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Application of the Elzaki iterative method to fractional partial differential equations

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Abstract

In this article, we present an iterative transformation method for solving fractional partial differential equations that combines the Elzaki transform and iterative methods. By this iterative transformation method, numerical solutions in the form of series are obtained. When we apply this method to the fractional linear Klein–Gordon equation, we find that it yields the same results, just like the Homotopy perturbation method. The procedures and results of this method for solving the new generalized fractional Hirota–Satsuma coupled KdV equation are given in the paper.

Keywords: Elzaki transform; Iterative method; Fractional differential equation

0 Introduction

In recent decades, fractional-order partial differential equations have been widely used and developed in physics, engineering, and fluid mechanics. Compared with integer-order partial differential equations, they are more suitable to portray complex phenomena and processes. Therefore, the method to solve fractional partial differential equations is also a relatively important problem. Now, there are methods to solve fractional-order partial differential equations. For example, in [1], the finite-difference methods, the Galerkin finite-element methods, and the spectral methods to solve fractional-order partial differential equations are mentioned; Gepreel uses the Homotopy perturbation method to obtain the solution of the fractional Klein–Gordon equation in [2]; Khalid used the Elzaki transform method to solve the equations [3]; in [4], Ziane used the fractional Elzaki variational iteration method to solve the equations; Jafari introduced the Iterative Laplace transform method in [5–7]; Tarig used the Sumudutrans form of the variational iteration method to solve linear homogeneous partial differential equations; Thabet [8] introduced a new analytic method to solve partial differential equations with fractional order, and El-Rashidy [9] used the method to obtain new traveling-wave solutions of the equations. Hosseini [10] used the modified Kudryashov method to obtain exact solutions of the coupled sine-Gordon equations. Mohammad Tamsir [11] employed a semianalytical approach to obtain the approximate analytical solution of the Klein–Gordon equations; The Klein–Gordon equation [12, 13] is a crucial equation in the study of relativistic quantum mechanics.

Many authors have solved the generalized Hirota–Satsuma coupled KdV equation utilizing various equations in [14–17], including the homotopy analysis approach [18]. This

is a significant class of equations in mathematics and physics. In this study, we employ the Elzaki transform [19, 20] in conjunction with an iterative approach [21] to generate approximations to partial differential equations with fractional order. The results demonstrate the method's validity, further, it also may be applied to other fractional-order partial differential equations.

1 Basic definition

Definition 1 The fractional Riemann–Liouville of operator D^p is as follows

$$D^p w(x) = \begin{cases} \frac{\partial^m w(x)}{\partial x^m}, & p = m, \\ \frac{\partial}{\Gamma(m-p)\partial x^m} \int_0^x \frac{w(x)}{(x-\xi)^{p-m+1}}\, d\xi, & m-1 < p < m, \end{cases} \tag{1}$$

where $m \in Z^+$, $p \in R^+$ when $0 < p \leq 1$,

$$D^p w(x) = \frac{1}{\Gamma(p)} \int_0^x \frac{w(x)}{(x-\xi)^{1-p}}\, d\xi.$$

Definition 2 The Riemann–Liouville integral operator with fractional order is defined as follows

$$I^p w(x) = \frac{1}{\Gamma(p)} \int_0^x (x-\xi)^{p-1} w(\xi)\, d\xi, \quad p > 0, \xi > 0. \tag{2}$$

Definition 3 The Caputo fractional derivative of $w(x)$ is defined as follows, $m \in N$

$$c_{D^p w(x)} = \begin{cases} I^{m-p}[\frac{\partial^m w(x)}{\partial x^m}], & m-1 < p < m, \\ \frac{\partial^m w(x)}{\partial x^m}, & p = m. \end{cases} \tag{3}$$

Definition 4 The Elzaki [22, 23] transform of $w(x)$ is defined as follows

$$E\big[w(x)\big] = T(v) = v \int_0^\infty w(x) e^{\frac{-x}{v}}\, dx, \quad x \geq 0, k_1 \leq v \leq k_2. \tag{4}$$

Definition 5 The fractional Caputo operator of the Elzaki transform is [24]

$$E\big[D_x^\alpha w(x)\big] = v^{-\alpha} E\big[w(x)\big] - \sum_{k=0}^{m-1} v^{2-\alpha+k} w^{(k)}(0), \quad \text{where } m-1 < \alpha < m. \tag{5}$$

2 Methodology of the Elzaki transform iterative method

To briefly describe this equation in detail, we consider the following fractional partial differential equations

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = M\big(w_1(x,t), w_2(x,t), \ldots, w_n(x,t)\big) + N\big(w_1(x,t), w_2(x,t), \ldots, w_n(x,t)\big), \tag{6}$$

where M and N are the nonlinear and linear operators from Banach space $\mathrm{B} \to \mathrm{B}$, respectively, $\alpha \in R^+$, $m-1 < \alpha \leq m$, $m = 0, 1, \ldots, n$,

subject to the initial condition

$$\frac{\partial^k w(x,0)}{\partial t^k} = w_k(x,0), \quad k = 0,1,\dots,m-1, m \in N. \tag{7}$$

