}_{x_1} w(x) = \frac{e^{-x}\sqrt{|w(x)|}+\sqrt{|{}^C_0\mathbf{D}^{\beta}_{x_1} w(x)|}}{30+x}, & 1<\beta\le 2, x_1\neq x\in[0,1], x_1=\frac{1}{2}, \\ \Delta w(\frac{1}{2}) = I_1(w(\frac{1}{2})) = \frac{\sin(|w(\frac{1}{2})|)}{20+\sin(|w(\frac{1}{2})|)}, \\ \Delta w'(\frac{1}{2}) = \bar{I}_1(w(\frac{1}{2})) = \frac{\cos(|w(\frac{1}{2})|)}{40+\cos(|w(\frac{1}{2})|)}, \\ w'(0)+cw(\frac{2}{3}) = 0, \qquad dw'(1)+w(\frac{5}{6}) = 0, \end{cases} \tag{21}$$

where $c = 1$, $d = 2$, $\beta = \frac{3}{2}$ and $\mathrm{z}_w(x) = {}^C_0\mathbf{D}^{\beta}_{x_1} w(x)$. Set

$$f(x,w,\mathrm{z}_w) = \frac{e^{-x}\sqrt{|w(x)|}+\sqrt{|\mathrm{z}_w(x)|}}{30+x}, \quad w,\mathrm{z}_w\in PC(\mathcal{J},\mathcal{R}).$$

It is clear that the mentioned function f is a continuous function.

Also, for every $w,\bar{w},\mathrm{z}_w,\bar{\mathrm{z}}_w\in PC(\mathcal{J},\mathcal{R})$, we have

$$\begin{aligned} \left|f(x,w,\mathrm{z}_w)-f(x,\bar{w},\bar{\mathrm{z}}_w)\right| &= \left|\frac{e^{-x}\sqrt{|w(x)|}+\sqrt{|\mathrm{z}(x)|}}{30+x}-\frac{e^{-x}\sqrt{|\bar{w}(x)|}+\sqrt{|\bar{\mathrm{z}}(x)|}}{30+x}\right| \\ &= \left|\frac{e^{-x}\sqrt{|w(x)|}-e^{-x}\sqrt{|\bar{w}(x)|}+\sqrt{|\mathrm{z}_w(x)|}-\sqrt{|\bar{\mathrm{z}}_w(x)|}}{30+x}\right| \\ &\le \left|\frac{e^{-x}(\sqrt{|w(x)|}-\sqrt{|\bar{w}(x)|})}{30+x}\right|+\left|\frac{\sqrt{|\mathrm{z}_w(x)|}-\sqrt{|\bar{\mathrm{z}}_w(x)|}}{30+x}\right|, \end{aligned}$$
$$\left|f(x,w,\mathrm{z}_w)-f(x,\bar{w},\bar{\mathrm{z}}_w)\right|\le\frac{1}{30}\left(|w-\bar{w}|+|\mathrm{z}_w-\bar{\mathrm{z}}_w|\right),$$

which satisfies the condition (A_2) with $K_f = L_f = \frac{1}{30}$. Now, set

$$\Delta w'\left(\frac{1}{2}\right) = \bar{I}_1\left(w\left(\frac{1}{2}\right)\right) = \frac{\cos(|w(\frac{1}{2})|)}{40+\cos(|w(\frac{1}{2})|)}, \quad w\in PC(\mathrm{J},\mathcal{R}).$$

Then, for each $w,\bar{w}\in PC(J,\mathcal{R})$, we have

$$\begin{aligned} \left|\bar{I}_1\left(w\left(\frac{1}{2}\right)\right)-\bar{I}_1\left(\bar{w}\left(\frac{1}{2}\right)\right)\right| &= \left|\frac{\cos(|w(\frac{1}{2})|)}{40+\cos(|w(\frac{1}{2})|)}-\frac{\cos(|\bar{w}(\frac{1}{2})|)}{40+\cos(|\bar{w}(\frac{1}{2})|)}\right| \\ &\le\frac{1}{40}|w-\bar{w}|. \end{aligned}$$

Hence, with $M = \frac{1}{40}$, the condition (A_3) is clearly satisfied. Also,

$$\begin{aligned} \bigl|f(x,w,\mathrm{z}_w)\bigr| &= \left|\frac{e^{-x}\sqrt{|w(x)|}+\sqrt{|\mathrm{z}_w(x)|}}{30+x}\right| \\ &\le \frac{e^{-x}}{30+x}\bigl|w(x)\bigr| + \frac{1}{30+x}\bigl|\mathrm{z}_w(x)\bigr|. \end{aligned}$$

Thus, condition (A_4) is satisfied with $a(x) = 0$, $b(x) = \frac{e^{-x}}{30+x}$, and $c(x) = \frac{1}{30+x}$. Let

$$\Delta w'\left(\frac{1}{2}\right) = \bar{I}_1\left(w\left(\frac{1}{2}\right)\right) = \frac{\cos(|w(\frac{1}{2})|)}{40+\cos(|w(\frac{1}{2})|)}, \quad w \in PC(\mathcal{J},\mathcal{R}).$$

Then, for every $w \in PC(\mathcal{J},\mathcal{R})$, we have

$$\begin{aligned} \left|\bar{I}_1\left(w\left(\frac{1}{2}\right)\right)\right| &= \left|\frac{\cos(|w(\frac{1}{2})|)}{40+\cos(|w(\frac{1}{2})|)}\right| \\ &\le \frac{1}{40}|w| + 1. \end{aligned}$$

Therefore, condition (A_5) is satisfied by $A^* = \frac{1}{40}$ and $B^* = 1$. By using Theorem 3.2 problem (21) has at least one solution on J.

5 Concluding remarks

We have obtained some appropriate results corresponding to the existence theory for nonlinear implicit impulsive FODEs with nonlocal four-point boundary conditions. The concerned problem has been investigated under a Caputo-type fractional-order derivative. The considered class is devoted to implicit-type FODEs under impulsive conditions. Implicit-type problems of FODEs have numerous applications in economics, optimization, etc. By the classical fixed-point theory, the respective results have been established. By proper examples, we have demonstrated the obtained analysis.

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