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The long-time behavior of solitary waves for the weakly damped KdV equation

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Abstract

In this paper, we first introduce the long-time behavior stability of solitary waves for the weakly damped Korteweg–de Vries equation. More concretely, solutions of the dissipative system with the initial values near a c_0-speed solitary wave, are approximated by a long curve on the family of solitary waves with the time-varying speed $|c(t) - c_0|$ being small, in the long-time period (i.e., $0 \le t \le O(\frac{1}{\epsilon^{\tau}})$). Meanwhile, the approximation difference in a suitably weighted space $H_a^1(\mathbb{R})$ is of the order of the damping coefficient and of some kind of exponential weight form. As a comparison, we also study the long-time behavior stability, i.e., for $0 \le t < +\infty$, the solutions are approximated by a long curve on the family of solitary waves with the exponential decay speed $c(t) = c_0 e^{-\beta t}$ $(0 < \beta \le 1)$, when the initial values are near a c_0-speed solitary wave. However, here, the approximation difference merely defined in $H^1(\mathbb{R})$ depends on the damping coefficient ϵ and the exponential decay coefficient β.

MSC: 35Q53; 35B35

Keywords: Perturbed KdV equation; Solitary waves; Stability

1 Introduction

This work mainly considers the long-time and long-time behavior stability for the weakly damped one-dimensional Korteweg–de Vries (KdV) equation

$$\begin{cases} u_t = -\partial_x [u_{xx} + \frac{1}{2} u^2] - \epsilon u, & t > 0, x \in \mathbb{R}, \\ u(x,t) = u(x,0), & t = 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a small damping parameter.

The authors in [1] first derived the KdV equation as a model for planar, unidirectional waves propagating in shallow water in 1895. Then, the authors in [2, 3] considered the KdV equation to feature wave motion for many other physical situations. Meanwhile, the initial value problems were studied in [4, 5] for the undamped KdV equation (i.e., $\epsilon = 0$) and in [6–8] for the damped case (i.e., $\epsilon \neq 0$). They showed that, in both cases, the solution $u(x,t)$ of the initial problem satisfies, for $\forall t > 0$, $u \in C([0,t], H^2) \cap C^1([0,t], H^{-1})$ and $e^{ax} u \in C([0,t], H^1) \cap C^1([0,t], H^{-3})$.

To give a more explicit picture, we describe the undamped and damped KdV equations separately.

Undamped Case: If $\epsilon = 0$ in equation (1.1), one can define the Hamiltonian

$$\mathcal{H}(u) = \int_{\mathbb{R}} \frac{1}{2}|u_x|^2 - \frac{1}{6}u^3\,dx, \tag{1.2}$$

and the impulse functional (see [9, 10])

$$\mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2\,dx. \tag{1.3}$$

Obviously, the profiles of traveling-wave solutions of the KdV equation are critical points of the Hamiltonian $\mathcal{H}$ for fixed values of $\mathcal{I}$, namely, relative equilibria (see [4]). The family of all traveling-wave profiles is called the manifold of relative equilibrium (MRE), which is the two-dimensional manifold of the form $u(x,t) = u_c(x - ct + \gamma)$ for all $c > 0$, $\gamma \in \mathbb{R}$. In addition, the profile of the solitary wave conforms to $u_c(y) \to 0$ as $|y| \to \infty$, i.e.,

$$u_c(y) = \alpha \operatorname{sech}^2 \varsigma y \quad \text{with } \alpha = 3c, \varsigma = \frac{1}{2}\sqrt{c}, \tag{1.4}$$

which uniquely (up to the space translations) satisfies the equation (see [11])

$$-\partial_y^2 u_c + cu_c - \frac{1}{2}u_c^2 = 0. \tag{1.5}$$

A solitary wave has a permanent phase shift or a different speed when a solitary wave acquires a small perturbation. Therefore, the orbital stability of solitary waves was introduced in [12–14]. Weinstein in [15, 16] and Bona, Souganidis, and Strauss in [17, 18] asserted that a solution that is initially close to a solitary wave $u_c(x - ct)$ in the Sobolev space $H^1(\mathbb{R})$, will forever remain close to the set of translates $u_c(x - ct + \gamma)$ of the wave. More precisely, for sufficiently small $\delta > 0$, one has

$$\inf_{\gamma} \left\| u(\cdot,t) - u_c(\cdot + \gamma) \right\|_{H^1} \le \delta, \quad \forall t > 0, \tag{1.6}$$

if the same quantity is small at the initial time $t = 0$.

In particular, Pego and Weistein showed the asymptotic stability of the traveling wave in [9] that if $u(x,t)$ is initially a small perturbation in the weighted norms space $H^2(\mathbb{R}) \cap H_a^1(\mathbb{R})$ of a given solitary wave $u_c(x - ct + \gamma)$, then

$$\left\| u(x,t) - u_{c_+}(x - c_+t + \gamma_+) \right\|_{H^2(\mathbb{R})\cap H_a^1(\mathbb{R})} \to 0 \quad \text{as } t \to +\infty, \tag{1.7}$$

for some c_+ near c and γ_+ near γ. Here, the exponential weights are of the form e^{ay} ($a > 0$) as follows:

$$L_a^2 = \left\{ v | e^{ay} v \in L^2(\mathbb{R}) \right\} \quad \text{with } \|v\|_{L_a^2} = \left\| e^{ay} v \right\|_{L^2}, \tag{1.8}$$

$$H_a^1 = \left\{ v | e^{ay} v \in H^1(\mathbb{R}) \right\} \quad \text{with } \|v\|_{H_a^1} = \left\| e^{ay} v \right\|_{H^1}. \tag{1.9}$$

Damped Case: If $\epsilon \neq 0$ in equation (1.1), one can deduce

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(u) &= \langle \mathcal{I}'(u), \partial_x \mathcal{H}(u) - \epsilon u \rangle = \langle \mathcal{I}'(u), \partial_x \mathcal{H}(u) \rangle - \langle \mathcal{I}'(u), \epsilon u \rangle \\ &= -\epsilon \int_{\mathbb{R}} u^2 \, dx = -2\epsilon \mathcal{I}(u),\end{aligned} \tag{1.10}$$

where $\partial_x \mathcal{H}(u) = -\partial_x[u_{xx} + \frac{1}{2}u^2]$. Clearly, $\mathcal{I}(u(t)) = \mathcal{I}(u(0))e^{-2\epsilon t}$. This implies that $\lim_{t\to+\infty} \mathcal{I}(u(t)) = 0$ and $\lim_{t\to+\infty} u(t,x) = 0$ almost everywhere in $\mathbb{R}$.

The authors in [19, 20] used the symmetry group to reduce the energy momentum and then obtained the stability of relative equilibria. In [21, 22], the authors analyzed the spectrum property of the self-adjoint operator generated by an energy functional, and then they found sharp conditions for the stability and instability of solitary waves or multisolitons. Specifically, Derks and Groesen in [23] considered the damped KdV equation in the bounded periodic domain $x \in [0, 2\pi]$. By applying the implicit theorem, they constructed an energy-decaying manifold $\overline{M}_\epsilon \sim O(e^{-2\varepsilon t})$, which is related to the damping coefficient ϵ, and then they obtained the long-time behavior stability of solutions near the constructing manifold $\overline{M}_\epsilon$, where the approximation difference is $O(\epsilon e^{-2\varepsilon t})$.

Here, inspired by the ideas about the spectral analysis given in [9] and the construction of the energy-decaying manifold given in [23], we study the long-time and long-time behavior, respectively, for the weakly damped equation (1.1) in the whole space $x \in \mathbb{R}$. Our first result is about the long-time behavior:

Theorem 1.1 *Let $u_c(x - ct + \gamma)$, $c > 0$, $\gamma \in \mathbb{R}$, be the solitary-wave solutions of the undamped KdV equation* (1.1) (*namely* $\epsilon = 0$). *Then, considering the initial problem for the weakly damped* $(0 < \epsilon \ll 1)$ *KdV equation* (1.1) *with data*

$$u(x,0) = u_{c_0}(x + \gamma_0) + v_0(x), \tag{1.11}$$

if the perturbation $v_0 \in H^2 \cap H_a^1$ with $\|v_0\|_{H^1} + \|v_0\|_{H_a^1} < \epsilon$, then for $0 \le t \le T(= O(\frac{1}{\epsilon^\tau}))$, we have

$$\begin{aligned}&\|u(\cdot,t) - e^{-\epsilon t}u_c(\cdot - ct + \gamma)\|_{H^1} \le C\epsilon^{1-2\tau}, \\ &\|u(\cdot + ct - \gamma, t) - e^{-\epsilon t}u_c\|_{H_a^1} \le C\epsilon e^{-\epsilon t}, \\ &\|u(\cdot + ct - \gamma, t) - e^{-\epsilon t}u_c\|_{H_a^1} \le C\epsilon^{1-\tau} e^{-(\epsilon^\tau + \epsilon)\cdot t}, \\ &|c(t) - c_0| \le C\epsilon^{1-2\tau} \quad \text{and} \quad |\gamma(t) - \gamma_0| \le C\epsilon^{1-2\tau},\end{aligned} \tag{1.12}$$

where $0 < a < \sqrt{\frac{c}{3}}$, $\tau < \frac{1}{2}$, C are constants.

Remark 1.1 1. The restriction $0 < a < \sqrt{\frac{c}{3}}$ is imposed in Theorem 1.1 since the expression $a(c - a^2)$ is maximized at $a = \sqrt{\frac{c}{3}}$ (see Proposition 2.5 in Ref. [9]).

2. It is natural to expect the solution to approximate the initial solitary wave as long as possible if the initial value has a slight perturbation. However, (1.10) implies that all solutions will vanish as $t \to +\infty$. Hence, it is valid to consider the stability near the initial solitary wave in the long-time period $0 \le t \le T = O(\frac{1}{\epsilon^\tau})$ satisfying that $0 < \epsilon \ll 1$, $\tau < \frac{1}{2}$,

where $T = O(\frac{1}{\epsilon^\tau})$ means the same order $T \approx \frac{1}{\epsilon^\tau}$ and the restraint on the quantity $\frac{1}{\epsilon^\tau}$ follows from (2.35).

3. To analyze the property of the damping condition and solitary wave, the solution to equation (1.1) will be formally expressed in the form

$$u(x,t) = e^{-\epsilon t} \cdot u_{c(t)}\big(x + \theta(t)\big) + v\big(x + \theta(t), t\big), \tag{1.13}$$

where $\theta(t) = \gamma(t) - \int_0^t c(s)\,ds$ and the leading (dominant) term $u_{c(t)}(x + \theta(t))$ is an exact solitary-wave solution of (1.1) with $\epsilon = 0$, when $c(t)$, $\gamma(t)$ are just near the initial c_0, γ_0.

4. Substitution of (1.13) into (1.1) yields an equation of the form

$$\partial_t v = \partial_y L_{c(t)} v - \epsilon v - (\dot{c}\partial_c + \dot{\gamma}\partial_y) u_{c(t)} + \Im(u_{c(t)}, v), \tag{1.14}$$

where $\Im(u_{c(t)}, v)$ will be given in (2.6) and

$$L_c = -\partial_y^2 + c - u_c. \tag{1.15}$$

Meanwhile, differentiating (1.5) with respect to y and c, we know that the operator $\partial_y L_c$ in L^2 is degenerate, i.e.,

$$\partial_y L_c \partial_y u_c = 0, \qquad \partial_y L_c \partial_c u_c = -\partial_y u_c. \tag{1.16}$$

These give rise to solutions $\partial_y u_c$ and $\partial_c u_c - t\partial_y u_c$ to the linearized problem

$$\partial_t v = \partial_y L_c v. \tag{1.17}$$

5. As in References [16, 24], to obtain more exponential decay, it is appropriate to require that the right-hand side of (1.14) is orthogonal to the 2-dimensional generalized kernel of the adjoint of $\partial_y L_c$. These constraints yield two coupled first-order differential equations for $c(t)$ and $\gamma(t)$ (called modulation equations), which are coupled to the infinite-dimensional dispersive evolution equation for $v(\cdot, t)$.