Then, the Ezaki transformation acts simultaneously on both sides of the equation, and we obtain

$$E\left[\frac{\partial^k w(x,0)}{\partial t^k}\right] = E\big[M\big(w_1(x,t), w_2(x,t),\dots,w_n(x,t)\big) + N\big(w_1(x,t), w_2(x,t),\dots,w_n(x,t)\big)\big], \tag{8}$$

hence,

$$v^{-k}E\big[w(x,t)\big] - \sum_{n=0}^{m-1} v^{2-k+n} w^{(n)}(x,0) = E\big[M\big(w_1(x,t), w_2(x,t),\dots,w_n(x,t)\big) + N\big(w_1(x,t), w_2(x,t),\dots,w_n(x,t)\big)\big]. \tag{9}$$

Through the use of the inverse Elzaki transform, we obtain

$$w(x,t) = E^{-1}\left[\sum_{n=0}^{m-1} v^{2+n} w^{(n)}(x,0)\right] + E^{-1}\big[v^k E\big[M\big(w_1(x,t), w_2(x,t),\dots,w_n(x,t)\big) + N\big(w_1(x,t), w_2(x,t),\dots,w_n(x,t)\big)\big]\big]. \tag{10}$$

The following iterative method is utilized

$$w(x,t) = \sum_{i=1}^{\infty} w_i(x,t). \tag{11}$$

The nonlinear operator M can be decomposed into

$$\begin{aligned} &M\big(w_1(x,t), w_2(x,t),\dots,w_n(x,t)\big) \\ &\quad = M\big(w_{10}(x,t), w_{20}(x,t),\dots,w_{n0}(x,t)\big) \\ &\qquad + \sum_{m=0}^{\infty}\left[M\left(\sum_{i=0}^{m} w_{1i}(x,t), \sum_{i=0}^{m} w_{2i}(x,t),\dots,\sum_{i=0}^{m} w_{ni}(x,t)\right)\right. \\ &\qquad \left. - M\left(\sum_{i=0}^{m-1} w_{1i}(x,t), \sum_{i=0}^{m-1} w_{2i}(x,t),\dots,\sum_{i=0}^{m-1} w_{ni}(x,t)\right)\right]. \end{aligned} \tag{12}$$

Then, we can obtain

$$\sum_{i=0}^{\infty} w_i(x,t) = E^{-1}\left[\sum_{n=0}^{m-1} v^{2+n} w^{(n)}(x,0)\right] + E^{-1} v^k\big[E\big[M\big(w_{10}(x,t), w_{20}(x,t),\dots,w_{n0}(x,t)\big) + N\big(w_{10}(x,t), w_{20}(x,t),\dots,w_{n0}(x,t)\big)\big]\big]$$

$$
\begin{aligned}
&+E^{-1}\left[v^{k}E\left[\sum_{m=0}^{\infty}\left[M\left(\sum_{i=0}^{m}w_{1i}(x,t),\sum_{i=0}^{m}w_{2i}(x,t),\ldots,\sum_{i=0}^{m}w_{ni}(x,t)\right)\right.\right.\right.\\
&\left.\left.\left.-M\left(\sum_{i=0}^{m-1}w_{1i}(x,t),\sum_{i=0}^{m-1}w_{2i}(x,t),\ldots,\sum_{i=0}^{m-1}w_{ni}(x,t)\right)\right]\right]\right]\\
&+E^{-1}\left[v^{k}E\left[\sum_{m=0}^{\infty}\left[N\left(\sum_{i=0}^{m}w_{1i}(x,t),\sum_{i=0}^{m}w_{2i}(x,t),\ldots,\sum_{i=0}^{m}w_{ni}(x,t)\right)\right.\right.\right.\\
&\left.\left.\left.-N\left(\sum_{i=0}^{m-1}w_{1i}(x,t),\sum_{i=0}^{m-1}w_{2i}(x,t),\ldots,\sum_{i=0}^{m-1}w_{ni}(x,t)\right)\right]\right]\right].
\end{aligned}
\tag{13}
$$

We make the following settings

$$
\begin{cases}
w_0 = E^{-1}[\sum_{n=0}^{m-1} v^{2+n}w^{(n)}(x,0)],\\
w_1 = E^{-1}v^{k}[E[M(w_{10}(x,t),w_{20}(x,t),\ldots,w_{n0}(x,t))\\
\qquad + N(w_{10}(x,t),w_{20}(x,t),\ldots,w_{n0}(x,t))]],\\
w_i = E^{-1}[v^{k}E[\sum_{i=0}^{\infty}[M(\sum_{i=0}^{m} w_{1i}(x,t),\sum_{i=0}^{m} w_{2i}(x,t),\ldots,\sum_{i=0}^{m} w_{ni}(x,t))\\
\qquad - M(\sum_{i=0}^{m-1} w_{1i}(x,t),\sum_{i=0}^{m-1} w_{2i}(x,t),\ldots,\sum_{i=0}^{m-1} w_{ni}(x,t))]]]\\
\qquad + E^{-1}[v^{k}E[\sum_{i=0}^{\infty}[N(\sum_{i=0}^{m} w_{1i}(x,t),\sum_{i=0}^{m} w_{2i}(x,t),\ldots,\sum_{i=0}^{m} w_{ni}(x,t))\\
\qquad - N(\sum_{i=0}^{m-1} w_{1i}(x,t),\sum_{i=0}^{m-1} w_{2i}(x,t),\ldots,\sum_{i=0}^{m-1} w_{ni}(x,t))]]].
\end{cases}
\tag{14}
$$

Finally, we obtain the approximate solution of the fractional-order partial differential equation

$$
w(x,t) \cong w_0(x,t) + w_1(x,t) + \cdots + w_m(x,t), \quad m = 1,2,\ldots. \tag{15}
$$

Theorem *B is the Banach space, if there exists $0 < K < 1$, $\|w_n\| \le K\|w_{n-1}\|$, for $\forall x \in N$, then the approximate solution $w(x,t)$ converges to A.*

Proof Define the sequence A_i, $i = 0,1,\ldots,n$

$$
\begin{cases}
A_0 = w_0,\\
A_1 = w_0 + w_1,\\
A_2 = w_0 + w_1 + w_2,\\
\ldots,\\
A_n = w_0 + w_1 + \cdots + w_n
\end{cases}
\tag{16}
$$

and prove that $(A_i)_{i\ge 0}$ is a Cauchy sequence, and we consider

$$
\|A_n - A_{n-1}\| \le \|w_n\| \le K^n w_0 \tag{17}
$$

for $m > n > 0 \in N$, we have

$$\begin{cases} \|A_n - A_m\| = \|A_n - A_{n-1} + A_{n-1} - A_{n-2} + \cdots + A_{m+1} - A_m\| \\ \quad \le \|A_n - A_{n-1}\| + \|A_{n-1} - A_{n-2}\| + \cdots + \|A_{m+1} - Am\| \\ \quad \le (K^n + K^{n-1} + \cdots + K^{m+1})A_0 \\ \quad \le \|\frac{K^{m+1}(1-K^{n-m})}{1-K}\|A_0, \end{cases} \tag{18}$$

where w_0 is bounded, and we have

$$\lim_{n,m\to\infty} \|A_n - A_m\| = 0. \tag{19}$$

Therefore, the sequence $(A_i)_{i\ge 0}$ is a Cauchy sequence in B, so the solution of Eq. (6) is convergent.

The error estimates are as follows:

$$\sup\left|w(x,t) - \sum_{i=0}^{m} w_i(x,t)\right| \le \frac{K^{m+1}}{1-K}\sup\left|w_0(x,t)\right|. \tag{20}$$

□

Remark Similar proofs can be found in [8].

3 Test example

Example 1 Consider the linear fractional Klein–Gordon equation [25]

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - u = 0, \quad 0 < \alpha \le 1 \tag{21}$$

subject to the initial condition:

$$u(x,0) = 1 + \sin x. \tag{22}$$

The Elzaki transform of the linear fractional Klein–Gordon equation [26] is

$$v^{-\alpha}E\big[u(x,t)\big] = v^{2-\alpha}u(x,0) + E\left[\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right]. \tag{23}$$

Using the inverse Elzaki transform of the above equation, we obtain

$$u(x,t) = E^{-1}\big[v^2 u(x,0)\big] + E^{-1}\left[v^\alpha E\left[\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right]\right], \tag{24}$$

then, we use the iterative method above, and we have

$$u_0(x,t) = E^{-1}\big[v^2(1+\sin x)\big], \tag{25}$$

$$u_0(x,t) = 1 + \sin x, \tag{26}$$

$$u_1(x,t) = E^{-1}\left[v^\alpha E\left[\frac{\partial^2 u_0(x,t)}{\partial x^2} + u_0(x,t)\right]\right], \tag{27}$$

$$u_1(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{28}$$

$$\begin{aligned} u_2(x,t) &= v^\alpha E\left[\frac{\partial^2 u_1(x,t) + u_0(x,t)}{\partial x^2} + u_1(x,t) + u_0(x,t)\right] \\ &\quad - v^\alpha E\left[\frac{\partial^2 u_0(x,t)}{\partial x^2} + u_0(x,t)\right], \end{aligned} \tag{29}$$