Next, we discuss the long-time behavior stability of solutions. In contrast to the restriction $c(t)$ near c_0 given in Theorem 1.1, we need that $c(t)$ decays exponentially to zero as $t \to +\infty$. This is presented as follows:

Theorem 1.2 *Let $u_{c(t)}(y)$, $y = x - \int_0^t c(s)\,ds + \gamma(t)$, be the solitary-wave solutions with $c(t) = c_0 e^{-\beta t} (0 < \beta \leq 1)$, of the undamped KdV equation (1.1) (namely, $\epsilon = 0$). Then, considering the initial problem for the weakly damped $(0 < \epsilon \ll 1)$ KdV equation (1.1) with data*

$$u(x,0) = u_{c_0}(x + \gamma_0) + v_0(x), \tag{1.18}$$

if the perturbation $v_0 \in H^2 \cap H_a^1$ with $\|v_0\|_{H^1} + \|v_0\|_{H_a^1} < \epsilon$, then for $0 \leq t < +\infty$, we have

$$\big\| u(\cdot, t) - e^{-\epsilon t} u_c(\cdot - ct + \gamma)\big\|_{H^1} \leq C\big(\epsilon + m(\epsilon, \beta, t)\big) e^{-\epsilon t}, \tag{1.19}$$

where C is a constant and $m(\epsilon,\beta,t)$ depends on ϵ, β, and t such that

$$m(\epsilon,\beta,t)=\begin{cases}O(\epsilon\sqrt{t}), & 0\le t\le 1,\\ O(\frac{\epsilon}{\beta\sqrt{t}}), & 1\le t<+\infty.\end{cases} \tag{1.20}$$

Remark 1.2 Note that here it is impossible to consider the long-time stability of solutions in weight space H_a^1 as in Theorem 1.1, since $a\to 0$ as $t\to +\infty$ follows from $a<\sqrt{\frac{c(t)}{3}}$ and $c(t)=c_0e^{-\beta t}(0<\beta\le 1)$.

Remark 1.3 The approximation exponent given in (1.12) of Theorem 1.1 and (1.19) of Theorem 1.2 strictly depends on the damping coefficient ϵ. This is in sharp contrast to the asymptotic stability (1.7) with the exponent weight $e^{-a(c-a^2)t}$ of decay given in Reference [9]. In other words, the weakly damped term will dominate the exponential decay rate.

The rest of this paper is organized as follows: In Sect. 2, we justify the representation (1.13) of the solution for nonlinear equations, and derive the equation of motion of the new variables $(c(t),\gamma(t),w(y,t))$. Moreover, we study the long-time behavior to finish the proof of Theorem 1.1. In Sect. 3, we also justify the new representation (3.1) and prove Theorem 1.2 for the long-time behavior stability. In the Appendix, we review the spectral analysis and certain smoothing and exponential decay estimates of the linearized operator $\partial_y L_c$ in (1.14).

2 The long-time behavior stability

2.1 Decomposition of the solution

Due to the weak damping term, we use time-dependent tubular coordinates in a neighborhood of solitary waves and skillfully represent solutions of the initial value problem (1.1) in the form (see also (1.13))

$$u(x,t)=e^{-\epsilon t}u_{c(t)}(y)+v(y,t), \tag{2.1}$$

where

$$y=y(x,t)=x-\int_0^t c(s)\,ds+\gamma(t) \tag{2.2}$$

and $u_{c(t)}(y)$ belongs to the family of traveling waves.

In order to achieve exponential decay for the perturbation $v(y,t)$ in the weighted space H_a^1, we wish to impose the constraint that

$$w(y,t)=e^{ay}v(y,t)\in \operatorname{range}(Q)=\ker(P), \tag{2.3}$$

where the projections P, Q are given in Proposition A.2 (see the Appendix). This requirement corresponds to the two scalar constraints $\langle w,\eta_k\rangle=0$, $k=1,2$, cf. (A.14), which follows the modulation equations, namely, two coupled first-order differential equations for $c(t)$, $\gamma(t)$ as $t>0$.

As this point, let us begin the proof of Theorem 1.1.

The solution $u(x,t)$ of the initial problem (1.1) satisfies, for $\forall t > 0$,

$$u \in C\big([0,t],H^2\big) \cap C^1\big([0,t],H^{-1}\big), \qquad e^{ax}u \in C\big([0,t],H^1\big) \cap C^1\big([0,t],H^{-3}\big). \tag{2.4}$$

Moreover, u is a classical solution of (1.1) for $t > 0$. Given the initial data in (1.11), if the perturbation $\|v_0\|_{H_a^1}$ is sufficiently small, it is easy to prove decomposition (2.1) exists in [0,t], with $(\gamma, c) \in C^1([0,t],\mathbb{R}^2)$.

We now derive evolution equations for $\gamma(t)$, $c(t)$, and $v(y,t)$ that are valid pointwise for $t > 0$. Substituting (2.1) into (1.1), we have

$$\begin{aligned}
0 &= \partial_t u + \partial_x^3 u + \partial_x\left(\frac{1}{2}u^2\right) + \epsilon u \\
&= \left[\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\right]\big(e^{-\epsilon t}u_{c(t)}(y) + v\big) \\
&\quad + \partial_y\left[\frac{1}{2}\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big)^2\right] + \epsilon\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big) \\
&= \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)e^{-\epsilon t}u_{c(t)}(y) \\
&\quad + \partial_y\left[\frac{1}{2}\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big)^2\right] + \epsilon\big[e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big] \\
&= \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)e^{-\epsilon t}u_{c(t)}(y) \\
&\quad + \partial_y\left[\frac{1}{2}\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big)^2\right] + \epsilon e^{-\epsilon t}u_{c(t)}(y) + \epsilon v(y,t) \\
&= \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + \left\{\dot{\gamma}\,\partial_y e^{-\epsilon t}u_{c(t)}(y) - \epsilon e^{-\epsilon t}u_{c(t)}(y) + e^{-\epsilon t}\frac{\partial u}{\partial c}\dot{c} + e^{-\epsilon t}\partial_t u_{c(t)}(y)\right. \\
&\quad \left. + \partial_y^3 e^{-\epsilon t}u_{c(t)}(y)\right\} + \partial_y\left[\frac{1}{2}\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big)^2\right] + \epsilon e^{-\epsilon t}u_{c(t)}(y) + \epsilon v(y,t) \\
&= \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + \dot{\gamma}\,\partial_y e^{-\epsilon t}u_{c(t)}(y) + e^{-\epsilon t}\frac{\partial u}{\partial c}\dot{c} \\
&\quad + e^{-\epsilon t}\big(\partial_t u_{c(t)}(y) + \partial_y^3 u_{c(t)}(y)\big) + \partial_y\left[\frac{1}{2}\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big)^2\right] + \epsilon v(y,t) \\
&= \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + \dot{\gamma}\,\partial_y e^{-\epsilon t}u_{c(t)}(y) + e^{-\epsilon t}\frac{\partial u}{\partial c}\dot{c} \\
&\quad + e^{-\epsilon t}\partial_y\big(-c(t)u_{c(t)}(y) + \partial_y^2 u_{c(t)}(y)\big) + \partial_y\left[\frac{1}{2}\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big)^2\right] + \epsilon v(y,t) \\
&= \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + \epsilon v(y,t) + \dot{\gamma}\,\partial_y e^{-\epsilon t}u_{c(t)}(y) + e^{-\epsilon t}\frac{\partial u}{\partial c}\dot{c} \\
&\quad + e^{-\epsilon t}\left(-\frac{1}{2}\partial_y\big(u_{c(t)}(y)\big)^2\right) + \partial_y\left[\frac{1}{2}\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big)^2\right] \\
&= \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + \epsilon v(y,t) + \partial_y(u_{c_0}v) + \dot{\gamma}\,\partial_y e^{-\epsilon t}u_{c(t)}(y) + e^{-\epsilon t}\frac{\partial u}{\partial c}\dot{c} \\
&\quad + e^{-\epsilon t}\left(-\frac{1}{2}\partial_y\big(u_{c(t)}(y)\big)^2\right) + \partial_y\left[\frac{1}{2}\big(e^{-\epsilon t}u_{c(t)}(y) + v(y,t)\big)^2\right] - \partial_y(u_{c_0}v) \\
&= \big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + \epsilon v(y,t) + \partial_y(u_{c_0}v) + \dot{\gamma}\,\partial_y e^{-\epsilon t}u_{c(t)}(y) + e^{-\epsilon t}\frac{\partial u}{\partial c}\dot{c}
\end{aligned}$$

$$+\,\partial_y\left[\frac{1}{2}e^{-2\epsilon t}u^2 + e^{-\epsilon t}uv + \frac{1}{2}v^2 - e^{-\epsilon t}\frac{1}{2}u^2 - u_{c_0}v\right]. \tag{2.5}$$

Hence,

$$\begin{aligned}\partial_t v &= \partial_y\left[-\partial_y^2 + c_0 - u_{c_0}\right]v - \epsilon v - e^{-\epsilon t}\left[\dot\gamma\,\partial_y u + \dot c\frac{\partial u}{\partial c}\right]\\&\quad - \partial_y\left[\left(\dot\gamma - c(t) + c_0\right)v\right] - \partial_y\left[\frac{1}{2}e^{-2\epsilon t}u^2 + e^{-\epsilon t}uv + \frac{1}{2}v^2 - e^{-\epsilon t}\frac{1}{2}u^2 - u_{c_0}v\right].\end{aligned} \tag{2.6}$$

Now, $w(y,t) = e^{ay}v(y,t)$ satisfies (and set $A_a = e^{ay}\partial_y L_{c_0}e^{-ay}$ with $L_{c_0} = -\partial_y^2 + c_0 - u_{c_0}$)

$$\partial_t w = A_a w - \epsilon w + \mathfrak{F}, \tag{2.7}$$

where we write

$$\begin{aligned}\mathfrak{F} &= -e^{-\epsilon t}e^{ay}(\dot c\partial_c + \dot\gamma\,\partial_y)u_{c(t)} - \dot\gamma\, e^{ay}\partial_y e^{-ay}w + \mathcal{F},\\ \mathcal{F} &= e^{ay}\partial_y\left(c(t) - c_0\right)e^{-ay}w - e^{ay}\partial_y\left[\frac{1}{2}e^{-2\epsilon t}u^2 - e^{-\epsilon t}\frac{1}{2}u^2\right]\\&\quad - e^{ay}\partial_y\left[e^{-\epsilon t}uv + \frac{1}{2}v^2 - u_{c_0}v\right]\\&= e^{ay}\partial_y\left(c(t) - c_0\right)e^{-ay}w - e^{ay}\partial_y\left[\frac{1}{2}e^{-2\epsilon t}u^2 - e^{-\epsilon t}\frac{1}{2}u^2\right]\\&\quad - e^{ay}\partial_y\left[e^{-\epsilon t}uv + \frac{1}{2}v^2 - e^{-\epsilon t}u_{c_0}v + \left(e^{-\epsilon t} - 1\right)u_{c_0}v\right].\end{aligned} \tag{2.8}$$

Meanwhile, (2.4) implies that this equation is initially justified in $C([0,t],H^{-3})$, but also holds in $C([0,t],L^2)$ and moreover is pointwise. The constraint $w \in \operatorname{range}(Q)$ in (2.3) now yields the following system of evolution equations for (w,γ,c):

$$\partial_t w = A_a w - \epsilon w + Q\mathfrak{F}, \quad P\mathfrak{F} = 0. \tag{2.9}$$

Written as an integral equation, the initial value problem for (2.9) becomes:

$$w(t) = e^{(A_a-\epsilon)t}w(0) + \int_0^t e^{(A_a-\epsilon)(t-s)}Q\mathfrak{F}(s)\,ds. \tag{2.10}$$

The equation $P\mathfrak{F} = 0$ yields equations for $\dot\gamma$, $\dot c$ as follows. Introduce the notation

$$\begin{aligned}e_1(y,t) &= e^{ay}\left(\partial_y u_{c(t)}(y) - \partial_y u_{c_0}(y)\right),\\ e_2(y,t) &= e^{ay}\left(\partial_c u_{c(t)}(y) - \partial_c u_{c_0}(y)\right),\end{aligned} \tag{2.11}$$

and note that $\langle e^{ay}\partial_y e^{-ay}w, \eta_k\rangle = -\langle v, \partial_y\tilde\eta_k\rangle$ for $k = 1,2$, by integration by parts. Then, by (A.14), the condition $P\mathfrak{F} = 0$ is equivalent to

$$0 = \left\langle\dot\gamma\left[e^{-\epsilon t}(\xi_1 + e_1) + (\partial_y - a)w\right] + \dot c e^{-\epsilon t}(\xi_2 + e_2) - \mathcal{F}, \eta_k\right\rangle, \quad k = 1,2. \tag{2.12}$$

Using the biorthogonality relation $\langle \xi_j, \eta_k \rangle = \delta_{jk}$, we obtain a system of equations for $\gamma(t)$ and $c(t)$:

$$\mathfrak{A}(t)\begin{pmatrix} \dot{\gamma} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \langle \mathcal{F}, \eta_1 \rangle \\ \langle \mathcal{F}, \eta_2 \rangle \end{pmatrix} \tag{2.13}$$

and

$$\mathfrak{A}(t) = \begin{pmatrix} e^{-\epsilon t} + e^{-\epsilon t}\langle e_1, \eta_1 \rangle - e^{-\epsilon t}\langle v, \partial_y \tilde{\eta}_1 \rangle, e^{-\epsilon t}\langle e_2, \eta_1 \rangle \\ e^{-\epsilon t}\langle e_1, \eta_2 \rangle - e^{-\epsilon t}\langle v, \partial_y \tilde{\eta}_2 \rangle, e^{-\epsilon t} + e^{-\epsilon t}\langle e_2, \eta_2 \rangle \end{pmatrix}. \tag{2.14}$$

The matrix $\mathfrak{A}(t)$ satisfies

$$\mathfrak{A}(t) = e^{-\epsilon t} I + O\big(\big|c(t) - c_0\big| + \|v\|_{L^2}\big) \quad \text{as } \big|c(t) - c_0\big| + \|v\|_{L^2} \to 0. \tag{2.15}$$

In order to obtain reversibility of the matrix $\mathfrak{A}(t)$, in some sense, we need the term $e^{-\epsilon t} I \approx I$. In other words, it is possible to consider stability in the long-time period $0 \le t \le T = O(\frac{1}{\epsilon^\tau})$ (given in (2.35)) instead of the long time "$t \to +\infty$". Otherwise, $e^{-\epsilon t} I \to 0$ as $t \to +\infty$.

2.2 The long-time behavior

In order to complete the proof of Theorem 1.1. It remains to establish the priori estimates from the evolution equations in (2.10)–(2.13). We have

Proposition 2.1 *There exist* $\delta_* > 0$, $\epsilon_0 > 0$, $C > 0$ *such that, if the decomposition* (2.10), (2.11), *and* (2.12) *exists for* $0 \le t \le T = O(\frac{1}{\epsilon^\tau})$ *with* $0 < \epsilon \ll 1, \tau < \frac{1}{2}$ *and satisfies*

$$e^{\epsilon t}\big\|w(t)\big\|_{H^1} + \big|c(t) - c_0\big| + \big\|v(\cdot, t)\big\|_{H^1} \le \delta_*, \quad 0 \le t \le T = O\bigg(\frac{1}{\epsilon^\tau}\bigg), \tag{2.16}$$

and if the perturbation $\|v_0\|_{H^1} + \|v_0\|_{H^1_a} < \epsilon < \epsilon_0$ *in* (1.11), *then*

$$\begin{aligned} &e^{\epsilon t}\big\|w(t)\big\|_{H^1} \le C\epsilon, \quad 0 \le t \le T = O\bigg(\frac{1}{\epsilon^\tau}\bigg), \\ &e^{(\epsilon^\tau + \epsilon)t}\big\|w(t)\big\|_{H^1} \le C\epsilon^{1-\tau}, \quad 0 \le t \le T = O\bigg(\frac{1}{\epsilon^\tau}\bigg), \\ &\big|c(t) - c_0\big| \le C\epsilon^{1-2\tau}, \quad 0 \le t \le T = O\bigg(\frac{1}{\epsilon^\tau}\bigg), \\ &\big|\gamma(t) - \gamma_0\big| \le C\epsilon^{1-2\tau}, \quad 0 \le t \le T = O\bigg(\frac{1}{\epsilon^\tau}\bigg), \\ &\big\|v(\cdot, t)\big\|_{H^1} \le C\epsilon^{1-2\tau}, \quad 0 \le t \le T = O\bigg(\frac{1}{\epsilon^\tau}\bigg). \end{aligned} \tag{2.17}$$

Proof The proof follows the two stages as given in Proposition 4.1 in Ref. [9] but with different detailed estimates.

(i) Local energy-decay estimate: Estimates of the weighted perturbation, $w(y,t) = e^{ay}v(y,t)$, in H^1, via the integral equation (2.10), the modulation equation (2.13), and the linear semigroup estimates of Lemma A.2 (see the Appendix).

If δ_* is sufficiently small and $0 \le t \le T = O(\frac{1}{\epsilon^\tau})$, then $\mathfrak{A}(t)$ defined in (2.13) has a bounded inverse, so we may estimate (2.13) to find

$$|\dot{\gamma}| + |\dot{c}| \le C\|\mathcal{F}\|_{L^2}. \tag{2.18}$$

From (2.8), using that $e^{ay}\partial_y e^{-ay} = \partial_y - a$ and the expression (1.4) (or the following estimate (3.19)), we obtain the estimates

$$\begin{aligned}
&\|\mathfrak{F}\| \le C\big(|\dot{\gamma}|\big(1+\|w\|_{H^1}\big)\big) + |\dot{c}| + \|\mathcal{F}\|_{L^2} \le C\big(1+\|w\|_{H^1}\big)\|\mathcal{F}\|_{L^2},\\
&\|\mathcal{F}\|_{L^2} \le C\big[\big(\big|c(t)-c_0\big| + \|v\|_{H^1} + \big(1-e^{-\epsilon t}\big)\big)\|w\|_{H_1} + \big(e^{-\epsilon t} - e^{-2\epsilon t}\big)\big]\\
&\qquad \le C\big(\delta_* + \big(1-e^{-\epsilon t}\big)\big)\|w\|_{H^1} + C\big(e^{-\epsilon t} - e^{-2\epsilon t}\big).
\end{aligned} \tag{2.19}$$

Now, we may choose b, b' with $b + \epsilon < b' + \epsilon < a(c - a^2) + \epsilon$, such that b, b', satisfies the condition of Lemma A.2. We may then estimate (2.10) as follows, for $t > 0$:

$$\begin{aligned}
&\big\|w(t)\big\|_{H^1}\\
&\quad \le Ce^{-(b'+\epsilon)t}\big\|w(0)\big\|_{H^1} + C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}\|\mathcal{F}\|_{L^2}\,ds\\
&\quad \le Ce^{-(b'+\epsilon)t}\big\|w(0)\big\|_{H^1} + C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}(1+\delta_*)\\
&\qquad \times \big[\big(\delta_* + \big(1-e^{-\epsilon s}\big)\big)\big\|w(s)\big\|_{H^1} + \big(e^{-\epsilon s} - e^{-2\epsilon s}\big)\big]\,ds.
\end{aligned} \tag{2.20}$$

Now, define

$$M_{w,b}(t) = \sup_{0\le s\le t} e^{(b+\epsilon)s}\big\|w(s)\big\|_{H^1}, \tag{2.21}$$

where the variable b is constrained in Remark A.1 (see the Appendix).

Then, multiplying (2.20) by $e^{(b+\epsilon)t}$, we find, for $t > 0$,

$$\begin{aligned}
&e^{(b+\epsilon)t}\big\|w(t)\big\|_{H^1}\\
&\quad \le Ce^{(b+\epsilon)t}e^{-(b'+\epsilon)t}\big\|w(0)\big\|_{H^1}\\
&\qquad + C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}(1+\delta_*)\big(\delta_* + \big(1-e^{-\epsilon s}\big)\big)e^{(b+\epsilon)t}\big\|w(s)\big\|_{H^1}\,ds\\
&\qquad + C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}e^{(b+\epsilon)t}\big(e^{-\epsilon s} - e^{-2\epsilon s}\big)\,ds\\
&\quad \le Ce^{-(b'-b)t}\big\|w(0)\big\|_{H^1}\\
&\qquad + C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}(1+\delta_*)\big(\delta_* + \big(1-e^{-\epsilon s}\big)\big)e^{(b+\epsilon)(t-s)}e^{(b+\epsilon)s}\big\|w(s)\big\|_{H^1}\,ds\\
&\qquad + C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}e^{(b+\epsilon)t}\big(e^{-\epsilon s} - e^{-2\epsilon s}\big)\,ds\\
&\quad \le C\big\|w(0)\big\|_{H^1} + C\delta_* M_{w,b}(t)\int_0^t (t-s)^{-1/2}e^{-(b'-b))(t-s)}\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}\big(1-e^{-\epsilon s}\big)e^{(b+\epsilon)(t-s)}e^{(b+\epsilon)s}\big\|w(s)\big\|_{H^1}\,ds \\
&+ C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}e^{(b+\epsilon)t}\big(e^{-\epsilon s}-e^{-2\epsilon s}\big)\,ds \\
\le{}& C\big\|w(0)\big\|_{H^1} + C\delta_* M_{w,b}(t)\int_0^t (t-s)^{-1/2}e^{-(b'-b))(t-s)}\,ds \\
&+ CM_{w,b}(t)\int_0^t (t-s)^{-1/2}e^{-(b'-b)(t-s)}\big(1-e^{-\epsilon s}\big)\,ds \\
&+ C\int_0^t (t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}e^{(b+\epsilon)t}\big(e^{-\epsilon s}-e^{-2\epsilon s}\big)\,ds.
\end{aligned}
\tag{2.22}
$$

It is sufficient to estimate the terms $\mathcal{A}(\epsilon)\triangleq\int_0^t(t-s)^{-1/2}e^{-(b'-b)(t-s)}(1-e^{-\epsilon s})\,ds$ and $\mathcal{B}_b(\epsilon)\triangleq\int_0^t(t-s)^{-1/2}e^{-(b'+\epsilon)(t-s)}e^{(b+\epsilon)t}(e^{-\epsilon s}-e^{-2\epsilon s})\,ds$ in (2.22) above. We first deal with the latter term $\mathcal{B}_b(\epsilon)$.

$$
\begin{aligned}
\mathcal{B}_b(\epsilon) &= \int_0^t (t-s)^{-\frac12}e^{-(b'+\epsilon)(t-s)}e^{(b+\epsilon)t}\big(e^{-\epsilon s}-e^{-2\epsilon s}\big)\,ds \\
&= \int_0^t (t-s)^{-\frac12}e^{-(b'-b)(t-s)}e^{(b+\epsilon)s}\big(e^{-\epsilon s}-e^{-2\epsilon s}\big)\,ds \\
&= \int_0^t (t-s)^{-\frac12}e^{-(b'-b)(t-s)}e^{bs}\big(1-e^{-\epsilon s}\big)\,ds.
\end{aligned}
\tag{2.23}
$$

If we set $b=0$ in (2.23), $\mathcal{B}_{b=0}(\epsilon)=\mathcal{A}(\epsilon)$. The substitution $t-s=\ell$ yields

$$
\begin{aligned}
\mathcal{B}_{b=0}(\epsilon) = \mathcal{A}(\epsilon) &= \int_0^t (t-s)^{-\frac12}e^{-(b'-b)(t-s)}\big(1-e^{-\epsilon s}\big)\,ds \\
&= \int_0^t (t-s)^{-\frac12}e^{-(b'-0)(t-s)}\big(1-e^{-\epsilon s}\big)\,ds \\
&= \int_0^t (t-s)^{-\frac12}e^{-b'(t-s)}\big(1-e^{-\epsilon s}\big)\,ds \\
&= \int_0^t \ell^{-\frac12}e^{-b'\ell}\big(1-e^{-\epsilon t}e^{\epsilon\ell}\big)\,d\ell.
\end{aligned}
\tag{2.24}
$$