$$u_2(x,t) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{30}$$

$$\begin{aligned} u_3(x,t) &= v^\alpha E\left[\frac{\partial^2 (u_2(x,t) + u_1(x,t) + u_0(x,t))}{\partial x^2} + u_2(x,t) + u_1(x,t) + u_0(x,t)\right] \\ &\quad - v^\alpha E\left[\frac{\partial^2 (u_1(x,t) + u_0(x,t))}{\partial x^2} + u_1(x,t) + u_0(x,t)\right] \end{aligned} \tag{31}$$

$$u_3(x,t) = \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{32}$$

$$\cdots$$

$$u_n(x,t) = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{33}$$

The result is

$$u(x,t) = 1 + \sin x + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots + \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots. \tag{34}$$

When $\alpha = 1$, the exact solution of the linear fractional Klein–Gordon equation is as follows:

$$u(x,t) = e^t + \sin x. \tag{35}$$

In Fig. 1, the approximate solution of u is depicted for the case where the value of α is 0.01, 0.002, and 0.1.

Example 2 Consider the new generalized fractional Hirota–Satsuma coupled KdV equation

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} &= -v_{xxx} + 3uv_x, \quad 0 < \alpha \le 1, \\ \frac{\partial^\alpha w}{\partial t^\alpha} &= -w_{xxx} + 3uw_x \end{aligned} \tag{36}$$

subject to the initial condition

$$\begin{aligned} u(x,0) &= \frac{1}{3}\left(\beta - 2k^2\right) + 2k^2 \tanh^2(kx), \\ v(x,0) &= -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1} \tanh(kx), \\ w(x,0) &= c_0 + c_1 \tanh(kx). \end{aligned} \tag{37}$$

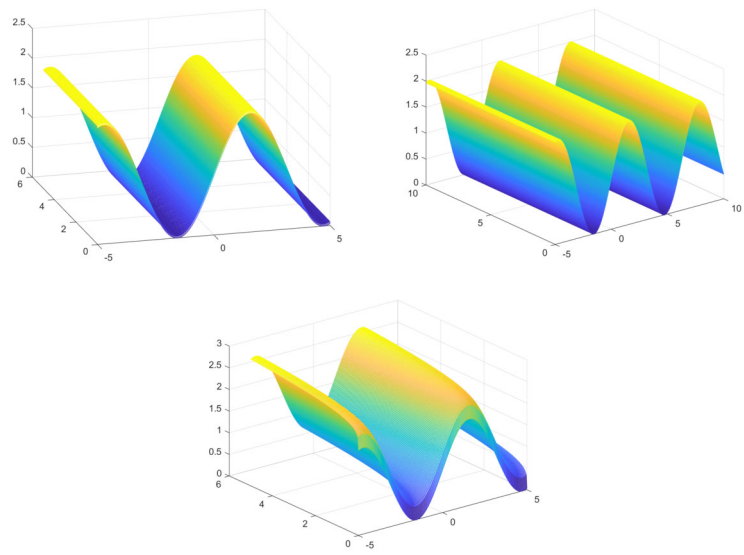

Figure 1 Graph of $u(x,t)$ at $\alpha = 0.01$, 0.02, and 0.1 of Example 1

When $\alpha = 1$, the exact results of the new generalized Hirota–Satsuma coupled KdV equation is as follows:

$$\begin{aligned} u(x,t) &= \frac{1}{3}\bigl(\beta - 2k^2\bigr) + 2k^2 \tanh^2\bigl(k(x+\beta t)\bigr), \\ v(x,t) &= -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)}{3c_1}\tanh\bigl(k(x+\beta t)\bigr), \\ w(x,t) &= c_0 + c_1\tanh\bigl(k(x+\beta t)\bigr). \end{aligned} \tag{38}$$

The Elzaki transform of the new generalized fractional Hirota–Satsuma coupled KdV equation is

$$\begin{aligned} v^{-\alpha}E\bigl[u(x,t)\bigr] &= v^{2-\alpha}u(x,0) + E\left[\frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x\right], \\ v^{-\alpha}E\bigl[v(x,t)\bigr] &= v^{2-\alpha}v(x,0) + E[-v_{xxx} + 3uv_x], \\ v^{-\alpha}E\bigl[w(x,t)\bigr] &= v^{2-\alpha}w(x,0) + E[-w_{xxx} + 3uw_x]. \end{aligned} \tag{39}$$

Using the inverse Elzaki transform, we obtain

$$\begin{aligned} u(x,t) &= E^{-1}\bigl[v^2u(x,0)\bigr] + E^{-1}\left[v^\alpha E\left[\frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x\right]\right], \\ v(x,t) &= E^{-1}\bigl[v^2v(x,0)\bigr] + E^{-1}\bigl[v^\alpha E[-v_{xxx} + 3uv_x]\bigr], \\ w(x,t) &= E^{-1}\bigl[v^2w(x,0)\bigr] + E^{-1}\bigl[v^\alpha E[-w_{xxx} + 3uw_x]\bigr]. \end{aligned} \tag{40}$$