Considering the long-time point $t\triangleq T_0(=O(\frac{\tau}{\epsilon}))$ in the first instance, we have

$$
\begin{aligned}
&\int_0^{T_0} \ell^{-\frac12}e^{-b'\ell}\big(1-e^{-\epsilon t}e^{\epsilon\ell}\big)\,d\ell \\
&\approx \int_0^{\frac{\tau}{\epsilon}} \ell^{-\frac12}e^{-b'\ell}\big(1-e^{-\epsilon\cdot\frac{\tau}{\epsilon}}e^{\epsilon\ell}\big)\,d\ell \\
&= \int_0^{\frac{\tau}{\epsilon}} \ell^{-\frac12}e^{-b'\ell}\big(1-e^{-\tau}e^{\epsilon\ell}\big)\,d\ell \quad \text{fix } s=\epsilon\ell \\
&= \int_0^{\tau} \left(\frac{s}{\epsilon}\right)^{-\frac12}e^{-b'\frac{s}{\epsilon}}\big(1-e^{-\tau}e^{s}\big)\frac{1}{\epsilon}\,ds \\
&= \int_0^{\tau} s^{-\frac12}e^{-\frac{b'}{\epsilon}s}\big(1-e^{-\tau}e^{s}\big)\epsilon^{-\frac12}\,ds.
\end{aligned}
\tag{2.25}
$$

Due to $\lim_{\epsilon\to 0^+} e^{-\frac{b'}{\epsilon}\tau}\epsilon^{-\frac{1}{2}} = 0$, integrating (2.25) by parts, we have

$$\begin{aligned}
0 &\le \lim_{\epsilon\to 0^+}\int_0^\tau s^{-\frac{1}{2}}e^{-\frac{b'}{\epsilon}s}\big(1-e^{-\tau}e^{s}\big)\epsilon^{-\frac{1}{2}}\,ds \\
&\le \lim_{\epsilon\to 0^+} 2\epsilon^{-\frac{1}{2}}\int_0^\tau s^{-\frac{1}{2}}e^{-\frac{b'}{\epsilon}s}\,ds \\
&= \lim_{\epsilon\to 0^+}\epsilon^{-\frac{1}{2}}\left\{s^{\frac{1}{2}}e^{-\frac{b'}{\epsilon}s}|_0^\tau + \frac{b'}{\epsilon}\int_0^\tau s^{\frac{1}{2}}e^{-\frac{b'}{\epsilon}s}\,ds\right\} \\
&= \lim_{\epsilon\to 0^+} b'\epsilon^{-\frac{3}{2}}\int_0^\tau s^{\frac{1}{2}}e^{-\frac{b'}{\epsilon}s}\,ds = 0,
\end{aligned}\tag{2.26}$$

where the last inequality follows from the monotone theorem and the fact that $\lim_{\epsilon\to 0^+} s^{\frac{1}{2}}e^{-\frac{b'}{\epsilon}s}\epsilon^{-\frac{3}{2}} = 0, \forall s\in[0,\tau]$.

After derivation to ϵ of $\mathcal{B}_{b=0}(\epsilon)$ defined in (2.24), we have

$$\mathcal{B}'_{b=0}(\epsilon) = \int_0^\tau s^{-\frac{1}{2}}e^{-\frac{b'}{\epsilon}s}\big(1-e^{-\tau}e^{s}\big)\epsilon^{-\frac{3}{2}}\,ds - \int_0^\tau s^{-\frac{1}{2}}e^{-\frac{b'}{\epsilon}s}\big(1-e^{-\tau}e^{s}\big)\epsilon^{-\frac{5}{2}}\cdot b's\,ds. \tag{2.27}$$

Similarly, as (2.25) and (2.26), it is easy to deduce

$$\lim_{\epsilon\to 0^+}\mathcal{B}'_{b=0}(\epsilon) = 0. \tag{2.28}$$

Therefore, one can deduce

$$\big|\mathcal{B}_{b=0}(\epsilon)\big| \le C(m)\epsilon^m, \quad \forall m\in\mathbb{N}. \tag{2.29}$$

Hence, inserting (2.29) into (2.22), we have for $b = 0$,

$$e^{(b+\epsilon)t}\big\|w(t)\big\|_{H^1} = e^{\epsilon t}\big\|w(t)\big\|_{H^1} \le C\big\|w(0)\big\|_{H^1} + C\big(\delta_* + \epsilon^m\big)M_{w,b=0}(T_0) + C(m)\epsilon^m. \tag{2.30}$$

Taking the supremum over $0\le t\le T_0(= O(\frac{\tau}{\epsilon}))$, we find that if δ_* is sufficient small, then

$$M_{w,b=0}(T_0) = \sup_{0\le t\le T_0} e^{\epsilon t}\big\|w(t)\big\|_{H^1} \le C\big\|w(0)\big\|_{H^1} + C(m)\epsilon^m. \tag{2.31}$$

Next, we estimate $|c(t)-c_0|$. Using (2.18) and (2.31), we find that

$$\begin{aligned}
&\big|c(t)-c_0\big| \\
&\quad\le \big|c(0)-c_0\big| + \int_0^t \big|\dot{c}(s)\big|\,ds \\
&\quad\le \big|c(0)-c_0\big| + \int_0^t C\big[\big(\big|c(t)-c_0\big| + \|v\|_{H^1} + \big(1-e^{-\epsilon s}\big)\big)\|w\|_{H_1} + \big(e^{-\epsilon s}-e^{-2\epsilon s}\big)\big]\,ds \\
&\quad\le \big|c(0)-c_0\big| + C\left(\delta_* + \int_0^t \big(e^{-\epsilon s}-e^{-2\epsilon s}\big)\,ds\right)M_{w,b=0}(t) + \int_0^t \big(e^{-\epsilon s}-e^{-2\epsilon s}\big)\,ds \\
&\quad\le \big|c(0)-c_0\big| + C\left(\delta_* + \frac{e^{-2\epsilon t}-2e^{-\epsilon t}+1}{2\epsilon}\right)M_{w,b=0}(t) + \frac{e^{-2\epsilon t}-2e^{-\epsilon t}+1}{2\epsilon}.
\end{aligned}\tag{2.32}$$

For fixed t, we have

$$\lim_{\epsilon\to 0^+}\frac{e^{-2\epsilon t}-2e^{-\epsilon t}+1}{2\epsilon}=0. \tag{2.33}$$

However, if we consider (2.33) on the long-time point $T_0(=O(\frac{\tau}{\epsilon}))$, we know that

$$\lim_{\epsilon\to 0^+}\frac{e^{-2\epsilon T_0}-2e^{-\epsilon T_0}+1}{2\epsilon}=\lim_{\epsilon\to 0^+}\frac{e^{-2\tau}-2e^{-\tau}+1}{2\epsilon}=\infty. \tag{2.34}$$

To obtain a small estimate $|c(t)-c_0|$, we need to consider the more appropriate long-time point $t\triangleq T(=O(\frac{1}{\epsilon^\tau}))$ (clearly, $< T_0$). Meanwhile, the estimates (2.24)–(2.31) are still valid in the short long-time period $0\le t\le T(=O(\frac{1}{\epsilon^\tau}))$.

By calculating, in the new long-time point $t\triangleq T(=O(\frac{1}{\epsilon^\tau}))$, one can deduce that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_0^T\left(e^{-\epsilon s}-e^{-2\epsilon s}\right)ds &\approx \lim_{\epsilon\to 0^+}\int_0^{\frac{1}{\epsilon^\tau}}\left(e^{-\epsilon s}-e^{-2\epsilon s}\right)ds\\ &=\lim_{\epsilon\to 0^+}\frac{e^{-2\epsilon^{1-\tau}}-2e^{-\epsilon^{1-\tau}}+1}{2\epsilon}\\ &=\lim_{\epsilon\to 0^+}\frac{(1-\tau)e^{-\epsilon^{1-\tau}}(1-e^{-\epsilon^{1-\tau}})}{\epsilon^\tau}\\ &=\lim_{\epsilon\to 0^+}\frac{(1-\tau)(1-e^{-\epsilon^{1-\tau}})}{\epsilon^\tau}\\ &=\lim_{\epsilon\to 0^+}\frac{(1-\tau)^2(e^{-\epsilon^{1-\tau}})}{\epsilon^{2\tau-1}}\\ &=\lim_{\epsilon\to 0^+}(1-\tau)^2\left(e^{-\epsilon^{1-\tau}}\right)\epsilon^{1-2\tau}.\end{aligned} \tag{2.35}$$

Obviously, it is sufficient to choose $\tau<\frac{1}{2}$ such that $\lim_{\epsilon\to 0^+}\int_0^{\frac{1}{\epsilon^\tau}}(e^{-2\epsilon s}-e^{-\epsilon s})\,ds=0$. Similarly, the fourth estimate of (2.17) holds.

Conversely, in the new long-time period $0\le t\le T(=O(\frac{1}{\epsilon^\tau}))$, we return to estimate the term $\mathcal{B}_b(\epsilon)$ with choosing $b=\epsilon^\tau$ in (2.23) instead of $b=0$ in (2.24). This supplies that the quantity $e^{(\epsilon^\tau+\epsilon)t}\|w(t)\|_{H^1}$ (i.e., (2.22) with $b=\epsilon^\tau$) has more exponential weight decay than $e^{\epsilon t}\|w(t)\|_{H^1}$, (i.e., (2.22) with $b=0$), that is

$$\begin{aligned}\mathcal{B}_{b=\epsilon^\tau}(\epsilon)&=\int_0^t(t-s)^{-\frac{1}{2}}e^{-(b'+\epsilon)(t-s)}e^{(b+\epsilon)t}\left(e^{-\epsilon s}-e^{-2\epsilon s}\right)ds\\ &=\int_0^t(t-s)^{-\frac{1}{2}}e^{-(b'-b)(t-s)}e^{(b+\epsilon)s}\left(e^{-\epsilon s}-e^{-2\epsilon s}\right)ds\\ &=\int_0^t(t-s)^{-\frac{1}{2}}e^{-(b'-b)(t-s)}e^{bs}\left(1-e^{-\epsilon s}\right)ds\\ &=\int_0^t(t-s)^{-\frac{1}{2}}e^{-(b'-\epsilon^\tau)(t-s)}e^{\epsilon^\tau s}\left(1-e^{-\epsilon s}\right)ds.\end{aligned} \tag{2.36}$$

Due to $\epsilon^\tau \ll b'$ and $0 \le t \le T(= O(\frac{1}{\epsilon^\tau}))$, by the Hölder inequality and the mean value principle, we have

$$\begin{aligned}
\mathcal{B}_{b=\epsilon^\tau}(\epsilon) &= \int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)(t-s)} e^{\epsilon^\tau s}\left(1-e^{-\epsilon s}\right) ds \\
&= \int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)(t-s)} e^{\epsilon^\tau s}\left(e^{-\epsilon\cdot 0}-e^{-\epsilon s}\right) ds \\
&\le C\int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)(t-s)}\, ds \cdot \sup_{s\in[0,t]}\left(e^{-\epsilon\cdot 0}-e^{-\epsilon s}\right) \\
&\le C\int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)(t-s)}\, ds \cdot \epsilon e^{-\epsilon\xi} s \\
&\le C_1\int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)(t-s)}\, ds \cdot \epsilon s \\
&= C_1\int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)(t-s)}\, ds \cdot \epsilon^{1-\tau}.
\end{aligned} \tag{2.37}$$

Also, the substitution $t-s=\ell$ follows

$$\begin{aligned}
&\int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)(t-s)}\, ds \\
&\quad= \int_0^t \ell^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)\ell}\, ds \\
&\quad= \int_0^1 \ell^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)\ell}\, ds + \int_1^t \ell^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)\ell}\, ds \\
&\quad\le \int_0^1 \ell^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)\ell}\, ds + \int_1^\infty \ell^{-\frac{1}{2}} e^{-(b'-\epsilon^\tau)\ell}\, ds \\
&\quad\le \int_0^1 \ell^{-\frac{1}{2}}\, ds + \int_1^\infty e^{-(b'-\epsilon^\tau)\ell}\, ds \\
&\quad= 2 + \frac{1}{b'-\epsilon^\tau} e^{-(b'-\epsilon^\tau)}.
\end{aligned} \tag{2.38}$$

Hence, by (2.36), (2.37), and (2.38), we have

$$\mathcal{B}_{b=\epsilon^\tau}(\epsilon) \le C\epsilon^{1-\tau}. \tag{2.39}$$

Hence, inserting (2.39) into (2.22), we have, for $b=\epsilon^\tau$,

$$e^{(\epsilon^\tau+\epsilon)t}\left\|w(t)\right\|_{H^1} \le C\left\|w(0)\right\|_{H^1} + C\left(\delta_* + \epsilon^m\right)M_{w,b=\epsilon^\tau}(T) + C\epsilon^{1-\tau}. \tag{2.40}$$

Taking the supremum over $0 \le t \le T(= O(\frac{1}{\epsilon^\tau}))$, we find that if δ_* is sufficient small, then

$$M_{w,b=\epsilon^\tau}(T) = \sup_{0\le t\le T} e^{(\epsilon^\tau+\epsilon)t}\left\|w(t)\right\|_{H^1} \le C\left\|w(0)\right\|_{H^1} + C\epsilon^{1-\tau}. \tag{2.41}$$

Remark 2.1 In some sense, there is a balance between the long-time point $T = O(\frac{1}{\epsilon^\tau})$ and the exponent weight $b=\epsilon^\tau$. In other words, if the long-time point is smaller, then the exponent weight of decay is larger. Here, we cannot obtain the exponent weight of decay $e^{-a(c-a^2)t}$ as in Ref. [9] due to perturbation estimates (2.23) and (2.36) caused by the weakly damped term.

Proof (ii) H^1 estimate: We make use of the damping quantity

$$\mathcal{E}(u) = \mathcal{H}(u) + c_0\mathcal{I}(u) = \int_{-\infty}^{\infty} \frac{1}{2}(\partial_x u)^2\,dx - \int_{-\infty}^{\infty} \frac{1}{6}u^3\,dx + \int_{-\infty}^{\infty} \frac{1}{2}c_0 u^2\,dx. \tag{2.42}$$

Since u_{c_0} is a critical point of the functional $\mathcal{E}$, we have for any $z \in H^1$,

$$\mathcal{E}(u_{c_0} + z) - \mathcal{E}(u_{c_0}) = \int_{-\infty}^{\infty} \frac{1}{2}(\partial_x z)^2 + \frac{1}{2}(c_0 - u_{c_0})z^2 - \frac{1}{6}z^3\,dx. \tag{2.43}$$

Now, we take $z = u(x,t) - u_{c_0}(y) = e^{-\epsilon t}u_{c(t)}(y) + v(y,t) - u_{c_0}(y)$ above, and observe that $\delta\mathcal{E}_0 = \mathcal{E}(u) - \mathcal{E}(u_{c_0})$ is decaying in time. Indeed,

$$\begin{aligned}
\frac{d\delta\mathcal{E}_0}{dt} &= \frac{d(\mathcal{E}(u) - \mathcal{E}(u_0))}{dt} \\
&= \frac{d\mathcal{E}}{dt} \\
&= \left\langle -\partial_{xx}u - \frac{1}{2}u^2 + c_0 u, -\partial_x\left(u_{xx} + \frac{1}{2}u^2\right) - \epsilon u\right\rangle \\
&= \left\langle -\left(\partial_{xx}u + \frac{1}{2}u^2\right), -\partial_x\left(u_{xx} + \frac{1}{2}u^2\right)\right\rangle + \left\langle c_0 u, -\partial_x\left(u_{xx} + \frac{1}{2}u^2\right)\right\rangle \\
&\quad + \left\langle -\left(\partial_{xx}u + \frac{1}{2}u^2\right), -\epsilon u\right\rangle + \langle c_0 u, -\epsilon u\rangle \\
&= -\epsilon\int_{\mathbb{R}} |u_x|^2\,dx + \frac{\epsilon}{2}\int_{\mathbb{R}} u^3\,dx - c_0\epsilon\int_{\mathbb{R}} u^2\,dx \\
&= -3\epsilon\mathcal{E}(u) + \epsilon\left(\int_{\mathbb{R}} \frac{1}{2}u_x^2\,dx + \frac{1}{2}c_0\int_{\mathbb{R}} u^2\,dx\right) \\
&= -3\epsilon\mathcal{E}(u) + \epsilon\left(\int_{\mathbb{R}} \frac{1}{2}u_x^2\,dx + \frac{1}{2}c_0\int_{\mathbb{R}} u^2\,dx\right) \\
&= -3\epsilon\big(\mathcal{E}(u) - \mathcal{E}(u_0)\big) - 3\epsilon\mathcal{E}(u_0) + \epsilon\left(\int_{\mathbb{R}} \frac{1}{2}u_x^2\,dx + \frac{1}{2}c_0\int_{\mathbb{R}} u^2\,dx\right) \\
&= -3\epsilon\delta\mathcal{E}_0 - 3\epsilon\mathcal{E}(u_0) + \epsilon\left(\int_{\mathbb{R}} \frac{1}{2}u_x^2\,dx + \frac{1}{2}c_0\int_{\mathbb{R}} u^2\,dx\right) \\
&= -3\epsilon\delta\mathcal{E}_0 - 3\epsilon C + \epsilon\left(\int_{\mathbb{R}} \frac{1}{2}u_x^2\,dx + \frac{1}{2}c_0\int_{\mathbb{R}} u^2\,dx\right).
\end{aligned} \tag{2.44}$$

Moreover, multiplying equation (1.1) by u_{xx}, one has

$$\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2(\mathbb{R})} = -\epsilon\|u_x\|_{L^2}. \tag{2.45}$$

Due to decaying estimates about $\|u\|_{L^2}$ and $\|u_x\|_{L^2}$ given in (1.10) and (2.45), one can deduce from (2.44) that for $0 \le t \le T = O(\frac{1}{\epsilon^\tau})$

$$\delta\mathcal{E}_0 \le e^{-3\epsilon t}\delta\mathcal{E}_0(0) + C\big(1 - e^{-3\epsilon t}\big)$$

$$\leq e^{-3\epsilon t}\delta\mathcal{E}_0(0) + C\left(1 - e^{-3\epsilon^{1-\tau}}\right). \tag{2.46}$$

At the same time, we estimate (2.43) as follows. Note that, for δ_* sufficiently small,

$$\begin{aligned}\left\|e^{-\epsilon t}u_{c(t)} - u_{c_0}\right\|_{H^1} &= \left\|e^{-\epsilon t}u_{c(t)} - e^{-\epsilon t}u_{c_0} + e^{-\epsilon t}u_{c_0} - u_{c_0}\right\|_{H^1} \\ &\leq C\left(\left|c(t) - c_0\right| + \left|e^{-\epsilon t} - 1\right|\right).\end{aligned} \tag{2.47}$$

Then, for some $k_1 > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{2}(\partial_y z)^2 + \frac{1}{2}c_0 z^2\,dy \leq k_1\|v\|_{H^1}^2 + C\left(\left|c(t) - c_0\right|^2 + \left|e^{-\epsilon t} - 1\right|^2\right). \tag{2.48}$$

Since $e^{-ay}u_{c_0}(y)$ is bounded in y, we may estimate

$$\begin{aligned}\int_{-\infty}^{\infty} u_{c_0}(y)z^2\,dy &\leq \sup_y\left|e^{-ay}u_{c_0}(y)\right|\|z\|_{L^2}\left\|e^{ay}z\right\|_{L^2} \\ &\leq C\left(\left|c(t) - c_0\right| + \left|e^{-\epsilon t} - 1\right| + \|v\|_{L^2}\right)\left(\left|c(t) - c_0\right| + \left|e^{-\epsilon t} - 1\right| + \|w\|_{L^2}\right) \\ &\leq \frac{1}{4}k_1\|v\|_{L^2}^2 + C\left[\left|c(t) - c_0\right|^2 + \|w\|_{L^2}^2 + \left|e^{-\epsilon t} - 1\right|^2\right],\end{aligned} \tag{2.49}$$

where we have used the estimate $ab \leq \delta a^2 + C(\delta)b^2$ for a suitably small δ. Finally, since $\|z\|_{H^1} \leq C(|c(t) - c_0| + \|v\|_{H^1} + |1 - e^{-\epsilon t}|) \leq C(\delta_* + |1 - e^{-\epsilon t}|)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{6}z^3\,dy \leq C\|z\|_{H^1}^3 &\leq C\left(\delta_* + \left|1 - e^{-\epsilon t}\right|\right)\left(\left|c(t) - c_0\right|^2 + \|v\|_{H^1}^2 + \left|e^{-\epsilon t} - 1\right|^2\right) \\ &\leq \frac{1}{4}k_1\|v\|_{L^2}^2 + C\left[\left|c(t) - c_0\right|^2 + \left|e^{-\epsilon t} - 1\right|^2\right].\end{aligned} \tag{2.50}$$

Hence, if δ_* is sufficiently small, (2.43) with (2.48), (2.49), and (2.50) yields

$$\frac{1}{2}k_1\|v\|_{H^1}^2 \leq \delta\mathcal{E}_0 + C\left[\left|c(t) - c_0\right|^2 + \left|e^{-\epsilon t} - 1\right|^2\right]. \tag{2.51}$$

Due to (2.35) and (2.46), we know that

$$\begin{aligned}\|v\|_{H^1} &\leq C\left(\sqrt{\delta\mathcal{E}_0} + \left|c(t) - c_0\right| + \left|e^{-\epsilon t} - 1\right|\right) \\ &\leq C\|v_0\|_{H^1} + C\epsilon^{1-\tau} + \left|c(t) - c_0\right| \\ &\leq C_1\epsilon + C_2\epsilon^{1-\tau} + C_3\epsilon^{1-2\tau} \\ &\leq C\epsilon^{1-2\tau}.\end{aligned} \tag{2.52}$$

This completes the proof of Proposition 2.1, which implies the conclusions of Theorem 1.1. □

3 The long-time behavior stability

3.1 A new decomposition of the solution

Note that, in the long-time stability case, the expression (2.15): $\mathfrak{A}(t) = e^{-\epsilon t}I + O(|c(t) - c_0| + \|v\|_{L^2})$ may not be reversible as $t \to +\infty$, which is derived by setting the form of solution

(2.1). Hence, we subtly analyze the following new form of the solution

$$u(x,t) = e^{-\epsilon t}\big[u_{c(t)}(y) + v(y,t)\big], \tag{3.1}$$

where

$$y = y(x,t) = x - \int_0^t c(s)\,ds + \gamma(t) \tag{3.2}$$

and $u_{c(t)}(y)$ belongs to the family of traveling waves with $c(t) = c_0 e^{-\beta t}(0 < \beta \leq 1)$.