Next, in terms of the iterative method above, we have

$$u_0(x,t) = E^{-1}\bigl[v^2u(x,0)\bigr],$$

$$v_0(x,t) = E^{-1}\left[v^2 u(x,0)\right], \tag{41}$$

$$\begin{aligned}
&w_0(x,t) = E^{-1}\left[v^2 u(x,0)\right],\\
&u_0(x,t) = \frac{1}{3}\left(\beta - 2k^2\right) + 2k^2\tanh^2(kx),\\
&v_0(x,t) = -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx),\\
&w_0(x,t) = c_0 + c_1\tanh(kx),
\end{aligned} \tag{42}$$

$$\begin{aligned}
&u_1(x,t) = E^{-1}\left[v^\alpha E\left[\frac{1}{2}u_{0_{xxx}} - 3u_0 u_{0_x} + 3(v_0 w_0)_x\right]\right],\\
&v_1(x,t) = E^{-1}\left[v^\alpha E[-v_{0_{xxx}} + 3u_0 v_{0_x}]\right],
\end{aligned} \tag{43}$$

$$\begin{aligned}
&w_1(x,t) = E^{-1}\left[v^\alpha E[-w_{0_{xxx}} + 3u_0 w_{0_x}]\right],\\
&u_1(x.t) = 4k^3\beta\,\mathrm{sech}^2(kx)\tanh(kx)\frac{t^{t^\alpha}}{\Gamma(1+\alpha)},\\
&v_1(x,t) = 4k^3\beta\left(\beta + k^2\right)\mathrm{sech}^2(kx)\frac{t^\alpha}{\Gamma(1+\alpha)},\\
&w_1(x,t) = c_1 k\beta\,\mathrm{sech}^2(kx)\frac{t^\alpha}{\Gamma(1+\alpha)},
\end{aligned} \tag{44}$$

$$\begin{aligned}
&u_2(x,t) = E^{-1}\left[v^\alpha E\left[\frac{1}{2}(u_0 + u_1)_{xxx} - 3u_0(u_0 + u_1)_x + 3\left((v_0 + v_1)(w_0 + w_1)\right)_x\right]\right],\\
&v_2(x,t) = E^{-1}\left[v^\alpha E\left[-(v_0 + v_1)_{xxx} + 3(u_0 + u_1)(v_0 + v_1)_x\right]\right],
\end{aligned} \tag{45}$$

$$\begin{aligned}
&w_2(x,t) = E^{-1}\left[v^\alpha E\left[-(w_0 + w_1)_{xxx} + 3(u_0 + u_1)(w_0 + w_1)_x\right]\right],\\
&u_2(x,t) = \left[96k^7\beta\,\mathrm{sech}^4(kx)\tanh(kx) - 144k^7\beta^2\tanh(kx)\,\mathrm{sech}^6(kx)\right.\\
&\qquad \left. - 16c_1 k^7\beta^2\,\mathrm{sech}^4(kx)\tanh(kx) - 16c_1 k^5\beta^3\,\mathrm{sech}^4(kx)\tanh(kx)\right]\\
&\qquad \times \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(1+\alpha)\Gamma(1+\alpha)}\\
&\qquad + \left[-48k^6\beta\,\mathrm{sech}^4(kx) + \left(\frac{c_0}{3c_1} - 8c_0\right)k^6\beta\tanh(kx)\,\mathrm{sech}^2(kx)\right.\\
&\qquad + (12c_1 + 72)k^6\,\mathrm{sech}^4(kx) + \left(\frac{-116}{3} + 8C_1\right)k^6\beta\,\mathrm{sech}^6(kx)\\
&\qquad + 8\beta^2 k^4\,\mathrm{sech}^4(kx)(c_1 - 1)\\
&\qquad + 8\left(\frac{c_0 k^4\beta^2}{3c^1} + \frac{k^2\beta c_1}{3} - c_0 k^4\beta^2\right)\mathrm{sech}^2(kx)\tanh(kx)\\
&\qquad \left. - 8\,\mathrm{sech}^2(kx)\left(\frac{4k^4\beta^2 + \beta^2}{3} + \beta^2 k^4 c_1\right)\right]\\
&\qquad \times \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left[\frac{68}{3}k^5\tanh(kx)\,\mathrm{sech}^2(kx) - 16k^5\tanh(kx)\right.\\
&\qquad \left. - 4k^3\beta\tanh(kx) - 4k^3\beta\,\mathrm{sech}^2(kx)\tanh(kx)\right]\frac{t^\alpha}{\Gamma(1+\alpha)},
\end{aligned} \tag{46}$$