Substituting (3.1) into (1.1), we similarly derive evolution equations for $\gamma(t)$, $c(t)$, and $v(y,t)$ as follows:

$$\begin{aligned}
0 &= \partial_t u + \partial_x^3 u + \partial_x\left(\frac{1}{2}u^2\right) + \epsilon u \\
&= e^{-\epsilon t}\big[\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big]\big(u_{c(t)}(y) + v\big) - \epsilon e^{-\epsilon t}\big(u_{c(t)}(y) + v(y,t)\big) \\
&\quad + \partial_y\left[\frac{1}{2}e^{-2\epsilon t}\big(u_{c(t)}(y) + v(y,t)\big)^2\right] + \epsilon e^{-\epsilon t}\big(u_{c(t)}(y) + v(y,t)\big) \\
&= e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)u_{c(t)}(y) \\
&\quad + \partial_y\left[\frac{1}{2}e^{-2\epsilon t}\big(u_{c(t)}(y) + v(y,t)\big)^2\right] \\
&= e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + e^{-\epsilon t}\big(\big(\dot{\gamma} - c(t)\big)\partial_y\big)u_{c(t)}(y) \\
&\quad + e^{-\epsilon t}\big(\partial_t u_{c(t)}(y) + \partial_y^3 u_{c(t)}(y)\big) + \partial_y\left[\frac{1}{2}e^{-2\epsilon t}\big(u_{c(t)}(y) + v(y,t)\big)^2\right] \\
&= e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + e^{-\epsilon t}\big(\big(\dot{\gamma} - c(t)\big)\partial_y\big)u_{c(t)}(y) \\
&\quad + e^{-\epsilon t}\partial_y\big(-c(t)u_{c(t)}(y) + \partial_y^2 u_{c(t)}(y)\big) + \partial_y\left[\frac{1}{2}e^{-2\epsilon t}\big(u_{c(t)}(y) + v(y,t)\big)^2\right] \\
&= e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + e^{-\epsilon t}\big(\big(\dot{\gamma} - c(t)\big)\partial_y\big)u_{c(t)}(y) \\
&\quad + e^{-\epsilon t}\left(-\frac{1}{2}\partial_y u_{c(t)}^2(y)\right) + \partial_y\left[\frac{1}{2}e^{-2\epsilon t}\big(u_{c(t)}(y) + v(y,t)\big)^2\right] \\
&= e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + e^{-\epsilon t}\left\{\dot{\gamma}\,\partial_y u_{c(t)}(y) + \frac{\partial u}{\partial c}\dot{c}\right\} \\
&\quad + e^{-\epsilon t}\left(-\frac{1}{2}\partial_y u_{c(t)}^2(y)\right) + \partial_y\left[\frac{1}{2}e^{-2\epsilon t}u_{c(t)}^2(y) + e^{-2\epsilon t}u_{c(t)}(y)v(y,t) + \frac{1}{2}e^{-2\epsilon t}v^2(y,t)\right] \\
&= e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + e^{-\epsilon t}\left\{\dot{\gamma}\,\partial_y u_{c(t)}(y) + \frac{\partial u}{\partial c}\dot{c}\right\} \\
&\quad + \partial_y\left[\frac{1}{2}\big(e^{-2\epsilon t} - e^{-\epsilon t}\big)u_{c(t)}^2(y) + e^{-2\epsilon t}u_{c(t)}(y)v(y,t) + \frac{1}{2}e^{-2\epsilon t}v^2(y,t)\right] \\
&= e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + e^{-\epsilon t}\partial_y(u_{c(t)}v) + e^{-\epsilon t}\left\{\dot{\gamma}\,\partial_y u_{c(t)}(y) + \frac{\partial u}{\partial c}\dot{c}\right\} \\
&\quad + \partial_y\left[\frac{1}{2}\big(e^{-2\epsilon t} - e^{-\epsilon t}\big)u_{c(t)}^2(y) + e^{-2\epsilon t}u_{c(t)}(y)v(y,t) + \frac{1}{2}e^{-2\epsilon t}v^2(y,t)\right] - e^{-\epsilon t}\partial_y(u_{c(t)}v)
\end{aligned}$$

$$\begin{aligned}
&= e^{-\epsilon t}\big(\partial_t + \big(\dot{\gamma} - c(t)\big)\partial_y + \partial_y^3\big)v + e^{-\epsilon t}\partial_y(u_{c(t)}v) + e^{-\epsilon t}\left\{\dot{\gamma}\,\partial_y u_{c(t)}(y) + \frac{\partial u}{\partial c}\dot{c}\right\} \\
&\quad + \partial_y\left[\frac{1}{2}\big(e^{-2\epsilon t} - e^{-\epsilon t}\big)u_{c(t)}^2(y) + e^{-2\epsilon t}u_{c(t)}(y)v(y,t)\right. \\
&\quad \left. + \frac{1}{2}e^{-2\epsilon t}v^2(y,t) - e^{-\epsilon t}(u_{c(t)}v)\right].
\end{aligned} \tag{3.3}$$

Hence,

$$\begin{aligned}
e^{-\epsilon t}\partial_t v &= e^{-\epsilon t}\partial_y\big[-\partial_y^2 + c(t) - u_{c(t)}\big]v - e^{-\epsilon t}\left[\dot{\gamma}\,\partial_y u + \dot{c}\frac{\partial u}{\partial c}\right] - e^{-\epsilon t}\partial_y[\dot{\gamma}v] \\
&\quad - \partial_y\left[\frac{1}{2}\big(e^{-2\epsilon t} - e^{-\epsilon t}\big)u_{c(t)}^2(y) + e^{-2\epsilon t}u_{c(t)}(y)v(y,t)\right. \\
&\quad \left. + \frac{1}{2}e^{-2\epsilon t}v^2(y,t) - e^{-\epsilon t}(u_{c(t)}v)\right].
\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}
\partial_t v &= \partial_y\big[-\partial_y^2 + c(t) - u_{c(t)}\big]v - \left[\dot{\gamma}\,\partial_y u + \dot{c}\frac{\partial u}{\partial c}\right] - \partial_y[\dot{\gamma}v] \\
&\quad - e^{\epsilon t}\partial_y\left[\frac{1}{2}\big(e^{-2\epsilon t} - e^{-\epsilon t}\big)u_{c(t)}^2(y) + e^{-2\epsilon t}u_{c(t)}(y)v(y,t)\right. \\
&\quad \left. + \frac{1}{2}e^{-2\epsilon t}v^2(y,t) - e^{-\epsilon t}(u_{c(t)}v)\right].
\end{aligned} \tag{3.5}$$

Since here the speeds $c(t)$ of the traveling wave will decay to zero, the exponential weight $a(<\sqrt{\frac{c(t)}{3}})$ will also decay. On the other hand, due to the fifth item in Remark 1.1, we cannot initially set the exponential weight $a = 0$ in H_a^1. Otherwise, it may follow more than a 2-dimensional generalized kernel. Hence, in contrast to the long-time stability case by setting (2.3) to prove Theorem 1.1, we need to set $w(y,t) = e^{a(t)y}v(y,t)$, $A_a(t) = e^{a(t)y}\partial_y L_{c(t)}e^{-a(t)y}$ and $L_{c(t)} = -\partial_y^2 + c(t) - u_{c(t)}$. Then, we deduce that

$$\partial_t w = \left[A_a(t) + \frac{da}{dt}\right]w + \mathfrak{F}, \tag{3.6}$$

where, for simplicity, writing $a = a(t)$ if there is no risk of confusion,

$$\begin{aligned}
&\mathfrak{F} = -e^{ay}(\dot{c}\partial_c + \dot{\gamma}\,\partial_y)u_{c(t)} - \dot{\gamma}e^{ay}\partial_y e^{-ay}w + \mathcal{F}, \\
&\mathcal{F} = -e^{\epsilon t}e^{ay}\partial_y\left[\frac{1}{2}\big(e^{-2\epsilon t} - e^{-\epsilon t}\big)u_{c(t)}^2\right] - e^{\epsilon t}e^{ay}\partial_y\left[\big(e^{-2\epsilon t} - e^{-\epsilon t}\big)u_{c(t)}v + \frac{1}{2}e^{-2\epsilon t}v^2\right].
\end{aligned} \tag{3.7}$$

Meanwhile, (2.4) implies that this equation is initially justified in $C([0,t],H^{-3})$, but also holds in $C([0,t],L^2)$ and moreover is pointwise.

As in the long-time behavior case above, we wish to impose the similar projections P, Q given in Proposition A.2

$$w(y,t) = e^{ay}v(y,t) \in \operatorname{range}(Q) = \ker(P). \tag{3.8}$$

However, here, we should denote them by $P(t), Q(t)$. Indeed, the current assumption of Proposition A.2: $0 < a(t) < \sqrt{\frac{c(t)}{3}}$ depends on t, from which it follows that the $\xi_j = \xi_j(t)$ and $\eta_k = \eta_k(t)$ depend on t for $j, k = 1, 2$. This requirement corresponds to the two scalar constraints $\langle w, \eta_k(t)\rangle = 0$, $k = 1, 2$, cf. (A.14), which also generates the modulation equations, namely, two coupled first-order differential equations for $c(t)$, $\gamma(t)$ as $t > 0$. Hence, the constraint $w \in \operatorname{range}(Q)$ in (3.8) now yields the following system of evolution equations for (w, γ, c):

$$\partial_t w = \left[A_a + \frac{da}{dt}\right] w + Q\mathfrak{F}, \quad P\mathfrak{F} = 0. \tag{3.9}$$

Written as an integral equation, the initial value problem for (3.9) becomes:

$$w(t) = e^{\int_0^t [A_a(s) + \frac{da}{dt}(s)]\,ds} w(0) + \int_0^t e^{\int_s^t [A_a(s) + \frac{da}{dt}(s)]\,ds} Q\mathfrak{F}(s)\,ds. \tag{3.10}$$

Then, similarly by (A.14), the condition $P\mathfrak{F} = 0$ is equivalent to

$$0 = \left\langle \dot{\gamma}\left[e^{ay}\partial_y u_{c(t)} + (\partial_y - a)w\right] + \dot{c}e^{ay}\partial_c u_{c(t)} - \mathcal{F}, \eta_k\right\rangle, \quad k = 1, 2. \tag{3.11}$$

Using the biorthogonality relation $\langle \xi_j, \eta_k\rangle = \delta_{jk}$, we obtain a system of equations for $\gamma(t)$ and $c(t)$:

$$\mathfrak{A}(t)\begin{pmatrix}\dot{\gamma}\\ \dot{c}\end{pmatrix} = \begin{pmatrix}\langle \mathcal{F}, \eta_1\rangle\\ \langle \mathcal{F}, \eta_2\rangle\end{pmatrix} \tag{3.12}$$

and

$$\begin{aligned}
\mathfrak{A}(t) &= \begin{pmatrix} \langle e^{ay}\partial_y u_{c(t)}, \eta_1\rangle + \langle (\partial_y - a)w, \tilde{\eta}_1\rangle, \langle e^{ay}\partial_c u_{c(t)}, \eta_1\rangle \\ \langle e^{ay}\partial_y u_{c(t)}, \eta_2\rangle + \langle (\partial_y - a)w, \tilde{\eta}_2\rangle, \langle e^{ay}\partial_c u_{c(t)}, \eta_2\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle e^{ay}\partial_y u_{c(t)}, \eta_1\rangle + \langle e^{ay}\partial_y e^{-ay} w, \tilde{\eta}_1\rangle, \langle e^{ay}\partial_c u_{c(t)}, \eta_1\rangle \\ \langle e^{ay}\partial_y u_{c(t)}, \eta_2\rangle + \langle e^{ay}\partial_y e^{-ay} w, \tilde{\eta}_2\rangle, \langle e^{ay}\partial_c u_{c(t)}, \eta_2\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle e^{ay}\partial_y u_{c(t)}, \eta_1\rangle, \langle e^{ay}\partial_c u_{c(t)}, \eta_1\rangle \\ \langle e^{ay}\partial_y u_{c(t)}, \eta_2\rangle, \langle e^{ay}\partial_c u_{c(t)}, \eta_2\rangle \end{pmatrix} + \begin{pmatrix} \langle e^{ay}\partial_y e^{-ay} w, \tilde{\eta}_1\rangle, \langle e^{ay}\partial_c u_{c(t)}, \eta_1\rangle \\ \langle e^{ay}\partial_y e^{-ay} w, \tilde{\eta}_2\rangle, \langle e^{ay}\partial_c u_{c(t)}, \eta_2\rangle \end{pmatrix}.
\end{aligned} \tag{3.13}$$

The matrix $\mathfrak{A}(t)$ satisfies

$$\mathfrak{A}(t) = I + O\big(\|v\|_{L^2}\big) \quad \text{as } \|v\|_{L^2} \to 0. \tag{3.14}$$

3.2 The long-time behavior

Now, we will estimate the weighted perturbation, $w(y, t) = e^{ay}v(y, t)$, in H^1, via the integral equation (3.10), the modulation equation (3.12), and the linear semigroup estimates of Lemma A.2.