$$
\begin{aligned}
v_2(x,t) = {} & \left(-96k^9\beta^2\operatorname{sech}^2(kx)\tanh^2(kx) - 96k^7\beta^3\operatorname{sech}^4(kx)\tanh^2(kx)\right) \\
& \times \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(1+\alpha)\Gamma(1+\alpha)} \\
& + \left[\frac{16k^8}{c_1}\beta\operatorname{sech}^4(kx)\tanh(kx) - 144k^8\beta\operatorname{sech}^4(kx)\tanh(kx)\right. \\
& - 152k^6\beta^2\operatorname{sech}^2(kx)\tanh(kx) - 96k^6\beta^2\operatorname{sech}^4(kx)\tanh(kx) \\
& \left. + \frac{16k^6\beta^2}{c_1}\operatorname{sech}^4(kx)\tanh(kx) - 8k^4\beta^3\operatorname{sech}^2(kx)\tanh(kx)\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
& \times \left[\frac{24k^7}{c_1} - \frac{24k^7}{c_1}\operatorname{sech}^4(kx) + \frac{8k^5\beta\operatorname{sech}^4(kx)}{3c_1} - \frac{8k^5\beta\operatorname{sech}^2(kx)}{3c_1}\right. \\
& + \frac{32k^5\beta}{c_1}\operatorname{sech}^2(kx)\tanh^2(kx) + \frac{4k^3\beta^2}{3c_1}\operatorname{sech}^2(kx) \\
& \left. - 4k^3\beta\left(\beta+k^2\right)\operatorname{sech}^2(kx)\right]\frac{t^{\alpha}}{\Gamma(1+\alpha)},
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
w_2(x,t) = {} & \left(-96k^7\beta^2\operatorname{sech}^4(kx)\tanh(kx) + 144k^7\beta^2\operatorname{sech}^6(kx)\tanh(kx)\right) \\
& \times \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(1+\alpha)\Gamma(1+\alpha)}t^{3\alpha} + \left(16k^6\beta\operatorname{sech}^3(kx) - 24k^6\operatorname{sech}^4(kx)\right. \\
& + 72k^6\operatorname{sech}^4(kx)\tanh^2(kx) - 8c_1k^4\beta\operatorname{sech}^2(kx)\tanh(kx) \\
& - 24c_1k^4\beta\operatorname{sech}^4(kx)\tanh(kx) \\
& \left. - 8\beta^2k^4\operatorname{sech}^2(kx) + 12\beta^2k^4\operatorname{sech}^4(kx)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
& \times \left[-8k^5\tanh(kx) + 8c_1k^3\operatorname{sech}^4(kx) - 4c_1k^3\operatorname{sech}^2(kx) + k^3\beta\tanh(kx)\right. \\
& \left. - 24k^3\tanh(kx)\operatorname{sech}^2(kx) - c_1k\beta\operatorname{sech}^2(kx)\right]\frac{t^{\alpha}}{\Gamma(1+\alpha)},
\end{aligned} \tag{48}
$$

$\cdots$

$$
\begin{aligned}
u_n(x,t) = {} & E^{-1}\left[\nu^{\alpha}E\left[\frac{1}{2}(u_0+u_1+\cdots+u_n)_{xxx} - 3u_0(u_0+u_1+\cdots+u_n)_x\right.\right. \\
& \left.\left. + 3\left((v_0+v_1+\cdots+v_n)(w_0+w_1+\cdots+w_n)\right)_x\right]\right] \\
& - E^{-1}\left[\nu^{\alpha}E\left[\frac{1}{2}(u_0+u_1+\cdots+u_{n-1})_{xxx} - 3u_0(u_0+u_1+\cdots+u_{n-1})_x\right.\right. \\
& \left.\left. + 3\left((v_0+v_1+\cdots+v_{n-1})(w_0+w_1+\cdots+w_{n-1})\right)_x\right]\right],
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
v_n(x,t) = {} & E^{-1}\left[\nu^{\alpha}E\left[-(v_0+v_1+\cdots+v_n)_{xxx}\right.\right. \\
& \left.\left. + 3(u_0+u_1+\cdots+u_n)(v_0+v_1+\cdots+v_n)_x\right]\right] \\
& - E^{-1}\left[\nu^{\alpha}E\left[-(v_0+v_1+\cdots+v_{n-1})_{xxx}\right.\right. \\
& \left.\left. + 3(u_0+u_1+\cdots+u_{n-1})(v_0+v_1+\cdots+v_{n-1})_x\right]\right],
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
w_n(x,t) = {} & E^{-1}\left[\nu^{\alpha}E\left[-(w_0+w_1+\cdots+w_n)_{xxx}\right.\right. \\
& \left.\left. + 3(u_0+u_1+\cdots+u_n)(w_0+w_1+\cdots+w_n)_x\right]\right] \\
& - E^{-1}\left[\nu^{\alpha}E\left[-(w_0+w_1+\cdots+w_{n-1})_{xxx}\right.\right. \\
& \left.\left. + 3(u_0+u_1+\cdots+u_{n-1})(w_0+w_1+\cdots+w_{n-1})_x\right]\right].
\end{aligned} \tag{51}
$$