Since $\|v\|_{L^2}$ decays to zero with respect to t, in the expression (3.14), $\mathfrak{A}(t)$ has a bounded inverse as $0 \le t < +\infty$. We may estimate (3.12) to find

$$|\dot{\gamma}| + |\dot{c}| \le C\|\mathcal{F}\|_{L^2}. \tag{3.15}$$

From (3.7), using that $e^{ay}\partial_y e^{-ay} = \partial_y - a$, we obtain the estimates

$$\begin{aligned}
\|\mathfrak{F}\| &\le C\big(|\dot\gamma|\big(1+\|w\|_{H^1}\big)\big) + |\dot c| + \|\mathcal{F}\|_{L^2} \le C\big(1+\|w\|_{H^1}\big)\|\mathcal{F}\|_{L^2},\\
\|\mathcal{F}\|_{L^2} &= \bigg\| -e^{\epsilon t}e^{ay}\partial_y\bigg[\frac{1}{2}\big(e^{-2\epsilon t}-e^{-\epsilon t}\big)u_{c(t)}^2\bigg]\\
&\quad - e^{\epsilon t}e^{ay}\partial_y\bigg[\big(e^{-2\epsilon t}-e^{-\epsilon t}\big)u_{c(t)}v + \frac{1}{2}e^{-2\epsilon t}v^2\bigg]\bigg\|_{L^2}\\
&= \bigg\| -e^{ay}\partial_y\bigg[\frac{1}{2}\big(e^{-\epsilon t}-1\big)u_{c(t)}^2\bigg] - e^{ay}\partial_y\bigg[\big(e^{-\epsilon t}-1\big)u_{c(t)}v + \frac{1}{2}e^{-\epsilon t}v^2\bigg]\bigg\|_{L^2}\\
&\le \frac{1}{2}\big(1-e^{-\epsilon t}\big)\big\|e^{ay}\partial_y u_{c(t)}^2\big\|_{L^2} + \big(1-e^{-\epsilon t}\big)\big\|e^{ay}\partial_y(u_{c(t)}v)\big\|_{L^2}\\
&\quad + \frac{1}{2}e^{-\epsilon t}\big\|e^{ay}\partial_y v^2\big\|_{L^2}.
\end{aligned} \tag{3.16}$$

Now, we may formally choose b' with $b < b' < a(c-a^2)$, such that b', as well as b, satisfies the condition of Lemma A.2. Meanwhile, we should note that in (3.10) the term $e^{\int_0^t [\frac{da}{dt}(s)]\,ds} \sim O(1)$ since $a < \sqrt{\frac{c}{3}} = \frac{1}{\sqrt{3}}e^{-\frac{\beta}{2}s} (\beta > 0)$. Hence, one can similarly estimate (3.10) as follows, for $t > 0$:

$$\begin{aligned}
\big\|w(t)\big\|_{H^1} &\le Ce^{-b't}\big\|w(0)\big\|_{H^1} + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}\|\mathcal{F}\|_{L^2}\,ds\\
&\le Ce^{-b't}\big\|w(0)\big\|_{H^1} + C\int_0^t \bigg\{(t-s)^{-1/2}e^{-b'(t-s)}\\
&\quad \times \bigg[\frac{1}{2}\big(1-e^{-\epsilon s}\big)\big\|e^{ay}\partial_y u_{c(t)}^2\big\|_{L^2} + \big(1-e^{-\epsilon s}\big)\big\|e^{ay}\partial_y(u_{c(t)}v)\big\|_{L^2}\\
&\quad + \frac{1}{2}e^{-\epsilon s}\big\|\partial_y v^2\big\|_{L^2}\bigg]\bigg\}\,ds.
\end{aligned} \tag{3.17}$$

Now, formally define

$$M_{w,b}(t) = \sup_{0\le s\le t} e^{b(t)s}\big\|w(s)\big\|_{H^1}, \tag{3.18}$$

where the variable $b = b(t)$ is similarly given in Remark A.1.

Moreover, the following crucial estimates follow from (1.4).

Lemma 3.1 *Assume that the solitary waves $u_{c(t)}(y)$ have the traveling speed $c(t) = c_0e^{-\beta t}$ as $0 \le t < +\infty$. Then,*

$$\begin{aligned}
&\big\|u_{c(t)}(y)\big\|_{L^2} \sim e^{-\frac{4}{3}\beta t}\big\|u_{c_0}(x,0)\big\|_{L^2},\\
&\big\|u_{c(t)}(y)\big\|_{L^\infty} \sim e^{-\beta t}\big\|u_{c_0}(x,0)\big\|_{L^\infty},\\
&\big\|\partial_y u_{c(t)}(y)\big\|_{L^\infty} \sim e^{-\beta t}\big\|\partial_y u_{c_0}(x,0)\big\|_{L^\infty}.
\end{aligned} \tag{3.19}$$

For simplicity, also writing $b = b(t)$ if there is no risk of confusion, and then multiplying (3.17) by e^{bt}, we find from (3.19) that, for $t > 0$,

$$e^{bt}\big\|w(t)\big\|_{H^1}$$

$$
\begin{aligned}
&\le Ce^{bt}e^{-b't}\left\|w(0)\right\|_{H^1}\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}\left(1-e^{-\epsilon s}\right)\left\|e^{ay}\partial_y u_{c(t)}^2\right\|_{L^2}\,ds\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}\left(1-e^{-\epsilon s}\right)\left\|e^{ay}\partial_y (u_{c(t)}v)\right\|_{L^2}\,ds\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}e^{-\epsilon s}\left\|e^{ay}\partial_y v^2\right\|_{L^2}\,ds\\
&\le Ce^{-(b'-b)t}\left\|w(0)\right\|_{H^1}\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}\left(1-e^{-\epsilon s}\right)\left\|e^{ay}\partial_y u_{c(t)}^2\right\|_{L^2}\,ds\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}\left(1-e^{-\epsilon s}\right)\left[\|\partial_y u_{c(t)}\|_{L^\infty}+\|u_{c(t)}\|_{L^\infty}\right]\left\|w(s)\right\|_{H^1}\,ds\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}e^{-\epsilon s}\|v\|_{L^\infty}\left\|w(s)\right\|_{H^1}\,ds\\
&\le Ce^{-(b'-b)t}\left\|w(0)\right\|_{H^1}\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}\left(1-e^{-\epsilon s}\right)e^{-\frac{3}{2}\beta s}\,ds\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}\left(1-e^{-\epsilon s}\right)\left[\|\partial_y u_{c(t)}\|_{L^\infty}+\|u_{c(t)}\|_{L^\infty}\right]e^{bt}\left\|w(s)\right\|_{H^1}\,ds\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{-\epsilon s}\|v\|_{L^\infty}e^{bt}\left\|w(s)\right\|_{H^1}\,ds\\
&\le Ce^{-(b'-b)t}\left\|w(0)\right\|_{H^1}\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}\left(1-e^{-\epsilon s}\right)e^{-\frac{3}{2}\beta s}\,ds\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}\left(1-e^{-\epsilon s}\right)e^{-\beta s}e^{bt}\left\|w(s)\right\|_{H^1}\,ds\\
&\quad + C\int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{-\epsilon s}\|v\|_{L^\infty}e^{bt}\left\|w(s)\right\|_{H^1}\,ds\\
&\le Ce^{-(b'-b)t}\left\|w(0)\right\|_{H^1}+I(\epsilon,\beta,t)+II(\epsilon,\beta,t)+III(\epsilon,t).
\end{aligned}
\tag{3.20}
$$

We first deal with the term I. For $0\le t\le 1$ and $0<\beta\le 1$, one can easily deduce that

$$
\begin{aligned}
I(\epsilon,\beta,t) &= \int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}\left(1-e^{-\epsilon s}\right)e^{-\frac{3}{2}\beta s}\,ds\\
&\le C\epsilon\int_0^t (t-s)^{-1/2}\,ds\\
&= C\epsilon\sqrt{t}.
\end{aligned}
\tag{3.21}
$$

On the other hand, for $1\le t<\infty$, one has

$$
I(\epsilon,\beta,t) = \int_0^t (t-s)^{-1/2}e^{-b'(t-s)}e^{bt}\left(1-e^{-\epsilon s}\right)e^{-\frac{3}{2}\beta s}\,ds
$$

$$
\begin{aligned}
&= \int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-b)(t-s)} e^{bs} \left(1-e^{-\epsilon s}\right) e^{-\frac{3}{2}\beta s}\, ds \\
&= \int_0^t (t-s)^{-\frac{1}{2}} e^{-(b'-b)(t-s)} e^{-(\frac{3}{2}\beta - b)s} \left(1-e^{-\epsilon s}\right) ds \\
&\le \epsilon \int_0^t (t-s)^{-\frac{1}{2}} e^{-(\beta-b)s}\, ds \quad \text{with } t-s=\ell \\
&= \epsilon \int_0^t \ell^{-\frac{1}{2}} e^{-(\beta-b)(t-\ell)}\, d\ell \\
&= \epsilon \int_0^{t/2} \ell^{-\frac{1}{2}} e^{-(\beta-b)(t-\ell)}\, d\ell + \epsilon \int_{t/2}^t \ell^{-\frac{1}{2}} e^{-(\beta-b)(t-\ell)}\, d\ell \\
&\le \epsilon \int_0^{t/2} \ell^{-\frac{1}{2}}\, d\ell \cdot e^{-(\beta-b)(t-\frac{t}{2})} + \epsilon \left(\frac{t}{2}\right)^{-\frac{1}{2}} \int_{t/2}^t e^{-(\beta-b)(t-\ell)}\, d\ell \\
&\le \epsilon \int_0^{t/2} \ell^{-\frac{1}{2}}\, d\ell \cdot e^{-(\beta-b)(t-\frac{t}{2})} + \epsilon \left(\frac{t}{2}\right)^{-\frac{1}{2}} \int_{t/2}^t e^{-(\beta-b)(t-\ell)}\, d\ell \\
&\le 2\epsilon \left(\frac{t}{2}\right)^{\frac{1}{2}} \cdot e^{-(\beta-b)(t-\frac{t}{2})} + \epsilon \left(\frac{t}{2}\right)^{-\frac{1}{2}} \frac{1}{\beta-b}\left[e^{-(\beta-b)(t-\frac{t}{2})} - 1\right].
\end{aligned} \tag{3.22}
$$

Since $b(t) \to 0$ as $t \to +\infty$, we can choose any $\beta > 0$ satisfying $\beta - b > 0$. Hence, one can deduce from (3.22) that $I(\epsilon,\beta,t) \sim O(\frac{\epsilon}{\beta\sqrt{t}})$ as $1 \le t < +\infty$.