The series-form solution is given as

$$\begin{aligned}
u(x,t) &= u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots + u_n(x,t),\\
v(x,t) &= v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots + v_n(x,t),\\
w(x,t) &= w_1(x,t) + w_2(x,t) + w_3(x,t) + \cdots + w_n(x,t),
\end{aligned} \tag{52}$$

$$\begin{aligned}
u(x,t) = {} & \frac{1}{3}\left(\beta - 2k^2\right) + 2k^2\tanh^2(kx) + \left[96k^7\beta\operatorname{sech}^4(kx)\tanh(kx)\right.\\
& - 144k^7\beta^2\tanh(kx)\operatorname{sech}^6(kx) - 16c_1k^7\beta^2\operatorname{sech}^4(kx)\tanh(kx)\\
& \left. - 16c_1k^5\beta^3\operatorname{sech}^4(kx)\tanh(kx)\right]\\
& \times \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(1+\alpha)\Gamma(1+\alpha)} + \left[-48k^6\beta\operatorname{sech}^4(kx)\right.\\
& + \left(\frac{c_0}{3c_1} - 8c_0\right)k^6\beta\tanh(kx)\operatorname{sech}^2(kx)\\
& + (12c_1 + 72)k^6\operatorname{sech}^4(kx) + \left(\frac{-116}{3} + 8C_1\right)k^6\beta\operatorname{sech}^6(kx)\\
& + 8\beta^2k^4\operatorname{sech}^4(kx)(c_1 - 1)\\
& + 8\left(\frac{c_0k^4\beta^2}{3c^1} + \frac{k^2\beta c_1}{3} - c_0k^4\beta^2\right)\operatorname{sech}^2(kx)\tanh(kx)\\
& \left. - 8\operatorname{sech}^2(kx)\left(\frac{4k^4\beta^2 + \beta^2}{3} + \beta^2k^4c_1\right)\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\
& + \left[\frac{68}{3}k^5\tanh(kx)\operatorname{sech}^2(kx) - 16k^5\tanh(kx)\right.\\
& \left. - 4k^3\beta\tanh(kx)\right]\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \cdots,
\end{aligned} \tag{53}$$

$$\begin{aligned}
v(x,t) = {} & -\frac{4k^2c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx)\\
& \times \left(-96k^9\beta^2\operatorname{sech}^2(kx)\tanh^2(kx) - 96k^7\beta^3\operatorname{sech}^4(kx)\tanh^2(kx)\right)\\
& \times \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(1+\alpha)\Gamma(1+\alpha)}\\
& + \left[\frac{16k^8}{c_1}\beta\operatorname{sech}^4(kx)\tanh(kx) - 144k^8\beta\operatorname{sech}^4(kx)\tanh(kx)\right.\\
& - 152k^6\beta^2\operatorname{sech}^2(kx)\tanh(kx) - 96k^6\beta^2\operatorname{sech}^4(kx)\tanh(kx)\\
& \left. + \frac{16k^6\beta^2}{c_1}\operatorname{sech}^4(kx)\tanh(kx) - 8k^4\beta^3\operatorname{sech}^2(kx)\tanh(kx)\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
& \times \left[\frac{24k^7}{c_1} - \frac{24k^7}{c_1}\operatorname{sech}^4(kx) + \frac{8k^5\beta\operatorname{sech}^4(kx)}{3c_1} - \frac{8k^5\beta\operatorname{sech}^2(kx)}{3c_1}\right.\\
& \left. + \frac{32k^5\beta}{c_1}\operatorname{sech}^2(kx)\tanh^2(kx) + \frac{4k^3\beta^2}{3c_1}\operatorname{sech}^2(kx)\right]\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \cdots,
\end{aligned} \tag{54}$$