For the term $II(\epsilon,\beta,t) = \int_0^t (t-s)^{-1/2} e^{-b'(t-s)}(1-e^{-\epsilon s})e^{-\beta s}e^{bt}\|w(s)\|_{H^1}\, ds$, as in (3.21) and (3.22), we have

$$
\begin{aligned}
II(\epsilon,\beta,t) &= \int_0^t (t-s)^{-1/2} e^{-b'(t-s)} \left(1-e^{-\epsilon s}\right) e^{-\beta s} e^{bt} \left\|w(s)\right\|_{H^1} ds \\
&\le \int_0^t (t-s)^{-1/2} e^{-b'(t-s)} \left(1-e^{-\epsilon s}\right) e^{-\beta s} e^{b(t-s)}\, ds \cdot \sup_{s\in[0,t]} e^{bs} \left\|w(s)\right\|_{H^1} \\
&\le \int_0^t (t-s)^{-1/2} e^{-(b-b')(t-s)} \left(1-e^{-\epsilon s}\right) e^{-\beta s}\, ds \cdot \sup_{s\in[0,t]} e^{bs} \left\|w(s)\right\|_{H^1} \\
&\le \int_0^t (t-s)^{-1/2} \left(1-e^{-\epsilon s}\right) e^{-\beta s}\, ds \cdot \sup_{s\in[0,t]} e^{bs} \left\|w(s)\right\|_{H^1} \\
&\le C\epsilon \cdot \sup_{s\in[0,t]} e^{bs} \left\|w(s)\right\|_{H^1}.
\end{aligned} \tag{3.23}
$$

For the term $III(\epsilon,t) = \int_0^t (t-s)^{-1/2} e^{-b'(t-s)} e^{-\epsilon s}\|v\|_{L^\infty} e^{bt}\|w(s)\|_{H^1}\, ds$,

$$
\begin{aligned}
III(\epsilon,t) &= \int_0^t (t-s)^{-1/2} e^{-b'(t-s)} e^{-\epsilon s} \|v\|_{L^\infty} e^{bt} \left\|w(s)\right\|_{H^1} ds \\
&\le \int_0^t (t-s)^{-1/2} e^{-b'(t-s)} e^{-\epsilon s} \|v\|_{L^\infty} e^{b(t-s)} e^{bs} \left\|w(s)\right\|_{H^1} ds \\
&\le \int_0^t (t-s)^{-1/2} e^{-b'(t-s)} e^{-\epsilon s} \|v\|_{L^\infty} e^{b(t-s)}\, ds \sup_{s\in[0,t]} e^{bs} \left\|w(s)\right\|_{H^1} \\
&\le \int_0^t (t-s)^{-1/2} e^{-(b'-b)(t-s)} e^{-\epsilon s} \|v\|_{L^\infty}\, ds \sup_{s\in[0,t]} e^{bs} \left\|w(s)\right\|_{H^1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^t (t-s)^{-1/2} e^{-(b'-b)(t-s)} e^{-\epsilon s} \|v\|_{L^\infty} \, ds \sup_{s\in[0,t]} e^{bs} \big\| w(s) \big\|_{H^1} \quad \text{with } t-s=\ell \\
&\leq \int_0^t \ell^{-1/2} e^{-(b'-b)\ell} e^{-\epsilon(t-\ell)} \|v\|_{L^\infty} \, d\ell \sup_{s\in[0,t]} e^{bs} \big\| w(s) \big\|_{H^1} \\
&\leq C\|v\|_{L^\infty} \sup_{s\in[0,t]} e^{bs} \big\| w(s) \big\|_{H^1} \\
&\leq C\epsilon \sup_{s\in[0,t]} e^{bs} \big\| w(s) \big\|_{H^1}.
\end{aligned}
\tag{3.24}
$$

In sum, (1.19) and (1.20) follows from the inequalities (3.20), (3.21), (3.22), (3.23), (3.24), and the fact that $H_a^1 = H^1$ if $a = 0$. The proof of Theorem 1.2 is finished.

Appendix

For the reader's convenience, we list out the spectral property and developing analysis of the operator $A_0 = \partial_y L_c$ given in (1.17) in the space L^2 and L_a^2. The interested reader is referred to References [9, 25, 26].

A.1 Spectral theory in L^2 and L_a^2

The spectrum of the operator $A_0 = \partial_y L_c$ on L^2 consists of a discrete spectrum (isolated eigenvalues of finite multiplicity) and an essential spectrum (everything else in the spectrum).

Lemma A.1 (Theorem 2.1 Ref. [9]) *A_0 has no isolated eigenvalues whose spectrum coincides with the imaginary axis.*

In fact, if λ is an eigenvalue of A_0 with L^2-eigenfunction $Y(y)$, then

$$
A_0 Y(y) = \partial_y L_c Y(y) = \partial_y \big[-\partial_y^2 + c - u_c(y) \big] Y(y) = \lambda Y(y). \tag{A.1}
$$

Since the solitary wave $u_c(y) \to 0$ at an exponential rate as $|y| \to \infty$ (see (1.4)), it follows that the constant coefficient equation is

$$
\partial_y \big(-\partial_y^2 + c \big) Y(y) = \lambda Y(y). \tag{A.2}
$$

Hence, the essential spectrum of $A_0 = \partial_y L_c$ is the imaginary axis and the corresponding eigenvalue function $Y(y)$ exponentially decays to zero as $y \to \infty$.

The following functions are also described in [9], in relation to the isolated eigenvalue $\lambda = 0$ of A_0 in the space L_a^2.

$$
\tilde{\xi}_1 = \partial_y u_c, \qquad \tilde{\xi}_2 = \partial_c u_c, \tag{A.3}
$$

$$
\tilde{\eta}_1 = \theta_1 \int_{-\infty}^y \partial_c u_c \, dx + \theta_2 u_c, \qquad \tilde{\eta}_2 = \theta_3 u_c. \tag{A.4}
$$

Here,

$$\theta_1 = \left(\frac{d}{dc}\mathcal{I}[u_c]\right)^{-1},$$
$$\theta_2 = \frac{1}{2}\left(\frac{d}{dc}\int_{-\infty}^{+\infty} u_c\,dx\right)^2\left(\frac{d}{dc}\mathcal{I}[u_c]\right)^{-2} \quad \text{and} \quad \theta_3 = -\theta_1. \tag{A.5}$$

The functions $\tilde{\xi}_1$, $\tilde{\xi}_2$, and $\tilde{\eta}_2$ decay exponentially as $|y| \to \infty$, at the rate $e^{-\sqrt{c}|y|}$. The function $\tilde{\eta}_1$ decays like $e^{\sqrt{c}y}$ as $y \to -\infty$, but is merely bounded as $y \to +\infty$. In addition, these functions have the following properties:

$$\partial_y L_c \tilde{\xi}_1 = 0, \qquad \partial_y L_c \tilde{\xi}_2 = -\tilde{\xi}_1,$$
$$L_c \partial_y \tilde{\eta}_1 = \tilde{\eta}_2, \qquad L_c \partial_y \tilde{\eta}_2 = 0, \tag{A.6}$$

and

$$\langle \tilde{\eta}_j, \tilde{\xi}_k \rangle = \delta_{jk}, \quad j,k = 1,2, \tag{A.7}$$

where $\langle u,v \rangle = \int_{-\infty}^{+\infty} u\bar{v}\,dx$.

Making a change of variables,

$$W(y) = e^{ay}Y(y), \tag{A.8}$$

the eigenvalue equation (A.1) is transformed into the equation

$$A_a W = e^{ay}\partial_y L_c e^{-ay} W = (\partial_y - a)\left[-(\partial_y - a)^2 + c - u_c\right]W = \lambda W. \tag{A.9}$$

Thus, the spectral theory of $A_0 = \partial_y L_c$ in L_a^2 is equivalent to the spectral theory of A_a in L^2. Since $u_c(y)$ and $\partial_y u_c(y)$ decay to zero at an exponential rate as $|y| \to \infty$, the essential spectrum of A_a also agrees with the spectrum of the constant coefficient operator

$$A_a^0 = (\partial_y - a)\left[-(\partial_y - a)^2 + c\right]. \tag{A.10}$$

Hence,

Proposition A.1 (Proposition 2.5 Ref. [9]) *For $0 < a < \sqrt{c/3}$, the essential spectrum of A_a is a curve parametrized by*

$$\begin{aligned}\tau \mapsto \varphi(i\tau - a) &= (i\tau - a)\left[-(i\tau - a)^2 + c\right] \\ &= i\tau^3 - 3a\tau^2 + \left(c - 3a^2\right)i\tau - a\left(c - a^2\right),\end{aligned} \tag{A.11}$$

where lies in the open left half-plane.

Define

$$\ker(A) = \left\{w \in \operatorname{dom}(A) | Aw = 0 \text{ in } L^2\right\}, \quad \ker_g(A) = \bigcup_{k=1}^{\infty} \ker\left(A^k\right). \tag{A.12}$$

For the generalized eigenspaces of A_a and its adjoint $A_a^* = -e^{-ay}L_c\partial_y e^{ay}$, one has:

Proposition A.2 (Proposition 2.8 Ref. [9]) *Assume* $\frac{d\mathcal{I}[u_c]}{dc} \neq 0$ *and* $0 < a < \sqrt{c/3}$. *Then,* $\lambda = 0$ *is the only eigenvalue for* A_a *with algebraic multiplicity two, and*

$$\ker_g(A_a) = \ker\bigl(A_a^2\bigr) = \operatorname{span}\{\xi_1, \xi_2\}, \qquad \ker_g\bigl(A_a^*\bigr) = \ker\bigl(A_a^{*2}\bigr) = \operatorname{span}\{\eta_1, \eta_2\}, \tag{A.13}$$

where $\xi_j = e^{ay}\tilde{\xi}_j$ *and* $\eta_j = e^{-ay}\tilde{\eta}_j$ *for* $j = 1, 2$, *i.e.,*

$$\xi_1 = e^{ay}\partial_y u_c, \qquad \xi_2 = e^{ay}\partial_c u_c, \tag{A.14}$$

$$\eta_1 = e^{-ay}\biggl(\theta_1 \int_{-\infty}^{y} \partial_c u_c\, dx + \theta_2 u_c\biggr), \qquad \eta_2 = e^{-ay}\theta_3 u_c, \tag{A.15}$$

where θ_1, θ_2, *and* θ_3 *are as in* (A.5). *In addition, the* ξ_j *and* η_k *are biorthogonal, with* $\langle \xi_j, \eta_k\rangle = \delta_{jk}$ *for* $j, k = 1, 2$. *Thus, the spectral projection* P *for* A_a, *associated with the eigenvalue* $\lambda = 0$, *and the complementary spectral projection* Q, *are given by*

$$Pw = \sum_{k=1}^{2} \langle w, \eta_k\rangle \xi_k, \qquad Qw = (I - P)w = w - \sum_{k=1}^{2} \langle w, \eta_k\rangle \xi_k, \tag{A.16}$$

for $w \in L^2$. *These projections satisfy* $PA_a w = A_a Pw$, $QA_a w = A_a Qw$, *for* $w \in \operatorname{dom} A_a$.

A.2 Decay of smoothing estimates

After the substitution

$$w(y, t) = e^{ay}v(y, t), \quad a > 0, \tag{A.17}$$

the linearized undamped evolution equation (1.1) becomes

$$\partial_t w = A_a w \quad \text{with } A_a = e^{ay}\partial_y L_c e^{-ay}. \tag{A.18}$$

Denote $A_a = A_a^0 + (\partial_y - a)u_c$ with

$$A_a^0 = (\partial_y - a)\bigl(-(\partial_y - a)^2 + c\bigr) = -\partial_y^3 + 3a\partial_y^2 + \bigl(c - 3a^2\bigr)\partial_y - a\bigl(c - a^2\bigr). \tag{A.19}$$

Since u_c exponentially decays to zero as $|y| \to \infty$, the coefficients in (A.18) converge to those of the free evolution equation

$$\partial_t w = A_a^0 w. \tag{A.20}$$

Using the Fourier transform, one obtains:

Proposition A.3 (Proposition 4.1 Ref. [9]) *For any integer* $n \geq 0$, *and* $0 < a < \sqrt{c/3}$, *there exists* $C = C(n, a)$ *such that, for any* $w \in L^2$ *and for all* $t > 0$,

$$\bigl\|\partial_y^n e^{A_a^0 t} w\bigr\|_{L^2} \leq Ct^{-n/2}e^{-a(c-a^2)t}\|w\|_{L^2}. \tag{A.21}$$

For the semigroup $e^{A_a t}$, by restraint on the invariant subspace range Q (see (A.16)) complementary to the generalized kernel of A_a, a decay and smoothing estimate is also valid:

Lemma A.2 (Theorem 4.2 Ref. [9]) *Let the assumptions of Proposition A.2 hold. Then, A_a is the generator of a C^0 semigroup on H^s for any real s, and, for any $b > 0$ such that the L^2-spectrum $\sigma(A_a) \subset \{\lambda | \operatorname{Re}\lambda < -b\} \cup \{0\}$, there exists C such that for all $w \in L^2$ and $t > 0$,*

$$\left\| e^{A_a t} Q w \right\|_{H^1} \leq C t^{-1/2} e^{-bt} \|w\|_{L^2}. \tag{A.22}$$

Remark A.1 The smoothing-decay estimate (A.22) will be used in the proofs of Theorem 1.1 and Theorem 1.2. Also, Lemma A.2 implies that for $0 < a < \sqrt{c/3}$, A_a has no eigenvalues in the open left half-plane. Therefore, $-b$, the exponential rate of local energy decay, can be taken to satisfy $-a(c - a^2) < -b \leq 0$.

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