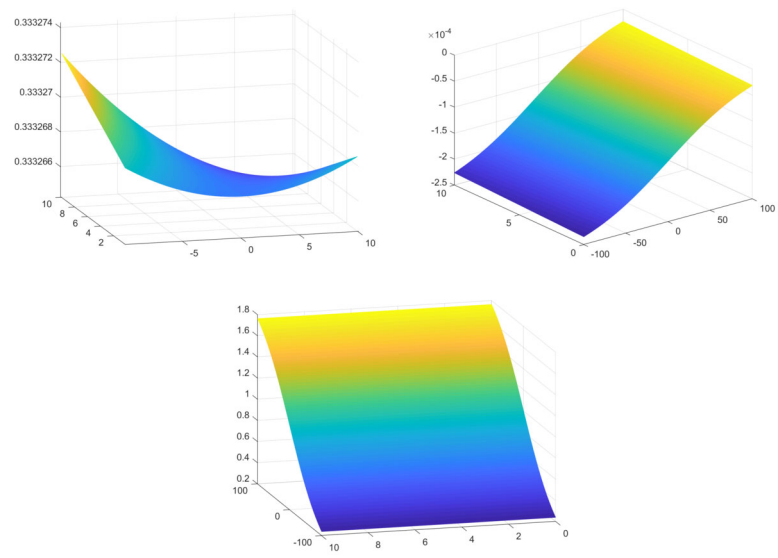

Figure 2 The graph of $u(x,t)$, $v(x,t)$, *and*$w(x,t)$ when $\alpha = c_1 = c_0 = 1$, $k = 0.01$ and $\beta = 1$

$$
\begin{aligned}
w(x,t) = {} & c_0 + c_1 \tanh(kx) + \big(-96k^7\beta^2 \operatorname{sech}^4(kx)\tanh(kx) \\
& + 144k^7\beta^2 \operatorname{sech}^6(kx)\tanh(kx)\big)\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(1+\alpha)\Gamma(1+\alpha)}t^{3\alpha} \\
& + \big(16k^6\beta \operatorname{sech}^3(kx) - 24k^6 \operatorname{sech}^4(kx) + 72k^6 \operatorname{sech}^4(kx)\tanh^2(kx) \\
& - 8c_1k^4\beta \operatorname{sech}^2(kx)\tanh(kx) - 24c_1k^4\beta \operatorname{sech}^4(kx)\tanh(kx) \\
& - 8\beta^2k^4 \operatorname{sech}^2(kx) + 12\beta^2k^4 \operatorname{sech}^4(kx)\big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
& \times \big[-8k^5\tanh(kx) + 8c_1k^3 \operatorname{sech}^4(kx) - 4c_1k^3 \operatorname{sech}^2(kx) + k^3\beta\tanh(kx) \\
& - 24k^3\tanh(kx)\operatorname{sech}^2(kx)\big]\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \cdots . \qquad (55)
\end{aligned}
$$

Figure 2 shows the analytic solution of $u(x,t)$, $v(x,t)$, and $w(x,t)$ where $\alpha = c_1 = c_0 = 1$, $k = 0.01$, and $\beta = 1$. We can see the exact solution and the approximate solution of the Elzaki iterative method for the case of $\alpha = 1$ from Table 1.

Remark The reasons for the complexity of the solution are as follows:
1. Selection of initial values; 2. More parameters.

4 Conclusion

In this article, we use the Elzaki transform with an iterative method to solve fractional partial differential equations. We find that the results using the homotopy perturbation method and the method in this article to the Klein–Gordon problem are the same. We see that the errors were not significant by picking specific values. Therefore, employing the Elzaki transform and the iterative method to solve fractional partial differential equations is effective.

Table 1 Approximate solutions, exact solutions, and error estimates for the Elzaki iterative transformation method when α is 1 for Example 2

t	x	E.S			A.S			E.E		
		u	v	w	u	v	w	u	v	w
0.1	−10	0.33	−1.47E−04	0.901	0.33	−1.47E−04	0.9	3.13E−05	4.39E−09	9.68E−04
	0	0.33	−1.33E−04	1.000	0.33	−1.33E−04	1	3.33E−05	1.33E−08	1.00E−03
	10	0.33	−1.99E−04	1.100	0.33	−1.20E−04	1.1	3.14E−05	−7.93E−05	1.03E−03
5	−10	0.33	−1.40E−04	0.950	0.33	−1.40E−04	0.9	−3.55E−06	−4.41E−09	4.97E−02
	0	0.33	−1.27E−04	1.050	0.33	−1.27E−04	1	1.00E−04	−4.80E−09	5.00E−02
	10	0.33	−1.13E−04	1.140	0.33	−1.13E−04	1.1	6.04E−05	−7.05E−08	4.92E−02
50	−10	0.33	−1.33E−04	1.000	0.33	−1.33E−04	0.9	−5.97E−06	2.35E−08	9.97E−02
	0	0.33	−1.20E−04	1.100	0.33	1.20E−04	1	1.99E−06	−2.40E−04	9.97E−02
	10	0.33	−1.07E−04	1.197	0.33	−1.07E−04	1	9.79E−06	−2.38E−07	9.78E−02

Source: The above data were obtained from the authors through matlab calculations, excel summaries and written in latex.
E.S denotes the exact solution.
A.S denotes the approximate solution.
E.E indicates the error estimate.

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Abbreviations
ES, denotes the exact solution; AS, denotes the approximate solution; EE, denotes the error estimate.

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Declarations

Ethics approval and consent to participate
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